Role of collective phenomena in high-energy multiple production*

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Beginning with qualitative ideas on two-particle correlations and rapidity cluster decomposition, collective phenomena are studied. These phenomena are found to be important at large multiplicities as well as at high transverse momentum. The discussion is given in terms of an explicit mathematical model which is formulated in terms of coupled nonlinear equations. These equations, which are shown to have nontrivial solutions, give constraints between experimental observables based upon an assumed pairing structure of the S-matrix elements. These equations provide a mechanism for relating the enhancement of the large-P_T cross sections to the low-multiplicity cross sections. Stronger predictions for P_T correlations are found if a nonlinear bootstrap condition is imposed. Some consequences of this bootstrap are studied.

I. INTRODUCTION

A. Perspective

The notion of a theory of hadron interactions can have at least three different meanings in current usage. It can be formal, being a description, for example, of hadrons through local fields. There are supposed to be a finite number of bare coupling constants and masses, together with a well-defined renormalization procedure. In such a formal theory, observables are, in principle, computable at all energies in terms of only a few parameters. At the opposite extreme, a purely phenomenological description is sometimes referred to as a theory. In such a theory, typically one phenomenon is parametrized in terms of a few parameters. Such a theory is supposed to be a good model of the data over a limited range of the relevant variables (energy, particle number, etc). Such a phenomenology is valuable when a formal theory is nonexistent, or if it describes an effect where computations in a normal theory are difficult or impractical.

Intermediate in scope and purpose between these two extremes, we wish to characterize a set of ideas or principles as a theory, provided these ideas admit a precise mathematical formulation, and allow quantitative predictions. As examples of the most successful of such theories we recall the usefulness of local current algebra, SU(3) and SU(6) symmetries, dual theories, and the Cabibbo theory of weak interactions. Such principles are intended to be precisely formulated and testable, although not necessarily exact or complete in a strictly formal sense. Moreover, they are expected to be applicable only in a limited domain.

A primary motive for studying such theories lies in the hope that they can be ultimately shown to be compatible with a more complete formal theory. A secondary motive, closely related, is a recognition of the role such intermediate theories play in many mature fields of science where a formal theory may be well established. The understanding of complex phenomena is often materially advanced when less than formal concepts can be used.

B. Description and motivation

In this paper we propose a set of principles to be satisfied by S-matrix elements. Precise formulation of these principles leads to coupled integral equations describing high-energy multiparticle production. The point of view we wish to emphasize in this work is the dynamic possibility of important collective phenomena in high-energy particle production. We have previously suggested[1] that local inclusive rapidity correlations and large multiplicity fluctuations as evidenced by the growth of f_2/f_1 might be due to such a collective mechanism. We conjecture here that the unexpectedly large inclusive cross sections at large transverse momentum may be due to similar phenomena.

To illustrate briefly what we intend by a collective mechanism, we review in outline a theory to describe high-energy multiplicity distributions.[1] In spirit and method this theory resembles the goals of the present article. The starting point consisted of the following qualitative ideas: (1) Rapidity correlations are (technically) long-range in that they fall off more slowly than an exponential. (2) The Pomeron, with intercept unity, is generated by means of summing over inelastic processes which proceed mainly through secondary exchanges, modified by the collective effect of final-state interactions. (3) Though the most singular part of the Pomeron factorizes, the inelastic sum need not produce a pole. Indeed, if the correlations are long-range [assumption (1)] one ex-

pects a critical behavior for the Pomeron singularity. This critical behavior has an analog in fluid physics; a simple qualitative description can be obtained in the Van der Waals theory. The above ideas lead to a phenomenology for the multiplicity distributions with two parameters, which are in fact determined by the assumptions, leaving a strong prediction for multiplicity distributions. This theory gives satisfactory quantitative predictions over Fermilab energies, and is expected to be valid at higher energies.

Besides giving a satisfactory description of available data, this simple phenomenology suggests at infinite energy that $f_2/f_1 \to \infty$, as a result of the long-range inclusive correlations. The importance of long-range correlations does not manifest itself dramatically at the exclusive level for low multiplicities; there the dominant effects are the secondary exchanges. The only dominant long-range effects come through the corrections due to final-state interactions. These effects, though individually small, can occur between all pairs of final particles. For a particular choice of couplings these effects can collectively add up in the sum over inelastic states to produce anomalous behavior inclusively. (At other values of the couplings, however, there may be only short-range correlations.) The collective effect is also manifested in a buildup of large-multiplicity ($n \approx f_1$) cross sections. Since $f_2/f_1 \to \infty$, the width of the multiplicity distribution is broader than it would be in short-range-order models. Thus as $f_1 \to \infty$, there is an enhanced cross section around the mean multiplicity.

We shall term the mechanism which accounts for the dramatic buildup of the effects of long-range correlations *collective*. It is an example, and not the only one we claim, of effects in a theory which in the sum are qualitatively different from the dominant behavior of any single term.

A natural question, to which we have no answer at present, is whether the observed rapidity correlation at short distances is due to a collective effect. At the first level of consideration it seems hardly necessary to invoke such an effect since many would argue that the correlations observed are due in a natural way to resonance production.[2] Still, there has been no proof, in fact, that this is so. It may be that a complete understanding of the correlations may require the inclusion of some collective mechanism in addition to a correct resonance spectrum. In this context collective effects include absorptive rescattering and stimulated emission processes.

One other place to look for some collective mechanism is in the large-P_T behavior at high energy. The inclusive cross sections above beam energies of 100 GeV show an enhancement at values of $P_T > 2$ GeV/c compared to a simple extrapolation of the lower P_T values.[3] In this paper we give reasons for supporting the idea that such behavior is collective.

In formulating a mathematical model in which to discuss collective phenomena, we are accepting a body of lore familiar in multiperipheral models.[4] We accept as the most important variables the rapidity of each produced particle and the particle number (multiplicity). Much of the dynamics is then a one-dimensional phenomenon. Underlying these statements is a belief that the multiparticle amplitude has a factorizable multi-Regge *limit* for all rapidities separated from each other by arbitrarily wide intervals, at least for the leading Regge singularities which contribute at or near the average multiplicities. We summarize this behavior as *rapidity cluster decomposition*. It is the assumption of a generalized Regge behavior; the multiperipheral form is not assumed at low subenergies.

In describing the transverse degrees of freedom we are influenced by the classical (eikonal) picture[5] of rescattering. In this spirit, we choose to hypothesize a factorization of the S-matrix elements for fixed impact parameters.

The plan of the remaining part of the paper is to introduce in Sec. II a mathematical model based on the idea that pairwise interactions dominate the exclusive production cross section. Section III describes a "natural" bootstrap equation which allows interesting and nontrivial solutions. The solutions are studied in Sec. IV and are found to relate inclusive large-P_T behavior to certain exclusive single-particle spectra. A discussion of the general results of the model are given in Sec. V. Various side issues are treated in depth in accompanying appendixes.

II. BASIC FORMALISM

The fundamental assumption we make is that an S-matrix theory can be constructed by proposing a mathematical algorithm for S-matrix elements. The algorithm would express higher-order elements in terms of a finite number of lower-order elements. Explicit justification from an underlying field theory is to be replaced by the requirement that the consequences of the theory conform to the experimentally known facts. Such algorithms must, of course, not grossly violate accepted principles such as crossing and unitarity. In the present formulation, we do not treat spin or internal quantum numbers. Because of the nonlinear nature of the type of model given below, these symmetries are expected to play an important role,

and are here ignored only to keep the mathematics manageable.

As one realization of this general program, we propose an ansatz for the amplitude of the process

$$a+b\to a'+b'+h_1+\cdots+h_n. \tag{2.1}$$

The matrix element $A_n(\vec{B}\,|\{y_i,\vec{b}_i\})$ depends on $\vec{B}$, the impact parameter of the initial state, and the rapidities y_i and impact parameters $\vec{b}_i$ of the n secondary particles; we combine these variables into a 3-vector $\vec{r}_i=(y_i,\vec{b}_i)$. The ansatz is

$$A_n(\vec{B}\,|\,\vec{r}_1\cdots\vec{r}_n)=h_0(\vec{B})\prod_j G(\vec{r}_j,\vec{B})\prod_{k\neq j}K(\vec{r}_j,\vec{r}_k,\vec{B}). \tag{2.2}$$

We do not explicitly indicate s dependences. The functions h_0, G, and K need to be specified for a complete prescription. We note here several considerations implicit in the ansatz (2.2):

(i) The impact parameter $\vec{b}_i$ is given the physical interpretation discussed by Webber.[6] In the high-energy limit, and taking the n produced secondaries to be centrally produced, we saturate the energy- and longitudinal-momentum-conservation rules by setting, in any phase-space integrals to be done over a' and b',

$$y_{a'}=y_a,$$

$$y_{b'}=y_b$$

$$[y_b-y_a\equiv Y \text{ and } y_a+y_b=0 \text{ (choosing c.m. frame)}].$$

In the same limit, we approximate the conservation of transverse angular momentum by

$$\vec{b}_{a'}=\vec{b}_a,$$

$$\vec{b}_{b'}=\vec{b}_b$$

$$[\vec{b}_b-\vec{b}_a\equiv\vec{B} \text{ and } \vec{b}_b+\vec{b}_a=0 \text{ (special choice of reference frame)}].$$

No other kinematical constraints are retained.

(ii) The semi-inclusive distribution for n produced particles is

$$\frac{d\sigma_n}{dy\,d^2P_T}=\frac{1}{(n-1)!}\int d^2b\int d^2b'e^{i\vec{P}_T\cdot(\vec{b}-\vec{b}')}\int d^3r_2\cdots d^3r_n\int d^2B A_n(\vec{B}\,|\,\vec{r},\vec{r}_2,\ldots,\vec{r}_n)A_n^*(\vec{B}\,|\,\vec{r}',\vec{r}_2,\ldots,\vec{r}_n), \tag{2.3a}$$

where $\vec{r}=(y,\vec{b})$ and $\vec{r}'=(y,\vec{b}')$. Similarly the two-particle semi-inclusive distribution is

$$\frac{d\sigma_n}{dy_1d^2P_{T_1}dy_2d^2P_{T_2}}=\frac{1}{(n-2)!}\int d^2b_1d^2b_1'd^2b_2d^2b_2'e^{i\vec{P}_{T_1}\cdot(\vec{b}_1-\vec{b}_1')}e^{i\vec{P}_{T_2}\cdot(\vec{b}_2-\vec{b}_2')}$$
$$\times\int d^3r_3\cdots d^3r_n\int d^2B A_n(\vec{B}\,|\,\vec{r}_1,\vec{r}_2,\ldots\vec{r}_n)A(\vec{B}\,|\,\vec{r}_1',\vec{r}_2',\ldots\vec{r}_n)^*. \tag{2.3b}$$

(iii) Our form for A_n generalizes models which argue for non-nearest neighbor two-body interactions,[1,7] as embodied in (2.2). The function $A_n(\vec{B}\,|\,\vec{r}_1\cdots\vec{r}_n)$ is assumed to be cluster decomposable in y and $\vec{b}$ separately. This is a constraint on the functions G and K; in particular $|K(\vec{r}_i,\vec{r}_j)|^2\to 1$ as $|y_i-y_j|\to\infty$. We recall that multiperipheral models can be expressed in a form such as (2.2), if G is a constant. Only for nearest neighbors in rapidity is K different from unity.[8] Alternatively, if $K\equiv 1$ and G is nontrivial, we obtain some of the eikonal models studied by Auerbach, Aviv, Blankenbecler, and Sugar.[9] Note that unless G and K are independent of $\vec{B}$, the *inclusive* densities will not obey rapidity cluster decomposition.

With the theory as presented, we can proceed in one of two ways: (a) For the unknown functions h_0, G, and K we can make some inspired physically motivated guesses[10]; or (b) the unknown functions themselves can be fixed by requiring them to satisfy certain constraints leading to deterministic equations.

We choose in Sec. III the second way of proceeding. For this we will need an effective approximation scheme. We proceed by first defining a formal generating functional

$$|h_0(\vec{B})|^2\mathfrak{F}_B[\phi]\equiv\sum_n\frac{1}{n!}\int d^3r_1\cdots d^3r_nA_n(\vec{B}\,|\,\vec{r}_1\cdots\vec{r}_n)A_n^*(\vec{B}\,|\,\vec{r}_1\cdots\vec{r}_n)\phi(\vec{r}_1)\cdots\phi(\vec{r}_n). \tag{2.4}$$

This functional $\mathfrak{F}_B[\phi]$ has, in general, an inclusive cluster expansion[11]

$$\ln\mathfrak{F}_B[\phi]=\int d^3r\,g_B(\vec{r})[\phi(\vec{r})-1]+\frac{1}{2!}\int d^3r_1d^3r_2u_B(\vec{r}_1,\vec{r}_2)[\phi(\vec{r}_1)-1][\phi(\vec{r}_2)-1]$$
$$+\sum_{n\geq 3}\frac{1}{n!}\int d^3r_1\cdots d^3r_nu_n(\vec{r}_1,\ldots,\vec{r}_n)[\phi(\vec{r}_1)-1]\cdots[\phi\vec{r}_n)-1]. \tag{2.5}$$

To determine g_B and u_B, given G and K, we must have the following two consistency conditions:

(i) By definition, $g_B(\vec{r})$ is the result of taking a functional derivative of $\mathfrak{F}$ by $\phi(\vec{r})$, setting $\phi = 1$ and dividing by $\mathfrak{F}[1]$:

$$g_B(\vec{r}) = \frac{1}{\mathfrak{F}[1]} \frac{\delta \mathfrak{F}_{[1]}}{\delta \phi(\vec{r})} \bigg|_{\phi=1}. \tag{2.6}$$

Using the explicit form (2.2) it can be seen that

$$g_B(\vec{r}) = |G(\vec{r}, \vec{B})|^2 \frac{\mathfrak{F}[K(\vec{r}, \cdot, \vec{B}) K^*(\vec{r}, \cdot, \vec{B})]}{\mathfrak{F}[1]}. \tag{2.7}$$

where the dot represents the integration variable of the functional integral [see (2.5) where the functional integral is defined].

(ii) Similarly one notes that by definition

$$u_B(\vec{r}_1, \vec{r}_2) + g_B(\vec{r}_1) g_B(\vec{r}_2) = \frac{1}{\mathfrak{F}[1]} \frac{\delta^2 \mathfrak{F}[1]}{\delta \phi(\vec{r}_1) \delta \phi(\vec{r}_2)} \bigg|_{\phi=1}. \tag{2.8}$$

The explicit form of A_n can then be used with (2.8) to obtain

$$u_B(\vec{r}_1, \vec{r}_2) + g_B(\vec{r}_1) g_B(\vec{r}_2) = |K(\vec{r}_1, \vec{r}_2, \vec{B})|^2 |G(\vec{r}_1, \vec{B})|^2 |G(\vec{r}_2, \vec{B})|^2 \times \frac{\mathfrak{F}[|K(\vec{r}_1, \cdot, \vec{B})|^2 |K(\vec{r}_2, \cdot, \vec{B})|^2]}{\mathfrak{F}[1]}. \tag{2.9}$$

Similar expressions can be derived for the higher-order functions $u_n(\vec{r}_1, \ldots, \vec{r}_n)$.

We now develop the theory assuming that $V \equiv K - 1$ is small compared to unity. Thus in the expressions for g_B, u_B, u_n, the arguments ϕ of the functionals $\mathfrak{F}[\phi]$ will always be near unity, so that we expect $u/g_B g_B$ to be of order V and higher correlations $u_n/g_B \cdots g_B$ to be of lower order. Allowing $g_B V$ to be at most of order unity, the first two terms in (2.5) will be of maximum order V, while all higher terms will be of smaller order. We thus arrive at the following truncated expansion for $\mathfrak{F}$:

$$\ln \mathfrak{F}_B[\phi] = \ln \mathfrak{F}_B[1] + \int d^3r\, g_B(\vec{r})[\phi(\vec{r}) - 1] + \frac{1}{2!} \int d^3r_1 d^3r_2 u_B(\vec{r}_1, \vec{r}_2) \times [\phi(\vec{r}_1) - 1][\phi(\vec{r}_2) - 1], \tag{2.10}$$

valid only when ϕ is near unity. A different way of understanding this truncation is discussed in Appendix A. A more general kind of approximation, valid when strong short-range correlations are important, is given in Appendix B.

Now if we substitute (2.10) into (2.7), and use $V \ll 1$, the g equation becomes

$$g(\vec{r}) = |G(\vec{r})|^2 \exp\left[2 \int d^3r' g(\vec{r}') V(\vec{r}, \vec{r}') \right]. \tag{2.11}$$

Terms of the form uVV, of order V, were dropped in the exponent of (2.11) relative to gV. The u equation (2.9) can similarly be obtained to leading order in V. Dividing $u(\vec{r}_1, \vec{r}_2)$ by the expression for $g(\vec{r}_1) g(\vec{r}_2)$ we obtain

$$1 + \frac{u}{gg} = |1 + V|^2 \frac{\mathfrak{F}[|K|^2 |K|^2] \mathfrak{F}[1]}{\mathfrak{F}[|K|^2] \mathfrak{F}[|K|^2]}. \tag{2.12}$$

The ratio of generating functionals on the right-hand side of (2.12) is an exponential whose argument is leading order V, all terms of order unity cancelling. Expanding the exponential one obtains

$$\frac{u(\vec{r}_1, \vec{r}_2)}{g(\vec{r}_1) g(\vec{r}_2)} = 2V(\vec{r}_1, \vec{r}_2) + 4 \int d^3r\, g(\vec{r}) V(\vec{r}, \vec{r}_1) V(\vec{r}, \vec{r}_2) + 2 \int d^3r\, d^3r' u(\vec{r}, \vec{r}') [V(\vec{r}_1, \vec{r}) V(\vec{r}_2, \vec{r}') + V(\vec{r}_2, \vec{r}) V(\vec{r}_1, \vec{r}')]. \tag{2.13}$$

The two equations (2.11) and (2.13) then accurately reflect the consequences of our basic assumption (2.2) with $|V| \ll 1$ but $|gV|$ of order unity.

To demonstrate the predictive power of the theory we note that once the functions h_0, G, and K are given, the consistency relations (2.7) and (2.9), which form two simultaneous integral equations for g_B and u_B [explicitly represented by (2.11) and (2.13) under our stated approximations] can be solved for each $\vec{B}$. Other inclusive sums can then be evaluated. For example, consider the inclusive $\vec{P}_T$ distribution. Using the form of A_n we find

$$\frac{d\sigma}{dy\, d^2P_T} = \int d^2b \int d^2b' e^{i\vec{P}_T \cdot (\vec{b} - \vec{b}')} \int d^2B\, |h_0(\vec{B})|^2 G(\vec{r}, \vec{B}) G(\vec{r}', \vec{B})^* \mathfrak{F}_B[K(\vec{r}, \cdot, \vec{B})]. \tag{2.14}$$

In view of (2.10), $d\sigma/dy\, d^2P_T$ can be computed in terms of $\mathfrak{F}_B[1]$, g_B, and u_B. Later, a more convenient expression for computation will be derived for this inclusive distribution.

In addition to the multiparticle inclusive distributions, it is straightforward to compute the overlap integral in terms of the generating function

$$\mathrm{Im}A_{ab\to ab}(\vec{B}) = |h_0(\vec{B})|^2 \mathfrak{F}_B[1] \tag{2.15a}$$

or

$$\sigma = \int d^2B\, |h_0(\vec{B})|^2 \mathfrak{F}_B[1]. \tag{2.15b}$$

The meaning of $h_0(\vec{B})$ can be obtained by direct consideration of σ_0, the $ab \to a'b'$ cross section, given by

$$\sigma_0 = \int d^2B\, |h_0(\vec{B})|^2. \tag{2.16}$$

We interpret h_0 as the "bare" or "skeleton" amplitude. We assume no bootstrap relation between h_0 and the "dressed" amplitude $A_{ab\to ab}$. It may be h_0 represents nondiffractive processes, whereas $A_{ab\to ab}$ describes diffractive processes.

The single-particle *exclusive* cross section depends upon the function G; with $y = y'$,

$$\frac{d\sigma_1}{dy\, d^2P_T} = \int d^2b\, d^2b' e^{i\vec{P}_T\cdot(\vec{b}-\vec{b}')} \int d^2B |h_0(\vec{B})|^2 G(\vec{r},\vec{B}) G(\vec{r}',\vec{B})^*. \tag{2.17}$$

A similar expression exists for the two-particle exclusive cross section, where the interaction kernel $K(\vec{r},\vec{r}',\vec{B})$ appears. Thus h_0 is determined by the bare $ab \to a'b'$ amplitude; the single- and double-particle exclusive cross sections are then determined respectively by G and K.

The form of the single-particle *inclusive* distributions has a form similar to (2.17):

$$\sigma\rho(y,\vec{P}_T) = \int d^2B\, |h_0(\vec{B})|^2 \int d^2b\, d^2b' e^{i\vec{P}_T\cdot(\vec{b}-\vec{b}')} \mathfrak{F}_B[K(\vec{r},\cdot,\vec{B}) K^*(\vec{r}',\cdot,\vec{B})] G(\vec{r},\vec{B}) G^*(\vec{r}',\vec{B}). \tag{2.18}$$

Similarly, the two-particle inclusive density is

$$\begin{aligned}\sigma\rho_2(y_1,y_2,\vec{P}_1,\vec{P}_2) = \int d^2B |h_0(\vec{B})|^2 \int d^2b_1 d^2b_1' d^2b_2 d^2b_2' e^{i\vec{P}_1\cdot(\vec{b}_1-\vec{b}_1')} e^{i\vec{P}_2\cdot(\vec{b}_2-\vec{b}_2')} \\ \times \mathfrak{F}_B[K(\vec{r}_1,\cdot,\vec{B}) K(\vec{r}_2,\cdot,\vec{B}) K^*(\vec{r}_1',\cdot,\vec{B}) K^*(\vec{r}_2',\cdot,\vec{B})] \\ \times G(\vec{r}_1,\vec{B}) G(\vec{r}_2,\vec{B}) G^*(\vec{r}_1',\vec{B}) G^*(\vec{r}_2',\vec{B}).\end{aligned} \tag{2.19}$$

We have now outlined the observables which can be computed in the theory. To proceed further, definite choices for the input functions h_0, G, and K need to be made.

III. BOOTSTRAP EQUATION FOR K

With the theory presented in Sec. II we can proceed in one of two ways: (a) For the unknown functions h_0, G, and K we can make some inspired physically motivated guesses[10] and then use the formalism of Sec. II to extract the predictions; or, (b) the unknown functions themselves can be fixed by requiring them to satisfy certain constraints, leading to deterministic equations. Some discussion of the first approach is given in Sec. V below. We discuss here the second alternative. This approach, we believe, is harder, but in the long run may lead to a deeper understanding.

We propose for study an additional equation which is meant to determine K; h_0 and G are left as free functions, e.g., to be taken from experimental data. We note that integral equations for *exclusive* quantities such as K are in some sense a natural way of complementing the integral equations derived in Sec. II for *inclusive* quantities such as g_B and u_B. The equation we propose is a proportionality between *inclusive* and *exclusive* correlation parameters at fixed $\vec{B}$.[12]

To keep the notation from being unduly cumbersome, at certain points of the following exposition we suppress notation for some of the variables in the functions, and concentrate only on the functional form of the integral equations. From context, these variables can always be supplied.

As discussed in Sec. II, we have assumed that

$$V = K - 1 \tag{3.1}$$

is small compared to unity, although the product gV can be of order unity. We have used these estimates to simplify the integral equations. One

consequence is that u/gg will be of order V.

We now assume the stronger statement that u/gg and V are exactly proportional:

$$\frac{u_B(\vec{r}_1,\vec{r}_2)}{g_B(\vec{r}_1)g_B(\vec{r}_2)} = cV(\vec{r}_1,\vec{r}_2,\vec{B}), \tag{3.2}$$

where c is a dimensionless constant of order unity. We assume for simplicity in this paper that V and c are real. The proportionality (3.2) provides a bootstrap equation for K.

Introducing this bootstrap equation into (2.13), we obtain our second equation relating g and V:

$$\left(\frac{c-2}{4}\right)V(\vec{r}_1,\vec{r}_2) = \int d^3r\, g(\vec{r})V(\vec{r},\vec{r}_1)V(\vec{r},\vec{r}_2) + \frac{c}{2}\int d^3r\, d^3r' g(\vec{r})g(\vec{r}')V(\vec{r},\vec{r}')[V(\vec{r}_1,\vec{r})V(\vec{r}_2,\vec{r}') + V(\vec{r}_2,\vec{r})V(\vec{r}_1,\vec{r}')]. \tag{3.3}$$

Given G, Eqs. (2.11) and (3.3) form a closed pair of nonlinear integral equations whose solutions must now be found. There is one obvious solution, the trivial one:

$$\text{If } V=0, \text{ then } g=|G|^2. \tag{3.4}$$

If G is independent of y, a natural class of nontrivial solutions to look for are those in which V is translation-invariant in y and $\vec{b}$, and in which g is independent of y. Translation invariance of V and constancy of g, both with respect to rapidity, are consequences of the assumption of rapidity cluster decomposition. Translation invariance of V with respect to $\vec{b}$ is not obviously required by our equations or assumptions, but we choose to look for such solutions for simplicity.

For such completely translation-invariant solutions $V(\vec{r},\vec{r}',\vec{B})$ is a function only of $\vec{r}-\vec{r}'$:

$$V(\vec{r},\vec{r}',\vec{B}) = V(\vec{r}-\vec{r}',\vec{B}). \tag{3.5}$$

If we wish to display y and $\vec{b}$ dependences explicitly we write

$$V(\vec{r},\vec{B}) = V(y,\vec{b},\vec{B}); \tag{3.6}$$

the meaning of the arguments of V will be clear from context. We consistently denote rapidities by y, impact parameters by $\vec{b}$ and the vector $(y,\vec{b})$ by $\vec{r}$. In what follows it will also be convenient to discuss the Fourier transform of $V(y,\vec{b},\vec{B})$;

$$v(\omega,\vec{q},\vec{B}) = \int dy\, d^2b\, e^{i(\omega y+\vec{q}\cdot\vec{b})}V(y,\vec{b},\vec{B}). \tag{3.7}$$

The notation adopted is that ω is conjugate to y and $\vec{q}$ is conjugate to $\vec{b}$. It is straightforward algebra to express (2.11) and (3.3) in terms of $V(y,\vec{b},\vec{B})$ or in terms of $v(\omega,\vec{q},\vec{B})$.

It is instructive at this point to note that there are nontrivial solutions to the coupled equations when $|G|^2$ is independent of $\vec{r}$. (We also expect solutions when the Fourier transform of G is a constant.) Taking $g(\vec{r})$ to be real and independent of $\vec{r}$, the Faltung theorem can be used to write (2.11) and (3.3) in terms of $v(\omega,\vec{q},\vec{B})$. Suppressing notation for $\vec{B}$, we have

$$g = |G|^2\exp[2gv(0,\vec{0})] \tag{3.8}$$

and

$$\left(\frac{c-2}{4}\right)v(\omega,\vec{q}) = gv(\omega,\vec{q})^2 + cg^2v(\omega,\vec{q})^3. \tag{3.9}$$

The second equation is algebraic and gives three solutions for each ω and $\vec{q}$:

$$gv = -\frac{1}{2}, 0, \left(\frac{c-2}{2c}\right). \tag{3.10}$$

In particular, these are the three possibilities at $\omega,\vec{q}=(0,\vec{0})$; so from (3.8), g has three possible values

$$\frac{g}{|G|^2} = e^{-1}, 1, \exp\left(\frac{c-2}{c}\right) \tag{3.11}$$

respectively. The corresponding $v(0,\vec{0})$ are

$$|G|^2v(0,\vec{0}) = -\tfrac{1}{2}e^1, 0, \left(\frac{c-2}{2c}\right)\exp\left(-\frac{c-2}{c}\right). \tag{3.12}$$

The last solution is extremely interesting since, for reasonable values of c, g is large (strong coupling) and the Fourier transform $\hat{u}$ of the correlation u is also large and positive:

$$\hat{u}(0,\vec{0}) = |G|^2\left(\frac{c-2}{2}\right)\exp\left(\frac{c-2}{c}\right). \tag{3.13}$$

The conditions (3.10) can be interpreted as sum rules on moments of $V(y)$.

We are now ready to proceed with the task of finding nontrivial, translation-invariant solutions to the coupled equations when $|G|^2$ is damped in $\vec{b}$, as expected in a realistic theory. We expect that some characteristic features of (3.10)–(3.13) will reappear in disguised form in our later analysis.

The equation (2.11) is a relatively straightforward equation of a type frequently encountered in mean-field theories in many-body equilibrium physics.[13] Equation (3.3) is considerably more

complicated; it resembles generally some equations employed by Wilson in his discussions of critical behavior in many-body systems.[14] We note there are no nontrivial *perturbative* solutions, $|G|^2 \ll 1$, of these two coupled equations. Experience with equations such as (3.3), e.g., in Wilson's work, suggests there are isolated, unstable solutions which must be cleverly located.

IV. STUDY OF ONE-DIMENSIONAL EQUATIONS

Since Eqs. (2.11) and (3.3) are very complicated in general, we leave to future work a complete discussion of their solutions. For the purpose of here extracting some qualitative conclusions, we simplify the equations by reducing $\vec{b}$ to one dimension. We may suppress the y dependence, since after Fourier transformation, the equations are diagonal in ω. In one dimension therefore we study the following coupled equations:

$$g(b) = \exp\left[\phi(b) + 2\int db' g(b') V(b-b')\right], \tag{4.1}$$

$$\left(\frac{c-2}{4}\right) V(b) = \int db' g(b') V\left(b' - \frac{b}{2}\right) V\left(b' + \frac{b}{2}\right) + c\int db'\, db'' g(b') g(b'') V(b'-b'') \times V\left(b' - \frac{b}{2}\right) V\left(b'' + \frac{b}{2}\right), \tag{4.2}$$

where $|G|^2 = e^{\phi(b)}$, and only translation-invariant solutions of (2.11) and (3.3) are considered.

Our strategy will be to reduce these equations to second-order nonlinear differential equations, which are coupled only through boundary conditions. Solutions to these equations then can be obtained by standard techniques.[15] We introduce the Fourier transform of $g(b)$,

$$\gamma(q) = \int db\, e^{-ibq} g(b).$$

Then the argument of the exponential in Eq. (4.1) can be written

$$\phi(b) + 2\int \frac{dq}{2\pi} \gamma(q) v(q) e^{-iqb} = \ln[g(b)], \tag{4.3}$$

where $v(q)$ is the Fourier transform of $V(b)$.

We can formally rewrite this expression as

$$\phi(b) + 2v\left(i\frac{\partial}{\partial b}\right) g(b) = \ln[g(b)], \tag{4.4}$$

where the operator is defined by the Taylor series of $v(q)$ around $q=0$. Equation (4.4) then is an infinite-order differential equation for g, which replaces (4.1), given ϕ and v.

Similarly, we may write (4.2) as a differential equation:

$$\left(\frac{c-2}{4}\right) v(q) = v\left(q + \frac{i}{2}\frac{\partial}{\partial b}\right) v\left(q - \frac{i}{2}\frac{\partial}{\partial b}\right) g(b)\Big|_{b=0} + cv\left(q + \frac{i}{2}\left(\frac{\partial}{\partial b_1} + \frac{\partial}{\partial b_2}\right)\right) v\left(q - \frac{i}{2}\left(\frac{\partial}{\partial b_1} + \frac{\partial}{\partial b_2}\right)\right) v\left(q + \frac{i}{2}\left(\frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2}\right)\right) g(b_1) g(b_2)\Big|_{b_1 = b_2 = 0}. \tag{4.5}$$

We recognize that such equations may have a wide range of different types of solutions. If some of these solutions have slowly varying functions $v(q)$ and $g(b)$, we can find such solutions by studying Eqs. (4.4) and (4.5), keeping only the lowest nontrivial derivative operators in the Taylor series expansion of v. Equation (4.4) then becomes

$$-v''(0) g''(b) + 2v(0) g(b) + \phi(b) = \ln[g(b)], \tag{4.6}$$

where we have taken $v'(0) = 0$, a consequence of $V(-b) = V(b)$ [cf. (4.2)].

In the case of (4.5), we have studied the resulting equation for v, and find the qualitative properties of the solution do not depend on the presence of the cubic term. Furthermore, the properties are not sensitive to the specific numerical value of c, provided $c > 2$. Thus, for the purpose of showing the method of finding, and displaying, the characteristic features of the solutions to the v equation, it is sufficient to consider the simpler equation[16]

$$v(q) = g(0) v^2(q) + \frac{g''(0)}{4}\{[v'(q)]^2 - v(q) v''(q)\}, \tag{4.7}$$

where we have taken $c = 6$. Note that coupling between (4.6) and (4.7) appears only through the values of $g(0)$, $g''(0)$, $v(0)$, and $v''(0)$. Thus we can study the equations separately. We first discuss (4.7).

For convenience of notation we introduce

$$f(q) = g(0) v(q) \tag{4.8}$$

and define

$$\lambda = \frac{-g''(0)}{8g(0)}. \tag{4.9}$$

Then a first integral of (4.7) can be written[17]

$$-\lambda f'^2(q) = f^2(q)\left\{\frac{1}{f(q)} + \ln[f(q)] - \frac{1}{f(0)} - \ln[f(0)]\right\}. \tag{4.10}$$

All solutions to (4.7) can be obtained from (4.10) by quadrature. The bracket has a minimum at $f(q)=1$; $-\lambda f'^2>0$ whenever $f(0)<1$. A solution which approaches zero at large q requires $f(0)<1$ and $\lambda<0$. We require this condition, since otherwise V would be unbounded at $b=0$, which violates our assumption that V be small.

We note the critical points[15] of Eq. (4.7), those with $f(q)=$ constant for all q, must satisfy

$$f=f^2 \tag{4.11}$$

whose solutions are $f=0,1$. These solutions should be compared with the solutions found for the integral equations (2.11) and (3.3), when ϕ is constant; if the cubic term has been kept in (4.5), the fixed points would be exactly (3.10), for each value of c.[18]

Given a solution $f(q)$, we now must show the g equation

$$\frac{-f''(0)}{g(0)}\, g''(b) + \frac{2f(0)}{g(0)}\, g(b) + \phi(b) = \ln[g(b)] \tag{4.12}$$

has solutions consistent with (4.7). We note that only $f(0)$ and $f''(0)$ couple to the g equation, and, moreover, only $g(0)$ and $g''(0)$ were coupled to the f equation (4.7). To verify the consistency of (4.7) and (4.12) we need only write equations coupling $f(0)$, $f''(0)$, $g(0)$, and $g''(0)$. These conditions are

$$f(0) - \frac{g''(0)}{4g(0)} f''(0) = 1 \tag{4.13}$$

from (4.7),

$$\frac{-f''(0)g''(0)}{g(0)} + 2f(0) + \phi(0) = \ln[g(0)] \tag{4.14}$$

from (4.12),

$$g''(0)>0 \text{ and } f(0)<1 \tag{4.15}$$

from the discussion after (4.10), and

$$g(b)\to 0 \text{ as } b\to\infty, \tag{4.16}$$

which is guaranteed if $\phi\to-\infty$ as $b\to\infty$, which in turn is true if $|G(b)|^2\to 0$ as $b\to\infty$. Thus given ϕ, $f(0)<1$, and $f''(0)<0$, the initial values $g(0)>0$ and $g''(0)>0$ are then determined by (4.13) and (4.14).

The remaining task is to show that for at least some reasonable functions $|G|^2$, Eq. (4.12) in fact has solutions for $f(0)<1$ and $f''(0)<0$. To demonstrate this, we define

$$R(z) = g(z[-f''(0)]^{1/2})/g(0), \tag{4.17}$$

$$z = b/[-f''(0)]^{1/2}, \tag{4.18}$$

and

$$\psi(z) = \phi(0) - \phi(z[-f''(0)]^{1/2}) + 4 - 2f(0). \tag{4.19}$$

Then (4.12) becomes

$$R'' + 2f(0)R - \ln R = \psi(z), \tag{4.20}$$

where all constraints are satisfied if the solution satisfies $R(0)=1$, $R'(0)=0$, and $R\to 0$ as $z\to\infty$, with $R''(0)=4[1-f(0)]>0$. There are clearly many functions R with these properties. For any such R, (4.20) defines an associated function ψ, and hence $|G|^2$, which satisfies our conditions. We conclude that the one-dimensional differential equations (4.6) and (4.7) have acceptable solutions for a wide class of functions $|G|^2$, for any choice of $0<f(0)<1$ and $f''(0)<0$.

We have studied the cubic version[16] of (4.7), and conclude that under similar conditions that there are consistent solutions to the coupled one-dimensional integral Eqs. (4.1) and (4.2).

We therefore believe it plausible that the coupled two-dimensional integral equations (2.11) and (3.3) also have acceptable solutions for suitably restrictive assumptions on $|G|^2$. Thus, for any given $|G|^2$, we anticipate there may be many solutions to the coupled equations. This means our theory will not predict unique correlation functions, but will rather predict certain constraints between different observables.

V DISCUSSION OF GENERAL RESULTS

We now survey the relations between physical observables implied in our theory. The expressions for observable distributions, in terms of g and V, take a relatively simple form if we introduce the Fourier transforms of G and $\sqrt{g}$. Let

$$\tilde{G}(\vec{P}_T, \vec{B}) = \int d^2b\, e^{i\vec{b}\cdot\vec{P}_T} G(\vec{b}, \vec{B}); \tag{5.1}$$

then (2.17) becomes

$$\frac{d\sigma_1}{dy\, d^2P_T} = \int d^2B |h_0(\vec{B})|^2 |\tilde{G}(\vec{P}_T, \vec{B})|^2. \tag{5.2}$$

If G is independent of $\vec{B}$, we obtain simply

$$\frac{1}{\sigma_0}\frac{d\sigma_1}{dy\, d^2P_T} = |\tilde{G}(\vec{P}_T)|^2, \tag{5.3}$$

where

$$\sigma_0 = \int d^2B |h_0(B)|^2.$$

We can derive a similar expression for $\sigma\rho \equiv d\sigma/dy\, d^2P_T$, if (2.11) is used together with (3.7). Defining

$$F(\vec{P}_T, \vec{B}) = \int d^2b\, e^{i\vec{b}\cdot\vec{P}_T} [g(\vec{b}, \vec{B})]^{1/2}, \tag{5.4}$$

we obtain [cf. (2.15a) and (2.18)]

$$\rho(y,\vec{P}_T)=\frac{1}{\sigma}\int d^2B\,|h_0(\vec{B})|^2\,\mathfrak{F}_B[1]\,|F(\vec{P}_T,\vec{B})|^2. \quad (5.5)$$

From this, we can find

$$\frac{\bar{n}}{Y}=\frac{1}{Y}\int dy\,d^2P_T\,\rho(y,\vec{P}_T)$$

$$=\frac{1}{\sigma}\int d^2B\,|h_0(\vec{B})|^2\,\mathfrak{F}_B[1]\int d^2b\,g(\vec{b},\vec{B}). \quad (5.6)$$

If g is independent of $\vec{B}$, we see

$$\rho(\vec{P}_T)=|F(\vec{P}_T)|^2 \quad (5.7)$$

and

$$\frac{\bar{n}}{Y}=\int d^2b\,g(\vec{b}). \quad (5.8)$$

For two-particle exclusive cross sections, we have

$$\frac{d\sigma_2}{dy_1d^2P_1dy_2d^2P_2}=\int d^2B\,|h_0(\vec{B})|^2\int d^2b_1\,d^2b_1'\,d^2b_2\,d^2b_2'\,e^{i\vec{P}_1\cdot(\vec{b}_1-\vec{b}_1')}$$

$$\times e^{i\vec{P}_2\cdot(\vec{b}_2-\vec{b}_2')}G(\vec{r}_1,\vec{B})G^*(\vec{r}_1',\vec{B})G(\vec{r}_2,\vec{B})G^*(\vec{r}_2',\vec{B})K(\vec{r}_1,\vec{r}_2,\vec{B})K^*(\vec{r}_1',\vec{r}_2',\vec{B}). \quad (5.9)$$

Let D be defined by

$$\frac{1}{\sigma_0}\frac{d\sigma_2}{dy_1d^2P_{T1}dy_2d^2P_{T2}}-\frac{d\sigma_1}{\sigma_0dy_1d^2P_{T1}}\frac{d\sigma_1}{\sigma_0dy_2d^2P_{T2}}=\frac{1}{\sigma_0}\int D(y_1,\vec{P}_{T1};y_2,\vec{P}_{T2};\vec{B})\,|h_0(\vec{B})|^2d^2B. \quad (5.10)$$

If $K=1+V$, with V small, G and K real and independent of $\vec{B}$,

$$D(y_1,P_{T1};y_2,P_{T2})=2\tilde{G}(\vec{P}_{T1})\tilde{G}(\vec{P}_{T2})\int\int d^2b_1d^2b_2\,e^{i(\vec{P}_{T1}\cdot\vec{b}_1+\vec{P}_{T2}\cdot\vec{b}_2)}\,G(\vec{r}_1)V(\vec{r}_1-\vec{r}_2)G(\vec{r}_2) \quad (5.11)$$

If we introduce v (the Fourier transform of V), $\tilde{D}$ as the Fourier transform of D in relative rapidity, and take G independent of y, we obtain

$$\tilde{D}(\omega,\vec{P}_{T1},\vec{P}_{T2})=2\tilde{G}(\vec{P}_{T1})\tilde{G}(\vec{P}_{T2})\int\frac{d^2q}{(2\pi)^2}\,v(\vec{q})\int\int d^2b_1d^2b_2\,e^{i[\vec{P}_{T1}\cdot\vec{b}_1+\vec{P}_{T2}\cdot\vec{b}_2-\vec{q}\cdot(\vec{b}_1-\vec{b}_2)]}G(\vec{b}_1)G(\vec{b}_2)$$

$$=2\tilde{G}(\vec{P}_{T1})\tilde{G}(\vec{P}_{T2})\int\frac{d^2q}{(2\pi)^2}\,v(\vec{q},\omega)\tilde{G}(\vec{P}_{T1}-\vec{q})\tilde{G}(\vec{P}_{T2}+\vec{q}). \quad (5.12)$$

With G and K independent of $\vec{B}$, D is exactly the two-particle exclusive correlation function.

For inclusive two-particle correlations, using (2.11) we find similar expressions with G replaced by $\sqrt{g}$. Let us define $\tilde{C}_2(\omega,\vec{P}_{T1},\vec{P}_{T2})$ as the relative rapidity Fourier transform of the inclusive two-particle correlation function

$$C_2=[\rho_2(y_1,\vec{P}_{T1};y_2,\vec{P}_{T2})-\rho_1(y,\vec{P}_{T1})\rho_1(y_2,\vec{P}_{T2})]. \quad (5.13)$$

Then if G and K do not depend on $\vec{B}$, we find

$$\tilde{C}_2(\omega;\vec{P}_{T1},\vec{P}_{T2})=2cF(\vec{P}_{T1})F(\vec{P}_{T2})\int\frac{d^2q}{(2\pi)^2}\,v(\vec{q},\omega)F(\vec{P}_{T1}-\vec{q})F(\vec{P}_{T2}+\vec{q}), \quad (5.14)$$

where F is defined in (5.4).

To illustrate a method for more explicit computations we now use the one-dimensional approximations of Sec. IV. We recall that in this approximation, the only free parameters are $v(0)$, $v''(0)$, and c. Proceeding as in Sec. IV, we expand $v(q)$ in powers of q, corresponding to our assumption of a sufficiently slowly varying $g(b)$. We can then express (5.14) in one transverse dimension as

$$\tilde{C}_2(\omega,P_{T1},P_{T2})=2cF(P_{T1})F(P_{T2})\Big\{\tilde{g}(P_{T1}+P_{T2})\Big[v(0,\omega)-\frac{v''(0,\omega)}{8}(P_{T2}-P_{T1})^2\Big]$$

$$-\frac{v''(0,\omega)}{4}\int db\,e^{ib\cdot(P_{T1}+P_{T2})}\{g^{1/2}(b)g''^{1/2}(b)-[g'^{1/2}(b)]^2\}\Big\}, \quad (5.15)$$

where $\tilde{g}(P_T)$ is the Fourier transform of $g(b)$. A similar reduction of (5.12) yields

$$\tilde{D}(\omega, P_{T1}, P_{T2}) = 2\tilde{G}(P_{T1})\tilde{G}(P_{T2})\Big\{\tilde{G}^2(P_{T1}+P_{T2})\Big[v(0,\omega) - \frac{v''(0,\omega)}{8}(P_{T2}-P_{T1})^2\Big] - \frac{v''(0,\omega)}{4}\int db\, e^{ib\cdot(P_{T1}+P_{T2})}\{G(b)G''(b) - [G'(b)]^2\}\Big\}, \tag{5.16}$$

where $\tilde{G}^2(P_T)$ is the Fourier transform of $G^2(b)$.

If we integrate over P_{T1} and P_{T2}, we can obtain the second multiplicity moment, but it is more convenient for our purposes to express f_2 in terms of functional derivatives with respect to a constant function z,

$$f_2 = \frac{\delta^2 \ln \mathfrak{F}(z)}{\delta z^2}\Big|_{z=1} = \int\int u(r_1, r_2)dr_1 dr_2.$$

We can write this as

$$\frac{f_2}{cY} = \int\int db_1 db_2 g(b_1)g(b_2) \times \int v(q,0)\frac{dq}{2\pi}e^{iq(b_1-b_2)}. \tag{5.17}$$

Expanding v around $q=0$, we obtain in our one-transverse-dimensional model

$$\frac{f_2}{cY} = v(0)\int db\, g^2(b) - \frac{v''(0)}{4}\int db\{g(b)g''(b) - [g'(b)]^2\}, \tag{5.18}$$

which may be compared with the corresponding formula for $\bar{n} = f_1$ in one transverse dimension,

$$\frac{f_1}{Y} = \int db\, g(b). \tag{5.19}$$

This relation can be used as a constraint, together with (5.17), to determine the parameters $f(0)$ and $f''(0)$ in terms of f_1/Y and f_2/Y, when $|G|^2$ is given. The latter can be determined from (5.3), if we *assume* no $\vec{B}$ dependence in G. The latter assumption can be tested through examination of leading-particle correlations, in principle.

Specific numerical results will depend on a choice of c. In terms of observables, the value of c is directly related to the magnitude of the two-particle exclusive correlation compared to two-particle inclusive correlations.

We believe that direct relationships, as expressed by Eqs (5.1)–(5.14) of this section, together with (2.11) and (3.3) between $\rho(\vec{P}_T)$, $\sigma_1(\vec{P}_T)$, f_1, and f_2, and the two-particle correlations, reflect essential features of a theory of multiparticle production dominated by pairing correlations, as hypothesized in Sec. II, and including the bootstrap assumption (3.2). Since our equations couple only $\omega = 0$ Fourier components, we obtain no predictions from the present work concerning y dependences of the correlations.

If given rapidity dependences, e.g., from two-body resonances, are to be imposed upon the correlations, the constraints from our theory play the role of sum rules. At each ω, certain moments (integrals over y) of the two-particle exclusive correlations V must obey equations analogous to (3.12) but for nonconstant $g(r)$, as obtained from (3.3). These constraints depend on solutions of (2.11), which involve only the $\omega = 0$ components of v. We plan to investigate these sum rules in a later work.

We conclude with a discussion of the prediction of this type of theory for large-P_T behavior. We expect to describe cross sections at P_T values much larger than the mean, but much smaller than $\sqrt{s}$, by such a mechanism. In other language, we consider only the kinematic regions appropriate to Feynman scaling (fixed P_T, but $s \to \infty$) rather than parton-theory scaling (fixed $x_\perp = P_T/\sqrt{s}$, but both P_T and $\sqrt{s} \to \infty$). We interpret the growth with s of cross sections at fixed but large P_T in hadronic inclusive reactions as a slow approach to a Feynman-scaling limit. We do not, and cannot, predict or discuss behavior of cross sections in the fixed $x_\perp$, $s \to \infty$ limit.

A measurement of $|G|^2$ from low-multiplicity final states [cf. Eq. (5.3)] provides the necessary input for the coupled integral Eqs. (2.11) and (3.3). A solution g then can be used to compute $\rho(\vec{P}_T)$; if $|G|^2$ is independent of $\vec{B}$, for example, then $\rho(\vec{P}_T)$ is given by (5.7), using (5.4). One mechanism for large-P_T inclusive enhancement (compared to single-particle exclusive cross sections) can be seen in Eq. (2.11). The correlation V, when positive, may drive g much larger than $|G|^2$, at small b. We speculate that the dramatic enhancement seen in data may be generated in this way.

There may be other collective enhancements as well. For example, in Appendix C we outline a different mechanism which enhances large P_T in a formalism which, although simplified, allows the role of long-range rapidity correlations to be esti-

mated. The connection between these two mechanisms seems a very interesting question to explore, and we plan to study this further.

APPENDIX A

The generating functional defined in (2.4) can be written

$$\mathfrak{F}_B[\phi]=\sum_n \frac{1}{n!}\int d^3r_1\cdots d^3r_n\phi(r_1)\cdots\phi(r_n) \times|G(\vec{r}_1,\vec{B})|^2\cdots|G(\vec{r}_n,\vec{B})|^2 \times\prod_{j\neq k}|K(\vec{r}_j,\vec{r}_k,\vec{B})|^2. \tag{A1}$$

Such a structure is identical in form to a grand-partition function of a system of interacting spins in thermal equilibrium. Each spin is represented by $\vec{b}_j$, located on a site y_j.[19]

The functions $|G|^2$ are the noninteracting probability distributions of each isolated spin. In an Ising-type model, G would be a function of y localized at a discrete set of sites $\{y_k\}$ on a lattice: $|\vec{b}|$ would take on only one value (e.g., 1), and only two orientations of $\vec{b}$ would be allowed (e.g., up and down). The kernel $|K|^2$ describes the spin-spin interaction in the analog system: $|K(\vec{r}_i,\vec{r}_j)|^2 = \exp[U(\vec{r}_i-\vec{r}_j)]$, where U is an exchange integral in a spin system.

A functional integral representation for $\mathfrak{F}$ can be written, for fixed $\vec{B}$, in the form

$$\mathfrak{F}[\phi]=\frac{\int\delta\psi\, e^{-F[\psi]}e^{\int\phi(\vec{r})\chi(\vec{r})d^3r}}{\int\delta\psi\, e^{-F[\psi]}}, \tag{A2}$$

where

$$\chi(\vec{r})=e^{i\psi(\vec{r})}, \tag{A3}$$

$$F[\psi]=\tfrac{1}{2}\int\int d^3r_1d^3r_2\Gamma(\vec{r}_1,\vec{r}_2)\times[\psi(\vec{r}_1)-\psi_0(\vec{r}_1)] \times[\psi(\vec{r}_2)-\psi_0(\vec{r}_2)], \tag{A4}$$

$\Gamma(\vec{r}_1,\vec{r}_2)$ can be shown to be the resolvent of $-2\ln|K|^2\equiv C(\vec{r}_1,\vec{r}_2)$ (see below), and

$$e^{i\psi_0(\vec{r})}=\chi_0(\vec{r})=|G(\vec{r})|^2.$$

Thus ψ_0 is imaginary. This form (A2) may be useful in making a correspondence with other many-body methods in high-energy hadron collisions, e.g., coherent-state phenomenology[20] and Reggeon field theories.[21]

The form (A2) can be obtained using arguments from Edwards and Lenard.[22] Let

$$\langle f[\psi]\rangle_0\equiv\frac{\int\delta\psi\, e^{-F[\psi]}f[\psi]}{\int\delta\psi\, e^{-F[\psi]}}, \tag{A5}$$

where $F[\psi]$ is defined in (A4). Then from Ref. 22,

$$\left\langle\exp\left[i\int J(\vec{r})\psi(\vec{r})d^3r\right]\right\rangle_0 = e^{i\int J\psi_0 d^3r}\langle e^{i\int J(\psi-\psi_0)d^3r'}\rangle_0 = e^{i\int J(\vec{r})\psi_0(\vec{r})d^3r} \times\exp\left[-\tfrac{1}{2}\int\int d^3r_1d^3r_2C(\vec{r}_1,\vec{r}_2)J(\vec{r}_1)J(\vec{r}_2)\right], \tag{A6}$$

where

$$\int d^3r\, C(\vec{r}_1,\vec{r})\Gamma(\vec{r},\vec{r}_2)=\delta^3(\vec{r}_1-\vec{r}_2) \tag{A7}$$

determines Γ in terms of C. In the special case $J(\vec{r})=\sum_k\delta^3(\vec{r}-\vec{r}_k)$, we obtain from (A6) the expression

$$e^{i\sum_k\psi_0(\vec{r}_k)}\exp\left[-\tfrac{1}{2}\sum_{k,l}C(\vec{r}_k,\vec{r}_l)\right]. \tag{A8}$$

Now let

$$i\psi_0(\vec{r})=\ln|G(\vec{r})|^2 \tag{A9}$$

and

$$e^{-C(\vec{r}_1,\vec{r}_2)/2}=|K(\vec{r}_1,\vec{r}_2)|^2; \tag{A10}$$

then we can write the diagonal density-matrix elements, putting $\phi=1$, as

$$\int d^3r_1\cdots d^3r_n|G(\vec{r}_1)|^2\cdots|G(\vec{r}_n)|^2\prod_{k,l}|K(\vec{r}_k,\vec{r}_l)|^2 = \int d^3r_1\cdots d^3r_n\langle e^{i[\psi(\vec{r}_1)+\cdots+\psi(\vec{r}_n)]}\rangle_0 = \left\langle\left(\int d^3r\, e^{i\psi(\vec{r})}\right)^n\right\rangle_0. \tag{A11}$$

The generating functional $\mathfrak{F}$ is obtained by returning ϕ and summing over all n. If we denote by $\langle\,\rangle_\phi$ the functional average with $\psi_0\to\psi_0+\ln\phi$,

$$\mathfrak{F}[\phi]=\sum_n\frac{1}{n!}\left\langle\left(\int d^3r\, e^{i\psi}\right)^n\right\rangle_\phi = \left\langle\exp\left(\int d^3r\, e^{i\psi(\vec{r})}\right)\right\rangle_\phi = \left\langle\exp\left(\int d^3r\,\phi(\vec{r})\chi(\vec{r})\right)\right\rangle_0, \tag{A12}$$

where $\chi(\vec{r})$ is given by (A3).

The "field" χ directly represents particle densities and correlations. The one- and two-particle exclusive cross sections are integrals over $\vec{B}$ of

$$\left.\frac{\delta\mathfrak{F}}{\delta\phi(\vec{r})}\right|_{\phi=0}=\langle\chi(\vec{r})\rangle_0=\chi_0(\vec{r})=e^{i\psi_0(\vec{r})} \tag{A13}$$

and

$$\left.\frac{\delta^2\mathcal{F}}{\delta\phi(\vec{r}_1)\delta\phi(\vec{r}_2)}\right|_{\phi=0} = \langle\chi(\vec{r}_1)\chi(\vec{r}_2)\rangle_0 , \tag{A14}$$

weighted by $|h_0(\vec{B})|^2$. The corresponding inclusives are similarly obtained from

$$\left.\frac{\delta\mathcal{F}}{\delta\phi(\vec{r})}\right|_{\phi=1} = \langle\chi(\vec{r})e^{\int\chi(\vec{r}')d^3r'}\rangle_0 \tag{A15}$$

and

$$\left.\frac{\delta^2\mathcal{F}}{\delta\phi(\vec{r}_1)\delta\phi(\vec{r}_2)}\right|_{\phi=1} = \langle\chi(\vec{r}_1)\chi(\vec{r}_2)\,e^{\int\chi(\vec{r}')d^3r'}\rangle_0 . \tag{A16}$$

Thus, $|K|^2$ is the propagator of χ, while C is the propagator of ψ.

The quadratic form of F with displaced expectation value for ψ, in the functional integral, suggests we could obtain our results Eq. (2.10) as a strong-coupling limit of a nonlinear field theory in the ψ "field," if the ψ "Lagrangian" generates a nonzero ground-state expectation value ψ_0 (cf. Sugar and Cardy's work[23]).

Such functional integral representations have been used by Wiegel[24] to show an equivalence between a Van der Waals–type fluid and the Ising model. It is shown in Ref. 24 that if $C(\vec{r}_1-\vec{r}_2)$ has exponential behavior in $|\vec{r}_1-\vec{r}_2|$ and a short-range repulsive core is present, then $F[\psi]$ takes on the Landau-Ginzberg form.

APPENDIX B: DISCUSSION OF MULTIPLICITY MOMENTS, SHORT-RANGE CORRELATION, AND CRITICALITY

If the dependence of A_n on $\vec{B}$ were completely factorizable, the multiplicity moments would be generated by $\alpha(z)=\ln[\mathcal{F}(z)]$;

$$f_1 = \left.\frac{d\ln[\mathcal{F}(z)]}{dz}\right|_{z=1} \tag{B1}$$

and

$$f_2 = \left.\frac{d^2\ln[\mathcal{F}(z)]}{dz^2}\right|_{z=1} . \tag{B2}$$

If $\mathcal{F}(z)$ is given by (2.5), these would have short-range-order behavior ($f_1, f_2 \sim Y$), provided the integrals of g and u grow only like Y for $Y\to\infty$.

We expect g to be independent of y, so $\int g(y,\vec{b})dy \sim Y$. If we also have

$$\iint u(y_1, y_2)\,dy_1 dy_2 \sim Y, \tag{B3}$$

i.e., $\int u(y)dy$ convergent where $y=(y_1-y_2)$, then $f_2\sim Y$ also. This would rule out dynamical critical phenomena associated with long-range correlations which, we believe, can play an important role in multiplicity distributions. In the present paper we will investigate the role of finite-range correlations, since even in this case we will find the potential for interesting effects at large P_T. We proceed then with approximation (2.10) under the assumption that the integral $\int dy\,u(y)$ is finite. In later work we plan to examine the explicit roles of long-range correlations induced by Y and $\vec{B}$ dependences in G. Note $\alpha(z)$ will have only a quadratic z dependence in approximation (2.10); for critical behavior higher powers are required, which implies the presence of higher-order correlation parameters.

The simple truncated cluster expansion (2.10) is most useful for estimating the cumulative effects of a weak pair correlation, provided we can argue the relative unimportance of higher-order parameters. If K contains a negative correlation which is strong but of short range in rapidity, e.g., between nearest neighbors only, as in a multiperipheral model with $t_{\min}$ effects,[25] it is possible to include this "repulsion" explicitly in $\mathcal{F}$ as in the Van der Waals theory, with only a slight increase in complexity. One of the advantages of such a modification is the possibility of then obtaining critical behavior without divergences in the integral of functions such as u. We now discuss such a modification, wherein only the weak long-range correlations appear in $u(\vec{r}_1,\vec{r}_2)$, and examine briefly the changes in our theory which would result.

If there is only a nearest-neighbor interaction present, the appropriate generating functional $\mathcal{F}_0$ can be expressed exactly in closed form. For simplicity, we first consider the problem without transverse coordinates, with G = constant. Let

$$p_0(z) = \frac{1}{Y}\ln[\mathcal{F}_0(z;Y)] . \tag{B4}$$

Then

$$\mathcal{F}_0(z;Y) = \sum_n \frac{z^n G^n}{n!}\int dy_1\cdots dy_n\,K_0(y_1-y_2)\cdots \times K_0(y_{n-1}-y_n) , \tag{B5}$$

where $K_0(y_i, y_{i+1})$ is unity except for y_i close to y_{i+1}, and $y_i \leqslant y_{i+1}$

If the Laplace transform of $\mathcal{F}_0$ with respect to Y is introduced, we can obtain a factorizable form of the integrals

$$\hat{\mathcal{F}}_0(z;p) = \int_0^\infty dY\,e^{-pY}\mathcal{F}_0(z;Y) \tag{B6}$$

and

$$\hat{K}_0(p) = \int_0^\infty dy\,e^{-py}K_0(y) . \tag{B7}$$

We obtain

$$\hat{\mathcal{F}}_0(z;p)=\sum_n z^n G^n \hat{K}_0(p)^n$$

$$=\frac{1}{1-z\,G\hat{K}_0(p)}\ . \tag{B8}$$

As $Y\to\infty$, the generating functional $\mathcal{F}_0(z;Y)$ will be dominated by the pole in p, which has smallest real part. Let $p_0(z)$ be determined then by the smallest root of

$$1-zG\hat{K}_0(p)=0\,. \tag{B9}$$

If we now consider the generating functional $\mathcal{F}_0[\phi]$, for a constant function ϕ it can be shown that as $Y\to\infty$, we obtain

$$\frac{1}{Y}\ln(\mathcal{F}_0[\phi;Y])\to p_0(\phi)\,. \tag{B10}$$

The presence of additional weak, long-range terms will then be included if we add to (2.10) the expression

$$Y[p_0(\phi)-p_0(1)]\,, \tag{B11}$$

with g and u to be determined by

$$K'=K/K_0 \tag{B12}$$

instead of K, and $G'=1$ instead of G.

If $K_0(y)$ is a θ function of $y-y_0$ (a rough representation of $t_{\min}$ effects), one obtains with (2.10) the Van der Waals form for $\alpha(z)$; $p_0(z)$ is the solution of

$$zG=p_0 e^{(p_0 y_0)}\ . \tag{B13}$$

If we then include g and u from (2.10) the resulting form of $\ln[\mathcal{F}(z)]$ will match that of Ref. 1.

If K_0 is explicitly treated in such a way, the bootstrap condition (3.2) must be relaxed, at least for the short-range parts of U. We would utilize integral equations, such as those given by (2.8), to determine inclusive correlations. Integral equations for inclusive correlation functions will now contain explicit short-range contributions from K_0 as well as the terms appearing previously.

Critical behavior can be induced by weak, long-range positive behavior in $V'=K'-1$, provided K_0 is included. Under such conditions we may find V' small everywhere, but of such long range that the Fourier components of two-particle correlations, $\tilde{C}_2(\omega)$, at $\omega=0$, diverge for $Y\to\infty$. Then other expansions for $\mathcal{F}$ may be more appropriate than (2.10), and it is possible to obtain integral equations for correlation functions whose solutions more accurately reflect the behavior of the exact $\mathcal{F}$. Such behavior is well-known in one-dimensional, long-range interaction fluid models, and those techniques are applicable to deducing the consequences of (2.2).[26]

APPENDIX C

We discuss here an alternative characterization of dynamics leading to collective effects in large-P_T behavior. Consider a noncollective model based on the intuitive notion that final-state rescattering or stimulated emission be described by the following postulate for single-particle semi-inclusive b distributions:

$$\frac{1}{\sigma_n}\frac{d\sigma_n}{dy\,d^2b}=\frac{n}{Y}[G_1(\vec{b})]^n\,, \tag{C1}$$

where $G_1(\vec{b})$ is an unknown function to be specified. The idea is that for n secondary-produced hadrons, the probability of seeing one at rapidity y and impact parameter $\vec{b}$ gets an independent contribution for each final secondary. The function $G_1(\vec{b})$ measures the strength with which each particle rescatters off (or is "produced by") all the others. The behavior represented by (C 1) is thus not an independent-emission model for produced secondaries, although G_1 can be determined from single-particle-production cross sections in this model. We assume G_1 to be a smooth continuous function of $\vec{b}$ with no singularities, as suggested by low-multiplicity data.

The model is specified by $G_1(\vec{b})$ and by a model for the $\{\sigma_n\}$. In what follows we assume $\{\sigma_n\}$ have the behavior characteristic of short-range-order models.[4] In terms of standard generating functions, this means that we expect

$$\alpha(z)=\frac{1}{Y}\ln\left[\sum_n z^n\sigma_n(Y)\right] \tag{C2}$$

to become independent of Y in the limit $Y\to\infty$, and that, furthermore, all the derivatives of $\alpha(z)$ exist at $z=1$ (i.e., the Mueller moments f_n increase at most linearly with Y).

The consequences of (C1) and (C2) for the single-particle inclusive distribution can then be worked out by summing the semi-inclusive distributions weighted by σ_n:

$$\sigma(Y)g(b)=\sum_n \sigma_n(Y)\frac{n}{Y}[G_1(b)]^n\,. \tag{C3}$$

The result can be expressed in terms of the generating function $\alpha(z)$:

$$g(b)=z\alpha'(z)\exp\{Y[\alpha(z)-\alpha(1)]\}\,|_{z=G_1(b)}\,. \tag{C4}$$

The model is very constrained then if we assume G_1 is independent of Y. To compare with the characteristic behavior seen in data one computes $\rho(\vec{P}_T)$ using (5.7) in terms of $F(\vec{P}_T)$, the Fourier transform of $\sqrt{g}$ [cf.(5.4)]:

$$F(\vec{P}_T)=\int d^2b\, e^{i\vec{b}\cdot\vec{P}_T}[G_1(\vec{b})\alpha'(G_1(\vec{b}))]^{1/2} \times \exp\{\tfrac{1}{2}Y[\alpha(G_1(\vec{b}))-\alpha(1)]\}. \quad \text{(C5)}$$

We assume that G_1 and, hence, g have no dependence on $\vec{B}$ or on Y. In the limit $Y\to\infty$ we find by the method of steepest descent that (C5) has the form

$$F(P_T)=[G_1(\langle b\rangle)\alpha'(G_1(\langle b\rangle))]^{1/2} \times \exp\tfrac{1}{2}Y[\alpha(G_1(\langle b\rangle))-\alpha(1) - \alpha'(G_1(\langle b\rangle))\vec{b}\cdot\vec{\nabla}G_1(\langle b\rangle)]\} \quad \text{(C6)}$$

where $\langle b\rangle$, as determined from the steepest descent equation, is a function only of P_T/Y. It then follows from (5.7) that $\rho(P_T)$ is the square of (C6) and has the form

$$\ln\rho(P_T)\sim Y\phi_1(P_T/Y)+\phi_2(P_T/Y). \quad \text{(C7)}$$

(We have suppressed throughout the possible dependence of G_1 on y. If this is included, ϕ_i will also depend on y.)

The striking prediction of this simple model is that the single-particle distribution should obey a scaling-type law. The current trend of data[3] argues against this type of behavior.[27] If $\phi_1 \sim -aP_T/Y$, then the small-P_T behavior is well described by $\rho\sim e^{-aP_T}$. In that case the scaling behavior is trivial. At large P_T, $\ln\rho/Y$ vs P_T/Y should approach the scaling from above. One proves this by taking some typical $g(y,\vec{b})$ and explicitly computing ϕ_1. The data show the antibehavior; the P_T/Y scaling is no better than simple Feynman scaling.

We remark that some qualitative effects we sought are exhibited by (C4); e.g., large-P_T slopes are smaller in high-multiplicity final states than in low-multiplicity states. However, no striking quantitative successes emerge. We could find no proof that a natural variant of the above model would allow the observed trend of large-P_T data, unless arbitrary Y dependences are allowed.

We do note however that some striking features could emerge if the short-range-order hypothesis on σ_n were dropped. This would allow for a collective-type model. In the language of the Feynman fluid analog, if the σ_n produce a phase transition[4] then $z\alpha'(z)$ as a function of z will have a discontinuous jump at some z_0. Conferring with (C6) we see this allows the possibiltity that at some P_T, $F(P_T)$ and hence $\rho(P_T)$ will similarly show a nonanalytic behavior. The effect will tend to enhance the large-P_T cross section [i.e., the small-$\vec{b}$ behavior of $g(\vec{b})$].

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[4]There are many references to such models, of which the more relevant can be found in the review by G. H. Thomas, in Proceedings of the Fifth International Conference on Many-Particle Hadrodynamics, Leipzig, Karl-Marx-Universität, 1974 (unpublished), p. 699; see also D. Horn and F. Zachariasen, *High Energy Collisions* (Benjamin, New York, 1973).

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[16]The full equation (4.5) is, to second order in derivatives,

$$\frac{c-2}{4}v(q)=g(0)v(q)^2+\frac{g''(0)}{4}\{[v'(q)]^2-v(q)v''(q)\} +c\{g(0)^2v(q)^3 + \frac{g(0)g''(0)}{4}[2v(q)v'(q)^2-3v''(q)v(q)^2]\}.$$

[17]For the general case (24), the first integral is

$$-\lambda f'(q)^2=\frac{1}{3}\int_{f(0)}^{f(q)}dt\left[\frac{f(q)}{t}\right]^2\left[\frac{1+3ct}{1+3cf(q)}\right]^{2/3} \times\frac{(t+\frac{1}{2})[t-(c-2)/2c]}{(t+1/3c)}.$$

When $0 < f(0) < (c-2)/2c$ and $f(q) < f(0)$ we are assured of $-\lambda f'^2 > 0$. As with the example in the text, we therefore require $\lambda < 0$.

[18]For the general case, $f(q) =$ constant satisfies

$$\frac{c-2}{4} f = f^2 + cf^3,$$

whose solutions are

$$f = -\frac{1}{2},\ 0,\ \frac{c-2}{2c}\ .$$

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