PHYSICAL REVIEW D VOLUME 13, NUMBER 2 15 JANUARY 1976

Particle emission rates from a black hole: Massless particles from an uncharged, nonrotating hole*

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(Received 18 August 1975)

Hawking has predicted that a black hole will emit particles as if it had a temperature proportional to its surface gravity. This paper combines Hawking's quantum formalism with the black-hole perturbation methods of Teukolsky and Press to calculate the emission rate for the known massless particles. Numerical results indicate that a hole of mass $M \gg 10^{17}$ g should emit a total power output of $2\times 10^{-4}\,\hbar c^6 G^{-2} M^{-2}$, of which 81% is in neutrinos, 17% is in photons, and 2% is in gravitons. These rates plus an estimate for the emission rates of massive particles from smaller holes allow one to infer that a primordial black hole will have decayed away within the present age of the universe if and only if its initial mass was $M < (5\pm 1)\times 10^{14}$ g.

I. INTRODUCTION

Hawking has calculated quantum mechanically[1] that a black hole will emit particles as if it were a hot body with a temperature T proportional to its surface gravity. Since the surface gravity is inversely proportional to the black-hole mass M, and the emitting area A is proportional to M^2, the luminosity or total power emitted is proportional to AT^4 or M^{-2}. As M decreases at this rate, the black-hole lifetime will be proportional to M^3. Dimensional arguments indicate that the lifetime will be less than the age of the universe only if $M \lesssim 10^{15}$ g. Consequently, the thermal emission is insignificant for black holes formed by the stellar collapse ($M \gtrsim M_\odot$, lifetime $\gtrsim 10^{66}$ yr), but it is of crucial importance for the small primordial black holes possibly formed by fluctuations in the early universe.[2-4]

This paper reports numerical calculations of the emission rates for massless particles. The spectra from the dominant angular modes are given for neutrinos, photons, and gravitons. The spectra are integrated to give the total number rate and power emitted in the various modes. From the total power emitted in all modes, the lifetime of a black hole is predicted. Essentially, this paper gives numerical coefficients for the dimensionally determined quantities of the preceding paragraph.

To simplify the notation, dimensionless units will be used such that

$$\hbar = c = G = k \text{ (Boltzmann's constant)} = 1. \tag{1}$$

That is, all quantities will be written in terms of the Planck mass ($[\hbar c/G]^{1/2} = 2.18\times 10^{-5}$ g), length ($[\hbar G/c^3]^{1/2} = 1.62\times 10^{-33}$ cm), time ($[\hbar G/c^5]^{1/2} = 5.39\times 10^{-44}$ sec), temperature ($[\hbar c^5/G]^{1/2}/k = 1.42\times 10^{32}$ °K), energy ($[\hbar c^5/G]^{1/2} = 1.96\times 10^{16}$ erg $= 1.22\times 10^{22}$ MeV), power ($c^5/G = 3.63\times 10^{59}$ erg sec^{-1}), charge ($[\hbar c]^{1/2} = 5.62\times 10^{-9}$ esu $= 11.7e$), etc. For example, the electron mass is $m_e = 4.19\times 10^{-23}$, the muon mass is $m_\mu = 8.65\times 10^{-21}$, the blackbody background temperature is $T_\gamma = 1.9\times 10^{-32}$, the age of the universe is $t_0 \approx 10^{61}$ (= 17 billion years), and the solar mass and luminosity are $M_\odot = 9.14\times 10^{37}$ and $L_\odot = 1.05\times 10^{-26}$, respectively.[5]

The present paper will limit itself to the known massless particles (ν_e, $\bar\nu_e$, ν_μ, $\bar\nu_\mu$, γ, and graviton) being emitted from an uncharged, nonrotating hole. Future papers in this series are being planned to consider rotating holes and the emission of massive particles. Massless particles will dominate the emission when $T \lesssim m_e$ (the smallest nonzero rest mass known). The approximation of zero rest mass should also be valid for $m_e \ll T \ll m_\mu$, in which case electrons and positrons will be emitted ultrarelativistically so that their rest mass can be ignored, whereas heavier particles will hardly be emitted at all. The approximation breaks down for the case $T \gtrsim m_\mu$ or $M \lesssim 5\times 10^{18} \approx 1\times 10^{14}$ g, which will not be considered.

Zaumen[6] and Gibbons[7] have shown that a black hole will discharge rapidly by a Schwinger-type pair-production process if

$$Q_* \equiv Q/M \gtrsim M m_e^2/e = 2.05\times 10^{-44} M = M/5.34\times 10^5 M_\odot . \tag{2}$$

Q_* is the charge parameter (dimensionless without setting $\hbar = 1$) that must be of order unity to affect significantly the geometry of a black hole and hence the emission of uncharged particles. Therefore, except for black holes above $10^5\, M_\odot$, which do not radiate at a significant rate anyway, the charge of the black hole can be ignored when analyzing the emission of uncharged particles. For a black hole small enough to be emitting electrons and positrons, the resulting random charge fluctuations are estimated to be of order unity. Such fluctuations do not affect the geometry significantly since only $M \gg 1 \approx 2\times10^{-5}$ g is being considered, but they do affect the coupling of the hole to electrons and positrons so that their average emission rates may be changed by a fraction of the order of the fine-structure constant. This effect will be ignored until a future paper.

The idealization of no rotation for the black hole is much less justified than the idealization of no charge, but there are two effects that *may* tend to make the rotation small. First there is the tendency of a rotating hole to emit more particles with angular momentum in the same direction as the hole than in the opposite direction. Indeed, for a hole rotating as fast as possible for a given mass, each particle emitted must decrease the angular momentum of the hole, and it appears that this decrease is characteristic of the total emission at any finite rotation. However, the classically dimensionless (no $\hbar$'s needed to make it dimensionless) rotation parameter that determines the shape of a black hole is

$$a_* \equiv J/M^2, \tag{3}$$

where J is the magnitude of the angular momentum. For $a_*=1$ (maximum rotation), it is easy to show that the emission leads to a decrease in a_*, but for a_* near zero, it is not yet known whether the angular momentum decreases fast enough compared with the mass to keep a_* decreasing, or whether $d\ln J/d\ln M = 2$ at some finite a_*, causing a_* to approach that value asymptotically rather than continuing to decrease toward zero.

The second effect which may tend to reduce the rotation is an instability to the exponential growth of massive scalar fields in a quasibound state around a rotating hole. Eardley has suggested this effect[8] as an analog of the "black-hole bomb,"[9] in which the rest mass of the field replaces the mirror to confine the field. This instability should rapidly drain angular momentum from the hole into orbiting particles, which then decay or radiate away their energy and angular momentum by gravitational radiation,[10] *if* (1) the size of the hole is roughly the Compton wavelength of one of these scalar particles (a pion, say), (2) the size of the particle itself is not too large compared with the size of the hole, and (3) it is possible to create many particles in the same mode so that the field can grow exponentially. (One might suppose that if a scalar particle were made of Fermi constituents, the exclusion principle for the constituents would prevent the scalar particles from piling up in the same mode by coherent amplification, so the drain of angular momentum would not occur at any exponentially large rate limited by the gravitation radiation from the mode but rather at a rate limited by the decay or interaction time, which would not be much, if any, faster than the direct emission mechanisms.)

In summary, this paper will consider the emission rates from an uncharged, nonrotating hole for massless particles of spin $\frac{1}{2}$, 1, and 2. This is meant to apply to neutrinos, photons, and gravitons (and possibly ultrarelativistic electrons and positrons from a hole small enough) being emitted from a primordial black hole that has been neutralized, if necessary, by $e^{\pm}$ emission and that somehow has little angular momentum.

II. THEORETICAL FORMALISM

According to Hawking's calculation, the expected number of particles of the jth species with charge e emitted in a wave mode labeled by frequency or energy ω, spheroidal harmonic l, axial quantum number or angular momentum m, and polarization or helicity p is

$$\langle N_{j\omega lmp}\rangle = \Gamma_{j\omega lmp}\{\exp[2\pi\kappa^{-1}(\omega - m\Omega - e\Phi)] \mp 1\}^{-1}. \tag{4}$$

Here the minus sign is for bosons and the plus sign is for fermions; $\Gamma_{j\omega lmp}$ is the absorption probability for an incoming wave of that mode (i.e., minus the fractional energy gain in a scattered classical wave, $-Z$ in the calculations of Teukolsky and Press[11]); κ, Ω, and Φ are the surface gravity, surface angular frequency, and surface electrostatic potential, respectively, of the black hole. The values of κ, Ω, and Φ are linked to the hole's mass M, area A, angular momentum J, and charge Q by the first law of black-hole mechanics[12]

$$dM = \frac{\kappa}{8\pi}\,dA + \Omega dJ + \Phi dQ. \tag{5}$$

The expected number emitted in each mode remarkably is the same as that of a thermal body whose absorptivity matches that of the hole and whose temperature is

$$T = \frac{\kappa}{2\pi}, \tag{6}$$

so $\frac{1}{4}A$ can be identified as the entropy of the black hole.[1] For a Kerr-Newman black hole with the horizon at radius

$$r_+ = M + (M^2 - Q^2 - a^2)^{1/2}, \tag{7}$$

the specific expressions for κ, Ω, and Φ are[13]

$$\kappa = \frac{4\pi(r_+ - M)}{A}$$

$$= \tfrac{1}{2}M^{-1}[1 + (1 - \tfrac{1}{2}Q_*^2)(1 - Q_*^2 - a_*^2)^{-1/2}]^{-1} \to \frac{1}{4M}, \tag{8}$$

$$\Omega = \frac{4\pi a}{A}$$
$$= \frac{a_*}{M}[2 - Q_*^2 + 2(1 - Q_*^2 - a_*^2)^{1/2}]^{-1}$$
$$\to \frac{a_*}{4M}\,, \tag{9}$$

$$\Phi = \frac{4\pi Q r_+}{A}$$
$$= Q_* \frac{1 + (1 - Q_*^2 - a_*^2)^{1/2}}{2 - Q_*^2 + 2(1 - Q_*^2 - a_*^2)^{1/2}}$$
$$\to \tfrac{1}{2}Q_*\,. \tag{10}$$

Here the quantities after the arrows are the leading terms for $a_* \equiv a/M \equiv J/M^2 \ll 1$ and $Q_* \equiv Q/M \ll 1$.

To convert from the expected number emitted per mode to the average emission rate per frequency interval, one counts the number of modes per frequency interval with periodic boundary conditions in a large container around the black hole and divides by the time it takes a particle to cross the container, finding

$$\frac{dN}{dt} = \langle N \rangle \frac{v\,dk}{2\pi} = \langle N \rangle \frac{d\omega}{2\pi} \tag{11}$$

for each j, l, m, p, and frequency interval $(\omega, \omega + d\omega)$. Since each particle carries off energy ω and angular momentum m about the axis of the hole, the mass and angular momentum of the hole decrease at the rates given by the total power and torque emitted:

$$-\frac{d}{dt}\binom{M}{J} = \sum_{j,l,m,p} \frac{1}{2\pi} \int \Gamma_{j\omega l m p} \{\exp[2\pi\kappa^{-1}(\omega - m\Omega - e\Phi)] \mp 1\}^{-1} \binom{\omega}{m} d\omega. \tag{12}$$

The nontrivial part of the calculation of the power and torque is the determination of the absorption probabilities Γ. Fortunately, Teukolsky has shown[14] that the fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations of an uncharged rotating black hole decouple into a single equation for each field, and furthermore that each of these equations is completely separable into ordinary differential equations. Teukolsky and Press[11] have developed analytic and numerical techniques for interpreting and solving these equations for gravitational and electromagnetic perturbations. Their techniques can be extended easily to the neutrino field, and I have simply modified their computer programs to cover neutrinos as well as gravitons and photons.

A check on the numerical computation can be given by the analytic form of Γ for small $M\omega$, which has been derived by Starobinsky and Churilov[15] for boson fields and which is extended in the Appendix to fermion fields obeying the Teukolsky equation. For a massless field with spin-s scattering off an uncharged hole, the formulas are

$$\Gamma_{s\omega l m p} = \left[\frac{(l-s)!\,(l+s)!}{(2l)!\,(2l+1)!!}\right]^2 \prod_{n=1}^{l} \left[1 + \left(\frac{\omega - m\Omega}{n\kappa}\right)^2\right] 2\left(\frac{\omega - m\Omega}{\kappa}\right)\left(\frac{A\kappa}{2\pi}\omega\right)^{2l+1}, \quad 2s \text{ even}, \tag{13}$$

$$\Gamma_{s\omega l m p} = \left[\frac{(l-s)!\,(l+s)!}{(2l)!\,(2l+1)!!}\right]^2 \prod_{n=1}^{l+1/2} \left[1 + \left(\frac{\omega - m\Omega}{n\kappa - \frac{1}{2}\kappa}\right)^2\right] \left(\frac{A\kappa}{2\pi}\omega\right)^{2l+1}, \quad 2s \text{ odd}, \tag{14}$$

with fractional errors of order $(A\kappa\omega)^{2l+1}$. Since $l \geq s$, the dominant contribution is from the $l = s$ modes, which give

$$\Gamma_{0\omega 0 0 p} = \frac{A}{\pi}\omega^2 = 8M[M + (M^2 - a^2)^{1/2}]\,\omega^2 \quad \text{for } s = 0, \tag{15}$$

$$\Gamma_{\frac{1}{2}\omega\frac{1}{2}mp} = \frac{1}{4}\left(1 + \frac{\Omega^2}{\kappa^2}\right)\left(\frac{A\kappa}{2\pi}\omega\right)^2 = M^2\omega^2 \quad \text{for } s = \tfrac{1}{2}, \tag{16}$$

$$\Gamma_{1\omega 1 m p} = \frac{4}{9}\frac{A}{\pi}[M^2 + (m^2 - 1)a^2](\omega - m\Omega)\omega^3 \quad \text{for } s = 1, \tag{17}$$

$$\Gamma_{2\omega 2 m p} = \frac{16}{225}\frac{A}{\pi}[M^2 + (m^2 - 1)a^2][M^2 + (\tfrac{1}{4}m^2 - 1)a^2](\omega - m\Omega)\omega^5 \quad \text{for } s = 2. \tag{18}$$

Here only the lowest-order term in ω has been kept, except for the $\omega - m\Omega$ factor for bosons which guarantees that in the superradiant regime $\omega < m\Omega$, the absorption probability for bosons in negative. [I.e., waves are amplified rather than absorbed. The thermal factor of Eq. (12) is also negative in this regime, so the quantum emission rate remains positive.]

From the behavior of these analytic absorption probabilities at low frequencies for the various angular modes, one can get the low-frequency ($M\omega \ll 1$) absorption cross section for a massless particle of spin s averaged over all orientations of the black hole:[16]

$$\sigma_s(\omega) = \pi\omega^{-2}\sum_{l,m}\Gamma_{s\omega lmp} \underset{\omega\to 0}{\sim} \begin{cases} A, & s=0 \\ 2\pi M^2, & s=\frac{1}{2} \\ \frac{4}{9}A(3M^2-a^2)\omega^2, & s=1 \\ \frac{16}{225}A(5M^2+\frac{5}{2}M^2a^2+a^4)\omega^4, & s=2. \end{cases} \tag{19}$$

At high frequencies ($M\omega \gg 1$) the angle-averaged cross section for each kind of particle must approach the geometrical-optics limit of $27\pi M^2$ for a nonrotating hole and roughly the same value for a rotating hole.[17] Thus the cross sections are smaller at low frequencies. As the frequency is reduced to zero, the cross sections retain finite values for neutrinos and hypothetical spin-0 massless particles and go to zero as the frequency squared for photons and as the frequency to the fourth power for gravitons.

Combining the low-frequency absorption probabilities (13) and (14) with the thermal factor (4) for a black hole with negligible rotation, one gets the emission rate in a given angular and polarization eigenstate for low frequencies,

$$\frac{d}{dtd\omega}N_{s\omega lmp} = \frac{\beta}{4\pi^2}\left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!!}\right]^2(2M\omega)^{2l+1}, \tag{20}$$

where $\beta=2$ for bosons and $\beta=\pi$ for fermions. The fractional errors are of order $M(\omega-m\Omega)$. Thus in each case the emission rate at low frequencies goes as ω^{2l+1}, and the power goes as ω^{2l+2}. This qualitative behavior causes the particles with lower spins (and thus lower l allowed, since $l \geq s$) to be emitted faster from a nonrotating hole, thereby dominating the low-frequency power drain from such a hole. However, the analytic expressions for low frequency break down long before the actual spectra peak, so numerical calculations are needed to determine whether and to what extent this effect holds also for the total power drain.

III. NUMERICAL CALCULATIONS

The particle emission rates were calculated by using Hawking's formula (4) and Eq. (11) with the absorption probabilities Γ computed by the method of Ref. 9, Sec. VII, using Bardeen's transformation discussed therein to allow stable integration of the Teukolsky equation from the horizon to infinity. A purely ingoing solution was chosen on the horizon, and after this solution was numerically integrated out to a sufficiently large radius, it was resolved into ingoing and outgoing waves at infinity. Then Γ was calculated as the ratio of the energy going down the hole to the energy of the ingoing wave at infinity, and the thermal factors were multiplied in to give the quantum emission rates. These rates were multiplied by the energy or angular momentum of each particle, integrated over frequency, and summed over all angular modes, polarizations, and species of particles to give the total power and torque emitted [cf. Eq. (12)].

The accuracy of the numerical result was limited by the step size in integrating the Teukolsky equation, the radius where the resolution into ingoing and outgoing waves is made, and the step size in integrating the spectra. To keep these three sources of error under control, variable step sizes were used with an error criterion for each step, and the resolution into ingoing and outgoing waves was required to be the same within a certain accuracy at two different radii. Thus the total error was governed by three accuracy criteria, and these were chosen for each mode to give roughly the same effect on the final result so that the result might have nearly the greatest accuracy possible for a given computer machine time.

The numerical calculations of the emission rates compared favorably with Eq. (20) at low frequencies, although departures from the extended Starobinsky-Churilov expression become significant at fairly small values of $M\omega$. For example, the actual value of Γ for neutrinos with $l=\frac{1}{2}$ becomes 50% larger than that given by Eq. (14) when $M\omega = 0.05$. This effect prevents one from getting an accurate estimate of the total power and torque emitted by inserting (13) and (14) into (12). [One might have expected such an estimate to be fairly accurate on grounds that the exponential of $8\pi M\omega$ (for a nonrotating hole) in the denominator of (12) might become large and make the integrand small before the expression for Γ develops serious errors.] In fact, such an estimate gave only 35% of the actual total power in neutrinos, 13% of the actual power in photons, and 5% of the actual power in gravitons, or 30% of the total in all massless particles.

IV. RESULTS

The power spectra for neutrinos, photons, and gravitons are given in Fig. 1. The integrated emission rates and power for the dominant angular modes are listed in Table I. The total in all of the known massless fields (four kinds of neutrinos with one helicity each and photons and gravitons with two helicities each) is $1.130\times10^{-3}\ c^3G^{-1}M^{-1}$ for the emission number rate and $2.011\times10^{-4}\ \times\hbar c^6G^{-2}M^{-2}$ for the power. One may compare these numerical results with the naive estimates of thermal emission from cross sections σ that are assumed to be independent of frequency. Then the power would be

$$P=acT^4\left[\tfrac{7}{16}\sigma(\nu_e)+\tfrac{7}{16}\sigma(\overline{\nu}_e)+\tfrac{7}{16}\sigma(\nu_\mu)+\tfrac{7}{16}\sigma(\overline{\nu}_\mu)+\sigma(\gamma)+\sigma(g)\right] \tag{21}$$

for emission of ν_e, $\overline{\nu}_e$, ν_μ, $\overline{\nu}_\mu$, γ, and g (gravitons). Here

$$a=\frac{\pi^2k^4}{15\hbar^3c^3} \tag{22}$$

is the radiation density constant,[5] and T is the temperature of the black hole, given by Eq. (6). If we take the high-frequency limit, all the cross sections go to $27\pi G^2M^2/c^4$, and the power estimate becomes $5.246\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$, which is a factor of 2.6 too large. If we take the low-frequency limit, Eq. (19) shows that the photon and graviton cross sections go to zero, whereas the neutrino cross sections go to $2\pi G^2M^2/c^4$, so the power estimate becomes $0.181\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$, which is a factor of 11 too small. (The thermally averaged cross sections turn out to be $18.05\pi M^2$ for photons, $6.492\pi M^2$ for photons and $0.742\pi M^2$ for gravitons.)

If the black hole is small enough that electrons and positrons are emitted ultrarelativistically (and thus at the same rate for each helicity as neutrinos) but not small enough for heavier particles to be emitted at a significant rate, the power is $3.65\times10^{-4}\,\hbar c^6G^{-2}M^{-2}$. The peak in the neutrino power spectrum (which should be the same as that for ultrarelativistic electrons) is at $\omega=0.18M^{-1}$; therefore, the assumption of only ultrarelativistic $e^{\pm}$ applies for

$$m_e=4.19\times10^{-23}\ll0.18M^{-1}\ll m_\mu=8.65\times10^{-21}, \tag{23}$$

which is true for the mass range

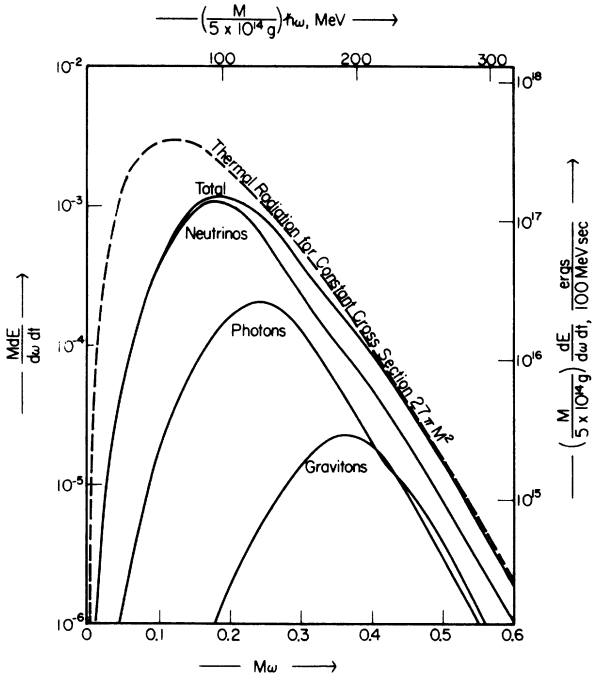

FIG. 1. Power spectra from a black hole, obtained by adding all angular modes for four kinds of neutrinos and for two polarization states (helicities) each of photons and gravitons. The lowest angular modes, $l=s$, dominate, but the $l=s+1$ modes can be seen coming in with a small "bump" in the neutrino spectrum at $M\omega\approx0.4$ and in the photon spectrum at $M\omega\approx0.5$. The total power spectrum can be seen at high frequencies to approach that of a thermal body with a cross section of $27\pi M^2$, but at low frequencies the spectrum drops below the Planck form as the cross section of the black hole is reduced.

$$2.1\times10^{19}=4.5\times10^{14}\,\text{g}\ll M\ll4.3\times10^{21}=9.4\times10^{16}\,\text{g}. \tag{24}$$

A black hole with $M\gg10^{17}$ g would emit virtually no known massive particles, and a hole with $M\lesssim5\times10^{14}$ g would emit muons and heavier particles at a significant rate.

Knowing the expression for the total power emitted from a nonrotating black hole, one can calculate the lifetime of such a hole. The power emitted causes the mass to decrease at the rate

$$\frac{dM}{dt}=-\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2}, \tag{25}$$

where α is a numerical coefficient (see above) that depends on which particle species can be emitted at a significant rate. Since most of the decay time of the hole is spent near the original mass M_0, α can be taken to be its value α_0 at

TABLE I. Emission rates and powers for the dominant angular modes.

$2s$ [a]	$2l$ [b]	δ [c]	ϵ [d]	ζ [e]	For each mode rate [f]	For each mode power [g]	g [h]	For each (s,l) rate [f]	For each (s,l) power [g]
1	1	6	4	8	1.191×10^{-4}	1.969×10^{-5}	8	9.531×10^{-4}	1.575×10^{-4}
1	3	5	3	8	1.12×10^{-6}	3.75×10^{-7}	16	0.180×10^{-4}	0.060×10^{-4}
1	5	4	2	9	9.5×10^{-9}	4.9×10^{-9}	24	0.002×10^{-4}	0.001×10^{-4}
2	2	6	3	8	2.44×10^{-5}	5.49×10^{-6}	6	1.463×10^{-4}	0.330×10^{-4}
2	4	5	2.4	8	1.63×10^{-7}	6.67×10^{-8}	10	0.016×10^{-4}	0.007×10^{-4}
2	6	4	2	9.4	1.1×10^{-9}	6.5×10^{-10}	14	0.0001×10^{-4}	0.0001×10^{-4}
4	4	5	2.4	8	1.10×10^{-6}	3.81×10^{-7}	10	0.110×10^{-4}	0.038×10^{-4}
4	6	4	2	9.4	4.7×10^{-9}	2.6×10^{-9}	14	0.0007×10^{-4}	0.0004×10^{-4}
Total rate and power for all modes								1.130×10^{-3}	2.011×10^{-4}

[a] s is the spin of the field, here doubled to give an integer; i.e., $2s=1$ for neutrinos, $2s=2$ for photons, and $2s=4$ for gravitons.

[b] l is the total angular momentum of the mode.

[c] $10^{-\delta}$ is the fractional error criterion for each step in the radial integration of the Teukolsky equation.

[d] $10^{-\epsilon}$ is the fractional error criterion for the resolution of a numerical solution of the Teukolsky equation into ingoing and outgoing waves.

[e] $10^{-\zeta}$ is the absolute error criterion for the integration over frequencies.

[f] Rate in units of $c^3G^{-1}M^{-1}=4.038\times10^{38}(M/\text{g})^{-1}\ \text{sec}^{-1}$.

[g] Power in units of $\hbar c^6G^{-2}M^{-2}=1.719\times10^{50}(M/\text{g})^{-2}\ \text{erg sec}^{-1}$.

[h] g is the number of modes for a given l and s, $(2l+1)\times$(number of particle species with the given s) $\times$(number of polarizations or helicities for each species).

that mass, if $\alpha(M)$ does not change rapidly with mass near M_0 (as it might for $M_0 \lesssim 5\times10^{14}$ g). Then the lifetime of the hole is

$$\tau \approx \frac{G^2}{\hbar c^4}\frac{M_0^3}{3\alpha_0}. \tag{26}$$

For $M \gg 10^{17}$ g, $\alpha = 2.011\times10^{-4}$, so

$$\begin{aligned}\tau &= 8.66\times10^{-27}(M_0/\text{g})^3\ \text{sec}\\ &= 2.16\times10^{66}(M_0/M_\odot)^3\ \text{yr}.\end{aligned} \tag{27}$$

For $5\times10^{14}\ \text{g} \ll M \ll 10^{17}$ g, $\alpha = 3.6\times10^{-4}$, so

$$\tau \approx 4.8\times10^{-27}\ (M_0/\text{g})^3\ \text{sec} = 1.5\times10^{-34}\ (M_0/\text{g})^3\ \text{yr}. \tag{28}$$

Since the lifetime of a black hole of stellar mass is so enormous, the decay is important only for black holes of much smaller mass, which cannot be formed by any processes (except for extremely rare quantum tunneling) that we know of in the present universe but which might have formed in the early universe.[2-4] It is of interest to determine what initial masses should have decayed away and what masses should still be around. Taking the lifetime of the black hole as the present age of the universe, say 16 billion years,[18] one finds that if only the known massless particles are emitted, $M_0 = 3.9\times10^{14}$ g. This is inconsistent with negligible emission of massive particles, so one must add ultrarelativistic $e^\pm$ emission, getting $M_0 = 4.7\times10^{14}$ g. This is at the mass where muon and pion emission are beginning to become important, so a somewhat larger mass should have decayed by now. However, unless the power is increased more than a factor of 2 due to the emission of muons and heavier particles (unlikely) and unless the universe age is outside 8–18 billion years[18] (also unlikely), probably $M_0 = (5\pm1)\times10^{14}$ g is the initial mass of a primordial nonrotating, uncharged black hole that just decays away at the present age of the universe by the emission of the known elementary particles.

In conclusion, the power emitted from an uncharged, nonrotating black hole of mass $M \gg 10^{17}$ g is

$$\begin{aligned}P &= 2.011\times10^{-4}\ \hbar c^6 G^{-2} M^{-2}\\ &= 3.458\times10^{46}(M/\text{g})^{-2}\ \text{erg sec}^{-1}\\ &= 2.28\times10^{-54}\ L_\odot (M/M_\odot)^{-2},\end{aligned} \tag{29}$$

of which 81.4% is in the four kinds of neutrinos, 16.7% is in photons, and 1.9% is in gravitons, assuming these are the only massless particles. For $5\times10^{14}\ \text{g} \ll M \ll 10^{17}$ g,

$$\begin{aligned}P &\approx 3.6\times10^{-4}\ \hbar c^6 G^{-2} M^{-2}\\ &= 6.3\times10^{16}(M/10^{15}\ \text{g})^{-2}\ \text{erg sec}^{-1},\end{aligned} \tag{30}$$

of which 45% is in electrons and positrons, 45%

is in neutrinos, 9% is in photons, and 1% is in gravitons. This assumes electrons and muons are the lightest particles with rest mass. The emission of particles is unimportant for stellar-mass black holes but should have caused any primordial black hole with an initial mass less than 4×10^{14} g (and perhaps somewhat greater values) to decay away by now.

ACKNOWLEDGMENTS

I would like to thank several people whose help and encouragement made this calculation possible. Among these are S. A. Teukolsky and W. H. Press, who gave me their computer programs for calculating the absorption probabilities of photons and gravitons, and who advised me on the modifications needed for neutrinos and on other numerical problems. B. A. Zimmerman, K. H. Despain, F. J. Nagy, and C. L. Rosenfeld assisted me in programming and using the Lawrence Berkeley Laboratory CDC-7600 computer via a remote terminal at Caltech. S. W. Hawking and B. J. Carr provided interesting discussions as the work was being written up. They and K. S. Thorne and C. M. Caves offered helpful comments on the manuscript.

APPENDIX

The absorption probability Γ at low frequencies can be calculated by analytically solving the Teukolsky equation with the approximation $M\omega \ll 1$ and finding what fraction of any ingoing wave from infinity gets reflected back out. In Boyer-Lindquist coordinates for an uncharged hole, a massless field of spin-weight s, frequency ω, and axial quantum number m obeys the radial Teukolsky equation[19]

$$\Delta^{-s+1}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right)+\left[(r^2+a^2)^2\omega^2-4aMr\omega m+a^2m^2+2ia(r-M)ms-2iM(r^2-a^2)\omega s+(2ir\omega s-\lambda)\Delta\right]R=0. \tag{A1}$$

Here

$$\Delta\equiv r^2-2Mr+a^2\equiv(r-r_+)(r-r_-), \tag{A2}$$

and λ is an eigenvalue of the angular equation

$$\frac{1}{\sin\Theta}\frac{d}{d\Theta}\left(\sin\Theta\frac{dS}{d\Theta}\right)+\left[(s-a\omega\cos\Theta)^2-\left(\frac{m+s\cos\Theta}{\sin\Theta}\right)^2-s(s-1)+\lambda-a^2\omega^2\right]S=0. \tag{A3}$$

(λ is the same as in Ref. 15 and is the same as $\lambda+2am\omega$ in Ref. 11.)

Following Starobinsky and Churilov[15] generally, define

$$x\equiv\frac{r-r_+}{2(r_+-M)}=\frac{r-M-(M^2-a^2)^{1/2}}{2(M^2-a^2)^{1/2}}, \tag{A4}$$

$$Q\equiv\frac{m\Omega-\omega}{2\kappa}=\frac{Mr_+}{r_+-M}(m\Omega-\omega), \tag{A5}$$

$$k\equiv2\omega(r_+-M)=2M\omega(1-a_*{}^2)^{1/2}. \tag{A6}$$

Then small $M\omega$ implies that the radial equation can be approximated as

$$x^2(x+1)^2\frac{d^2R}{dx^2}+(s+1)x(x+1)(2x+1)\frac{dR}{dx}+\left[k^2x^4+2iskx^3-\lambda x(x+1)+isQ(2x+1)+Q^2\right]R=0, \tag{A7}$$

with $k\ll1$. Small $M\omega$ also implies $a^2\omega^2\ll1$, in which case the angular eigenvalue becomes very nearly

$$\lambda=(l-s)(l+s+1), \tag{A8}$$

where $l-s$ is a non-negative integer. (In the limit of $a_*\to0$, l is the total angular momentum of the mode.)

For $kx\ll l+1$, the first two terms inside the square brackets of Eq. (A7) can be dropped, leading to an equation with three regular singular points. A solution obeying the ingoing boundary conditions at the horizon[19] is[20]

$$R=x^{-s+iQ}(x+1)^{-s-iQ}\,{}_2F_1(-l-s,\ l-s+1;\ 1-s+2iQ;\ -x). \tag{A9}$$

Here ${}_2F_1(a,b;c;z)$ is the hypergeometric function. For $x\gg|Q|+1$, the last two terms inside the square brackets can be dropped, and $x+1$ can be replaced by x, leading to an equation with one regular and one irregular singular point. The solution is, if $2l$ is not an integer,[21]

$$R = C_1 e^{-ikx} x^{l-s} {}_1F_1(l-s+1;\ 2l+2;\ 2ikx) + C_2 e^{-ikx} x^{-l-s-1} {}_1F_1(-l-s;\ -2l;\ 2ikx). \tag{A10}$$

Here ${}_1F_1(a; c; z)$ is the confluent hypergeometric function. [To avoid solutions with logarithmic terms, and to simplify the matching procedure, we will henceforth assume $2l$ is nearly, but not exactly, integral. This is actually the case when $a^2\omega^2 \neq 0$ if we use Eq. (A8) as the definition of l when λ is given from Eq. (A3) rather than as an approximate formula for λ when $l-s$ is given as a non-negative integer.]

By matching the two solutions in the overlap region $|Q|+1 \ll x \ll (l+1)/k$, one can get

$$C_1 = \frac{\Gamma(2l+1)\,\Gamma(1-s+2iQ)}{\Gamma(l-s+1)\Gamma(l+1+2iQ)},\quad C_2 = \frac{\Gamma(-2l-1)\Gamma(1-s+2iQ)}{\Gamma(-l-s)\Gamma(-l+2iQ)}\ . \tag{A11}$$

Then the asymptotic form of the confluent hypergeometric functions can be used to get the solution in the form

$$R = Y_{\text{in}}\, e^{-ikx} r^{-1} + Y_{\text{out}}\, e^{ikx} r^{-2s-1} \tag{A12}$$

for $kx \gg 1$, where

$$\begin{aligned} Y_{\text{in}} &= \frac{\Gamma(2l+1)\Gamma(2l+2)\Gamma(1-s+2iQ)}{\Gamma(l-s+1)\Gamma(l+s+1)\Gamma(l+1+2iQ)}\,\frac{k}{\omega}\,(-2ik)^{-l+s-1} + \frac{\Gamma(-2l)\Gamma(-2l-1)\Gamma(1-s+2iQ)}{\Gamma(-l-s)\Gamma(-l+s)\Gamma(-l+2iQ)}\,\frac{k}{\omega}\,(-2ik)^{l+s},\\ Y_{\text{out}} &= \frac{\Gamma(2l+1)\Gamma(2l+2)\Gamma(1-s+2iQ)}{[\Gamma(l-s+1)]^2\,\Gamma(l+1+2iQ)}\left(\frac{k}{\omega}\right)^{2s+1}(2ik)^{-l-s-1} + \frac{\Gamma(-2l)\Gamma(-2l-1)\Gamma(1-s+2iQ)}{[\Gamma(-l-s)]^2\,\Gamma(-l+2iQ)}\left(\frac{k}{\omega}\right)^{2s+1}(2ik)^{l-s}. \end{aligned} \tag{A13}$$

To obtain the ratio of outgoing to ingoing fluxes, one can either calculate the normalization factors of Ref. 11 to apply to $|Y_{\text{out}}/Y_{\text{in}}|^2$, or one can use the following trick: Solve the radial equation with s replaced by $-s$ to get the asymptotic form

$$R_{-s} = Z_{\text{in}}\, e^{-ikx} r^{-1} + Z_{\text{out}} e^{ikx} r^{2s-1} \tag{A14}$$

[i.e., Z_{in} and Z_{out} are the same as Y_{in} and Y_{out}, respectively, in Eq. (A13) above with s replaced by $-s$]. Then the reflection coefficient is (cf. Ref. 11)

$$1-\Gamma = \frac{dE_{\text{out}}}{dE_{\text{in}}} = \left|\frac{Y_{\text{out}}\, Z_{\text{out}}}{Y_{\text{in}}\ Z_{\text{in}}}\right|. \tag{A15}$$

After some algebra, one finds that with a fractional error of order k^{2l+1},

$$\Gamma_{s\omega lmp} = \text{Re}\left\{4e^{i\pi(s-1/2)}\cos[\pi(l-s)]\,\frac{\Gamma(-2l)\Gamma(-2l-1)}{\Gamma(2l+1)\Gamma(2l+2)}\left[\frac{\Gamma(l-s+1)}{\Gamma(-l-s)}\right]^2\frac{\Gamma(l+1+2iQ)}{\Gamma(-l+2iQ)}\,(2k)^{2l+1}\right\}\ . \tag{A16}$$

Now one can keep $2s$ exactly integral and take the limit as $l-s$ approaches a non-negative integer. Then

$$\Gamma_{s\omega lmp} = \text{Re}\left\{\left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!}\right]^2\frac{\Gamma(l+1+2iQ)}{\Gamma(-l+2iQ)}\,(2ik)^{2l+1}\right\}\ . \tag{A17}$$

Taking the cases of integral or half-integral spins separately (corresponding to $2l$ even or odd, respectively) to express the quotient of the two Γ functions as a finite product, and rewriting Q and k in terms of ω, m, Ω, κ, and A, one obtains Eqs. (13) and (14). The result for integral spins was given by Starobinsky and Churilov, though not the result for half-integral spins.

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*Work supported in part by the National Science Foundation under Grant No. MPS75-01398, and by NSF and Danforth Foundation predoctoral fellowships.

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