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Resonance contributions to finite-energy sum rules*

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A general formula is derived for the contribution of an s-channel resonance to finite-energy sum rules for t-channel helicity amplitudes. All amplitudes of the form $R' + 3 \to R'' + 4$ are considered, where particles 3 and 4 have arbitrary spin and R', R'' are Reggeons or particles. Guided by the requirement that the nucleon and $\Delta(1232)$ contributions to the zeroth-moment sum rules should cancel, the formula is brought into a simple and transparent form. It is shown that the algebraic expressions have a natural interpretation in terms of spin diagrams. An application of the sum rules yields predictions for the relative magnitude and helicity structure of all couplings $R34$, for all Reggeons $R = \pi\text{-}B$, $\rho\text{-}A_2, A_1$ and particles $3,4 = N(938)$ or $\Delta(1232)$.

I. INTRODUCTION

Experience has shown that s-channel resonances are highly correlated to the high-energy behavior of hadronic amplitudes. Technically this is expressed as a semilocal saturation of finite energy sum rules (FESR's). The FESR's have traditionally[1] been used mainly to constrain the high-energy Regge parameters in reactions for which phase-shift analyses exist at low energy. However, the FESR's also impose restrictions on the resonances themselves and on their couplings. This becomes apparent when one considers that the given set of resonances must conform to duality in many different amplitudes. For example, one may write down FESR's for amplitudes where the external particles are resonances or Reggeons. Although such amplitudes are in general not directly measurable, duality may enforce restrictions on the couplings of the resonances that contribute to the FESR. These couplings may then be experimentally tested in some other reaction. Thus it would seem that a systematic investigation of how the FESR's are satisfied for many different amplitudes could lead to a better understanding of duality phenomena. It should also provide us with a number of experimentally testable predictions for the resonance couplings.

An example of how duality can be used to determine resonance couplings is provided by a study[2] of the $\pi\Delta \to \pi\Delta$ charge-exchange reaction. From the requirement that the prominent resonance contributions vanish at $t \simeq -0.6$ it was, e.g., found that the $\Delta(1232)$ produced through the decay $\Delta\frac{7}{2}^+(1950) \to \pi\Delta$ should have helicity $\pm\frac{3}{2}$. This was later strikingly confirmed by the data.[3]

The work of Gell *et al.*[2] also illustrates the obvious technical difficulties associated with a study of duality in amplitudes having high-spin external particles. A given s-channel resonance, in general, has several kinematically allowed couplings to the initial and final states. Furthermore, the contribution of the resonance to specific t-channel helicity amplitudes is expressed through a complicated s-t helicity crossing matrix. If there does exist a simple duality connection between s-channel resonances and t-channel Regge exchange one might guess that it can only hold for a special type of resonance coupling. The challenge is thus to understand whether semilocal duality *can* be valid for general amplitudes, and to develop a formalism which makes transparent the regularities in the resonance contributions to t-channel amplitudes.

In studying the duality properties of general amplitudes it is advantageous to consider, in addition to ordinary quasi-two-body amplitudes, amplitudes with external Reggeons. Let us emphasize from the beginning that we shall never need to specify what we mean by a Reggeon "state"—our Reggeon amplitudes are always defined through appropriate high-energy limits of multiparticle amplitudes. Figure 1(a) shows the general type of Reggeon amplitude with which we shall be dealing in this paper. An s-channel resonance contribution to the discontinuity of this amplitude is shown in Fig. 1(b). It can clearly be expressed in terms of the amplitudes for the (quasi-two-body) processes $2+3 \to 1+s$ and $5+4 \to 6+s$.

The analytic structure of the six-point amplitude in the high-energy limit of Fig. 1(a) has been studied, using both the dual resonance (B_6) model and a general class of Feynman diagrams. It has been shown that ordinary FESR's can be derived both in the forward limit[4] (corresponding to inclusive cross sections) and in the more general case of nonvanishing momentum transfers.[5,6] The analyticity of the Reggeon amplitude is, however, equivalent to that of ordinary four-point ampli-

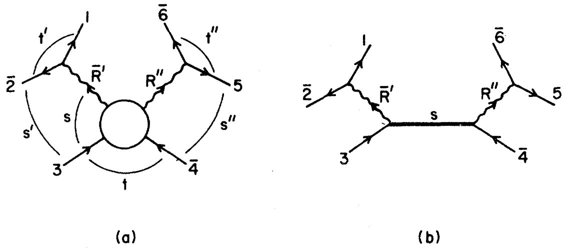


FIG. 1. (a) Reggeized six-point amplitude; (b) s-channel resonance contribution to discontinuity of the six-point amplitude.

tudes only when certain subsidiary conditions are satisfied, which specify the value of the azimuthal (Toller) angles of particles 1, 2 and 5, 6 in Fig. 1. These conditions are automatically satisfied in the forward limit, but must be explicitly imposed in general. In the examples considered in this paper we shall clearly see how the conditions simplify the expressions for the resonance contributions, and make the dual structure of the Reggeon amplitude analogous to that of particle amplitudes.

From the theoretical (FESR) point of view, there is thus every reason to include Reggeon amplitudes in a study of duality in general amplitudes. This conclusion is further supported by the results of a considerable number of phenomenological applications of the Reggeon FESR's to multiparticle data.[7] All these studies indicate that the FESR's are satisfied in the same semilocal sense for Reggeon amplitudes as they are in ordinary meson-nucleon scattering. The diversity of phenomena is, of course, much richer for Reggeon amplitudes, since more quantum numbers and kinematic variables are available.

Once we do demand that Reggeon amplitudes be dual, the constraints on the resonance couplings become much stronger. This is well illustrated by the work[6] on the $N(938)$ and $\Delta(1232)$ contributions to FESR's for the $\pi N \to \pi N$, $\rho N \to \pi N$, and $\rho N \to \rho N$ amplitudes (where ρ is a Reggeon). Based on fairly general arguments, it was shown that the N and Δ contributions must approximately cancel in zeroth-moment FESR's. This requirement led directly to a constraint on the ρNN coupling: It must be dominantly helicity flip. Thus, by considering only the lowest-mass contributions in Reggeon FESR's, one already derives a result which has important implications for the higher-mass contributions to, e.g., $\pi N \to \pi N$ FESR's: The $I_t = 1$ helicity-flip amplitude ($\nu B^{(-)}$) built up from the resonances must be much larger than the helicity-nonflip amplitudes ($A'^{(-)}$).

The calculation[6] of the N and Δ contributions also gives us a good example of how, for certain values of the couplings, the s-channel resonance contribution to t-channel helicity amplitudes simplifies. It is by no means obvious that there exists any set of ρNN and $\rho N\Delta$ couplings for which the N and Δ contributions cancel in all zeroth-moment FESR's for the amplitudes $\rho N \to \pi N$ and $\rho N \to \rho N$. In fact, such a requirement imposes many more conditions than there are adjustable couplings. In the calculation of Ref. 6, where invariant amplitudes and couplings were used to evaluate the contributions, the solution was made possible by a large number of seemingly accidental cancellations. It became apparent that the N-Δ cancellation mechanism could provide an excellent testing ground for developing a formalism in which the existence of couplings that satisfy the duality constraints can be more easily understood.

The primary aim of this paper is to derive an expression for the s-channel resonance contributions to t-channel helicity amplitudes that makes it transparent how semilocal duality can be satisfied. We consider all amplitudes of the type shown in Fig. 1. Particles 3 and 4, as well as the s-channel resonance s, can be mesons or baryons of arbitrary spin and isospin. R' and R'' denote any meson trajectory. By factorization, the spins of particles 1, 2 and 5, 6 are unimportant; for simplicity we shall assume that these particles are spinless. Any of the Reggeons R', R'' can be replaced by a π (or K) meson by going to the particle pole on the corresponding trajectory. Thus our formulas will be applicable also to the familiar pseudoscalar meson-baryon amplitudes.

The formula we derive will be completely general; no approximations or assumptions of a dynamical nature will be made. Since we want to treat arbitrary spins, it is clear that we have to use the helicity formalism.[8] Note, however, that we also have dynamical reasons to choose the helicity basis for the couplings. The solution to the N-Δ cancellation problem[6] discussed above was a simple helicity structure (flip) at the ρNN vertex. Similarly, in the $\Delta\frac{7}{2}^+(1950) \to \pi\Delta$ decay only a single helicity ($\pm\frac{3}{2}$) of the Δ contributes.[2]

As we already indicated above, we shall consider FESR's for t-channel, rather than s-channel, c.m. helicity amplitudes. Again, this is in accordance with standard practice. The kinematic singularities in s of the t-channel amplitudes are easily removed. Furthermore, for couplings such as πNN where only one t-channel helicity coupling is kinematically allowed, there are several nonvanishing s-channel couplings (as specified by the crossing matrix). Finally, it is only for t-channel amplitudes that N, Δ contributions can be made to cancel[6] in all zeroth-moment FESR's.

The over-all framework of our formalism can thus be fixed on quite general grounds. The derivation of our formula would hardly been possible, however, without the explicit test case of N-Δ cancellation. A central requirement on our expressions was that they should simply and naturally explain how the cancellation can simultaneously take place in so many amplitudes. This is indeed accomplished by our final formulas. The important point to note here is that we achieved this within the general framework outlined above. The formulas can therefore readily be applied to other processes and, we hope, will prove equally useful for describing the corresponding duality phenomena.

The derivation of our formula for an s-channel resonance contribution to a t-channel helicity amplitude is given in Sec. II. Because of the general case we consider, the expressions are sometimes rather lengthy. Fortunately, however, the derivation has a very simple diagrammatical interpretation. In fact, one could derive the final result (possibly up to a phase factor) directly in terms of the spin diagrams. It is also satisfying that spin and isospin are treated in a completely analogous manner.

In Sec. III we apply our formula to study the N-Δ cancellation. We extend the analysis of Ref. 6 by considering the complete set of amplitudes for which the cancellation may be expected to occur. Thus R', R'' in Fig. 1 can be any $I=1$ meson trajectory: ρ-A_2, π-B, or A_1. The external particles 3, 4 can be NN, $N\Delta$, or $\Delta\Delta$. All together we therefore study a total of 18 reactions in terms of 212 crossing-odd helicity amplitudes. Because of the efficiency of our method this requires only a little more effort than counting the number of amplitudes. We find that the cancellation is indeed a very general phenomenon. Within our approximation (we neglect the N-Δ mass difference and momentum transfers compared to the baryon masses) it can be imposed on all amplitudes except those where both external baryons are Δ's, in which case there is only a partial cancellation. The constraints lead to definite predictions for the Reggeon couplings, several of which can be compared to experiment. This and other aspects of our work are discussed in Sec. IV.

In deriving our formula we make extensive use of the symmetry, reality, and crossing properties of helicity amplitudes. We found it necessary to make a comprehensive study of these. The discussions given in the literature, in general, address themselves only to a part of the relations we need, and often use different conventions. We studied the symmetries by investigating the properties of effective Lagrangians, expressed in terms of the $(2j+1)$-component fields constructed by Weinberg.[9] Appendix A gives an account of our results. In Appendix B we study the kinematic singularities of the helicity amplitudes that are relevant for our FESR applications. The singularities of the Reggeon (and, more generally, multiparticle) amplitudes differ in some interesting respects from those of standard two-body amplitudes. Finally, in Appendix C we give the relation between the Weinberg fields for spin $\frac{1}{2}$ and 1 and the more commonly used spin-$\frac{1}{2}$ Dirac field and the vector field. We also express some familiar Lagrangians in terms of the Weinberg fields. This makes it possible to translate our discussion in Appendix A to a more standard language (in the special case of low spin).

II. GENERAL FORMULA FOR RESONANCE CONTRIBUTION

A. Wigner rotations and velocity diagrams

The contribution of a particular resonance to the discontinuity of the six-point amplitude in the variable $s\equiv(p_3-p_1-p_2)^2$ (Fig. 1) will be denoted by $\mathcal{T}$. The purpose of this section will be to treat the kinematics, spin, and isospin of $\mathcal{T}$ in a way which differs as little as possible from one set of spins and isospins to another. Moreover, our treatment of spin and isospin will be closely related to each other. As discussed in the Introduction, the derivation of our formula for a resonance contribution $\mathcal{T}$ is guided both by general consistency arguments and by the requirement that it should give a simple explanation of the N,Δ cancellation.[6] In order to make these arguments clear, we shall start by giving a general outline of our approach.

In the high-energy Regge limit of the six-point amplitude shown in Fig. 1, where $s'=(p_3-p_2)^2$ and $s''=(p_5-p_4)^2\to\infty$, a given Regge-pole contribution is expected to factorize in a manner described by Fig. 1. Thus we can think of the lower part of this diagram as describing the $2\to 2$ process Reggeon (R') + particle (3) $\to$ Reggeon (R'') + particle (4). (The angles describing the azimuthal orientation of the momenta 1, 2, 5, and 6 with respect to the 3-s-4 reaction plane require further discussion, to be given later in this section.) As is commonly done for $2\to 2$ amplitudes, we shall impose the FESR constraint on t-channel helicity amplitudes. We therefore consider the s-channel resonance contributions [Fig. 1(b)] to the discontinuity of the six-point helicity amplitudes $\mathcal{T}^{(t)}_{\lambda_3\lambda_4}$, where the helicities of particles 3 and 4 are defined in the t-channel c.m., with $t=(p_3+p_4)^2$. Particles 1, 2, 5, and 6 serve merely as the source of the Reggeons and will always be assumed spinless. (Baryon trajectories will not be considered here.)

The aim of our approach is to obtain dynamical information about the couplings at the 3-s-R' and s-4-R'' Reggeon vertices by imposing FESR constraints on the six-point amplitude. Such couplings are defined by the factorized 4-point amplitudes for $2+3\to 1+s$ and $4+5\to 6+s$. The consistent use of t-channel helicity amplitudes for describing $2\to 2$ processes would therefore dictate that the Reggeon vertex 3-s-R' be described in terms of helicity couplings defined in the c.m. of the process $3+\bar{s}\to 1+\bar{2}$, i.e., in the t'-channel c.m. frame, where $t'=(p_1+p_2)^2$. Similarly, the s-4-R'' vertex should be expressed in terms of t''-channel c.m. helicity couplings, where $t''=(p_5+p_6)^2$.

The helicity states of particles 3 and 4 are initially defined in the over-all t-channel c.m. Since the helicity label of the resonance s is summed over, its axis of quantization is immaterial. For the present discussion let us take the states of s to be t-channel c.m. helicity states. The resonance contribution $\mathcal{T}^{(t)}$, written in terms of R' and R'' helicity couplings in the t' and t'' c.m. frames, respectively, will therefore involve Wigner rotations[10] corresponding to boosts from the t-channel c.m. to the t'-channel c.m. and from the t-channel c.m. to the t''-channel c.m. All the relevant Wigner rotation angles can be collected on a single velocity diagram shown in Fig. 2, which is identical to the diagram[10] often used to discuss crossing properties of 4-point amplitudes. On this diagram each point represents a frame, each line represents a boost (in general, a complex boost defined by appropriate analytic continuation of the boost parameter), and all angles can be computed in terms of invariants by using the rules of hyperbolic geometry. The Wigner rotation induced on the wave function of a particle by a given boost is the angle on the velocity diagram which is subtended by the boost in the rest frame of that particle. From Fig. 2 it can be seen that, in boosting the four-point amplitude for $3+\bar{s}\to 1+\bar{2}$ from the t-channel c.m. (T) to the t'-channel c.m. (12), the wave functions of particles 3 and $\bar{s}$ acquire rotations of ω_3 and θ', respectively. Similarly, the wave functions of particles $\bar{4}$ and s in the amplitude for $\bar{4}+s\to 5+\bar{6}$ are rotated by ω_4 and θ'', respectively, in boosting from T to 56. Note that the angles ω_3 and ω_4 are just the familiar crossing angles for going from the s-channel to the t-channel center of mass of the Reggeon particle amplitude, and that the difference $\theta'-\theta''$ is the s-channel c.m. scattering angle θ_s.

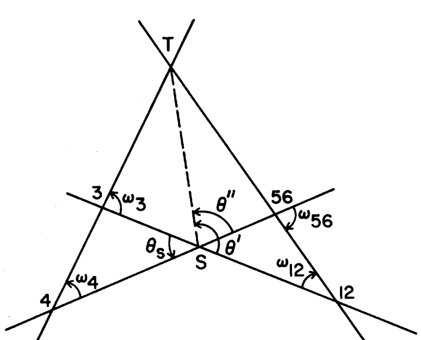


FIG. 2. s-channel velocity diagram.

It is instructive to study the angles of Fig. 2 in the kinematic regime relevant to the N-Δ cancellation mentioned in the Introduction. Thus, consider $m_3=m_4=m_N$ (nucleon mass) and $\sqrt{s}=m_N$ or $m_\Delta\approx m_N$, and suppose that the momentum transfers t, t', and t'' are small compared to the nucleon mass. Under these conditions, the triangle 3-4-S on the velocity diagram, Fig. 2, becomes small in the sense that all three of its sides represent boost velocities $\ll 1$, i.e., particles 3 and 4 are both moving slowly in the s-channel c.m. The "hyperbolic excess" of this triangle (analogous to the spherical excess of a triangle inscribed on the surface of a sphere) is therefore vanishingly small, and a certain relationship obtains between ω_3, ω_4, and θ_s, namely $\theta_s+(\pi-\omega_3)+\omega_4=\pi$ or $\omega_3-\omega_4=\theta_s$. In addition, because $m_3=m_4=m_N$ and t', t'' are small, the velocity of the boost ST tends to infinity.[10] If we again neglect the curvature over the short distances $s3$ and $s4$ in Fig. 2, this implies $\theta'=\omega_3$ and $\theta''=\omega_4$. Thus the two Wigner rotation angles for the 3-s-R' vertex are equal, as are the rotations acting on the s-4-R'' vertex. This motivates us to define certain irreducible combinations of the t'- and t''-channel helicity couplings at the Reggeon vertices which lead to a Clebsch-Gordan reduction of the two rotation matrices acting on each vertex. Again we are led by consistency to define similar irreducible combinations of the over-all t-channel helicity amplitudes $\mathcal{T}$.

The procedure we use to define irreducible amplitudes resembles that of an LS coupling scheme where one defines channel spins by vector addition of the spins of the particles in the initial or final state of a given channel. However, the representation labels which characterize our irreducible amplitudes (to be called l, l', and l'' for the over-all amplitude $\mathcal{T}$, the 3-s-R' vertex, and the s-4-R'' vertex, respectively) are *not* channel spins. The representation label we define for, say, the 3-s-R' vertex (l') is, in fact, related to the spins of particles 3 and s by vector subtraction, i.e., l' describes the angular momentum which is added to the spin of particle 3 to give the spin of particle s. For example, the label for the $\pi N\Delta$ vertex (which has only one allowed coupling for a real pion) is just the orbital angular momentum in the decay $\Delta \to N\pi$, which must be P wave. The $\pi N\Delta$ vertex is therefore pure $l = 1$ by our definition. For the same reason, the πNN vertex is also $l = 1$. A key to the aforementioned cancellation conspiracy among the π, ρ, N, and Δ is that the correct couplings of the ρ Reggeon which satisfy all FESR constraints (pure flip for ρNN and Stodolsky-Sakurai $M1$ coupling for $\rho N\Delta$) are also pure $l = 1$.

The final step in our treatment of spin is to write a resonance contribution to the over-all irreducible amplitudes in terms of the irreducible vertices and observe that a certain triangular combination of Clebsch-Gordan coefficients is proportional to a 6-j symbol containing l, l', l'', and the spins of particles 3, 4 and s. This provides a pleasant simplification of the formula, and may in itself be seen as a motivation for our choice of amplitudes. It should nevertheless be stated that our motivation for this choice comes primarily from experience with the N and Δ, and any more general physical reasons are unclear at present.

The treatment of the isospin of $\mathcal{T}$ is straightforward except for the ubiquitous problem of phase conventions for crossing, which is discussed in Appendix A. We consider six-point amplitudes with definite isospin in the t' and t'' channels (the isospin of the Reggeons, to be called I' and I''). A given resonance contribution also has a definite isospin in the s channel. It is related to amplitudes with definite isospin I in the t channel by an isospin crossing matrix or, equivalently,[11] a 6-j symbol, containing I, I', I'', and the isospins of particles 3, 4 and s. Thus, the treatments of isospin and spin bear a considerable resemblance to each other.

B. The azimuthal angles

It has been shown previously[5,6] that the FESR for the Reggeon process $R' + 3 \to R'' + 4$ (Fig. 1) is only valid when a certain kinematic restriction is satisfied by the momenta of particles 1, 2, 5, and 6 in the six-point amplitude. The purpose of this subsection is to translate this kinematic condition, which was originally stated in terms of invariants, into a restriction on the angles ϕ' and ϕ'', where ϕ' is the azimuthal angle of particles 1 and 2 with respect to the 3-s-4 reaction plane (taken to be the x-z plane) in the t' center of mass, and ϕ'' is a similarly defined quantity in the t''-channel c.m. The dependence of the 4-point amplitudes $3+\bar{s} \to 1+\bar{2}$ and $\bar{4}+s \to 5+\bar{6}$ on these azimuthal variables can then be separated out trivially.

To state the restriction on the six-point amplitude necessary for the validity of the FESR, let us define s_{ij} and s_{ijk} to be the invariant mass in the ij or ijk channel in Fig. 1, e.g., $s_{156} = (p_1+p_5+p_6)^2$, $s_{23} = (p_3-p_2)^2 \equiv s'$, etc. The $R' + 3 \to R'' + 4$ amplitude will be FESR analytic in the Regge limit s_{23}, $s_{45} \to \infty$ if the ratios s_{156}/s_{23}, s_{126}/s_{45}, and $s_{16}/s_{23}s_{45}$ are all vanishingly small. The fact that these conditions are compatible with the four dimensionality of space-time (which imposes one condition on the nine s_{ij} variables) has been verified[6] by explicitly constructing a set of four-momenta in the t-channel c.m. which satisfy the conditions:

$$
\begin{aligned}
p_3 &= (\tfrac{1}{2}\sqrt{t}\,, 0, 0, p),\\
p_4 &= (\tfrac{1}{2}\sqrt{t}\,, 0, 0, -p),\\
p_{12} &\equiv p_1+p_2 = \left(\frac{t+t'-t''}{2\sqrt{t}}, q\sin\theta_t, 0, q\cos\theta_t\right),\\
p_{56} &\equiv p_5+p_6 = \left(\frac{t-t'+t''}{2\sqrt{t}}, -q\sin\theta_t, 0, -q\cos\theta_t\right),\\
p_0 &\equiv \frac{p_1}{s_{23}} = \frac{-p_2}{s_{23}} = \frac{p_5}{s_{45}} = \frac{-p_6}{s_{45}}\\
&= \frac{1}{2p\sin\theta_t}(0, \cos\theta_t, -i, -\sin\theta_t) + O\left(\frac{1}{s_{23}}, \frac{1}{s_{45}}\right).
\end{aligned} \tag{2.1}
$$

Note that these momenta are all defined in the t-channel c.m. To calculate the azimuthal angle ϕ', we must boost p_1 and p_2, hence p_0, to the t' c.m. This can, of course, be done by brute force, but there is a much simpler method which makes use of an intriguing property of the vector p_0. Consider a hypothetical spin-one particle with mass m and four-momentum $k_\mu = (E, k\sin\theta, 0, k\cos\theta)$, and define the polarization vectors for the three helicity states of this particle in the usual way:

$$
\begin{aligned}
\epsilon_\mu(k, \pm 1) &= \mp\frac{1}{\sqrt{2}}(0, \cos\theta, \pm i, -\sin\theta),\\
\epsilon_\mu(k, 0) &= \frac{1}{m}(k, E\sin\theta, 0, E\cos\theta).
\end{aligned} \tag{2.2}
$$

It is now seen that the vector p_0 can be written

$$(p_0)_\mu = \frac{1}{\sqrt{2}p\sin\theta_t}\epsilon_\mu(p_{12}, -1). \tag{2.3}$$

The physical significance of this fact is unclear and may deserve further investigation. We use it here only as a mathematical convenience which allows us to boost p_0 from one frame to another with ease, because of the property of the polarization vectors

$$\Lambda_\mu{}^\nu \epsilon_\nu(k, \lambda) = \epsilon_\mu(\Lambda k, \lambda') D^1_{\lambda'\lambda}(R_W(k, \Lambda)). \tag{2.4}$$

(A sum over repeated helicity indices is implied.)

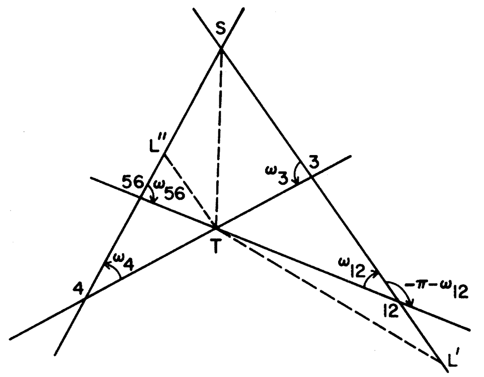


FIG. 3. t-channel velocity diagram.

Here Λ is some arbitrary Lorentz transformation, D^1 is the usual $j=1$ rotation matrix, and $R_W(k,\Lambda)$ is the Wigner rotation[10] induced by the transformation Λ on a particle of four-momentum k.

If we let Λ be the transformation from T to L' on the t-channel velocity diagram in Fig. 3, then in L' $\vec{p}_{12}$ will be parallel to $\vec{p}_3$, which we take to define the $+z$ axis. It can also be seen from Fig. 3 that the Wigner rotation $R_W(p_{12},\Lambda)$ is simply related to the ordinary S-T crossing angle ω_{12} of particle 12. The azimuthal angle ϕ' is given by

$$\begin{aligned}\tan\phi' &= \frac{(\Lambda p_0)_y}{(\Lambda p_0)_x}\\ &= \frac{\epsilon_y(\Lambda p_{12},\lambda)d^1_{\lambda,-1}(-\pi-\omega_{12})}{\epsilon_x(\Lambda p_{12},\lambda)d^1_{\lambda,-1}(-\pi-\omega_{12})}\\ &= \frac{i}{\cos\omega_{12}}\,. \end{aligned}\tag{2.5a}$$

Equation (2.5a) determines ϕ' up to an additive factor of π, which we fix by the convention

$$\cos\phi' = +i\cot\omega_{12},\qquad \sin\phi' = -\frac{1}{\sin\omega_{12}}\,. \tag{2.6}$$

Since the crossing angle ω_{12} is real in the s- (and t-) channel physical region, ϕ' must have the form

$$\phi' = \frac{\pi}{2} + i\psi', \tag{2.7}$$

where ψ' is real. It follows from Eq. (2.6) that

$$e^{\psi'} = -\frac{1+\cos\omega_{12}}{\sin\omega_{12}}\,. \tag{2.8}$$

A similar method can be used to find the azimuthal angle ϕ'' of particles 5,6 in the t'' c.m. The boost Λ now takes the T frame into L'' in Fig. 3. The result is

$$\tan\phi'' = \frac{i}{\cos\omega_{56}}\,. \tag{2.5b}$$

Consistency with the convention (2.6) requires

$$\phi'' = \frac{\pi}{2} - i\psi'', \tag{2.9}$$

where ψ'' is real and satisfies

$$e^{\psi''} = -\frac{1-\cos\omega_{56}}{\sin\omega_{56}}\,. \tag{2.10}$$

The Wigner angles in Eqs. (2.8) and (2.10), as well as all other angles on Figs. 2 and 3, are calculated from the law of cosines for hyperbolic geometry. For our purposes, this law is most conveniently stated in terms of invariants by noting that each of the frames of interest is characterized by the fact that the spatial components of a particular four-momentum vector vanish in that frame (i.e., it is the rest frame of a certain particle or channel). In general, if frames 1, 2, and 3 are the frames in which the momenta k_1, k_2, and k_3, respectively, are purely timelike, then the Wigner angle subtended at frame 1 by the boost from frame 2 to frame 3 is given by

$$\cos\omega_1 = \frac{4[(k_1\cdot k_2)(k_1\cdot k_3) - k_1^2(k_2\cdot k_3)]}{\lambda^{1/2}((k_1+k_2)^2, k_1^2, k_2^2)\lambda^{1/2}((k_1+k_3)^2, k_1^2, k_3^2)}\,, \tag{2.11}$$

where

$$\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc\,. \tag{2.12}$$

For the angles ω_{12} and ω_{56} we obtain in this way

$$\begin{aligned}\cos\omega_{12} &= -\frac{(s+t'-m_3^2)(t+t'-t'')+2t'(m_3^2-m_4^2-t'+t'')}{\lambda^{1/2}(s,m_3^2,t')\lambda^{1/2}(t,t',t'')}\,, \quad \sin\omega_{12} = -\frac{2(t'\Phi)^{1/2}}{\lambda^{1/2}(s,m_3^2,t')\lambda^{1/2}(t,t',t'')}\,,\\ \cos\omega_{56} &= \frac{(s+t''-m_4^2)(t+t''-t')-2t''(m_3^2-m_4^2-t'+t'')}{\lambda^{1/2}(s,m_4^2,t'')\lambda^{1/2}(t,t',t'')}\,, \quad \sin\omega_{56} = -\frac{2(t''\Phi)^{1/2}}{\lambda^{1/2}(s,m_4^2,t'')\lambda^{1/2}(t,t',t'')}\,,\end{aligned}\tag{2.13}$$

where Φ is the usual Kibble function[10] that vanishes on the boundary of the physical regions.

C. Spin diagrams

The steps involved in the subsequent derivation can be greatly clarified by a diagrammatic approach to representation theory. It seems likely that the diagrammatic rules and manipulations to be described in this section could be formulated precisely and the entire derivation carried out diagrammatically, but we will not attempt to do so here. We will instead present a more conventional algebraic derivation and supplement the formulas with their diagrammatic interpretation.

Our convention[12] for the 3-j symbol in terms of the usual Clebsch-Gordan coefficient is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-\lambda_3}}{(2j_3+1)^{1/2}} \langle j_1, j_2; \lambda_1, \lambda_2 | j_3 - \lambda_3 \rangle. \tag{2.14}$$

The 3-j symbol, as depicted in Fig. 4(a), is a basic component of the spin diagrams.[13] The familiar $(2j+1)$-dimensional unitary matrix representation of the rotation group $D^j(R)$ and its specialization to rotations around the y axis $d^j(\omega)$ will be portrayed by a line with a coil wrapped around it, as shown in Fig. 4(b). In both of these symbols, the lines themselves carry representation labels j. An unadorned line, Fig. 4(c), can be thought of as the $2j+1$-dimensional unit matrix and used, for example, to connect the legs of two 3-j symbols.

We have found it convenient to employ Wigner's[12] covariant notation for spin indices, in which a helicity index is written down (up) according to whether it is covariant (contravariant). The meaning of covariance here is abstracted from the transformation properties of spin states and is best explained in context. The Lorentz transformation property of a single particle spin-j helicity ket state is described by the D^j matrix representation of the Wigner rotation. A bra helicity state transforms via the complex-conjugate representation. The helicity indices of ket and bra states are called covariant and contravariant, respectively. Accordingly, the "metric tensor" which converts a contravariant index into a covariant one is just the unitary matrix which transforms D^j into its complex conjugate, explicitly

$$C^j_{\lambda\sigma} = (-1)^{j+\lambda}\delta_{\lambda,-\sigma} = (-1)^{2j} C^j_{\sigma\lambda}, \tag{2.15a}$$

$$C_j^{\lambda\sigma} = (-1)^{j+\sigma}\delta_{\lambda,-\sigma} = (-1)^{2j} C_j^{\sigma\lambda}. \tag{2.15b}$$

(Here and elsewhere we avoid confusion due to the changing position of indices by always using lower case Latin letters $j, l, \ldots$ to denote the representation and using lower case Greek letters $\lambda, \mu, \nu, \sigma, \ldots$ for helicity indices.) The matrices $D^j_{\lambda\lambda'}$ themselves should be thought of as having one contravariant index λ and one covariant index λ', although in this case we will conform to the standard notation and write both indices in the down position. The phase in the definition of the 3-j symbol (2.14) is chosen so that all three of its indices are covariant, as exemplified by its fundamental invariance property

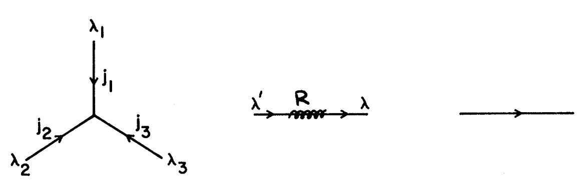
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} D^{j_1}_{\lambda_1\lambda_1'}(R) D^{j_2}_{\lambda_2\lambda_2'}(R) D^{j_3}_{\lambda_3\lambda_3'}(R) = \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{pmatrix}. \tag{2.16}$$

(A summation over repeated helicity indices is always implied.) This invariance is depicted in Fig. 5(a). 3-j symbols with one or more contravariant indices are obtained by applying the metric tensor (2.15) from the left, e.g.,

$$\begin{pmatrix} j_1 & j_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & j_3 \end{pmatrix} \equiv C^{\lambda_3\lambda_3'}_{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3' \end{pmatrix}. \tag{2.17}$$

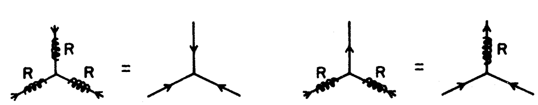
Another useful form of Eq. (2.16) is obtained by applying in inverse rotation to one of the free indices and using the unitarity of the D^j matrices,

$$\begin{pmatrix} \lambda_1 & j_2 & j_3 \\ j_1 & \lambda_2 & \lambda_3 \end{pmatrix} D^{j_2}_{\lambda_2\lambda_2'}(R) D^{j_3}_{\lambda_3\lambda_3'}(R) = D^{j_1}_{\lambda_1\lambda_1'}(R) \begin{pmatrix} \lambda_1' & j_2 & j_3 \\ j_1 & \lambda_2' & \lambda_3' \end{pmatrix}. \tag{2.18}$$



(a) (b) (c)

FIG. 4. (a) The 3-j symbol; (b) irreducible representation of a rotation, $D^j_{\lambda\lambda'}(R)$; (c) $(2j+1)$-dimensional unit matrix.



(a) (b)

FIG. 5. (a) Rotational invariance property of a 3-j symbol; (b) a convenient spin-diagram manipulation.

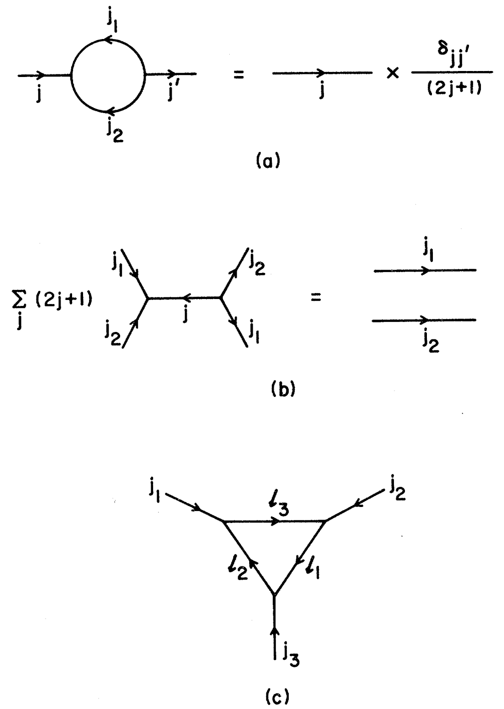


FIG. 6. (a) Orthogonality property of 3-j symbols; (b) completeness property of 3-j symbols. [Note: In Figs. 6a and 6b a factor (±1), depending on the representations involved, has been suppressed.] (c) a 6-j symbol times a 3-j symbol.

This equation, pictured in Fig. 5(b), expresses the fact that when a coil appears on two legs of a 3-j symbol it may be pushed through onto the third leg. The 3-j symbol also obeys two orthogonality relations,

$$\begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda' \\ j_1 & j_2 & j' \end{pmatrix} = \frac{\delta_{jj'}\delta_{\lambda\lambda'}}{2j+1} \tag{2.19}$$

shown in Fig. 6(a), and

$$\sum_j (2j+1)\begin{pmatrix} j_1 & j_2 & j \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix}\begin{pmatrix} \lambda_1' & \lambda_2' & \lambda \\ j_1 & j_2 & j \end{pmatrix} = \delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'} \tag{2.20}$$

shown in Fig. 6(b). Some familiar formulas can be derived by simple combinations of these diagrams. For example, by applying rotations to both sides of Fig. 6(b) and using the identity in Fig. 5(b), one obtains the Clebsch-Gordan series for reducing the direct product of two D matrices. Another familiar combination is the triangular contraction of three 3-j symbols shown in Fig. 6(c). By repeated use of Fig. 5(b), it is easy to see that Fig. 6(c) has the same rotational properties as the 3-j symbol and is thus proportional to it. The representation-dependent proportionality constant is just the 6-j symbol or Racah coefficient. Algebraically, we define the 6-j symbol by[12]

$$\begin{pmatrix} j_1 & l_2 & \sigma_3 \\ \lambda_1 & \sigma_2 & l_3 \end{pmatrix}\begin{pmatrix} \sigma_1 & j_2 & l_3 \\ l_1 & \lambda_2 & \sigma_3 \end{pmatrix}\begin{pmatrix} l_1 & \sigma_2 & j_3 \\ \sigma_1 & l_2 & \lambda_3 \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}. \tag{2.21}$$

D. Derivation of the formula

In the first part of this derivation we treat spin, and consequently suppress all isospin indices. We consider the six-point helicity amplitude (Fig. 1)

$$T^{(t)}_{\lambda_3\lambda_4} = \langle p_1; p_2; p_5; p_6 | T | p_3, \lambda_3; p_4, \lambda_4\rangle, \tag{2.22}$$

where all quantities are evaluated in the t-channel c.m. frame $\vec{p}_3 = -\vec{p}_4$. In accordance with the previous discussion, we have assumed that particles 1, 2, 5, and 6 are spinless. We do not include a "particle-2" phase[8] for the spinning particles.

In order to evaluate a resonance contribution to the discontinuity of the amplitude (2.22) in the variable $s = (p_3 - p_1 - p_2)^2$ we will use crossing symmetry to express (2.22) in terms of the s-channel process $2+3+6 \to 1+4+5$, for which the s-channel resonance appears as a term in the unitarity sum. The crossing properties of helicity amplitudes have been extensively investigated,[10] usually in the context of $2\to 2$ processes. The results of these investigations can be applied straightforwardly to the case we consider here. The crossing relation between the t-channel and s-channel c.m. amplitudes is thus [cf. Eq. (A18)]

$$T^{(t)}_{\lambda_3\lambda_4} = e^{i\pi j_4}(-1)^{j_4-\lambda_4} T^{(s)}_{\lambda_3'\lambda_4'} d^{j_3}_{\lambda_3'\lambda_3}(\omega_3)\, d^{j_4}_{\lambda_4'\lambda_4}(\omega_4). \tag{2.23}$$

The expressions for the crossing angles ω_3, ω_4 are standard.[10] Alternatively, the angles can be read off the velocity diagram in Fig. 2 by observing that the d matrices in Eq. (2.23) must rotate the spin quantization axes of the s-channel amplitude into those of the t-channel amplitude. Using Eq. (2.11) we get in this way

$$\cos\omega_3 = \frac{(s+m_3^2-t')(t+m_3^2-m_4^2) - 2m_3^2(m_3^2-m_4^2-t'+t'')}{\lambda^{1/2}(s,m_3^2,t')\lambda^{1/2}(t,m_3^2,m_4^2)},$$

$$\cos\omega_4 = -\frac{(s+m_4^2-t'')(t+m_4^2-m_3^2) + 2m_4^2(m_3^2-m_4^2-t'+t'')}{\lambda^{1/2}(s,m_4^2,t'')\lambda^{1/2}(t,m_3^2,m_4^2)} \tag{2.24}$$

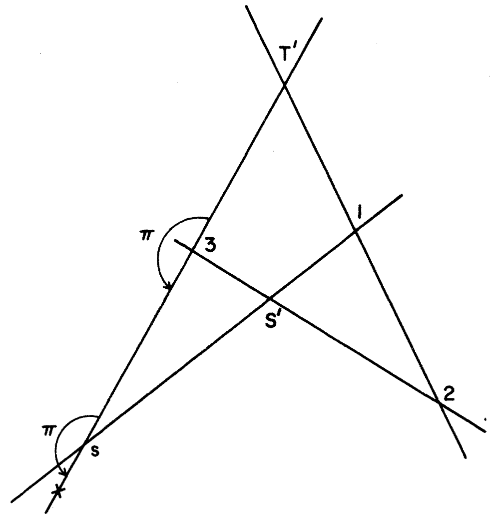


FIG. 7. Velocity diagram for the s' channel.

with $\sin\omega_3 > 0$ and $\sin\omega_4 > 0$.

Our next step is to write down a particular spin-j_s resonance contribution to the generalized unitarity sum[14] for $T^{(s)}$. Thus, we write

$$\mathrm{Disc}_s T^{(s)}_{\lambda_3\lambda_4} = (2\pi)\delta(s - M^{*2})\mathcal{T}^{(s)}_{\lambda_3\lambda_4} + \cdots, \tag{2.25}$$

where the ellipsis stands for other terms contributing to the discontinuity in s. Here M^* is the mass of the resonance, and $\mathcal{T}^{(s)}$ is given in terms of 4-point amplitudes by

$$\begin{aligned}\mathcal{T}^{(s)}_{\lambda_3\lambda_4} &= \langle q_4, \lambda_4; p_5 | T^\dagger | p_s, \lambda_s; q_6\rangle\langle p_s, \lambda_s; p_1 | T | p_3, \lambda_3; q_2\rangle \\ &= \langle p_s, \lambda_s; q_6 | T | q_4, \lambda_4; p_5\rangle^* \langle p_s, \lambda_s; p_1 | T | p_3, \lambda_3; q_2\rangle.\end{aligned} \tag{2.26}$$

In this expression $q_i = -p_i$, where p_i are the momenta in the t channel (Fig. 1). We have not yet specified the axis of quantization for the spin j_s of particle s. The s-channel c.m. frame is just the Gottfried-Jackson (GJ) frame for the process $2+3 \to 1+s$. Similarly, it can be seen from Fig. 2 that the s-channel c.m. frame rotated by an angle θ_s is the GJ frame for the process $5+4 \to 6+s$. Thus, we may write

$$\mathcal{T}^{(s)}_{\lambda_3\lambda_4} = T^{(\mathrm{GJ})*}_{\lambda_4\lambda'_s} d^{j_s}_{\lambda'_s\lambda_s}(-\theta_s) T^{(\mathrm{GJ})}_{\lambda_3\lambda_s}. \tag{2.27}$$

Finally, we want to relate the amplitudes in the GJ frame to the t'- and t''-channel c.m. amplitudes. The crossing relation for the amplitude $T_{\lambda_3\lambda_s}$ is again of the form (2.23); however, the crossing angles should now be read off the velocity diagram in Fig. 7. Since the spin of the resonance s is quantized along the direction of the incoming momentum p_3 in the GJ frame, the boost connects a frame X slightly below s in Fig. 7 to the t'-channel c.m. T'. The crossing angles for particles 3 and s are both seen to be equal to π. The crossing relation between the GJ amplitude and the t'-channel c.m. amplitude is thus

$$T^{\mathrm{GJ}}_{\lambda_3\lambda_s} = e^{-i\pi j_s}(-1)^{j_3+\lambda_3} T_{-\lambda_3,-\lambda_s}. \tag{2.28}$$

From now on, an unsuperscripted amplitude will represent the c.m. amplitude for either $3+\bar{s} \to 1+\bar{2}$, $4+\bar{s} \to \bar{5}+6$, or $3+\bar{4} \to 1+\bar{2}+5+\bar{6}$. (The subscripts on the helicity labels will always be sufficient to distinguish the three possiblilities.) Now, combining Eqs. (2.27) and (2.28) with Eqs. (2.23) and (2.25), we obtain the resonance contribution to the disconitnuity of the six-point t-c.m. amplitude in terms of the four-point t'-c.m. and t''-c.m. amplitudes,

$$\mathcal{T}_{\lambda_3-\lambda_4} = e^{i\pi j_4} d^{j_4}_{\mu_4\lambda_4}(\omega_4) T^*_{\mu_4-\mu_s} e^{-i\phi''(\mu_4+\mu_s)} d^{j_s}_{\mu_s\nu_s}(-\theta_s) T_{\nu_3-\nu_s} e^{i\phi'(\nu_3+\nu_s)} d^{j_3}_{\nu_3\mu_3}(\omega_3) C^{j_3}_{\mu_3\lambda_3}. \tag{2.29}$$

Here we have explicitly exhibited the dependence of the four-point amplitudes on the azimuthal angles ϕ' and ϕ'' defined in Eqs. (2.7)–(2.10). Note that, although the azimuthal angles needed for the validity of the FESR are complex, ϕ'' in (2.29) has not been complex-conjugated. This is because the unitarity relation (2.26) must first be expressed in terms of physical states (i.e., real angles ϕ', ϕ''), and then ana-

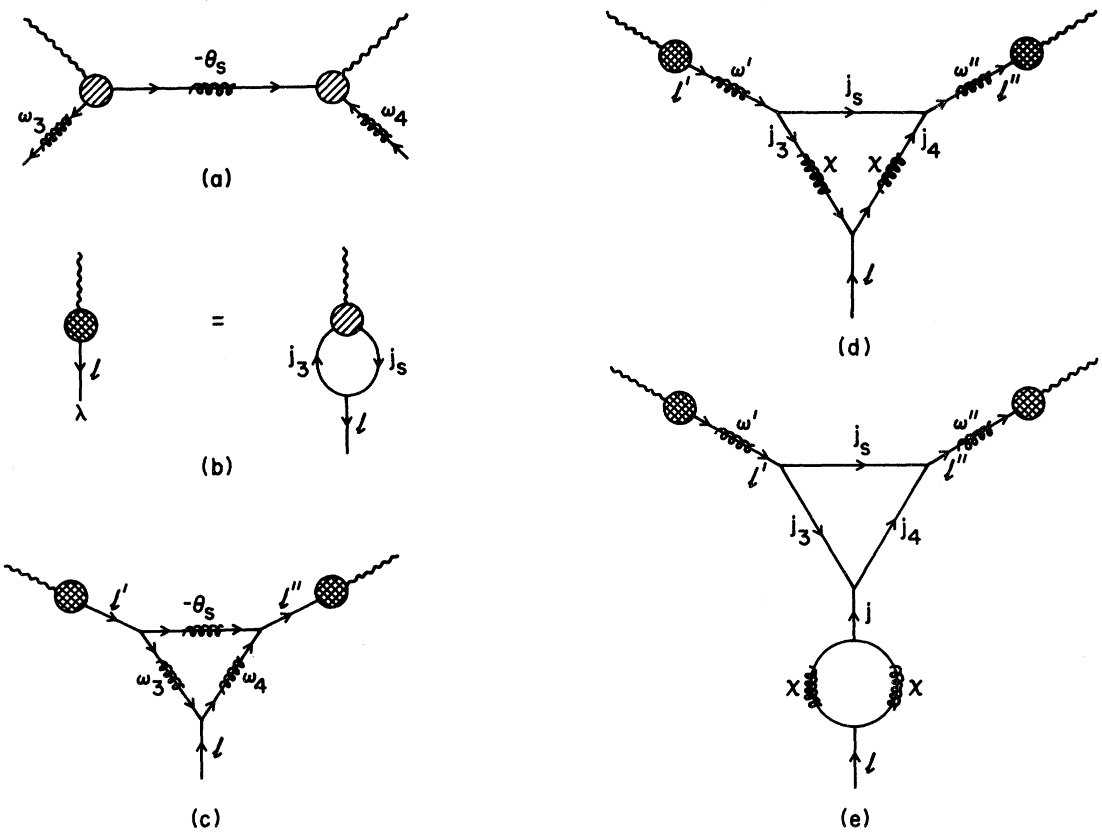


FIG. 8. Spin-diagram derivation of the s-channel resonance formula: (a) t-channel six-point amplitude in terms of t'- and t''-channel four-point amplitudes; (b) definition of reduced amplitudes; (c) reduced six-point amplitude in terms of reduced four-point amplitudes; (d) result of using Fig. 5b; (e) result of using Fig. 6b.

lytically continued to the unphysical region.

The expression (2.29) can be represented as a spin diagram by Fig. 8(a). Because all the rotations here are around the y-axis, it is not obvious which indices should be considered covariant and which contravariant. The arrows in Fig. 8(a) are drawn heuristically to indicate the subsequent development. Guided by our previous discussion regarding the magnitudes of the Wigner angles in the N, Δ case, we define the reduced amplitudes for $3+\overline{s}\rightarrow 1+\overline{2}$ by

$$T_{3s}(l',\lambda')=(2l'+1)\begin{pmatrix} j_3 & j_s & \lambda' \\ \lambda_3 & \lambda_s & l' \end{pmatrix} e^{i\pi\lambda'/2}T_{\lambda_3-\lambda_s} \tag{2.30}$$

with a similar set of amplitudes $T_{4s}(l'',\lambda'')$ to describe $4+\overline{s}\rightarrow\overline{5}+6$. This reduction is represented graphically by Fig. 8(b). The over-all six-point amplitude is reduced in an identical way:

$$\mathcal{T}(l,\lambda)=(2l+1)\begin{pmatrix} j_3 & j_4 & \lambda \\ \lambda_3 & \lambda_4 & l \end{pmatrix} e^{i\pi\lambda/2}\mathcal{T}_{\lambda_3-\lambda_4}. \tag{2.31}$$

The phase factors here and in Eq. (2.30) are not essential, but have been chosed for later convenience. Note that the parity relation for the reduced amplitudes is, according to Eq. (A28),

$$T_{3s}(l',\lambda')=\sigma'\eta_P^3\eta_P^s(-1)^{l'}T_{3s}(l',-\lambda'), \tag{2.32}$$

where σ' is the naturality of the Reggeon, and η_P^3, η_P^s are the parities of particles 3 and s.

Now, inverting Eq. (2.30) by the orthogonality relation Eq. (2.20) or Fig. 6(b), Eq. (2.29) is written in terms of reduced amplitudes as (noting that l, l', and l'' will always be integers)

$$\mathcal{T}(l,\lambda)=(2l+1)e^{i\pi j_4}e^{i\pi\lambda/2}\sum_{l',l''}\begin{pmatrix} j_3 & \lambda_4 & l \\ \lambda_3 & j_4 & \lambda\end{pmatrix}\begin{pmatrix} \mu_4 & \mu_s & l'' \\ j_4 & j_s & \lambda''\end{pmatrix}\begin{pmatrix} \nu_3 & \nu_s & l' \\ j_3 & j_s & \lambda'\end{pmatrix}$$
$$\times d^{j_4}_{\mu_4\lambda_4}(\omega_4)d^{j_s}_{\mu_s\nu_s}(-\theta_s)\,d^{j_3}_{\nu_3\lambda_3}(\omega_3)T_{3s}(l',\lambda')T^*_{4s}(l'',\lambda'')e^{-\psi'\lambda'-\psi''\lambda''}, \tag{2.33}$$

where ψ' and ψ'' are defined by Eqs. (2.7) and (2.9). The diagram for this expression is shown in Fig. 8(c).

Next we define a new set of angles ω', ω'', and χ by

$$\begin{aligned}\omega' &= \pi-\omega_3+\chi, \\ \omega'' &= \omega_4+\chi, \\ \omega'+\omega'' &= \pi-\theta_s.\end{aligned} \tag{2.34}$$

This particular choice of angles is dictated by the requirement of simplicity for the N-Δ case. Note that χ is just one half the hyperbolic excess of the triangle 3-4-S in Fig. 2, and is thus small when this triangle is nonrelativistic. In fact, this choice seems somewhat more auspicious because, for a momentum transfer t small compared to the external masses $m_3 \approx m_4$, one side (3-4) of the triangle is always short, and χ therefore never becomes very large in the s-channel physical region. In the equal-mass case $m_3=m_4=m$ [see Eq. (3.1) of Sec. III], $\sin\chi$ is of order $\sqrt{-t}/2m$ even as $s\to\infty$.

Replacing ω_3, ω_4, and θ_s in Eq. (2.33) by ω', ω'', and χ, it is seen that a rotation of π-ω' acts on the j_3 and j_s legs of the j_3-j_s-l' 3-j symbol in Fig. 8(c) and can thus be "pushed through" onto the l' leg by virtue of Eq. (2.18) [Fig. 5(b)]. Similarly, a rotation of ω'' can be pushed through the j_4-j_s-l'' symbol onto the l'' leg. This leaves only rotations by the hyperbolic excess χ remaining on the central triangle in Fig. 8(d) or, algebraically,

$$\mathcal{T}(l,\lambda)=(2l+1)e^{i\pi j_4}e^{i\pi\lambda/2}\sum_{l',l''}\begin{pmatrix} j_3 & \lambda_4 & l \\ \lambda_3 & j_4 & \lambda\end{pmatrix}\begin{pmatrix} \lambda_4' & \nu_s' & l'' \\ j_4 & j_s & \mu''\end{pmatrix}\begin{pmatrix} j_3 & j_s & l' \\ \lambda_3' & \nu_s' & \mu'\end{pmatrix}$$
$$\times d^{l'}_{\mu'\lambda'}(\omega')d^{l''}_{\lambda''\mu''}(\omega'')d^{j_4}_{\lambda_4\lambda_4'}(\chi)d^{j_3}_{\lambda_3'\lambda_3}(\chi)T_{3s}(l',\lambda')T^*_{4s}(l'',\lambda'')e^{-\lambda'\psi'-\lambda''\psi''}. \tag{2.35}$$

We wish to identify the central triangle in Fig. 8(d) as a 6-j symbol. By Eq. (2.21) [Fig. 6(c)], this is automatic when $\chi\to 0$. For the more general case, we are led to define the following functions of χ,

$$\mathcal{B}^{jl}_{\mu\lambda}(j_3,j_4;\chi)=\begin{pmatrix} j_3 & \lambda_4' & j \\ \lambda_3' & j_4 & \mu\end{pmatrix}\begin{pmatrix} \lambda_3 & j_4 & l \\ j_3 & \lambda_4 & \lambda\end{pmatrix}d^{j_3}_{\lambda_3'\lambda_3}(\chi)d^{j_4}_{\lambda_4\lambda_4'}(\chi)(-1)^{2j_3}. \tag{2.36}$$

This looks almost like the Clebsch-Gordan reduction of two d matrices but the indices are tied together in reverse order, and it therefore does not reduce to a single representation. We will extend the covariant-contravariant notation to include the functions (2.36). Thus for example,

$$\mathcal{B}^{\mu l}_{j\lambda}(j_3,j_4;\chi)=C^{\mu\nu}_j\mathcal{B}^{jl}_{\nu\lambda}(j_3,j_4;\chi), \tag{2.37}$$

etc. Note that when the angle χ is set to zero, we get

$$\mathcal{B}^{\mu l}_{j\lambda}(j_3,j_4;0)=\frac{\delta_{jl}\delta_{\mu\lambda}}{2j+1}. \tag{2.38}$$

Now, when a "completeness sum" of two 3-j symbols [Eq. (2.20) or Fig. 6(b)] is inserted in Eq. (2.35), the χ dependence takes the form (2.36), and the remaining three 3-j symbols form a 6-j symbol by Eq. (2.21). This corresponds to the graphical operation which takes us from Fig. 8(d) to Fig. 8(e). In this way, Eq. (2.35) becomes

$$\mathcal{T}(l,\lambda)=\sum_{j,l',l''}e^{i\pi j_4}(2l+1)(2j+1)e^{i\pi\lambda/2}(-1)^{j_s-j_3+j+l''}\begin{Bmatrix} l' & l'' & j \\ j_4 & j_3 & j_s\end{Bmatrix}\begin{pmatrix} \mu' & \mu'' & j \\ l' & l'' & \mu\end{pmatrix}$$
$$\times d^{l'}_{\mu'\lambda'}(\omega')d^{l''}_{\lambda''\mu''}(\omega'')\mathcal{B}^{\mu l}_{j\lambda}(j_3,j_4;\chi)T_{3s}(l',\lambda')T^*_{4s}(l'',\lambda'')e^{-\lambda'\psi'-\lambda''\psi''}. \tag{2.39}$$

We now consider the isospin dependence which has been suppressed in Eq. (2.39) and the preceding analysis. Our procedure will be to define an isospin-reduced six-point amplitude $\mathcal{T}(I,I',I'')$ in terms of the s-channel amplitude $\mathcal{T}^{(s)}_{a_1a_4a_5;a_2a_3a_6}$ for $2+3+6\to 1+4+5$ where a_i is the isospin index for particle i. [This is, of course, the same amplitude that appears, for example, in Eq. (2.26) except that now helicity indices are suppressed.]

$$\mathcal{T}(I,I',I'')\equiv(-1)^{2I_1+2I_4+2I_5}\mathcal{T}^{(s)}_{a_1a_4a_5;a_2a_3a_6}\begin{pmatrix} a_3 & I_4 & a \\ I_3 & a_4 & I\end{pmatrix}\begin{pmatrix} I_1 & a_2 & a' \\ a_1 & I_2 & I'\end{pmatrix}\begin{pmatrix} I_5 & a_6 & a'' \\ a_5 & I_6 & I''\end{pmatrix}\begin{pmatrix} I & I' & I'' \\ a & a' & a''\end{pmatrix}. \tag{2.40}$$

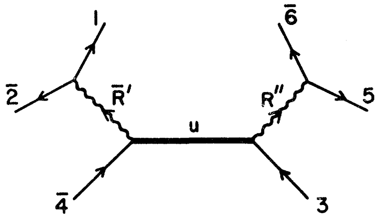


FIG. 9. u-channel resonance contribution to discontinuity of the six-point amplitude.

It is not difficult to show, by using the isospin crossing rule derived in Appendix A, that $\mathcal{T}(I,I',I'')$ is that combination of amplitudes which has definite isospin in the t, t', and t'' channels. The phase factor in (2.40) has been chosen so that when this equation is inverted, it expresses $\mathcal{T}^{(s)}$ as a sum over the channel isospins of $\mathcal{T}(I,I',I'')$ multiplied by 3-j symbols of appropriate covariance, without extraneous phase factors. In a similar way, we define the reduced four-point amplitudes for $2+3\to 1+s$ and $5+4\to 6+s$ as

$$T_{3s}(I')=(-1)^{2I_1+2I_3}\begin{pmatrix} I_1 & a_2 & a' \\ a_1 & I_2 & I' \end{pmatrix}\begin{pmatrix} I' & a_3 & I_s \\ a' & I_3 & a_s \end{pmatrix} T^{(s')}_{a_1a_s;a_2a_3} \tag{2.41}$$

and

$$T_{4s}(I'')=(-1)^{2I_4+2I_6}\begin{pmatrix} I_6 & a_5 & a'' \\ a_6 & I_5 & I'' \end{pmatrix}\begin{pmatrix} I'' & a_4 & I_s \\ a'' & I_4 & a_s \end{pmatrix} T^{(s'')}_{a_6a_s;a_4a_5}. \tag{2.42}$$

Now the procedure which led from Eq. (2.26) to Eq. (2.39) can be carried out on the isospin-reduced amplitudes by inserting Eq. (2.26) into the right-hand side of Eq. (2.40). Then writing the four-point amplitudes in Eq. (2.26) in terms of their isospin-reduced counterparts Eq. (2.41) and Eq. (2.42), one obtains the reduced s-channel resonance contribution to the discontinuity of the six-point amplitude, $\mathcal{T}(I,I',I'')$. This expresses $\mathcal{T}(I,I',I'')$ as a product of $T_{3s}(I')$, $T^*_{4s}(I'')$, and eight 3-j symbols. The two pairs of 3-j symbols involving the isospins of particles 1,2 and 5,6 can be contracted away by the orthogonality relation (Eq. 2.19). The remaining four 3-j symbols can be manipulated into the familiar tetrahedral form of a 6-j symbol. This manipulation involves two convenient symmetries of the 3-j symbol, namely, that it is symmetric (antisymmetric) under permutation of two columns if the sum of the three isospins is an even (odd) integer, and that a fully covariant 3-j symbol is equal to a fully contravariant one. Since the mechanics of this calculation are straightforward, and the result differs little from the familiar isospin crossing matrix for the four-point amplitude, we omit the details. The expression obtained for the reduced $\mathcal{T}$ is

$$\begin{aligned}\mathcal{T}(l,\lambda;I,I',I'')=(-1)^{I_3-I_s+I+I_6-I_5}\sum_{j,l',l''}&(2j+1)(2l+1)e^{i\pi\lambda/2}e^{i\pi j_4}(-1)^{j_s-j_3+j+l''}\\ &\times\begin{Bmatrix} I' & I'' & I \\ I_4 & I_3 & I_s \end{Bmatrix}\begin{Bmatrix} l' & l'' & j \\ j_4 & j_3 & j_s \end{Bmatrix}\begin{pmatrix} \mu' & \mu'' & j \\ l' & l'' & \mu \end{pmatrix}\\ &\times d^{l'}_{\mu'\lambda'}(\omega')d^{l''}_{\lambda''\mu''}(\omega'')\mathcal{B}^{\mu l}_{j\lambda}(j_3,j_4;\chi)T_{3s}(l',\lambda';I')\\ &\times T^*_{4s}(l'',\lambda'';I'')e^{-\lambda'\psi'-\lambda''\psi''}.\end{aligned} \tag{2.43}$$

This expression and a similar expression for a resonance contribution to the discontinuity in the u-channel (see below) will be used extensively in Sec. III to discuss the cancellation between the nucleon and $\Delta(1232)$ contributions to FESR's and to derive constraints on their couplings to $I=1$ Reggeons.

In general, a given resonance or set of resonances will contribute to an FESR via both the s-channel and the u-channel discontinuities. Therefore our discussion would not be complete without a formula analogous to Eq. (2.43) for the FESR contribution of a u-channel resonance (shown in Fig. 9), which we will denote by $\overline{\mathcal{T}}$. The pattern of the derivation for $\overline{\mathcal{T}}$ is quite similar to that which led to Eq. (2.43). That is, we use crossing symmetry to write the c.m. amplitude for the t-channel process $3+\overline{4}\to 1+\overline{2}+5+\overline{6}$ in terms of the u-channel c.m. amplitude for the process $2+\overline{4}+6\to 1+\overline{3}+5$. A u-channel resonance then appears as a term in the generalized unitarity equation for the u-channel c.m. helicity amplitude. (See Fig. 9.) It is determined by the four-point amplitudes for $\overline{4}+u\to 1+\overline{2}$ and $\overline{3}+u\to\overline{5}+6$, and hence by the $\overline{4}+u\to R'$ and $\overline{3}+u\to R''$ vertices. For the applications which will be considered in Sec. III, the vertices which appear in the u-channel resonance contributions can be related to those which appear in the s-channel contributions by charge-conjugation invariance of the Reggeon couplings.

After going through the detailed derivation of the s-channel resonance formula [Eq. (2.43)], it will be most instructive to obtain the u-channel formula by analogy. This will serve to emphasize the slight but important differences between the two expressions and, in particular, to trace the origin of the relative phase between the s- and u-channel discontinuities which is of paramount importance for FESR applications. (Recall that FESR's are satisfied by amplitudes which are odd under $s \to u$ crossing.) Consider first the spinology of a u-channel resonance, suppressing isospin. The steps leading to Eq. (2.29) can be repeated for the u-channel as described in the previous paragraph, leading to the analogous formula

$$\overline{\mathcal{T}}_{-\lambda_3\lambda_4} = e^{i\pi j_3} d^{j_3}_{\mu_3\lambda_3}(\overline{\omega}_4)\overline{T}^*_{\mu_3-\mu_u} e^{-i\overline{\Phi}''(\mu_3+\mu_u)} d^{j_u}_{\mu_u\nu_u}(-\theta_u)\overline{T}_{\nu_4-\nu_u} e^{i\overline{\Phi}'(\nu_4+\nu_u)} d^{j_4}_{\nu_4\mu_4}(\overline{\omega}_3)\, C^{j_4}_{\mu_4\lambda_4}. \tag{2.44}$$

Here $\overline{\mathcal{T}}_{\lambda_3\lambda_4}$ is the u-channel resonance contribution to the t-channel c.m. six-point amplitude, $\overline{T}_{\mu_3\mu_u}$ and $\overline{T}_{\nu_4\nu_u}$ are the c.m. amplitudes for $\overline{3}+u \to \overline{5}+6$ and $\overline{4}+u \to 1+\overline{2}$, respectively, and the $t \to u$ crossing angles which appear in (2.44) can be obtained by making the following substitutions on both sides of Eq. (2.24):

$$\begin{aligned} &\omega_3 \to \overline{\omega}_3,\\ &\omega_4 \to \overline{\omega}_4,\\ &s \to u,\\ &m_3 \leftrightarrow m_4. \end{aligned} \tag{2.45}$$

In Eq. (2.44), θ_u is the u-channel scattering angle. Note that for the case of equal external masses $m_3 = m_4$, the contribution of the same resonance to the s- and u-channel discontinuities (e.g., the Δ° and $\overline{\Delta}^{++}$ contributions to $\pi^- p \to \pi^- p$) involves the same angles, i.e., $\overline{\omega}_3 = \omega_3$, and $\overline{\omega}_4 = \omega_4$ and $\theta_u = \theta_s$. This provides a considerable simplification for the N, Δ discussion of Sec. III.

Except for some obvious substitutions, Eq. (2.44) for $\overline{\mathcal{T}}_{-\lambda_3\lambda_4}$ is identical in form to Eq. (2.29) for $\mathcal{T}_{\lambda_3-\lambda_4}$. We of course want to define the reduced $\overline{\mathcal{T}}(l,\lambda)$ in the same way as we defined the reduced $\mathcal{T}(l,\lambda)$. (They are, after all, discontinuities of the same amplitude.) Hence, we write

$$\overline{\mathcal{T}}(l,\lambda) = (2l+1)\begin{pmatrix} j_3 & j_4 & \lambda \\ \lambda_3 & \lambda_4 & l \end{pmatrix} e^{i\pi\lambda/2}\overline{\mathcal{T}}_{\lambda_3-\lambda_4}. \tag{2.46}$$

Using symmetries of the 3-j symbol, Eq. (2,46) can be written (letting l = integer)

$$\begin{aligned} \overline{\mathcal{T}}(l,-\lambda) &= (2l+1)(-1)^{j_3+j_4+l}\begin{pmatrix} j_3 & j_4 & \lambda \\ \lambda_3 & \lambda_4 & l \end{pmatrix} e^{-i\pi\lambda/2}\overline{\mathcal{T}}_{-\lambda_3\lambda_4} \\ &= (2l+1)\begin{pmatrix} j_4 & j_3 & \lambda \\ \lambda_4 & \lambda_3 & l \end{pmatrix} e^{-i\pi\lambda/2}\overline{\mathcal{T}}_{-\lambda_3\lambda_4}. \end{aligned} \tag{2.47}$$

Thus, by analogy with Eq. (2.39), we obtain

$$\begin{aligned} \overline{\mathcal{T}}(l,-\lambda) = \sum_{j,l',l''} & e^{i\pi j_3}(2l+1)(2j+1)e^{-i\pi\lambda/2}(-1)^{j_u-j_4+j+l''}\begin{Bmatrix} l' & l'' & j \\ j_3 & j_4 & j_u \end{Bmatrix}\begin{pmatrix} \mu' & \mu'' & j \\ l' & l'' & \mu \end{pmatrix} \\ & \times d^{l'}_{\mu'\lambda'}(\overline{\omega}')d^{l''}_{\lambda''\mu''}(\overline{\omega}'')\mathcal{B}^{\mu l}_{j\lambda}(j_4,j_3;\overline{\chi})\overline{T}_{4u}(l',\lambda')\overline{T}^*_{3u}(l'',\lambda'')e^{-\lambda'\overline{\phi}'-\lambda''\overline{\phi}''}, \end{aligned} \tag{2.48}$$

where $\overline{\omega}'$, $\overline{\omega}''$, and $\overline{\chi}$ are defined in terms of $\overline{\omega}_3$, $\overline{\omega}_4$, and θ_u exactly as their s-channel counterparts, i.e., by the obvious analog of Eq. (2.34). In Eq. (2.48) $\overline{T}_{4u}$ and $\overline{T}_{3u}$ are, respectively, the amplitudes for $\overline{4}+u \to 1+\overline{2}$ and $\overline{3}+u \to \overline{5}+6$. Now we want to change Eq. (2.48) to an expression for $\overline{\mathcal{T}}(l,\lambda)$ by changing the sign of λ on both sides and then pushing the minus sign of λ on the right-hand side through the $\mathcal{B}$ function, through the 3-j symbol, through the d matrices, and into the four-point amplitudes. To do this we employ the following identities (with j, l = integers):

$$\mathcal{B}^{-\mu l}_{j-\lambda}(j_4,j_3;\overline{\chi}) = (-1)^{l-j}(-1)^{\lambda-\mu}\mathcal{B}^{\mu l}_{j\lambda}(j_4,j_3;\overline{\chi}), \tag{2.49}$$

$$\begin{pmatrix} -\mu' & -\mu'' & j \\ l' & l'' & -\mu \end{pmatrix} = (-1)^{j+l'+l''}\begin{pmatrix} \mu' & \mu'' & j \\ l' & l'' & \mu \end{pmatrix}, \tag{2.50}$$

$$d^{l'}_{-\mu'-\lambda'}(\overline{\omega}')d^{l''}_{-\lambda''-\mu''}(\overline{\omega}'') = (-1)^{\mu'+\mu''-\lambda'-\lambda''}d^{l'}_{\mu'\lambda'}(\overline{\omega}')d^{l''}_{\lambda''\mu''}(\overline{\omega}''). \tag{2.51}$$

Noting that $\mu'+\mu''=\mu$ because of the 3-j symbol, and assuming that λ is an integer (always true if particles

1, 2, 5, and 6 are spinless), it is seen that the net phase factor from all these operations is $(-1)^{l+l'+l''}$ $(-1)^{\lambda+\lambda'+\lambda''}$. Hence

$$\overline{\mathcal{T}}(l,\lambda)=\sum_{j,l',l''} e^{i\pi j_3}(2l+1)(2j+1)e^{i\pi\lambda/2}(-1)^{j_u-j_4+j+l''}\begin{Bmatrix} l' & l'' & j \\ j_3 & j_4 & j_u\end{Bmatrix}\begin{pmatrix} \mu' & \mu'' & j \\ l' & l'' & \mu\end{pmatrix}$$
$$\times d^{l'}_{\mu'\lambda'}(\overline{\omega}')d^{l''}_{\lambda''\mu''}(\overline{\omega}'')\mathcal{B}^{\mu l}_{j\lambda}(j_4,j_3;\overline{\chi})\overline{T}_{4u}(l',-\lambda')\,\overline{T}^*_{3u}(l'',-\lambda'')\,e^{\lambda'\overline{\psi}'+\lambda''\overline{\psi}''}(-1)^{l+l'+l''}(-1)^{\lambda+\lambda'+\lambda''}. \tag{2.52}$$

We can use the parity relations Eq. (2.32) for the reduced four-point amplitudes to change $\overline{T}_{4u}(l',-\lambda')$ to $\overline{T}_{4u}(l',\lambda')$ and similarly for $\overline{T}_{3u}$. It is also advantageous to employ charge-conjugation invariance at the Reggeon vertices to rewrite $\overline{T}_{3u}$ and $\overline{T}_{4u}$, which refer to $\overline{3}+u\to\overline{5}+6$ and $\overline{4}+u\to 1+\overline{2}$, in terms of T_{3u} and T_{4u}, i.e., the amplitudes for $3+\overline{u}\to\overline{5}+6$ and $4+\overline{u}\to 1+\overline{2}$. In the discussion of the N and Δ in Sec. III, the s- and u-channel contributions will then involve the same set of Reggeon couplings. The implementation of parity and charge-conjugation invariance is discussed in detail in Appendix A. Using formula (A32) for charge conjugation at a vertex, along with the parity relation Eq. (2.32), we can write

$$\overline{T}_{4u}(l',-\lambda')\overline{T}^*_{3u}(l'',-\lambda'')=C'\sigma'C''\sigma''\eta^3_C\eta^3_P\eta^4_C\eta^4_P(-1)^{l'+l''}T_{4u}(l',\lambda')T^*_{3u}(l'',\lambda''), \tag{2.53}$$

where C',C'' are the charge-conjugation parities of Reggeons R',R'', and σ',σ'' are their naturalities. The η_P and η_C phase factors are the parity and charge-conjugation parity of particles 3 and 4. At this point, the $\mathcal{CPT}$ theorem plays an essential role in resolving the apparent arbitrariness of these latter phases when, for example, particles 3 and 4 are distinct baryons. The result which is needed is derived in Appendix A and is given by (A24). This allows us to make the replacement

$$\eta^3_C\eta^3_P\eta^4_C\eta^4_P=(-1)^{I_3-I_4}\eta^3_T\eta^4_T$$
$$=(-1)^{I_3-I_4}. \tag{2.54}$$

(Here we are assuming that the t-channel isospin exchange is integer. The more general case is easily treated along the same lines.) The last step in Eq. (2.54) employed the choice of time-reversal phase (A26). As discussed in Appendix A, a different choice of η^3_T and η^4_T would also imply a different relative phase for the Reggeon couplings, thus rendering all the subsequent discussion independent of phase conventions.

The isospin dependence of the u-channel resonance formula can also be obtained most easily by analogy with the s-channel formula Eq. (2.43). Except for a phase factor, whose origin will be discussed presently, the u-channel isospin dependence can be obtained from Eq. (2.43) simply by the interchange $I_3\leftrightarrow I_4$, $I_s\leftrightarrow I_u$. Aside from interchange of I_3 and I_4 in the isospin 6-j symbol, this simply replaces the isospin factor in front of Eq. (2.43) by

$$(-1)^{I_4-I_u+I+I_6-I_5}. \tag{2.55}$$

However, note that by the isospin crossing rule (A19), when the six-point amplitude is written in terms of the isospin-reduced amplitude Eq. (2.40), **the ordering of the lines in the 3-j symbols must** be the same in all channels. In particular, the I_3-I_4-I symbol will be the same in both the s- and u-channels. Hence, if we want to cast the u-channel unitarity equation in a form which is analogous to the s channel but with $I_3\leftrightarrow I_4$ everywhere, we must interchange the lines of the I_3-I_4-I symbol. This, along with a factor $(-1)^{2I_3}$ because the u channel involves an outgoing antiparticle [see Appendix A, Eq. (A19)] gives an extra isospin factor

$$(-1)^{2I_3}(-1)^{I_3+I_4+I}. \tag{2.56}$$

Finally, we note that the particular form of Eq. (A32), and hence, of Eq. (2.53) for charge conjugation at a vertex resulted from the convention that the ordering of lines in the isospin 3-j symbol is always $B\overline{B}R$. Thus the amplitudes $\overline{T}_{3u}$ and $\overline{T}_{4u}$ will involve isospin 3-j symbols at the $B\overline{B}R$ vertex which have different ordering from those which appear for T_{3s} and T_{4s} in the s channel. In putting the right-hand side of the unitarity equation in a form analogous to the s channel, we incur a factor

$$(-1)^{I_4+I'+I_u}(-1)^{I_3+I''+I_u}. \tag{2.57}$$

Collecting all these factors, including the one which arose from the $\mathcal{CPT}$ theorem Eq. (2.54), we find that the net isospin factor for the u channel is

$$(-1)^{I_4-I_u+I+I_6-I_5}(-1)^{I_3-I_4}(-1)^{2I_3}(-1)^{I_3+I_4+I}\times(-1)^{I_4+I'+I_u}(-1)^{I_3+I''+I_u}=(-1)^{I_3-I_u+I'-I''+I_6-I_5}. \tag{2.58}$$

With the expression for the isospin factor and inserting Eq. (2.53) (sans $\mathcal{CP}$ factors for 3 and 4) into Eq. (2.52), we obtain the u-channel resonance formula

$$\overline{\mathcal{T}}(l,\lambda;I,I',I'')=C'\sigma'C''\sigma''(-1)^{I_3-I_u+I'-I''+I_6-I_5}\times\sum_{j,l',l''}(2j+1)(2l+1)e^{i\pi\lambda/2}e^{i\pi j_3}(-1)^{j_u-j_4+j+l''}\times\begin{Bmatrix}I' & I'' & I\\ I_3 & I_4 & I_u\end{Bmatrix}\begin{Bmatrix}l' & l'' & j\\ j_3 & j_4 & j_u\end{Bmatrix}\begin{pmatrix}\mu' & \mu'' & j\\ l' & l'' & \mu\end{pmatrix}d^{l'}_{\mu'\lambda'}(\overline{\omega}')d^{l''}_{\lambda''\mu''}(\overline{\omega}'')\times\mathcal{B}^{\mu l}_{j\lambda}(j_4,j_3;\overline{\chi})T_{4u}(l',\lambda')T^*_{3u}(l'',\lambda'')e^{\lambda'\overline{\psi}'+\lambda''\overline{\psi}''}(-1)^{l}(-1)^{\lambda+\lambda'+\lambda''}. \quad (2.59)$$

Before going on to discuss applications of Eq. (2.43) and Eq. (2.59), we must mention one problem which arises when one tries to form the combination of s- and u-channel discontinuities which enters into FESR's, i.e., which is odd under $\nu\to-\nu$ where $\nu=s-u$. The difficulty is that discontinuities $\mathcal{T}$ and $\overline{\mathcal{T}}$ are related not by interchanging s and u, but by interchanging p_3 and p_4 (compare Figs. 1 and 9), which also causes $s_{23}\leftrightarrow s_{24}$ and $s_{35}\leftrightarrow s_{45}$. Fortunately, we are saved by the kinematic restriction [Eq. (2.1)] which was necessary for FESR analyticity of the six-point amplitude. Because of this restriction $s_{23}=-s_{24}$ and $s_{35}=-s_{45}$. Hence $s_{23}\leftrightarrow s_{24}$ and $s_{35}\leftrightarrow s_{45}$ is equivalent to $s_{23}\leftrightarrow -s_{23}$ and $s_{45}\leftrightarrow -s_{45}$, which introduces signature factors τ' and τ'' for the Reggeons R' and R''. Thus, the combination of amplitudes which is odd under $\nu\to-\nu$ is $\mathcal{T}-\tau'\tau''\overline{\mathcal{T}}$.

III. APPLICATION TO THE N AND Δ CONTRIBUTIONS

In this section we shall show how the general formulas derived in Sec. II can be used to give a simple description of the cancellation between the nucleon N and $\Delta(1232)$ contributions to FESR's. As we already mentioned in the Introduction, this problem, in fact, served as a guide in developing our formalism. A detailed discussion of the dynamical reasons for the cancellation may be found in Ref. 6. Let us only mention here that the cancellation is expected to be accurate to within 20% in zeroth-moment FESR's. We consider all amplitudes of the form $R'B_3\to R''B_4$, where B_3 and B_4 can be either N or Δ. The Reggeons R' and R'' can be any $I=1$ meson trajectory: ρ-A_2, π-B, or A_1. We thus get predictions for couplings that were not considered in Ref. 6, such as $\rho\Delta\Delta$, $\pi N\Delta$, $A_1N\Delta$, etc. Some of these predictions can be compared with experiment.

A. The approximations

The small mass difference between the N and the Δ is an important ingredient for understanding the proportionality of their FESR contributions. In fact, it was found in Ref. 6 that neglecting the N-Δ mass difference and the momentum transfers in comparison with the baryon masses, in general, changed the exact results by less than 20%. We shall next discuss the implication of this approximation for the angles ω, χ, and ψ that appear in the formulas of Sec. II.

A prominent feature in the expression for the angles is the presence of kinematic factors that vanish at or near s-channel threshold. Thus, e.g., the formula for ω_3 in Eq. (2.24) has a factor $\lambda^{-1/2}(s,m_3^2,t')$ which blows up at $\sqrt{s}=m_3\pm\sqrt{t'}$. Similarly, ψ' is singular on the physical region boundary $\Phi=0$ according to Eqs. (2.8) and (2.13). It is evident that an approximation like $\sqrt{s}=m_\Delta\approx m_N$ would not make sense if applied directly to these angles.

The resolution to this dilemma is that the full six-point amplitude does *not* have any kinematic singularities in s. The singularities must therefore completely cancel each other. Consider first the threshold and pseudothreshold singularities arising from the factors $\lambda(s,m_3^2,t')$ and $\lambda(s,m_4^2,t'')$. It is a well-known[10] property of t-channel helicity amplitudes (see also Appendix B) that they do not have s-channel threshold singularities. In the expression (2.39) for $\mathcal{T}$ the singularities in the angles ω' and ω'' must thus be canceled by corresponding singularities in the vertices $T(l',\lambda')$ and $T(l'',\lambda'')$. In fact, the t'-channel helicity amplitude $T(l',\lambda')$ *is* singular at the t'-channel pseudothreshold $t'=(m_3-\sqrt{s})^2$ (which is the same as $\sqrt{s}=m_3\pm\sqrt{t'}$). The behavior of $T(l',\lambda')$ at pseudothreshold is discussed in Appendix B. We have explicitly verified that these singularities indeed cancel out in Eq. (2.39).

Suppose now that we substitute the regularized amplitudes $\hat{T}(l',\lambda')$, $\hat{T}(l'',\lambda'')$ defined by Eq. (B7) for $T(l',\lambda')$ and $T(l'',\lambda'')$ in the expression (2.39) for $\mathcal{T}$. All the singularities can then be eliminated, and the approximation $m_\Delta\approx m_N$ can be applied to the remaining regular expression. This is, in effect, the approach chosen in Ref. 6. For notational simplicity we prefer in this paper to apply the approximations directly to the expression (2.39) for $\mathcal{T}$.

According to the above discussion this is legitimate provided we interpret the relations obtained for $T(l',\lambda')$ to hold, in fact, only for $\hat{T}(l',\lambda')$. The true relations for $T(l',\lambda')$ can then be reconstructed from Eq. (B7) by using the exact values for the masses in the kinematic factor.

The situation is analogous with respect to the singularity at the physical region boundary $\Phi=0$. The singularity in ψ' and ψ'' now appears in the full six-point amplitude, however. [The vertices $T(l',\lambda')$ and $T(l'',\lambda'')$ are, of course, regular at $\Phi=0$.] The behavior of the six-point function at $\Phi=0$ is discussed in Appendix B and given in Eq. (B5). Since this singularity is always removed from the amplitude before writing the FESR, the divergent behavior of ψ' and ψ'' is automatically canceled. All expressions are thus smooth around $\Phi=0$, and the approximation $m_N \approx m_\Delta$ can be expected to be reasonable.

We shall assume then, that we can put $s=m^2$ for both the N and the Δ FESR contributions, where $m\approx 1$ GeV is some average mass $m_N \lesssim m \lesssim m_\Delta$. This implies that all the angles ω, ψ, and χ will be the same for N and Δ. Furthermore, the fact that m is close to s-channel threshold means that χ and ψ are both small angles. In Sec. II we already pointed out that χ is the "hyperbolic excess" angle for a triangle whose sides are proportional to the s-channel momenta. Close to threshold the area of the triangle becomes vanishingly small and so $\chi \to 0$. We can verify this by calculating χ explicitly from Eq. (2.34) and (2.24). In the equal-mass case $m_3=m_4$, $t'=t''$ the exact value of $\sin\chi$ is

$$\sin\chi=\frac{\Phi^{1/2}}{[(\sqrt{s}+m_3)^2-t'](4m_3{}^2-t)^{1/2}}\,, \tag{3.1}$$

where

$$\Phi=-t[(s-m_3{}^2)^2-2t'(s+m_3{}^2)+st+t'^2]$$

is the Kibble function for the Reggeon amplitude. Even if we disregard the smallness of the numerator $\Phi^{1/2}$ (which may be canceled by a $\Phi^{-1/2}$ from ψ, as we argued above), the denominator is large enough to make $\chi=0$ a good approximation.

According to Eqs. (2.8) and (2.13) the exact expression for $\sinh\psi'$ is, in the equal-mass case,

$$\sinh\psi'=\frac{-t(s-m_3{}^2+t')}{2(t'\Phi)^{1/2}}\,. \tag{3.2}$$

The factor $\Phi^{-1/2}$ can be disregarded as it will cancel out in the expression for $\mathcal{T}$. For $s=m_3{}^2=m^2$ and small momentum transfers t, t' it then follows from Eq. (3.2) that $\psi\approx 0$.

The approximate relations for the angles which make the proportionality between the N and Δ FESR contributions possible can thus be summarized by

$$\begin{aligned}&\omega'_N=\omega'_\Delta,\quad \omega''_N=\omega''_\Delta,\\ &\chi_N=\chi_\Delta=0,\\ &\psi'_N=\psi''_N=\psi''_\Delta=\psi''_\Delta=0.\end{aligned} \tag{3.3}$$

In the remainder of this section we shall assume that Eqs. (3.3) hold. It follows from the above discussion that this approximation should be reasonable. It is, of course, more difficult to estimate quantitively the error it induces in $\mathcal{T}$. We rely in this respect on the numerical calculations of Ref. 6, which indicates that Eq. (3.3) changes the exact results generally by less than 20%.

B. The $R'N\to R''N$ FESR's

We first consider amplitudes for which the external baryons B_3 and B_4 in Fig. 1 are nucleons. Since we want to write zeroth-moment FESR's we have to form a combination of the s- and u-channel discontinuities $\mathcal{T}$ and $\overline{\mathcal{T}}$ [Eqs. (2.43) and (2.59)] which is antisymmetric in the variable s-u. As discussed in Sec. II, this antisymmetric combination is $\mathcal{T}-\tau'\tau''\overline{\mathcal{T}}$, where τ',τ'' are the Reggeon signatures. Furthermore, we have to remove the kinematic singularities in s of $\mathcal{T}$ and $\overline{\mathcal{T}}$. The singularities all occur at the physical region boundary $\Phi=0$ and are explicitly given by Eq. (B5) [when both Reggeons are spinless particles one may use Eq. (B1) instead]. Since $\sin\theta_t$ has the same value for the N and Δ FESR contributions in our approximation $m_N=m_\Delta=m$, we need not explicitly remove the common factor $(\sin\theta_t)^{-|\lambda|}$ from our expressions. However, because $\sin\theta_t$ has the opposite sign in $\mathcal{T}$ and $\overline{\mathcal{T}}$, the proper antisymmetric contribution is

$$\mathcal{T}^{(-)}(l,\lambda;\,I)\equiv\mathcal{T}(l,\lambda;\,I)-\tau'\tau''(-1)^\lambda\overline{\mathcal{T}}(l,\lambda;\,I). \tag{3.4}$$

Substituting $B_3=B_4=N$ and $I'=I''=1$ in the expressions (2.43) and (2.59) for $\mathcal{T}$ and $\overline{\mathcal{T}}$ we get, in the approximation (3.3),

$$\mathcal{T}^{(-)}(l,\lambda;I)=i\sum_{l',l''}(-1)^{l''}(2l+1)\begin{Bmatrix}l' & l'' & l\\ \frac{1}{2} & \frac{1}{2} & j_s\end{Bmatrix}\begin{Bmatrix}1 & 1 & I\\ \frac{1}{2} & \frac{1}{2} & I_s\end{Bmatrix}T^{R'}_{Ns}(l',\lambda')\,T^{R''*}_{Ns}(l'',\lambda'')\begin{pmatrix}\mu' & \mu'' & l\\ l' & l'' & \lambda\end{pmatrix}$$
$$\times\, d^{l'}_{\mu'\lambda'}(\omega')\,d^{l''}_{\lambda''\mu''}(\omega'')\,e^{i\pi\lambda/2}[(-1)^{l+I}-\sigma'\sigma''C'\tau'C''\tau''(-1)^{\lambda'+\lambda''}]. \tag{3.5}$$

The subscript s in Eq. (3.5) is N or Δ, depending on which contribution we are considering. In order to isolate a given naturality σ in the t channel we form the combinations [cf. Eq. (2.32)]

$$\mathcal{T}_{\sigma}^{(-)}(l,\lambda;\, I) \equiv \mathcal{T}^{(-)}(l,\lambda;\, I) + \sigma(-1)^{l}\mathcal{T}^{(-)}(l,-\lambda;\, I). \tag{3.6}$$

We find then

$$\mathcal{T}_{\sigma}^{(-)}(l,\lambda;\, I) = i\sum_{l',l''}(-1)^{l''}(2l+1)\begin{Bmatrix} l' & l'' & l \\ \frac{1}{2} & \frac{1}{2} & j_s \end{Bmatrix}\begin{Bmatrix} 1 & 1 & I \\ \frac{1}{2} & \frac{1}{2} & I_s \end{Bmatrix} T_{Ns}^{R'}(l',\lambda')T_{Ns}^{R''*}(l'',\lambda'')\begin{pmatrix} \mu' & \mu'' & l \\ l' & l'' & \lambda \end{pmatrix}$$
$$\times\, d^{l'}_{\mu'\lambda'}(\omega')d^{l''}_{\lambda''\mu''}(\omega'')e^{i\pi\lambda/2}[(-1)^{l+I} - \sigma C'\tau' C''\tau''][1+\sigma\sigma'\sigma''(-1)^{\lambda'+\lambda''}]. \tag{3.7}$$

This equation gives the contribution of the N and Δ resonances to the zeroth-moment FESR for the amplitude $R'N \to R''N$, with t-channel quantum numbers characterized by l, λ, I, σ and $\tau = -\tau'\tau''(-1)^{\lambda}$. A sum over $l, l'', \lambda', \lambda'', \mu'\mu''$ is implied. The main advantage of this formula is that, as we shall next explore, the vertices $T(l', \lambda')$ and $T(l'', \lambda'')$ are nonzero only for a single value of l', l'', $|\lambda'|$, and $|\lambda''|$. The sum therefore in practice need only be done over μ' and μ''.

1. $\pi N \to \pi N$

Let us first apply our formula (3.7) to the familiar case of πN elastic scattering. Since the π is spinless, we must have $\lambda' = \lambda'' = 0$. The parity condition (2.32) requires $l' = l'' = 1$ for both the πNN and $\pi N\Delta$ vertices. This ensures that the dependence on $\omega' = \omega''$ will be the same for $s = N$ and $s = \Delta$. The only difference between the N and Δ contributions is thus in the over-all normalization, which is determined by the relative magnitudes of $T_{NN}^{\pi}(1,0)$ and $T_{N\Delta}^{\pi}(1,0)$ together with the j_s, I_s dependence of the 6-j symbols. We want to fix the value of the single parameter $|T_{N\Delta}^{\pi}/T_{NN}^{\pi}|$ so that the two contributions cancel in all nonzero amplitudes.

The last bracket in Eq. (3.7) requires $\sigma = +$, as expected. The other bracket then implies

$$(-1)^{l+I} = -1.$$

Since l and I both can be either 0 or 1, it follows that there are only two nonzero antisymmetric amplitudes, having $l=0$, $I=1$ and $l=1$, $I=0$, respectively. The $l=1$ amplitude must have $|\lambda| = 1$ according to Eq. (3.6). The two amplitudes of course correspond to $A'^{(-)}$ and $B^{(+)}$ in standard notation.

We can now understand why a single value of the parameter $|T_{N\Delta}^{\pi}/T_{NN}^{\pi}|$ guarantees the cancellation in two amplitudes. Since $l' = l'' = 1$ and $j_s = I_s$ for $s = N$ and Δ, the product of the two 6-j symbols is the same for both amplitudes. From either amplitude we thus get, by substituting the values of the 6-j symbols,[15]

$$|\hat{T}_{N\Delta}^{\pi}(1,0)| = \sqrt{2}\,|\hat{T}_{NN}^{\pi}(1,0)|. \tag{3.8}$$

As discussed above, the cancellation constraint relates the regularized amplitudes $\hat{T}$, defined by Eq. (B7). The relation (3.8) is very well satisfied by the known πNN coupling and the $\pi N\Delta$ coupling measured by the $\Delta \to \pi N$ decay width.[6]

With no further effort we can also see how the cancellation works when the pions are Reggeized. Because of the $\mathcal{C}\tau$ symmetry relation (A35), there is still only one allowed πNN coupling, $T_{NN}^{\pi}(1,0)$. However, there are four possible couplings at the Reggeized $\pi N\Delta$ vertex. As in the above on-shell case, the N and Δ contributions will be proportional provided $T_{N\Delta}^{\pi}(1,0)$ is the only nonvanishing coupling, even for Reggeized pions. Since unnatural parity exchange ($\sigma = -$) is now allowed in the t channel, we have to study four $\pi N \to \pi N$ amplitudes. Because of the couplings we have chosen, $\lambda' = \lambda'' = 0$ in Eq. (3.7), and so the last bracket shows that neither the N nor the Δ resonance appears in the $\sigma = -$amplitudes. This is, of course, a special way of ensuring that the $N + \Delta$ contribution to the $\sigma = -$ FESR's is small. For the $\sigma = +$ FESR's the cancellation works exactly as in the on-shell case. The relation (3.8) must therefore hold also for Reggeized pions at moderate momentum transfers $|t'|, |t''| \lesssim 0.6$.

From the above argument it is, of course, not clear that the set of physical couplings we obtained is the only solution to the cancellation constraints. We strongly suspect that this is so, however, because the equations are overconstrained. Furthermore, in the case of the ρNN and $\rho N\Delta$ couplings it was explicitly shown[6] that there is only one solution, namely the one which emerges naturally from Eq. (3.7) (see below). We shall not attempt to establish the uniqueness of our solutions in the present paper.

2. $\rho N \to \rho N$

The kinematically allowed values of l' in the $T_{NN}^{\rho}(l', \lambda')$ and $T_{N\Delta}^{\rho}(l', \lambda')$ couplings are $l' = 0, 1$ and $l' = 1, 2$, respectively. For the N and Δ contributions to be proportional we must therefore have $l' = 1$ in both couplings. The parity relation

(2.32) then requires $|\lambda'|=1$. Thus the ρNN coupling must be helicity flip. It can also be easily verified that the $\rho N\Delta$ coupling is of the $M1$ type. Both couplings are known to be strongly favored by the data.

The fact that $|\lambda'|=|\lambda''|=1$ in the last bracket of Eq. (3.7) ensures that the N, Δ cancellation works precisely as in the $\pi N\to\pi N$ case. Thus neither resonance contributes to the $\sigma=-$ FESR's. Cancellation in the $\sigma=+$ FESR's requires again

$$|\hat{T}^{\rho}_{N\Delta}(1,1)|=\sqrt{2}\,|\hat{T}^{\rho}_{NN}(1,1)|. \tag{3.9}$$

For this coupling we actually have $T=\hat{T}$, according to Eqs. (B7) and (B9). The relation (3.9) then implies

$$\frac{d\sigma}{dt}(\pi^+p\to\pi^0\Delta^{++})=\frac{3}{2}\frac{d\sigma}{dt}(\pi^-p\to\pi^0 n), \tag{3.10}$$

which is in good agreement[16] with the data.

3. $\pi N\to\rho N$

All the relevant couplings have been fixed (up to a sign) by our consideration of the elastic processes above. The $\pi N\to\rho N$ reaction thus serves as a consistency check on our approach. In fact, the only difference in Eq. (3.7), compared to the previous cases, is that $\sigma'\sigma''=-1$ and $(-1)^{\lambda'+\lambda''}=-1$ have changed sign. The two sign changes compensate each other, and the cancellation thus works as before. The relative phase between the $R'NN$ and $R'N\Delta$ couplings, not fixed by Eqs. (3.8) and (3.9), must be the same for $R'=\pi$ and $R'=\rho$.

4. $A_1N\to A_1N$

The $C\tau$ symmetry relation (A35) implies that $T^{A_1}_{NN}(1,1)=T^{A_1}_{NN}(1,-1)$ is the only allowed A_1NN coupling. We are thus led to require that the only nonvanishing $A_1N\Delta$ coupling $T^{A_1}_{N\Delta}(l',\lambda')$ has $l'=1$, $|\lambda'|=1$. The main difference between the A_1 trajectory and the π, ρ trajectories treated above is that $C\tau=-1$ for A_1. For the elastic reaction this quantity enters squared in Eq. (3.7), and so the cancellation is once again guaranteed by

$$|\hat{T}^{A_1}_{N\Delta}(1,1)|=\sqrt{2}\,|\hat{T}^{A_1}_{NN}(1,1)|. \tag{3.11}$$

5. $\pi N\to A_1N$

This process provides a consistency check on the previously derived couplings, given by Eqs. (3.8) and (3.11). The cancellation works out differently from the cases considered above, because the product $\sigma'\sigma''(-1)^{\lambda'+\lambda''}=-1$ has changed sign. The last bracket in Eq. (3.7) now shows that the N and Δ resonances do not contribute to natural-parity-exchange ($\sigma=+$) FESR's. For $\sigma=-$, the first bracket in Eq. (3.7) requires $l+I$ to be odd. Since $l=0$ is a natural-parity-exchange amplitude [cf. Eq. (3.6)], we must have $l=1$, $I=0$. The two $\sigma=-$ amplitudes correspond to $\lambda=0$ and $|\lambda|=1$. The product of the 6-j symbols in Eq. (3.7) is thus the same in both amplitudes, and furthermore is equal to what it was in all the cases previously considered. The N and Δ therefore cancel in the $\sigma=-$ FESR's when Eqs. (3.8) and (3.11) hold, provided only that the relative phase between the $R'NN$ and $R'N\Delta$ couplings is the same for $R'=\pi$ and $R'=A_1$.

6. $\rho N\to A_1N$

The cancellation is very similar to that for the process $\pi N\to A_1N$, as the few sign differences compensate each other in Eq. (3.7). Thus the N and Δ do not contribute to $\sigma=+$ FESR's, and cancel in the $\sigma=-$ FESR's for the couplings established above.

C. The $R'N\to R''\Delta$ FESR's

The requirement of N, Δ cancellation in the $R'N\to R''\Delta$ amplitudes imposes constraints on the $T^{R'}_{N\Delta}$, $T^{R'}_{\Delta N}$, and $T^{R'}_{\Delta\Delta}$ couplings of Sec. II. The first two couplings are related by $C\tau$ symmetry [cf. Eq. (A34)]:

$$T^{R'}_{N\Delta}(l',\lambda')=-\sigma'C'\tau'(-1)^{l'+\lambda'}T^{R'}_{\Delta N}(l',\lambda'). \tag{3.12}$$

For the physical couplings derived above, the phase factor in Eq. (3.12) is -1, for all trajectories $R'=\pi, \rho, A_1$.

The expression for the contribution of the N and Δ resonances to the $R'N\to R''\Delta$ FESR is simplified by the ansatz that the only nonvanishing couplings have $l'=1$ and satisfy

$$\hat{T}^{R'}_{\Delta s}(1,\lambda')=\beta_s\hat{T}^{R'}_{Ns}(1,\lambda')\quad(s=N,\Delta). \tag{3.13}$$

Owing to our phase conventions [cf. Eqs. (A25) and (A26)], the factors β_s are real. We furthermore assume that β_s does not depend on R', t', or λ'. From the $R'N\to R''N$ FESR's

$$\beta_N=\pm\sqrt{2}. \tag{3.14}$$

The derivation of the resonance contributions to the amplitudes $\mathcal{T}^{(-)}_{\sigma}(l,\lambda;I)$ proceeds just as in Sec. III B. With our assumptions $l'=l''=1$ and Eq. (3.13) the result is

$$\mathcal{T}^{(-)}_{\sigma}(l,\lambda;I)=ie^{i\pi\lambda/2}(2l+1)\begin{Bmatrix}1 & 1 & l\\ \frac{1}{2} & \frac{3}{2} & j_s\end{Bmatrix}\begin{Bmatrix}1 & 1 & I\\ \frac{1}{2} & \frac{3}{2} & I_s\end{Bmatrix}T^{R'}_{Ns}(1,\lambda')T^{R''*}_{\Delta s}(1,\lambda'')$$
$$\times\begin{pmatrix}\mu' & \mu'' & l\\ 1 & 1 & \lambda\end{pmatrix}d^1_{\mu'\lambda'}(\omega')d^1_{\lambda''\mu''}(\omega'')[(-1)^{l+I}-\sigma C'\tau'C''\tau''][1+\sigma\sigma'\sigma''(-1)^{\lambda'+\lambda''}]. \tag{3.15}$$

The structure of this equation is very similar to that of Eq. (3.7) for the $R'N \to R''N$ amplitudes. Thus the only nonzero contributions are to amplitudes having $(-1)^{l+I} = -1$, i.e., $(l, I) = (1, 2)$ or $(2, 1)$. The N and Δ cancel in all FESR's provided the couplings satisfy Eq. (3.13) with

$$\beta_N \beta_\Delta = 2. \tag{3.16}$$

With Eq. (3.16) the relative magnitudes of all couplings $T_{34}^{R'}(l', \lambda')$ have been specified, for $R' = \pi, \rho, A_1$ and $3, 4 = N$ or Δ. The helicity structure of each coupling depends only on R': $R' = \pi$ has $l' = 1$, $\lambda' = 0$, while $R' = \rho$ and A_1 have $l' = 1$, $|\lambda'| = 1$. The coupling strengths satisfy

$$\hat{T}_{NN}^{R'} : \hat{T}_{N\Delta}^{R'} : \hat{T}_{\Delta\Delta}^{R'} = 1 : \pm\sqrt{2} : 2. \tag{3.17}$$

There is no constraint on the relative magnitudes of couplings with different R'.

D. The $R'\Delta \to R''\Delta$ FESR's

The $R'\Delta \to R''\Delta$ amplitudes logically belong to the set of reactions we are studying. All couplings having been fixed by the considerations in Sec. III B and III C, it is of considerable interest to see whether the N and Δ cancel also in the $R'\Delta \to R''\Delta$ FESR's. The expression for the resonance contributions can be derived as in Sec. III B. In the approximation (3.3) we obtain the by now familiar-looking formula

$$\mathcal{T}_\sigma^{(-)}(l, \lambda; I) = -i \sum_{l', l''} (-1)^{l''} e^{i\pi\lambda/2}(2l+1) \begin{Bmatrix} l' & l'' & l \\ \frac{3}{2} & \frac{3}{2} & j_s \end{Bmatrix} \begin{Bmatrix} 1 & 1 & I \\ \frac{3}{2} & \frac{3}{2} & I_s \end{Bmatrix} T_{\Delta s}^{R'}(l'\lambda') T_{\Delta s}^{R''*}(l''\lambda'')$$
$$\times \begin{pmatrix} \mu' & \mu'' & l \\ l' & l'' & \lambda \end{pmatrix} d^{\,l'}_{\mu'\lambda'}(\omega') d^{\,l''}_{\lambda''\mu''}(\omega'')[(-1)^{l+I} - \sigma C'\tau' C''\tau''][1 + \sigma\sigma'\sigma''(-1)^{\lambda'+\lambda''}]. \tag{3.18}$$

The parameters l, I that characterize the $R'\Delta \to R''\Delta$ amplitudes have the range $l = 0, 1, 2, 3$ and $I = 0, 1, 2$. Now from the 3-j symbol in Eq. (3.18) it is clear that neither the N nor the Δ contributes to the $l = 3$ amplitudes when $l' = l'' = 1$. Thus the FESR constraints are trivially satisfied for these amplitudes.

With the couplings established above, the N and Δ contribute only to amplitudes $\mathcal{T}_\sigma^{(-)}(l, \lambda; I)$ with $(-1)^{l+I} = -1$, i.e., $(l, I) = (0, 1), (1, 0), (1, 2), (2, 1)$. As usual, the relative size of the contributions is unchanged when $l \leftrightarrow I$. By substituting the values of the 6-j symbols and Eq. (3.17) we find the ratio of the two contributions to be

$$\frac{N}{\Delta} = \frac{5}{4} \quad (l, I) = (0, 1), (1, 0),$$
$$\frac{N}{\Delta} = -\frac{25}{16} \quad (l, I) = (1, 2), (2, 1). \tag{3.19}$$

Thus in FESR's with $(l, I) = (0, 1)$ and $(1, 0)$ the N and Δ contributions add, while for $(l, I) = (1, 2)$ and $(2, 1)$ they partially cancel.

To see the significance of the noncancellation, in particular for the $(l, I) = (0, 1)$ and $(1, 0)$ amplitudes, let us briefly recall the reason for expecting a cancellation. It stems mainly from the corresponding $\pi N \to \pi N$ amplitudes $A'^{(-)}$ and $B^{(+)}$ where all resonance and Regge contributions are accurately known. In those amplitudes, the N and Δ contributions are almost an order of magnitude larger than the resonance contributions at higher masses. Furthermore, the N and Δ separately would oversaturate the FESR's by a large factor, even with a reasonable cutoff around $\sqrt{s} = 2$ GeV (for a graphical illustration of this, see Figs. 3 and 4 of Ref. 6). The validity of the FESR's therefore requires a rather exact N, Δ cancellation, which actually is observed.

Comparing Eqs. (3.7) and (3.18) for $(l, I) = (0, 1)$ and $(1, 0)$, we can see that the N, Δ contributions have a similar magnitude and angular dependence in the $\pi N \to \pi N$ and $\pi\Delta \to \pi\Delta$ amplitudes. However, as observed above, while they *cancel* each other in $\pi N \to \pi N$, they *add* in $\pi\Delta \to \pi\Delta$. According to the above argument we should then expect that the higher-mass resonance contributions and/or the Regge terms are considerably larger in $\pi\Delta \to \pi\Delta$ than in $\pi N \to \pi N$. By factorization arguments and Eq. (3.17), however, the high-energy amplitudes should be comparable to the $\pi N \to \pi N$ amplitudes. The only possibility then seems to be that there are resonances that couple strongly to $\pi\Delta$ and cancel the combined N, Δ contributions to the $\pi\Delta \to \pi\Delta$ FESR's.

Finally, it is interesting to observe that the $N + \Delta$ contribution is small in the $l = 3$ and (to a lesser extent) in the $l = 2$ $\pi\Delta \to \pi\Delta$ amplitudes. The Regge terms for these amplitudes should be small for all isospins (and helicities). Thus, since the resonances cannot build up a Regge exchange in FESR's of any moment, we may expect duality to be more locally satisfied.

IV. DISCUSSION

The main objective of this paper is to develop a convenient framework for calculating s-channel resonance contributions to FESR's for t-channel helicity amplitudes. The amplitudes we considered are quite general, in that the external legs can have arbitrary spin or be Reggeons. We imposed certain general conditions on the formalism, which one may expect to be necessary in any duality bootstrap scheme. However, the most important constraint in practice was that the formalism should give a simple description of a special but nontrivial duality phenomenon: the cancellation between the N and Δ contributions to zeroth-moment FESR's.[6] The formula derived in Sec. II fulfills these objectives. Moreover, the generalization of the formalism from the particular example just mentioned to resonances of arbitrary mass and spin seems so natural and unique that we believe the methods developed here will be applicable to many different phenomena. This will, of course, provide the definite test of the usefulness of the ideas presented in this paper.

One of the general consistency requirements we imposed was that all three-point vertices should be described in terms of analogously defined couplings, namely t-channel helicity couplings. Our special example clearly showed, moreover, that it was advantageous to form certain irreducible combinations $T(l, \lambda)$ of the helicity amplitudes, defined by Eq. (2.31). For an (s-channel) vertex $R+3 \to 4$, in a frame where all momenta are collinear, λ is the helicity of R (=t-channel helicity) and l represents the spin that is added to the spin of 3 to give the spin of 4. Such couplings are, in fact, familiar from other applications. E.g., they are identical to the multipole couplings commonly used in electromagnetic transitions (R = photon).[17] Also, since the pion is spinless, it is clear that l is the orbital angular momentum in decays like $4 \to \pi + 3$ (particles 3 and 4 having arbitrary spin). It is encouraging that our approach naturally leads us to use couplings that have a simple physical interpretation.

In deriving our formulas we found it very illuminating to give a diagrammatic interpretation to the quantities appearing in the algebraic expressions (3-j and 6-j symbols, D^j functions). Such symbols can be joined to form spin diagrams, as long as any two lines that are joined have the same sense. Algebraic identities such as orthogonality and completeness for the 3-j symbols, and the Clebsch-Gordan reduction of D^j functions, correspond to simple and natural rules for the spin diagrams. These rules make it possible to do all derivations directly in terms of the diagrams. This has the considerable advantage that the structure of any expression always is clear, and the steps that lead to a simplification become evident.

The spin-diagram rules that we formulated in Sec. II C made it possible to give a diagrammatic interpretation of the derivation and final structure of our formula. However, the rules were not precise enough to account for all helicity-independent phase factors. We therefore used the diagrams mainly to guide the algebraic derivation. In the future it should prove useful to find a complete set of rules that makes it possible to write down the final expression directly from the spin diagram. Long and tedious algebraic manipulations would then become superfluous in a variety of spin-related problems.[18]

Having derived the general formula for a resonance contribution to the FESR, we applied it to study N, Δ cancellation for all amplitudes $R' + 3 \to R'' + 4$. Here R', R'' stands for any $I = 1$ meson trajectory, i.e., π-B, ρ-A_2, or A_1. The external particles 3, 4 can be N or Δ. A subset of these reactions, namely those with $R', R'' = \pi$ or ρ and $3, 4 = N$, were studied previously[6] using a more cumbersome technique. It was found that the cancellation requirement uniquely fixes the helicity structure and relative magnitudes of the πNN, $\pi N\Delta$, ρNN, and $\rho N\Delta$ couplings. The predicted couplings are in very good agreement with the experimental data.[6]

In the present paper we extended the results of Ref. 6 by considering off-shell π (or B) and A_1 Reggeons. We also studied amplitudes where one or both of the external baryons are Δ's. We did not prove that the couplings we obtain are the only solution to the cancellation requirement. However, the equations are so severely overconstrained that the existence of another solution seems highly unlikely.

All couplings RNN, $RN\Delta$, and $R\Delta\Delta$ were determined by imposing N, Δ cancellation on the amplitudes $R'N \to R''N$ and $R'N \to R''\Delta$ only. The helicity structure of each coupling depends only on the Reggeon R. In terms of the reduced couplings $T(l, \lambda)$ of Eq. (2.31), all couplings have $l = 1$. The t-channel helicity flip is $\lambda = 0$ for the π-B and $|\lambda| = 1$ for the ρ-A_2 and A_1 trajectories. The relative magnitudes of the couplings for a given Reggeon are specified by Eq. (3.17).

Apart from the couplings already considered in Ref. 6, there is one coupling ($BN\Delta$) obtained in this paper that can be confronted with experiment. Data on the reaction $\pi^+ p \to \omega \Delta^{++}$ at 7 GeV/c were recently analyzed[19] in terms of the same amplitudes $T(l, \lambda)$ that we used in this paper. The bulk of the cross section was found to be due to one natural-parity- (ρ) exchange amplitude $T(1,1)$ and one unnatural-parity- (B) exchange amplitude $T(1,0)$. These are precisely the couplings predicted to be important in our scheme.

From a more theoretical point of view, we find it very satisfying that the simple couplings we obtained guarantee an N, Δ cancellation in the large number of zeroth-moment FESR's that can be formed from the $R'N \to R''N$ and $R'N \to R''\Delta$ amplitudes. This indicates that the sum rules are satisfied in an analogous way for many different amplitudes, with the external lines being either particles or Reggeons. The final set of amplitudes, describing the reactions $R'\Delta \to R''\Delta$, served as a consistency check, since all relevant couplings had already been determined. We found that the $N+\Delta$ contribution indeed is small for all $R'\Delta \to R''\Delta$ amplitudes with $l=3$. For the $l=2$ amplitudes there is a partial (64%) cancellation. For the $l=0$ and $l=1$ amplitudes (that correspond to the $A'^{(-)}$ and $B^{(+)}$ amplitudes in the $\pi N \to \pi N$), however, the N and Δ contributions have the same sign. Thus for these two amplitudes the FESR's for the $R'N \to R''N$ and $R'\Delta \to R''\Delta$ processes must be satisfied differently, higher-mass resonances being relatively more important in the latter process.

It is interesting to ask[6] whether the coupling systematics obtained here is consistent also with the first-moment ($n=1$) FESR's. If the N and Δ cancel in an $n=0$ FESR they, in general, will not cancel in the $n=1$ FESR, which involves the s-u symmetric amplitude. Thus, if $N+\Delta$ is dual in a semilocal sense to a t-channel Regge exchange, that exchange must be nonzero. On the other hand, in amplitudes where N and Δ vanish separately in the $n=0$ FESR, they will vanish also in the $n=1$ FESR. For these amplitudes, then, the t-channel exchange must also vanish.

It is straightforward to study the $n=1$ FESR's using the formulas of Sec. III (but taking the symmetric combination of s- and u-channel contributions). For l, I being 0 or 1 (which are the only allowed values in $R'N \to R''N$) one finds that the $n=1$ FESR's are fully consistent with the couplings established above. Thus the t-channel exchange is nonzero in precisely the same amplitudes where the $N+\Delta$ contribution is nonvanishing. In particular, from the results of Sec. III it is immediately clear that the N and Δ will contribute to $n=1$ FESR's only when $(-1)^{l+I}=+1$. This agrees with the trajectories studied in this paper (π, ρ, A_1), which all had $I=l=1$. Furthermore, it predicts that the $I=0$ trajectories f, ω must have $l=0$, i.e., a helicity-nonflip coupling to nucleons. Conversely, the N and Δ do build up a nonzero exchange in the $l=I=0$ amplitudes.

In view of the contested status of the A_1 resonance and exchange,[20] we want to point out that neither the $n=0$ nor the $n=1$ FESR constraints considered above determine the magnitude of the A_1 couplings. The only amplitude in which the N, Δ s-channel resonances are dual to A_1 exchange (with the proper coupling) is $\rho N \to A_1 N$ with $(l, I) = (1,1)$. However, in this reaction the A_1 also appears as an external particle. Hence both the s-channel and t-channel couplings involve the A_1, allowing a solution where all the A_1 couplings vanish.

In conclusion, we feel that the compact and transparent formula we have developed, together with the success of the applications considered here, makes an investigation of FESR duality constraints for other reactions and resonances both feasible and attractive. By considering simultaneously the resonance contributions to several different FESR's one may hope to gain an understanding of how the sum rules are satisfied, and how the resonances must couple to each other. Eventually, other resonance parameters such as mass, spin, and parity should be constrained by the requirement of over-all self-consistency. Already in the example at hand, it is quite difficult to conceive of any resonance, with parameters different from those of the Δ, that would be able to cancel the N contribution in the FESR's we considered.

APPENDIX A: SYMMETRY PROPERTIES OF HELICITY COUPLINGS

1. The Lagrangian

We shall investigate the properties of helicity couplings by means of the effective Lagrangian that describes the coupling. Since we want to treat particles of arbitrary spin j, the Lagrangian is most conveniently expressed in terms of the fields constructed by Weinberg.[9] These fields are written in terms of creation and annihilation operators of helicity states, and transform according to the $(j,0)$ or $(0,j)$ representation of the Lorentz group. In addition, we shall assume that our fields belong to some irreducible representation I of SU_2 (= isospin). The presentation here will be self-contained, but we refer to the original papers[9] for a more complete discussion of the fields. In Appendix C we also give the relation between the Weinberg fields and the more conventional representations for $j=\frac{1}{2}$ and 1.

We shall denote the creation operator for a single-particle state $|\vec{p}, j, \lambda; I, r\rangle$ of momentum $\vec{p}$, spin j, helicity λ, isospin I and $I_Z=r$ by $a_r^+(\vec{p}, \lambda)$ (the spin and isospin labels will be suppressed). The normalization of these operators is such that

$$[a_r(\vec{p},\lambda), a^\dagger_{r'}(\vec{p}',\lambda')]_\pm = (2\pi)^3 2p^0 \delta^3(\vec{p}-\vec{p}')\delta_{rr'}\delta_{\lambda\lambda'}, \tag{A1}$$

where $p^0=(m^2+\vec{p}^{\,2})^{1/2}$ and $\pm$ denotes a commutator or anticommutator. The helicity states are as usual defined by

$$|\vec{p},j,\lambda;\ I,r\rangle = U(\Lambda_p)|\vec{0},j,\lambda;I,r\rangle, \tag{A2}$$

$$\Lambda_p = R_z(\phi)R_y(\theta)R_z(-\phi)B_z(|\vec{p}|) \equiv R(\theta,\phi)B_z(|\vec{p}|),$$

where $\vec{p}=(|\vec{p}|,\theta,\phi)$. From the well-known transformation properties of the helicity states[10] it follows that under a general Lorentz transformation Λ that takes $\vec{p}$ into $\vec{p}'$,

$$U(\Lambda)a^\dagger_r(\vec{p},\lambda)U^{-1}(\Lambda) = D^j_{\lambda'\lambda}(\Lambda_{p'}^{-1}\Lambda\Lambda_p)a^\dagger_r(\vec{p}',\lambda'). \tag{A3}$$

Here and in the following a sum on repeated indices is always implied. Equivalently,

$$U(\Lambda)a_r(\vec{p},\lambda)U^{-1}(\Lambda) = D^j_{\lambda\lambda'}(\Lambda_p^{-1}\Lambda^{-1}\Lambda_{p'})a_r(\vec{p}',\lambda'). \tag{A4}$$

Similarly, under an isospin rotation R_I we have

$$U(R_I)a_r(\vec{p},\lambda)U^{-1}(R_I) = D^I_{rr'}(R_I^{-1})a_{r'}(\vec{p},\lambda). \tag{A5}$$

We shall take our antiparticle creation and annihilation operators $b^\dagger_r(\vec{p},\lambda)$, $b_r(\vec{p},\lambda)$ to obey the same relations (A1) and (A3)–(A5). Note that the antiparticle operators are sometimes defined with an extra helicity-dependent phase factor. We leave this out—thus our antiparticle states transform exactly as the particle states.

Next we want to define field operators that transform according to the $(j,0)$ or $(0,j)$ representation of the Lorentz group. We denote the corresponding representation matrices by $D^j(\Lambda)$ and $\bar{D}^j(\Lambda)$, respectively. The generators of rotations and boosts are represented by

$$D^j:\ \vec{J}=\vec{J}^{(j)},\quad \vec{K}=-i\vec{J}^{(j)},$$
$$\bar{D}^j:\ \vec{J}=\vec{J}^{(j)},\quad \vec{K}=+i\vec{J}^{(j)}, \tag{A6}$$

where $J^{(j)}$ is the standard spin-j representation of angular momentum. Thus for pure rotations the matrices D^j and $\bar{D}^j$ are the same; in general, the relation is

$$\bar{D}^j(\Lambda) = D^{j\dagger}(\Lambda^{-1}). \tag{A7}$$

This can also be written

$$\bar{D}^{j*}(\Lambda) = C^jD^j(\Lambda)C_j, \tag{A8}$$

where C^j and $C_j=(C^j)^{-1}$ are the $(2j+1)\times(2j+1)$ matrices defined by Eqs. (2.15) of the text.

The fields $\phi^j_{\sigma,r}(x)$ and $\chi^j_{\sigma,r}(x)$ which belong respectively to the $(j,0)$ and $(0,j)$ representations of the Lorentz group and carry isospin I are defined by

$$\phi^j_{\sigma,r}(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2p^0}[D^j_{\sigma\lambda}(\Lambda_p)a_r(\vec{p},\lambda)e^{-ip\cdot x} + D^j_{\sigma\lambda}(\Lambda_p)C^{\lambda\lambda'}_jC^{rr'}_Ib^\dagger_{r'}(\vec{p},\lambda')e^{ip\cdot x}],$$
$$\chi_{\sigma,r}(x) = \int\frac{d^3\vec{p}}{(2\pi)^3 2p^0}[\bar{D}^j_{\sigma\lambda}(\Lambda_p)a_r(\vec{p},\lambda)e^{-ip\cdot x} + (-1)^{2j}\bar{D}^j_{\sigma\lambda}(\Lambda_p)C^{\lambda\lambda'}_jC^{rr'}_Ib^\dagger_{r'}(\vec{p},\lambda')e^{ip\cdot x}]. \tag{A9}$$

It is straightforward to verify the transformation laws:

$$U(\Lambda)\phi^j_{\sigma,r}(x)U^{-1}(\Lambda) = D^j_{\sigma\sigma'}(\Lambda^{-1})\phi^j_{\sigma',r}(\Lambda x), \tag{A10}$$

$$U(\Lambda)\chi^j_{\sigma,r}(x)U^{-1}(\Lambda) = \bar{D}^j_{\sigma\sigma'}(\Lambda^{-1})\chi^j_{\sigma',r}(\Lambda x). \tag{A11}$$

Under isospin rotations,

$$U(R_I)\phi^j_{\sigma,r}(x)U^{-1}(R_I) = D^I_{rr'}(R_I^{-1})\phi^j_{\sigma,r'}(x), \tag{A12}$$

with an identical relation for $\chi^j_{\sigma,r}(x)$.

We can now write down the general form of the Lagrangian that describes the interaction of three particles of spin j_k and isospin I_k $(k=1,2,3)$:

$$\mathcal{L}(x) = \chi^{j_1\dagger}_{\alpha,r}(x)R^{\alpha\beta\gamma}(i\vec{\partial},i\vec{\partial})\phi^{j_2}_{\beta,s}(x)\phi^{j_3}_{\gamma,t}(x)\begin{pmatrix} r & I_2 & I_3\\ I_1 & s & t\end{pmatrix} + \text{H.c.} \tag{A13}$$

Any invariant trilinear Lagrangian can be brought into this form by utilizing the derivative relations that connect fields (of given spin) belonging to different representations of the Lorentz group. Examples of these relations and of the Lagrangian (A13) are given in Appendix C. The differential operators in $R^{\alpha\beta\gamma}$ act on the fields χ^{j_1} and ϕ^{j_2} only; this can always be arranged by partial integration. Invariance of the Lagrangian under Lorentz transformations imposes certain conditions on $R^{\alpha\beta\gamma}$, which can be worked out using Eqs. (A10)–(A11). We shall not need those here. Finally, isospin invariance is explicitly guaranteed by the 3-j symbol in Eq. (A13). For a discussion of the definition and properties of the 3-j symbols see Sec. II C of the text. In particular, using Eq. (2.16), it can easily be verified that the Lagrangian is invariant under isospin rotations.

We shall make use of Eq. (A13) for two purposes. First, we want to find the precise connection between amplitudes in different channels that are related by crossing symmetry. The crossing properties of helicity amplitudes have, of course, been extensively studied previously.[10] Here we shall see how isospin can be naturally included in the same formalism. Second, we take advantage of the simple behavior of the fields (A9) under $\mathcal{CPT}$ symmetry to derive the relation between the charge-conjugation, parity, and time-reversal quantum numbers of any particle. A correct treatment of both of these questions is essential for the derivation of our formula in Sec. II.

2. The crossing relation

From the form (A9) of the fields it is clear that an amplitude with an incoming particle of momentum p is related to the amplitude with an outgoing antiparticle of momentum $-p$. Since all other factors in the interaction (A13) are identical for the two amplitudes (in particular, $R^{\alpha\beta\gamma}$ is the same function of the momenta) the connection is determined by the wave functions multiplying the creation and annihilation operators of the fields. We shall first discuss the helicity-dependent part of the wave functions, and then go on to the isospin dependence.

To be specific, consider the amplitude that has an incoming particle of spin j_2, helicity λ, and a physical momentum p ($p^0 > m$). The particle state is annihilated by the field $\phi^{j_2}_{\beta;s}$ of Eq. (A13), leaving the wave function

$$D^{j_2}_{\alpha\lambda}(\Lambda_p). \tag{A14}$$

The same field annihilates the antiparticle of helicity μ and physical momentum p' ($p^{0'} > m$) in the amplitude where the antiparticle is outgoing. The helicity-dependent part of the wave function is

$$D^{j_2}_{\alpha\mu'}(\Lambda_{p'})C^{\mu'\mu}_{j_2} = (-1)^{j_2+\mu}D^{j_2}_{\alpha,-\mu}(\Lambda_{p'}). \tag{A15}$$

Our task is now to find the relation between the wave functions when p' is analytically continued to the unphysical value $-p$. As is well known, the relation depends on the path of the analytic continuation.[10,21] We shall choose the path

$$\begin{aligned} &p' \to -p: \\ &(\vec{p}'^2)^{1/2} \to (\vec{p}^2)^{1/2}, \\ &\theta' \to \pi - \theta, \\ &\phi' \to \phi + \pi. \end{aligned} \tag{A16}$$

In terms of the boost parameter ζ',

$$E' = m\cosh\zeta',$$
$$|\vec{p}'| = m\sinh\zeta',$$

the continuation (A16) (together with $E' \to -E$) implies[22] $\zeta' \to i\pi - \zeta$. The transformation Λ_{-p} for the helicity state with four-momentum $-p$ is thus

$$\Lambda_{-p} = e^{-i(\phi+\pi)J_z}e^{-i(\pi-\theta)J_y} \times e^{-i(\phi+\pi)J_z}e^{-i(i\pi-\zeta)K_z},$$

where we have defined the continuation of the third Euler angle so that Λ_{-p} is independent of ϕ when $\theta = 0$. After the continuation $p' \to -p$ the antiparticle wave function (A15) is

$$(-1)^{j_2+\mu}e^{-i(\phi+\pi)\alpha}d^{j_2}_{\alpha,-\mu}(\pi-\theta)e^{i(\phi+\pi)\mu}e^{(i\pi-\zeta)\mu} = e^{i\pi j_2}(-1)^{j_2-\mu}D^{j_2}_{\alpha\mu}(\Lambda_p). \tag{A17}$$

Comparing Eqs. (A14) and (A17) we have the following rule.

Helicity crossing rule: The amplitude $T_\lambda(p)$ describing a vertex with an incoming particle of spin j, helicity λ, and four-momentum p is related to the amplitude $T^c_\lambda(-p)$ for the same vertex with the particle replaced by the outgoing antiparticle of the same helicity λ and momentum $-p$ by

$$T^c_\lambda(-p) = e^{+i\pi j}(-1)^{j-\lambda}T_\lambda(p) \quad (p^0 > m), \tag{A18}$$

when the path of the analytic continuation is given by Eq. (A16).

It can be readily verified that the same rule holds when the particle is annihilated by a χ field in the Lagrangian. On the other hand, if "particle" and "antiparticle" are interchanged everywhere in the above rule, so that the states are annihilated by the fields $\phi^\dagger$ or $\chi^\dagger$, there is an additional factor $(-1)^{2j}$ on the right-hand side of Eq. (A18). Finally, "incoming particle" may be interchanged with "outgoing antiparticle" without any change in Eq. (A18).

Turning next to the isospin part of the wave function, we want to find a simple rule that

(i) expresses the vertex in each channel as a product of an isospin reduced vertex and a 3-j symbol, and

(ii) gives the connection implied by crossing symmetry between the so defined reduced vertices.

The simplest rule is obtained if we associate the isospin-dependent part of the wave functions with the 3-j symbol. The isospin-reduced vertices will then, by construction, obey the helicity crossing rule (A18). Thus the only problem is to specify how to write the 3-j symbol in each channel.

This becomes clear if we consider a few examples. For $\bar{1}+2+3 \to$ vacuum, i.e., when all lines are incoming with $I_z = r'$, s', and t', respectively, we see from Eqs. (A9) and (A13) that the isospin factor is

$$C_{I_1}^{rr'}\begin{pmatrix} r & I_2 & I_3 \\ I_1 & s' & t' \end{pmatrix} = \begin{pmatrix} I_1 & I_2 & I_3 \\ r' & s' & t' \end{pmatrix} .$$

Similarly, in the case of $2+3\to 1$ we have

$$\begin{pmatrix} r' & I_2 & I_3 \\ I_1 & s' & t' \end{pmatrix},$$

whereas $\bar{1}+2\to\bar{3}$ gives

$$C_{I_1}^{rr'}C_{I_3}^{tt'}\begin{pmatrix} r & I_2 & I_3 \\ I_1 & s' & t \end{pmatrix} = \begin{pmatrix} I_1 & I_2 & t' \\ r' & s' & I_3 \end{pmatrix}(-1)^{2I_3}.$$

These examples adequately illustrate the following rule.

Isospin crossing rule: In each channel of a three-point vertex write the amplitude as a reduced amplitude multiplied by a 3-j symbol with the I_z components in the "down" position for incoming lines, in the "up" position for outgoing lines. Include an extra factor $(-1)^{2I}$ for each outgoing antiparticle of isospin I. Then the crossing relation between the reduced amplitudes in different channels is given by the helicity crossing rule (A18). The ordering of the lines in the 3-j symbol is arbitrary, but should be the same in all channels.

(A19)

The above crossing rules were derived for the special case of three-point vertices. However, it is clear from the derivation that the rules can be applied also to amplitudes with more than three external particles. For example, the isospin dependence of the reaction $1+2\to 3+4$ can be described by summing over the isospin I_s of the $12\equiv s$ system. There are two 3-j symbols for each term, corresponding to the vertices $(I_1I_2I_s)$ and $(I_sI_3I_4)$. Crossing to the t channel $2+\bar{3}\to\bar{1}+4$ can be done according to the above rules by treating each vertex separately. The amplitude is still expanded over the s-channel isospin, however. An expansion in terms of t-channel isospin can be obtained by means of the usual s-t isospin crossing matrix.

As a corollary to the rule (A19) let us write down the crossing relation for the full amplitude (i.e., including the isospin 3-j factors). If T_r is the amplitude for a vertex with an incoming particle of isospin I and $I_z=r$, then the amplitude $T_{r'}^c$ for the same vertex with the particle replaced by its outgoing antiparticle that has $I_z=r'$ is given by

$$T_{r'}^c=(-1)^{2I}C_I^{r'r}T_r. \tag{A20}$$

If we had crossed an incoming antiparticle into an outgoing particle, the factor $(-1)^{2I}$ in Eq. (A20) would have been absent. Here we have suppressed all the dependence, given by Eq. (A18), on the spin and helicity of the particle.

3. The $\mathcal{CPT}$ connection

An important property of the Weinberg fields (A9) is that they are essentially invariant under $\mathcal{CPT}$ conjugation. This makes a proof of the $\mathcal{CPT}$ theorem straightforward, and establishes the relation between the $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ quantum numbers of any particle.

We define the quantum numbers of the particle states under the discrete symmetries by

$$\begin{aligned} \mathcal{C}a_r^\dagger(\vec{p}_z,\lambda)\mathcal{C}^{-1}&=\eta_C(-1)^{I-v}C_I^{rr'}b_{r'}^\dagger(\vec{p}_z,\lambda) ,\\ \mathcal{P}a_r^\dagger(\vec{p}_z,\lambda)\mathcal{P}^{-1}&=\eta_PC_{\lambda\lambda'}^{j}a_r^\dagger(-\vec{p}_z,\lambda') ,\\ \mathcal{T}a_r^\dagger(\vec{p}_z,\lambda)\mathcal{T}^{-1}&=\eta_Ta_r^\dagger(-\vec{p}_z,\lambda) .\end{aligned} \tag{A21}$$

Here $\vec{p}_z$ is a momentum along the positive z axis. The helicity state created by $a_r^\dagger(-\vec{p}_z,\lambda)$ is defined by Eq. (A2) with $\theta=\pi$, $\phi=0$. $v=0$ $(\frac{1}{2})$ for integer (half-integer) values of I. The phase factors η_C, η_P, and η_T can be chosen to be ± 1. We define the phases $\bar{\eta}_C$, $\bar{\eta}_P$, and $\bar{\eta}_T$ for the antiparticle states exactly as in Eq. (A21), with $a^\dagger\to b^\dagger$ everywhere. The requirement that the annihilation and creation parts of the fields (A9) transforms in the same way under $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ then leads to the relations

$$\begin{aligned} \bar{\eta}_C&=(-1)^{2I}\eta_C ,\\ \bar{\eta}_P&=(-1)^{2j}\eta_P ,\\ \bar{\eta}_T&=\eta_T .\end{aligned} \tag{A22}$$

The transformation properties of the fields can be worked out using Eqs. (A21) and (A22) and the relation (A8) between the $(j,0)$ and $(0,j)$ representations. Here we only need the combined result

$$\begin{aligned} \mathcal{CPT}\,\phi_{\sigma,r}(x)\mathcal{T}^{-1}\mathcal{P}^{-1}\mathcal{C}^{-1}&=(-1)^{I-v}\eta_C\eta_P\eta_T\phi_{\sigma,r}^\dagger(-x) ,\\ \mathcal{CPT}\cdot\chi_{\sigma,r}(x)\mathcal{T}^{-1}\mathcal{P}^{-1}\mathcal{C}^{-1}&=(-1)^{2j}(-1)^{I-v}\eta_C\eta_P\eta_T\chi_{\sigma,r}^\dagger(-x) .\end{aligned} \tag{A23}$$

The $\mathcal{CPT}$ invariance of the Lagrangian is guaranteed provided

$$(-1)^{I-v}\eta_C\eta_P\eta_T=1 \tag{A24}$$

for each field. Indeed, we see from Eq. (A13) that the combined operation of a $\mathcal{CPT}$ transformation and a reversal of the ordering of the fields [which takes care of the factor $(-1)^{2j}$ in Eq. (A23)] amounts to

$$\mathcal{L}(x)\to\mathcal{L}^\dagger(-x)$$

and thus leaves the action unchanged.

For mesons that are eigenstates of $\mathcal{C}$, Eq. (A24) determines η_T. For baryons, on the other hand, η_C must be fixed by convention. Eq. (A24) then

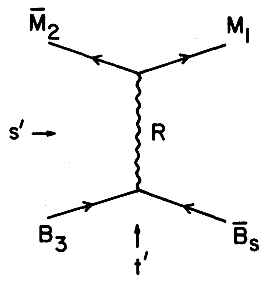


FIG. 10. Four-point amplitude for $B\bar{B}\to M\bar{M}$.

specifies the value of η_T associated with a given convention. This is the fact that we need in our derivation in Sec. II. The u-channel resonance contribution in Fig. 9 can be related to the s-channel contribution (Fig. 1) by charge conjugation. When particles 3 and 4 are (distinct) baryons, it might seem that the result depends on the convention chosen for η_C. However, by Eq. (A24) different conventions for η_C correspond to different values for η_T. This means that the phases of the vertices involving baryons 3 and 4 depend on the convention. For example, using Eq. (A21) we get for the $3sR'$ vertex (when the Reggeon is on a particle pole)

$$\begin{aligned}\langle R'|\mathcal{L}\,|3s\rangle &= [(\langle R'|\mathcal{T}^\dagger e^{-i\pi J_y})\mathcal{L}e^{i\pi J_y}\,\mathcal{T}|3s\rangle]^* \\ &= \eta_T^{R'}\,\eta_T^3\eta_T^s\langle R'|\mathcal{L}\,|3s\rangle^*\ .\end{aligned}\qquad\text{(A25)}$$

Thus $\eta_T^3 \to -\eta_T^3$ implies $\langle R'|\mathcal{L}|3s\rangle \to i\langle R'|\mathcal{L}|3s\rangle$. It can be easily seen that this phase change, together with the change in sign of η_C^3, makes the calculation in Sec. II convention-independent.

In Sec. II we shall for definiteness assume that

$$\eta_T^3 = \eta_T^4\ . \qquad\text{(A26)}$$

This choice is always possible when particles 3 and 4 are baryons. In applications where all external particles are mesons Eq. (A26) may not hold. The relative sign between the s- and u-channel resonance contributions would then have to be reversed.

4. Applications to helicity amplitudes

As an illustration of the above methods, we shall derive some symmetry relations for the Reggeon couplings that we need in the text. We consider the high-energy Regge pole limit of a c.m. helicity amplitude for the process $B_3\bar{B}_s\to M_1\bar{M}_2$ (Fig. 10). M_1 and M_2 are spinless mesons. B_3 and B_s will be baryons in our applications, but can also be thought of as mesons. The spin, helicity, isospin, and I_z component of particle k will be denoted by j_k, λ_k, I_k, and a_k. The Reggeon R is characterized by its naturality σ, signature τ, isospin I, $I_z=a$, and charge conjugation C.

In accordance with the above discussion of crossing we define the isospin-reduced amplitude by writing the complete amplitude as

$$T(M_1\bar{M}_2B_3\bar{B}_s) = T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s)\begin{pmatrix} a_1 & a_2 & I\\ I_1 & I_2 & a\end{pmatrix}\begin{pmatrix} I_3 & I_s & a\\ a_3 & a_s & I\end{pmatrix}(-1)^{2I_2}\ . \qquad\text{(A27)}$$

The ordering of the lines in the 3-j symbols is conventionally fixed to be $M\bar{M}R$ and $B\bar{B}R$ (note that the ordering will therefore be different in amplitudes related by charge conjugation). We do not include a "particle-2" phase[8] in the definition of our helicity states. We want to find the symmetry relations imposed on the reduced amplitude by parity, charge conjugation, and signature.

Parity. The standard formulas give

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s) = \sigma\eta_P^3\eta_P^s(-1)^{j_3+j_s}(-1)^{\lambda_3-\lambda_s}T_{-\lambda_3-\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s)\ , \qquad\text{(A28)}$$

where $\sigma=\eta_P^1\eta_P^2$. Note that η_P^s is the parity of B_s, not that of $\bar{B}_s$.

Charge conjugation. The transformation properties of the states under $\mathcal{C}$ are given by Eqs. (A21) and (A22). The relation between the complete amplitudes is thus (we indicate I_z of each particle in parentheses)

$$T[M_1(a_1)\bar{M}_2(a_2)B_3(a_3)\bar{B}_s(a_s)] = \prod_k[(-1)^{I_k-v}\eta_C^k\,C_{I_k}^{a_ka_k'}](-1)^{2I_2+2I_s}T[\bar{M}_1(a_1')M_2(a_2')\bar{B}_3(a_3')B_s(a_s')]\ . \qquad\text{(A29)}$$

It is now straightforward to deduce the relation between the isospin-reduced amplitudes. Taking into account the different ordering in the 3-j symbols and the extra phase for outgoing antiparticles we find, assuming I to be integer,

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s) = \eta_C^1\eta_C^2\eta_C^3\eta_C^s\,T_{\lambda_3\lambda_s}(\bar{M}_1M_2\bar{B}_3B_s). \qquad\text{(A30)}$$

Note that all phases η_C^k are again defined for *particle* states, according to Eq. (A21). For half-integer I there is an extra factor $(-1)^{2I_2+2I_s}$ on the right-hand side of Eq. (A30).

Charge conjugation at one vertex. The above formulas (A28) and (A30) are valid irrespective of the exchange mechanism. For a single Regge-pole exchange that is an eigenstate[23] of $\mathfrak{C}$ each vertex can be separately charge-conjugated. For $a_1 + a_2 = a_3 + a_s = 0$ we have then

$$T[M_1(a_1)\bar{M}_2(a_2)B_3(a_3)\bar{B}_s(a_s)] = (-1)^{I+I_3+I_s} C\eta_C^3\eta_C^s C_I^{00} C_{I_3}^{a_3 a'_3} C_{I_s}^{a_s a'_s} T[M_1(a_1)\bar{M}_2(a_2)\bar{B}_3(a'_3)B_s(a'_s)] \ , \tag{A31}$$

which gives

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s) = C\eta_C^3\eta_C^s\, T_{\lambda_3\lambda_s}(M_1\bar{M}_2\bar{B}_3B_s). \tag{A32}$$

$\mathfrak{C}\tau$ symmetry. When we combine the charge-conjugation relation (A32) with signature symmetry

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s) = \tau\, T_{\lambda_s\lambda_3}(M_1\bar{M}_2\bar{B}_sB_3) \tag{A33}$$

we get

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B_3\bar{B}_s) = C\tau\eta_C^3\eta_C^s\, T_{\lambda_s\lambda_3}(M_1\bar{M}_2B_s\bar{B}_3). \tag{A34}$$

In the special case when $B_3 = B_s = B$, Eq. (A34) simplifies to

$$T_{\lambda_3\lambda_s}(M_1\bar{M}_2B\bar{B}) = C\tau\, T_{\lambda_s\lambda_3}(M_1\bar{M}_2B\bar{B}) \ . \tag{A35}$$

Combined with the parity relation (A28), Eq. (A35) forces certain Reggeon couplings to vanish. Thus for $B = N$ we get the well-known result that the π trajectory coupling to $N\bar{N}$ is pure helicity nonflip, while the coupling of the A_1 trajectory is pure helicity flip.

APPENDIX B: KINEMATIC SINGULARITIES

There are two types of kinematic singularities that we have to consider in this paper. First, since we want to write FESR's for the six-point helicity amplitude in Fig. 1a, we have to remove its kinematic singularities in s. Second, in our application of the FESR's to the N, Δ cancellation problem in Sec. III we make use of the small mass difference between the two contributions by putting $m_N = m_\Delta$. This is a good approximation only for amplitudes that do not have kinematic singularities at the t-channel pseudothreshold $t = (m_\Delta - m_N)^2$. We therefore need to know the (pseudo-) threshold behavior of the amplitudes $T(l', \lambda')$ defined in Eq. (2.30). In this Appendix we discuss these two different aspects of kinematic singularities.

1. Kinematic singularities of the six-point amplitude

Consider first the analogous problem for the four-point amplitude $B_3\bar{B}_4 \to \bar{M}_{12}M_{56}$ (i.e., with the Reggeons on particle poles). As is well known,[10] all the kinematic singularities in s of the t-channel helicity amplitudes for this reaction occur on the boundary of the physical region. When M_{12} and M_{56} are spinless, the singularities can be removed by writing

$$T_{\lambda_3\lambda_4}(\bar{M}_{12}M_{56}B_3\bar{B}_4) = (\sin\theta_t)^{|\lambda_3-\lambda_4|}\hat{T}_{\lambda_3\lambda_4}(\bar{M}_{12}M_{56}B_3\bar{B}_4) \ , \tag{B1}$$

where θ_t is the t-channel scattering angle and $\hat{T}$ is free of kinematic singularities in s.

We want to find the analog of Eq. (B1) for amplitudes with four spinless particles in the final state (cf. Fig. 1a). One could try to solve this problem using the general methods developed for studying the kinematic singularities of N-point amplitudes.[24] For our special application, however, we find it more illuminating to generalize the derivation given in Ref. 6, which was based on invariant amplitudes.

We shall first give the general (somewhat heuristic) argument, and then check the result using the explicit calculations of Ref. 6. Consider the amplitude for the process $B_3\bar{B}_4 \to M_1\bar{M}_2M_5\bar{M}_6$ of Fig. 1a evaluated in the t-channel c.m. In the standard reference frame, where B_3 and $\bar{B}_4$ move along the z axis with $\vec{p}_1 + \vec{p}_2$ and $\vec{p}_5 + \vec{p}_6$ lying in the xz plane, the particle momenta are given by Eq. (2.1) of the text. The energy which we disperse in is $s = (p_3 - p_1 - p_2)^2$, and the angle θ_t between $\vec{p}_3$ and $\vec{p}_1 + \vec{p}_2$ is the t-channel scattering angle for the Reggeon amplitude. Note that the conditions[6] required for FESR analyticity have already been imposed in Eq. (2.1). They imply that the momenta p_1, p_2, p_5, and p_6 are all proportional to the single four-vector p_0. As we shall see, this greatly simplifies the kinematic structure of the amplitude.

The amplitude consists of the wave functions for the two spinning particles B_3 and $\bar{B}_4$ multiplied by various combinations of the particle momenta, as required by Lorentz invariance, and by functions (= invariant amplitudes) that depend only on the invariants. As can be seen from Eq. (A9), the wave functions do not have s-dependent singularities. We shall assume that the invariant amplitudes are likewise free of kinematic singularities. The singularities in s must therefore come from the four-momenta.

Now by dotting the momentum p of any particle into $p_3 + p_4$ and $p_3 - p_4$, one can easily see that the expressions for p^0 and p^z in terms of the invariants are regular in s. Thus we only need concern ourselves with the dependence on p^x and p^y. This is dictated by the invariance of the amplitude under rotations $R_z(\phi)$ about the z axis. The product of the wave functions gets multiplied by a factor $e^{-i(\lambda_3-\lambda_4)\phi}$ under $R_z(\phi)$; this must be compensated for by a proper combination of p^x and p^y. Since

$$p^x \pm ip^y \xrightarrow[R_z(\phi)]{} e^{\pm i\phi}(p^x \pm ip^y),$$

the dependence on p^x, p^y must be contained in a factor

$$(p^x \pm ip^y)^{|\lambda_3-\lambda_4|}\ , \tag{B2}$$

where the + (−) sign corresponds to $\lambda_3-\lambda_4$ being >0 (<0).

In general, the helicity amplitude can be expanded into a sum of terms, each of which is multiplied by a factor (B2) with different choices for the momentum p. We want to extract a common kinematic factor that makes all the factors (B2) regular in s. In the case of our six-point amplitude, p could be either $p_{12} \equiv p_1 + p_2$ or any of the momenta p_1, p_2, p_5, p_6. Since $p_{12}^y = 0$ and $p_{12}^x \propto \sin\theta_t$ the first choice gives a factor

$$(\sin\theta_t)^{|\lambda_3-\lambda_4|}\ , \tag{B3}$$

This would be the only possible choice for a four-point amplitude, and we see that the result indeed agrees with Eq. (B1).

When any of the vectors p_1, p_2, p_5, or p_6 is substituted for p in Eq. (B2), the situation is, in general, complicated by the fact that p^y is a nontrivial function of the invariants. However, in the special case we consider here all these momenta are proportional to p_0 [cf. Eq. (2.1)]. Both p_0^x and p_0^y behave like $(\sin\theta_t)^{-1}$; thus Eq. (B2) becomes

$$(\sin\theta_t)^{-|\lambda_3-\lambda_4|}. \tag{B4}$$

The factor (B3) can be viewed as a special case of Eq. (B4), since factors of $(\sin\theta_t)^2$ are regular in s when they occur in the numerator. We have therefore

$$T_{\lambda_3\lambda_4}(B_3\bar{B}_4 \to M_1\bar{M}_2M_5\bar{M}_6) = (\sin\theta_t)^{-|\lambda_3-\lambda_4|}\hat{T}_{\lambda_3\lambda_4}(B_3\bar{B}_4 \to M_1\bar{M}_2M_5\bar{M}_6)\ , \tag{B5}$$

where $\hat{T}$ has no kinematic singularities in s.

The result (B5) implies that there is an important kinematic difference between Reggeon amplitudes and ordinary four-point amplitudes [which obey Eq. (B1)]. To establish this we should still make sure that T in Eq. (B5) cannot, in general, be factorized into a power of $(\sin\theta_t)^2$ multiplying a regular function of s [consistency with Eq. (B1) requires that this happen at least when both Reggeons are put on a spin-zero particle pole]. Let us therefore analyze the explicit calculation of Ref. 6 from this point of view.

From Eqs. (10a) and (10b) of Ref. 6 we learn, first of all, that the kinematic singularities of multiparticle amplitudes, in general, appear not only in factors of $\sin\theta_t$, but also in terms involving the antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. The fact that such terms are not present in Eq. (B5) is due to the constraint imposed on the momenta in Sec. II. When the momenta p_1, p_2, p_5 and p_6 are all proportional to the single vector p_0, the only invariant that can be formed with the ϵ symbol is

$$\epsilon_{\mu\nu\rho\sigma}p_0^\mu p_{12}^\nu p_3^\rho p_4^\sigma = \tfrac{1}{2}iq\sqrt{t}\ .$$

Here q is the final-state momentum in the t-channel c.m. [cf. Eq. (2.1)] and so does not depend on s. This shows again how the constraint on the momenta, originally derived[5,6] from the requirement that the invariant amplitudes should be FESR-analytic, actually simplifies the kinematic singularities as well.

All the helicity-flip amplitudes calculated in Ref. 6 have a *negative* power of $\sin\theta_t$ in the kinematic factor, as predicted by Eq. (B5). The coefficients of the invariant amplitudes do not have a common factor of $(\sin\theta_t)^2$ that could cancel this negative power. Finally, the invariant amplitudes themselves are not proportional to $(\sin\theta_t)^2$. This can be seen, e.g., by calculating the s-channel nucleon Born term. Equations (30) and (31) of Ref. 6 show that the residue of the Born term is constant (i.e., independent of s) in all invariant amplitudes.

We can also see from Eqs. (10d) and (15b) of Ref. 6 why the power of $\sin\theta_t$ is positive when both Reggeons are put on a spin-zero particle pole. From the definition of the invariant amplitudes, it is clear that only $\mathcal{Q}$ and $\mathcal{B}$, respectively, have the spin-zero pole. The coefficients of both of these amplitudes, however, are proportional to $(\sin\theta_t)^2$.

We conclude, then, that the explicit calculations of Ref. 6 support the general arguments that led to Eq. (B5). The striking fact that the kinematic singularities of Reggeon amplitudes must be extracted as a negative power of $\sin\theta_t$ may well have dynamical consequences. It means, in effect, that the low-mass contributions are, in general, more suppressed relative to the higher-mass contributions in Reggeon FESR's than they are in the corresponding particle FESR's.

2. *Pseudothreshold behavior of $T(l',\lambda')$*

In Sec. II we describe the contribution of a given resonance B_s to the discontinuity $\mathcal{T}$ of the six-point amplitude (Fig. 1) in terms of helicity amplitudes for the processes $B_3\bar{B}_s \to M_1\bar{M}_2$ and $B_4\bar{B}_s \to \bar{M}_5M_6$. The energy s that we disperse in thus enters as an external mass in these helicity amplitudes. Now it is well known[10] that helicity amplitudes have kinematic singularities at threshold and pseudothreshold. The t'-channel helicity amplitude $T_{\lambda_3\lambda_s}$ shown in Fig. 10 is therefore singular in s at $s=(m_3\pm\sqrt{t'})^2$, where t' is the Reggeon mass. We know that the full six-point discontinuity does not have these singularities in s (see subsection 1 of this Appendix). Hence they must be canceled by corresponding singularities in the Wigner angles (we discuss this in more detail in Sec. III). This is why we can make the approximation $\sqrt{s}=m_\Delta\simeq m_N$ in the expression for $\mathcal{T}$. However, we must be careful to apply that approximation only to helicity amplitudes $\hat{T}_{\lambda_3\lambda_s}$ which are regular at $s=(m_3\pm\sqrt{t'})^2$. Here we want to investigate the precise behavior of our helicity amplitudes at these values of s.

We shall use the method for evaluating threshold singularities proposed by Jackson and Hite.[25] Their procedure has the advantage of exposing the physical origin of the singularities (a mismatch between J and L). It can furthermore be applied with equal ease to a general four-point amplitude and to the special case when the Reggeon is a spin-zero particle (π or K). This is important, because the singularity structure is different in the two cases.

In Sec. II we describe the process $B_3\bar{B}_s\to M_1\bar{M}_2$ (Fig. 10) in terms of the amplitudes $T(l',\lambda')$ defined by Eq. (2.30) to be

$$T(l',\lambda')=(2l'+1)\begin{pmatrix} j_3 & j_s & \lambda' \\ \lambda_3 & -\lambda_s & l' \end{pmatrix} e^{i\pi\lambda'/2}T_{\lambda_3\lambda_s}. \tag{B6}$$

The definition of the 3-j symbol is given by Eqs. (2.14) and (2.17). Remember also that the helicity amplitudes $T_{\lambda_3\lambda_s}$ are defined without a "particle-2" phase.[8] Since

$$t'-(m_3-\sqrt{s})^2=-[\sqrt{s}-(m_3+\sqrt{t'})][\sqrt{s}-(m_3-\sqrt{t'})],$$

we can summarize the behavior of $T(l',\lambda')$ at $\sqrt{s}=m_3\pm\sqrt{t'}$ by its pseudothreshold behavior

$$T(l',\lambda')=[t'-(m_3-\sqrt{s})^2]^{-\beta/2}\,\hat{T}(l',\lambda'), \tag{B7}$$

where $\hat{T}(l',\lambda')$ is regular at $t'=(m_3-\sqrt{s})^2$.

For $\sqrt{s}=m_3\pm\sqrt{t'}$ the energies E_3, E_s of particles B_3 and $\bar{B}_s$ are, in the t' c.m. system,

$$E_3=\frac{t'+m_3{}^2-s}{2\sqrt{t'}}=\mp m_3,$$

$$E_s=\frac{t'-m_3{}^2+s}{2\sqrt{t'}}=\pm(m_3\pm\sqrt{t'})=\pm\sqrt{s}\,.$$

Thus for either value of $\sqrt{s}$ one of the particles is related to its rest frame by a complex boost $i\pi$ along the z axis. For example, when $\sqrt{s}=m_3-\sqrt{t'}$ we have $E_s=-\sqrt{s}$. According to Ref. 25 we should then construct the total spin S and angular momentum L, using the pseudoamplitudes

$$T^P_{\lambda_3\lambda_s}=(-1)^{j_s-\lambda_s}T_{\lambda_3\lambda_s}. \tag{B8}$$

In addition, the parity of $\bar{B}_s$ should be reversed if it is a fermion. It can easily be checked that (since l' is always an integer) the construction of S and L is completely equivalent in the case when $\sqrt{s}=m_3+\sqrt{t'}$. The pseudothreshold behavior of the amplitude is therefore the same at $\sqrt{s}=m_3\pm\sqrt{t'}$, as we have indeed already implied in Eq. (B7).

Substituting Eq. (B8) into Eq. (B6) we see that $l'=S$, i.e., the amplitudes $T(l',\lambda')$ correspond to a definite spin state S of particles B_3 and B_s in the crossed (s') channel. The calculation of the pseudothreshold behavior of $T(l',\lambda')$ then becomes very simple. The exponent β in Eq. (B7) is the maximum value of J-L, which, in general, is equal to S or $S-1$, depending on the parities:

$$\beta=l'-\tfrac{1}{2}[1-\sigma'\eta^3_P\eta^s_P(-1)^{l'}]. \tag{B9}$$

Here σ' is the naturality of the Reggeon, and η^3_P, η^s_P are the parities of B_3 and B_s. The result (B9) can be simply obtained also using the approach of Trueman,[26] or by direct inspection of the behavior of the s'-t' helicity crossing matrix.

When the Reggeon is on a $J=0$ particle pole, Eq. (B9) for β does not apply. Instead, we have $L=S=l'$ and so J-$L=-l'$. Thus

$$\beta=-l' \tag{B10}$$

when $J=0$ is the only allowed value of the total angular momentum. According to Eq. (B7) this means that $T(l',\lambda')$ in this case is constrained to vanish at pseudothreshold (for $l'>0$).

APPENDIX C: $(2j+1)$-COMPONENT FIELDS

In this appendix we present a brief discussion of the relationship between $(2j+1)$-component fields for $j=\frac{1}{2}$ and 1 and the more common Dirac and vector fields. Much of this discussion is contained either explicitly or implicitly in Weinberg's original papers.[9] For illustrative purposes, we will also translate a few familar Lagrangians into the $(2j+1)$-component formalism. Since isospin is not germane to this discussion, we write the $(j,0)$ and $(0,j)$ fields (A9) as

$$\phi^j_\sigma(x)=\int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}[D^j_{\sigma\lambda}(\Lambda_p)a(\vec{p},\lambda)e^{-ip\cdot x}+D^j_{\sigma\lambda}(\Lambda_p)C^{\lambda\lambda'}_j b^\dagger(\vec{p},\lambda')e^{ip\cdot x}], \tag{C1a}$$

$$\chi^j_\sigma(x)=\int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}[\bar{D}^j_{\sigma\lambda}(\Lambda_p)a(\vec{p},\lambda)e^{-ip\cdot x}+(-1)^{2j}\bar{D}^j_{\sigma\lambda}(\Lambda_p)C^{\lambda\lambda'}_j b^\dagger(\vec{p},\lambda')e^{ip\cdot x}]. \tag{C1b}$$

As shown by Weinberg, these two fields are related by a differential operator

$$\phi_\sigma^j(x) = \Pi_{\sigma\sigma'}^j(i\partial)\chi_{\sigma'}^j(x), \tag{C2}$$

where $\Pi_{\sigma\sigma'}^j(i\partial)$ is a polynomial of order $2j$ in the four components of $i\partial_\mu \equiv i(\partial/\partial x^\mu)$. It is easy to show that Eq. (C2) along with the Lorentz transformation properties of the fields (A10) and (A11) require that

$$\Pi_{\sigma\sigma'}^j(\Lambda p) = D_{\sigma\lambda}^j(\Lambda)\Pi_{\lambda\lambda'}^j(p)\bar{D}_{\lambda'\sigma'}^j(\Lambda^{-1}). \tag{C3}$$

The explicit form of $\Pi_{\sigma\sigma'}^j$ is then obtained by considering the special case $p = p(m) \equiv (m, 0, 0, 0)$ and $\Lambda = R =$ pure rotation. Since $Rp(m) = p(m)$, the matrix $\Pi_{\sigma\sigma'}^j(p(m))$ commutes with all the generators $J_i^{(j)}$ and is therefore a multiple of the unit matrix. Fixing the constant by inspection of (C1) and (C2) gives

$$\Pi_{\sigma\sigma'}^j(p) = D_{\sigma\lambda}^j(\Lambda_p)\bar{D}_{\lambda\sigma'}^j(\Lambda_p^{-1}). \tag{C4}$$

[In this entire discussion we are tacitly assuming that $\Pi(i\partial)$ is being applied to fields in the interaction picture, and we therefore only need to consider $\Pi(p)$ for on-mass-shell values of $p, p^2 = m^2$.] We may verify (C2) by direct substitution of (C1) and (C4). Note that in order to verify that $\Pi_{\sigma\sigma'}^j(i\partial)$ converts the antiparticle term of the χ field into the corresponding term of the ϕ field, we need the identity

$$\Pi_{\sigma\sigma'}^j(-p) = (-1)^{2j}\Pi_{\sigma\sigma'}^j(p). \tag{C5}$$

This can be obtained by continuing the boost and rotation parameters in (C4) just as we did in (A16) and (A17), which gives

$$D_{\mu\nu}^j(\Lambda_{-p}) = D_{\mu\nu'}^j(\Lambda_p)C_{\nu'\nu}^j e^{-i\pi\nu}, \tag{C6}$$

$$\bar{D}_{\nu\lambda}^j(\Lambda_{-p}^{-1}) = (-1)^{2j}e^{i\pi\nu}C_j^{\nu\lambda'}\bar{D}_{\lambda'\lambda}^j(\Lambda_p^{-1}), \tag{C7}$$

from which (C5) follows. Since $\Pi_{\sigma\sigma'}^j(p)$ is a polynomial in the components of p,[9] the relation (C5) is actually independent of the path of continuation.

Now consider an ordinary four-component Dirac field for a spin-$\frac{1}{2}$ particle,

$$\psi_\alpha(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}[u_\alpha(p,\lambda)a(\vec{p},\lambda)e^{-ip\cdot x} + (-1)^{j-\lambda}v_\alpha(p,\lambda)b^\dagger(\vec{p},\lambda)e^{ip\cdot x}]. \tag{C8}$$

(Here and elsewhere, a sum over repeated indices is implied.) The extra phase factor in the second term of (C8) is included so that the antiparticle states may have the same Lorentz transformation properties as particle states. The spinors in (C8) are obtained from

$$u_\alpha(p,\lambda) = S_{\alpha\beta}(\Lambda_p)u_\beta(p(m),\lambda), \tag{C9}$$

$$v_\alpha(p,\lambda) = S_{\alpha\beta}(\Lambda_p)v_\beta(p(m),\lambda), \tag{C10}$$

where $S_{\alpha\beta}(\Lambda)$ is the usual Dirac matrix representation of the Lorentz group, and the rest spinors $u(p(m),\lambda)$ and $v(p(m),\lambda)$ satisfy the rotational property

$$S_{\alpha\beta}(R)u_\beta(p(m),\lambda')D_{\lambda'\lambda}^{1/2}(R^{-1}) = u_\alpha(p(m),\lambda) \tag{C11}$$

$$S_{\alpha\beta}(R)v_\beta(p(m),-\lambda')D_{\lambda'\lambda}^{1/2}(R^{-1}) = v_\alpha(p(m),-\lambda). \tag{C12}$$

Using (C9)–(C12) along with (A3) and (A4), it is easy to show that

$$U(\Lambda)\psi_\alpha(x)U^{-1}(\Lambda) = S_{\alpha\beta}(\Lambda^{-1})\psi_\beta(\Lambda x). \tag{C13}$$

Now, proceeding along the same lines that led to Eq. (C4), we can show that the Dirac field can be written in terms of the 2-component field $\phi^{1/2}$ by

$$\psi_\alpha(x) = \Pi_{\alpha\sigma}^D(i\partial)\phi_\sigma^{1/2}(x), \tag{C14}$$

where

$$\Pi_{\alpha\sigma}^D(p) = u_\alpha(p,\lambda)D_{\lambda\sigma}^{1/2}(\Lambda_p^{-1}). \tag{C15}$$

Again, (C14) may be formally verified by substituting (C1a), (C8), and (C15) and noting the crossing property of Π^D,

$$\Pi_{\alpha\sigma}^D(-p) = v_\alpha(p,\lambda)(-1)^{1/2-\lambda}C_{\lambda\lambda'}^{1/2}D_{\lambda'\sigma}^{1/2}(\Lambda_p^{-1}), \tag{C16}$$

which can be derived by crossing both the wave function and the D matrix in (C15) (see Appendix A).

The connection between the Dirac field and the 2-component ϕ and χ fields can be made more explicit if, following Weinberg,[9] we introduce a particular representation of Dirac matrices in which $S_{\alpha\beta}(\Lambda)$ is explicitly reduced to 2×2 block form [corresponding to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation content of the Dirac field]:

$$\gamma^0 = \begin{pmatrix} 0 & \underline{1} \\ \underline{1} & 0 \end{pmatrix},$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{C17}$$

$$\gamma^5 = \begin{pmatrix} -\underline{1} & 0 \\ 0 & \underline{1} \end{pmatrix},$$

It is easy to show that the Lorentz-transformation matrices are reduced to

$$S(\Lambda) = \begin{pmatrix} D^{1/2}(\Lambda) & 0 \\ 0 & \bar{D}^{1/2}(\Lambda) \end{pmatrix} \tag{C18}$$

and therefore, in matrix notation

$$\Pi^D(p) = S(\Lambda_p)u(p(m))D^{1/2}(\Lambda_p^{-1}) = \begin{pmatrix} \underline{1} \\ \overline{\Pi}^{1/2}(p) \end{pmatrix}, \quad \text{(C19)}$$

where

$$\overline{\Pi}^j_{\sigma\sigma'}(p) = \overline{D}^j_{\sigma\lambda}(\Lambda_p)D^j_{\lambda\sigma'}(\Lambda_p^{-1}), \quad \text{(C20)}$$

and $u(p(m))$ is the 4×2 matrix whose components are $u_\alpha(p(m),\lambda) = \delta_{\alpha,\lambda} + \delta_{\alpha-2,\lambda}$. Note that $\overline{\Pi}^j(p)$ is just the inverse of Π^j and, hence

$$\chi^j_\sigma(x) = \overline{\Pi}^j_{\sigma\sigma'}(i\partial)\phi^j_{\sigma'}(x). \quad \text{(C21)}$$

It is now straightforward to express interactions involving Dirac bilinears in terms of the corresponding 2-component fields. For example, the Yukawa coupling of a nucleon to a pseudoscalar meson can be put in the form (A13) (suppressing isospin)

$$\overline{\psi}\gamma^5\psi\pi = \chi^{(1/2)\dagger}_\alpha R_{\alpha\beta}(i\overleftarrow{\partial}, i\overrightarrow{\partial})\phi^{1/2}_\beta\pi, \quad \text{(C22)}$$

where $R_{\alpha\beta}$ is a 2×2 matrix

$$R_{\alpha\beta}(p_1,p_2) = -\Pi^{1/2}_{\alpha\gamma}(p_1)\overline{\Pi}^{1/2}_{\gamma\beta}(p_2) - \delta_{\alpha\beta}. \quad \text{(C23)}$$

By a straightforward generalization of the preceding discussion, one obtains the relationship between the 3-component fields (C1) for $j=1$ and the 4-component vector field, which transforms by the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group. Define the vector field as

$$\rho_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}[\epsilon_\mu(p,\lambda)a(\vec{p},\lambda)e^{-ip\cdot x} + \epsilon^*_\mu(p,\lambda)b^\dagger(\vec{p},\lambda)e^{ip\cdot x}]. \quad \text{(C24)}$$

It can then be shown that

$$\rho_\mu(x) = \Pi^V_{\mu\sigma}(i\partial)\phi^1_\sigma(x), \quad \text{(C25)}$$

where

$$\Pi^V_{\mu\sigma}(p) = \epsilon_\mu(p,\lambda)D^1_{\lambda\sigma}(\Lambda_p^{-1}). \quad \text{(C26)}$$

Familiar interactions between Dirac and vector fields can now be translated into the $(2j+1)$-component formalism. For example, the coupling of a vector field to left- and right-handed currents becomes

$$\tfrac{1}{2}\overline{\psi}(x)\gamma^\mu(1+\gamma^5)\psi(x)\rho_\mu(x) = \chi^{(1/2)\dagger}_\alpha(x)\tau^\mu_{\alpha\beta}\chi^{1/2}_\beta(x)\Pi^V_{\mu\sigma}(i\overrightarrow{\partial})\phi^1_\sigma(x), \quad \text{(C27)}$$

$$\tfrac{1}{2}\overline{\psi}(x)\gamma^\mu(1-\gamma^5)\psi(x)\rho_\mu(x) = \phi^{(1/2)\dagger}_\alpha(x)\overline{\tau}^\mu_{\alpha\beta}\phi^{1/2}_\beta(x)\Pi^V_{\mu\sigma}(i\overrightarrow{\partial})\phi^1_\sigma(x), \quad \text{(C28)}$$

where

$$\tau^0_{\alpha\beta} = \overline{\tau}^0_{\alpha\beta} = \delta_{\alpha\beta}, \quad \text{(C29)}$$

$$\tau^i_{\alpha\beta} = -\overline{\tau}^i_{\alpha\beta} = (\sigma_i)_{\alpha\beta}. \quad \text{(C30)}$$

With a partial integration and the use of (C2) and (C21), Eqs. (C27) and (C28) can be put in the form (A13).

In this paper the $(2j+1)$-component formalism was used primarily to obtain symmetry and crossing properties of helicity amplitudes which were needed for the derivation in Sec. II. However, it can also be used in a more direct fashion to derive Feynman rules for helicity amplitudes which will then automatically satisfy the constraints imposed by local field theory.[9] The graphical rules obtained in this way lead naturally to the use of spin diagrams, Wigner rotations, and t-channel (i.e., collinear) helicity couplings. These matters will not be pursued further here.

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