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Crossing relations for infinitely rising trajectories: An asymptotic expression of local duality*

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Asymptotic crossing relations for the Regge amplitudes are proposed as a model-independent expression of local duality. They allow local duality to be formulated solely in terms of Regge trajectories without any violation of unitarity or neglect of resonance widths. The crossing relations connect the direct- and crossed-channel Regge-pole amplitudes for the elastic scattering of two equal-mass particles as $s \to \infty$ in a given domain of the real st plane. The main purpose of this paper is to determine the type of domain in which such crossing relations will possess nontrivial solutions that are both self-consistent and in keeping with the known properties of the Regge trajectories and residues. An extensive and systematic study of all domains involving large, positive s and values of t in some interval $[t_1, t_2]$ along the positive t axis is presented. We consider both intervals of fixed length and intervals whose length is increasing with s. Only one type of interval is found that is satisfactory, namely, intervals consisting of values of t that are increasing in proportion to s, i.e., for any given s, $t \in [s/r_1, s]$ for some $r_1 > 1$. The reasons are the following. It is the only type of interval (1) in which we can provide an existence proof for the solutions to the crossing relations, the proof being valid for asymptotically parallel trajectories; (2) for which the generalization to unequal-mass scattering encounters no obvious inconsistencies; and (3) for which the residues calculated from the crossing relations have an asymptotic form of the type found in the Veneziano model. Furthermore, we can prove the validity of the proposed crossing relations over this type of interval if the modified s-channel background integral B_s' satisfies the bound $\ln|B_s'| < \sqrt{2as^p/p}$, where $\mathrm{Re}\alpha(s) \to as^p$ as $s \to \infty$ for $p > 0.802$. The analogous proof for any of the other types of domains considered requires the bound $\ln|B_s'| < N \ln s$ for some N, which is a much stronger bound. The proof to which we refer assumes only that the contributions of Regge branch cuts to the scattering amplitude are negligible compared to the contributions of Regge poles, $\alpha_n(s)$, for s positive and sufficiently large. In the indicated domain, the crossing relations imply certain homogeneous integral equations that the residues must satisfy. Although a complete solution is not given, we show that as $s \to \infty$ the residues behave as $\exp(-\gamma as^p)$, where γ is a logarithmic function of r_1 and p. From the expression for γ we obtain the upper bound $\gamma < \ln(3 + 2\sqrt{2}) - \sqrt{2/p}$ (for $p = 1$, $\gamma < 0.348$). The constant analogous to γ in the Veneziano model is 0.38. We add that if the crossing relations proposed here are not valid, then either the background integrals must generate the crossed-channel Regge terms or the background integrals in concert with the direct-channel Regge terms must do so.

I. INTRODUCTION

The particles and resonances of strong-interaction physics are known to lie anong the rising portions of Regge trajectories. If the trajectories continue their upward climb indefinitely,[1] the infinity of resonances generated in their wake should be related by duality to the crossed-channel trajectories which control the asymptotic growth of the scattering amplitude. In brief, we would say that crossed-channel trajectories are built out of direct-channel resonances. However, the exact meaning of such a statement and hence of local duality is not clear. At present these ideas are well defined primarily in the context of the narrow-width approximation in which case they find expression in the well-known Veneziano[2] and related models.[3]

In this paper we investigate the possibility of formulating a model-independent definition of local duality for the elastic scattering of two equal-mass particles as suggested in an earlier paper.[4] By local duality we refer to the idea that a sum of s-channel resonances should be Regge-behaved for large s at each value of t in some, as yet, unspecified domain of the t plane. One approach to this problem is to attempt to unitarize the Veneziano model directly and in this way move away from the narrow-width restriction. Here we assume that something of this kind is possible, that a sum over the s-channel resonances, i.e.,

$$\sum_{l=0}^{\infty} (2l+1) P_l(z_s) \sum_n r_{ln}(s), \tag{1.1}$$

does indeed behave as $s^{\alpha(t)}$, or at least as a sum of t-channel Regge-pole terms as $s \to \infty$. In Eq. (1.1), $r_{ln}(s)$ is a resonance amplitude for a resonance of mass $M_n - \frac{1}{2}i\Gamma_n$ and spin l. Also $z_s \equiv \cos\theta_s = 1 + 2t/(s - 4\mu^2)$, where θ_s is the s-channel scattering angle and μ is the mass of the external particle. Primarily we are concerned with what types of trajectories, residues, and resonances are capable of satisfying such a condition once it has been unambiguously specified.

The ambiguity which we have in mind resides in the specification of the domain in the st plane in which asymptotic duality should be exhibited and also in the resonance amplitudes themselves, the problem in the latter being that there is no universally acknowledged way to define a resonance amplitude away from the resonance position. We postpone consideration of the appropriate domain and consider the second ambiguity. To resolve it we simply choose a specific, yet not overly restrictive, form for the resonance amplitudes, namely

$$r_{ln}(s) \equiv \tfrac{1}{2}\,[1+(-1)^l\,\sigma_n]\left(\frac{2\alpha_n+1}{l+\alpha_n+1}\right)\left[\frac{\beta_n(s)}{l-\alpha_n(s)}\right], \tag{1.2}$$

where σ_n is the signature of the nth trajectory. This form is suggested by the fact that resonances lie along Regge trajectories, and, in fact, Eq. (1.2) is simply the lth partial-wave projection of the nth Regge term. This is the basic assumption of the paper. It is to be compared with the more common tendency to express the resonance amplitude in terms of the poles and residues in the complex s plane, $r_{ln}(s)=g_n(l,s)[s-M_n^2+iM_n\Gamma_n]^{-1}$, where $g_n(l,M_n^2-iM_n\Gamma_n)$ is the residue of the pole.

It is more convenient to parameterize the resonance amplitudes by the Regge trajectories and residues than by the unfamiliar functions $g_n(l,s)$. The explicit l dependence of (1.2) allows (1.1) to be analytically continued to arbitrarily large values of t in a particularly simple manner. Recall that the sum over l in (1.1) only converges for t inside the Lehman ellipse. However, it can be explicitly summed for $t<0$ using (1.2) and Dougall's formula.[5] The result, of course, is a sum of the usual s-channel Regge amplitudes $\sum R_s^{(n)}$, where

$$R_s^{(n)} \equiv -2\pi\beta_n(s)(\alpha_n+\tfrac{1}{2})\left[\frac{P_{\alpha_n}(-z_s)+\sigma_n P_{\alpha_n}(z_s)}{2\sin\pi\,\alpha_n(s)}\right] \tag{1.3}$$

The choice of (1.2) as the appropriate resonance amplitudes leads to a mathematical statement of local duality of the form

$$\sum R_s^{(n)} \sim R_t^{(L)} \text{ or } \sum_n R_t^{(n)} \tag{1.4}$$

for large s, where $R_t^{(n)}$ is the nth t-channel Regge amplitude defined by (1.3) with $s \longleftrightarrow t$ and $R_t^{(L)}$ is the amplitude for the leading trajectory in the t channel. The appropriate choice in (1.4) will depend on the particular domain in which an asymptotic expression of duality can occur. If the domain is such that $(t/s)\to 0$ as $s\to\infty$, $R_t^{(L)}$ will suffice; otherwise the sum over the lower-lying trajectories will be needed. At any rate, the s-channel trajectories and residues must produce trajectories and residues in the crossed channel that are functionally identical to themselves in the case of elastic scattering. This suggests that Eq. (1.4) be regarded as providing an asymptotic bootstrap mechanism for the Regge parameters. However, there is the difficulty that the s- and t-channel trajectories in Eq. (1.4) may enter at different values of their arguments. Whether or not this proves to be a problem depends on the particular domain of validity chosen for Eq. (1.4).

Another important question in this approach regards the number of s-channel trajectories in Eq. (1.4) needed to generate crossed-channel Regge terms. In general we will assume that a finite but increasing (with s) number of s-channel trajectories is sufficient.

A complicating factor in these considerations is the presence of Regge cuts which we have ignored in the previous statements. Our attitude is that Regge poles plus unitarity probably necessitate Regge cuts and any attempt to unitarize the Veneziano model or otherwise give meaning to local duality outside the narrow-width approximation must take Regge cuts into account. This is a serious problem for duality, particularly if Regge cuts lie above their respective Regge poles. In that case, the Regge-cut amplitudes will dominate the Regge-pole amplitudes at sufficiently high energies, and there is evidence that Regge cuts are important even at low energies. If this is the case, Eq. (1.4) should presumably contain contributions from Regge cuts as well as Regge poles. Our statement of local duality would then have the form

$$C_s+\sum R_s^{(n)} \sim C_t+\sum R_t^{(n)} \tag{1.5}$$

as $s\to\infty$, where C_s (C_t) are the s- $(t$-$)$ channel Regge-cut amplitudes. Although our motivation for suggesting Eq. (1.5) has been the desire to define local duality, we should notice that (1.5) follows immediately from the assumption that the Regge background integrals, as defined by (1.7), are negligible at large s since crossing requires that for all s and t

$$C_s+\sum R_s^{(n)}+B_s=C_t+\sum R_t^{(n)}+B_t, \tag{1.6}$$

where

$$B_s(s,t)\equiv -\frac{1}{2i}\int_{-1/2-i\infty}^{-1/2+i\infty} dl\,(2l+1)a_l(s) \times\left[\frac{P_l(-z_s)+\sigma P_l(z_s)}{2\sin\pi l}\right] \tag{1.7}$$

and similarly for B_t.

Admittedly Eq. (1.5) is not as appealing as (1.4) but experimental evidence makes it difficult to justify the omission of C_t for very large energies, especially for t away from the forward direction. However, this conclusion applies only to negative values of t, i.e., the physical region of the s channel. Perhaps there are other domains in the t plane in which C_t and C_s are not as large as the Regge pole terms as $s\to\infty$. That this is indeed the case is suggested by the familiar formula[6] for the position of the branch point for negative t generated by the exchange of N_B Regge trajectories,

$$\alpha^B(t)=N_B\alpha(t/N_B^2)-N_B+1. \tag{1.8}$$

If this is valid for large, positive values of t, it implies that any trajectory increasing as fast or faster than $\sqrt{t}(\ln t)^R$ for $R>0$ will eventually lie above the leading Regge branch point as $t\to\infty$. The leading term in the asymptotic expansion of the leading Regge trajectory, $\mathrm{Re}\alpha_L(t)$, can be written as

$$\mathrm{Re}\alpha_L(t)\approx at^p(\ln t)^R+\cdots \tag{1.9}$$

for large t. Then by Eq. (1.8)

$$\mathrm{Re}\alpha^B(t)=\frac{a}{(N_B)^{2p-1}}\,t^p(\ln t)^R-\frac{a}{(N_B)^{2p-1}}\,t^p(\ln N_B^2)^R+\cdots. \tag{1.10}$$

If $p>\frac{1}{2}$, the first term in (1.10) will be smaller than (1.9) as $t\to\infty$; if $p=\frac{1}{2}$, then Eq. (1.10) is of the form $\mathrm{Re}\alpha^B\to a\sqrt{t}(\ln t)^R-a\sqrt{t}(\ln N_B^2)^R$, which is smaller than (1.9) for $p=\frac{1}{2}$ as long as $R>0$. In this paper we will assume that for some $\epsilon>0$

$$\lim_{t\to\infty}\frac{\sqrt{t}(\ln t)^\epsilon}{\mathrm{Re}\alpha_n(t)}=0, \tag{1.11}$$

and similarly for the s-channel trajectories. This argument makes it plausible that there is some positive number T_0 such that as $s\to\infty$ for $t>T_0$ the contributions from the Regge cuts are negligible compared to those from the higher-lying Regge trajectories. In such a case Eq. (1.5) reduces to

$$\sum_{n=1}^{K(s)}R_s^{(n)}\sim\sum_{n=1}^{K(t)}R_t^{(n)} \tag{1.12}$$

as $s\to\infty$ for $t>T_0$, where $K(s)$ is the number of trajectories lying above the leading Regge branch point at a given s; $K(t)$ of course is similarly defined. We assume that $K(s)\to\infty$ as $s\to\infty$. In general, we expect T_0 to be at least less than threshold $(4\mu^2)$ and possibly negative or zero since Eq. (1.8) implies that both $\alpha(t)$ and its derivative are greater than or equal to $\alpha^B(t)$ and its derivative, respectively, at $t=0$, i.e., $\alpha(0)\geqslant\alpha^B(0)$ and

$$\left(\frac{d\alpha}{dt}\right)_{t=0}=N_B\left(\frac{d\alpha^B}{dt}\right)_{t=0}. \tag{1.13}$$

Therefore, it is quite likely that for large s and positive t the background integrals will be larger than the Regge-cut amplitudes. We will assume that this is the case.

Let us pause to compare this with the situation for intermediate to large values of s with $t\leqslant 0$. Here duality is most naturally expressed in the context of finite energy sum rules (FESR).[7] From a calculational point of view it is necessary to assume that the integrals in the FESR are saturated by a finite number of resonances, a procedure which is useful although necessarily approximate in nature. In this case there is no limiting point which allows the approximations involved to be made arbitrarily accurate by approaching sufficiently closely to the limit point; in particular, if s is too large, Regge cuts will become important and eventually dominate the trajectories. On the other hand, comparison with experimental data is possible since we are in the physical region of the s channel; this allows the observation of the unexpected correlation between the crossed-channel Regge terms and the semilocal average of the direct-channel resonances.[8] A very obvious distinction between the two expressions of duality is that FESR give averaged relationships whereas for $t>T_0$ the relationship postulated here is local, i.e., pointwise.

Equation (1.12) is the basic equation which we propose to study in this paper. In Sec. II we trace briefly the development of the study of the consistency of crossing with infinitely rising trajectories. We define the modified background integral B_s' and indicate the type of behavior the partial waves would have to exhibit in order that B_s' is bounded either by a power of s or by an increasing exponential as $s\to\infty$. In Sec. III the kinematical domain in which Eq. (1.12) is expected to hold is discussed. Three different types of intervals $[t_1, t_2]$ are defined. Existence proofs are constructed for the crossing relations for certain particular cases. Bounds for the ratio of the expansion coefficients, i.e., residues, and for the spacing of the trajectories are obtained. In Sec. IV the asymptotic behavior of the residue functions is derived from the crossing relations for the three different intervals introduced in Sec. III. In Sec. V we consider briefly the generalization of the crossing relations to the case of external particles of unequal mass. In Sec. VI a brief summary of the results and assumptions is given. In Appendix A we derive some useful mathematical relations, including an asymptotic expansion in

ν for Legendre function, $P_\nu(z)$, valid for a large range of z, including z arbitrarily near $z = \pm 1$. In Appendix B we obtain an estimate for the asymptotic behavior or the modified background integral in terms of the large s and large Iml behavior of the partial-wave amplitude.

II. CONSISTENCY OF CROSSING WITH INFINITELY RISING TRAJECTORIES

The study of this subject was initiated by Khuri,[9] who pointed out that if $\text{Re}\alpha(s) \to \infty$ as $s \to \infty$, the s-channel Regge term $R_s^{(n)}$ might become embarrassingly large as $s \to \infty$, i.e., it might increase exponentially with s. The reason for this is that as $s \to \infty$ for fixed $t > 0$, $P_\alpha(z_s)$ increases as $\exp[2\alpha(s)\sqrt{t}/\sqrt{s}]$, as can be seen from Eq. (A15). This raises the possibility of a conflict with crossing since it is the t-channel Regge terms $R_t^{(n)}$ that control, via crossing, the asymptotic behavior of the scattering amplitude and give the well-known behavior $s^{\alpha(t)}$ (for fixed t). Khuri's approach was to determine under what conditions one could prevent $R_s^{(n)}$ from increasing exponentially as $s \to \infty$. He concluded that this could not be done without sacrificing some of the commonly accepted properties of $\alpha(s)$, $\beta(s)$, or $a_l(s)$, the latter being the partial-wave amplitude. In other words, the consistency of infinitely rising trajectories and crossing had been brought into question, and the outlook did not look promising.

In a subsequent paper, Jones and Teplitz[10] suggested that the least objectionable property to relinquish was the assumption that the residue $\beta(s)$ was bounded by a power of s for large $|s|$. If, instead, β had an essential singularity at infinity, it could decrease sufficiently rapidly to offset the growth of $P_{\alpha(s)}$ as $s \to \infty$ for all fixed positive values of t. In that case $R_s^{(n)}$ would not increase exponentially—rather, it would vanish for large s and fixed t.

The author of this paper later pointed out[11] that Khuri had overlooked one possibility, namely that as $s \to \infty$

$$\text{Re}\alpha_n(s) \to a\sqrt{s}\,\ln(s). \tag{2.1}$$

This behavior is consistent with Khuri's assumptions and causes $R_s^{(n)}$ to be bounded by a polynomial in s even if $\beta(s)$ does not have an essential singularity at infinity.

Actually there is a basic inconsistency present in Refs. 9, 10, and 11. According to Eq. (3) of Ref. 9

$$\lim_{s\to\infty} \left| \frac{B_s}{s^{N_0}} \right| = 0 \tag{2.2}$$

for fixed t, where B_s is the s-channel background integral. The constant N_0 in Eq. (2.2) is the same constant that appears in both Eq. (2) and assumption (iii) of Ref. 9 and is clearly independent of t. Now the inconsistency arises in trying to require that $R_s^{(n)}$ also be bounded by s^N. The fact is that if Eq. (2.2) is correct as the authors of Refs. 9, 10, and 11 clearly assumed, crossing requires in the case of infinitely rising trajectories that $\sum R_s^{(n)}$ increase faster than s^N for any N. The same can be said for $R_s^{(n)}$ unless it increases exponentially with increasing n. We prove this by assuming the contrary, i.e., that $|R_s^{(n)}| < n^\lambda s^{N_1}$. Ignoring Regge cuts, we have

$$T = B_s + \sum_{n=1}^{H(s)} R_s^{(n)}, \tag{2.3}$$

where T is the scattering amplitude and $H(s)$ is the number of trajectories having $\text{Re}\alpha(s) > -\frac{1}{2}$. Thus

$$|T| \leq |B_s| + \sum_{n=1}^{H} |R_s^{(n)}| \leq s^{N_0} + s^{N_1} \sum_1^H n^\lambda \leq s^N, \tag{2.4}$$

assuming that $H(s)$ increases no factor than a power of s. But we know from crossing that as $s \to \infty$

$$T \approx s^{\alpha_L(t)} \tag{2.5}$$

and that for t sufficiently large $\text{Re}\alpha_L(t) > N$ for any N. Hence we have established a contradiction, and the statement is proven. Although we have assumed that $H(s) \leq s^N$ to make this particular point, it is not assumed anywhere else in this paper. Also the point in question is not critical to any subsequent conclusions.

This brings us to the question of the validity of Eq. (2.2). Unfortunately, the proof given by Khuri[9] is not sufficient to establish Eq. (2.2). The problem with his proof is that the background integral, B_s, can be bounded by s^{N_0} as $s \to \infty$ for $\cos\theta_s$ fixed between 0 and 1—see Eq. (2) of Ref. (9)—without implying either that its discontinuities across its cuts in $\cos\theta_s$ are bounded or that it is bounded for $\cos\theta > 1$. A simple example of such a case would be

$$B_s \equiv \sum_{n=1}^{2} e^{-i\pi\lambda_n}(z_s - 1)^{\lambda_n}, \tag{2.6}$$

where $\lambda_1 = is - \frac{1}{2}$ and $\lambda_2 = -is - \frac{1}{2}$. For $z_s < 1$ and real this can be written as $2(1 - z_s)^{-1/2}\cos[s\ln(1 - z_s)]$, which is clearly bounded as $s \to \infty$. However, its discontinuity across the cut in z_s from 1 to ∞ is

$$\text{disc}B_s = -2i \sum_{n=1}^{2} (z_s - 1)^{\lambda_n} \sin(\pi\lambda_n) \tag{2.7}$$

and this increases exponentially as $s \to \infty$.

At this point we express the background integral in a manner more suitable for our purposes. In particular we let B_s' be the part of the background integral that does not give rise to any Regge poles and we let R_s' be the part that does. The background integral will be a sum of these two terms, $B_s = B_s' + R_s'$. Naturally such a decomposition is not unique. However, it can be accomplished by writing the partial-wave amplitude $a_l(s)$ in B_s as

$$a_l(s) = \sum_{n=H+1}^{\infty} \left[\frac{\alpha_n(s)}{l}\right]\left[\frac{\beta_n(s)}{l-\alpha_n(s)}\right] \times \exp\{\xi_n[\alpha_n(s)+\tfrac{1}{2}-(l+\tfrac{1}{2})]\} + A_l(s), \quad (2.8)$$

where $\beta_n(s)$ is the residue of $a_l(s)$ at $l=\alpha_n(s)$ and ξ_n is the smallest positive function of n which increases sufficiently rapidly as $n\to\infty$ that the infinite series in (2.8) converges. Note that the real part of the argument of the above exponential is always negative since $l+\frac{1}{2}$ is imaginary in B_s and $\mathrm{Re}\alpha_n(s) < -\frac{1}{2}$ by the definition of the background integral. Equation (2.8) serves as the definition of $A_l(s)$ which clearly has no poles at $l=\alpha_n(s)$ for $n \geq H(s)+1$. Thus B_s' will be given by Eq. (1.7) with $a_l(s)$ replaced by $A_l(s)$, and R_s' will be given Eq. (1.7) with $a_l(s)$ replaced by

$$\sum_{n=H+1}^{\infty} \left[\frac{\alpha_n(s)}{l}\right]\left[\frac{\beta_n(s)}{l-\alpha_n(s)}\right] \times \exp\{\xi_n[\alpha_n+\tfrac{1}{2}-(l+\tfrac{1}{2})]\}. \quad (2.9)$$

The factor $[\alpha_n/l]$ in (2.9) may not be necessary but is included to assure the convergence of the integration over $\mathrm{Im}l$. For any s the scattering amplitude may be written as

$$T = B_s' + \left(R_s' + \sum_{n=1}^{H} R_s^{(n)}\right). \quad (2.10)$$

It can be shown that if $|\beta_n(s)\alpha_n(s)| < M_n g(s)$ for large s, then $|R_s'| < g(s)$ as $s\to\infty$. As s increases, successive Regge terms emerge from R_s' and enter the sum over n in (2.10). The advantage of this decomposition is that both B_s' and the sum $(R_s' + \sum R_s^{(n)})$ are analytic functions of s since the boundary of integration in B_s' is not crossed by any Regge poles as s is increased.

Since we cannot justify the use of the bound, (2.2), obtained in Ref. 9, it is necessary to either derive a valid bound for the modified background integral or to make some assumption regarding its asymptotic behavior. Our procedure will be to link the asymptotic behavior of B_s' to that of $A_l(s)$ and consider various possibilities for their behavior. First we define a function $c_l(s)$ by the equation

$$A_l(s) \equiv c_l(s) l_2^{-1/2-\epsilon} \quad (2.11)$$

for some $\epsilon > 0$ where $l_2 \equiv \mathrm{Im}l$. We assume that the behavior of $A_l(s)$ is essentially the same as that of $a_l(s)$ as $l_2 \to \pm\infty$, that is, we assume that $c_l(s)$ vanishes at $l_2 = 0$ but is bounded and nonvanishing as $l_2\to\pm\infty$. We know that $a_l(s)$ must decrease at least as rapidly as indicated by these statements and (2.11) since otherwise the background integral would not exist. Also we let $c(s)$ be the least upper bound of $c_l(s)$, i.e.,

$$|c_l(s)| < c(s) \quad (2.12)$$

for l on the verticle line, $l = -\frac{1}{2} + il_2$.

In Appendix B we show that the modified background integral obeys the asymptotic bound

$$|B_s'| < Msc(s)/t. \quad (2.13)$$

The derivation makes no assumption regarding the value of t except that t be positive. This is particularly interesting since it implies that if $A_l(s)$ satisfies the bound (2.11), then $|B_s'|$ is bounded by (2.13) not only for large s and fixed t but also for the case when both s and t are large. If the least upper bound of $A_l(s)$ is a power of s, say s^{N_0-1}, then $|B_s'|$ will be bounded by s^{N_0}, and Eq. (1.12) will apply for all t greater than t_0, where $\mathrm{Re}\alpha_L(t_0) = N_0$. However, if $A_l(s)$ increases exponentially, B_s' will almost surely do the same. Then Eq. (1.12) can not be valid for any fixed value of t, and we must restrict our attention to values of t that are increasing with s. Suppose, for example, that as both s and t increase

$$\Delta B \to s^{N_0} \exp[b_0 s^{M_0} (\ln s)^{R_0}].$$

Assume that $\mathrm{Re}\alpha_n(s)$ is asymptotically linear for simplicity. Then it will be shown later that if $M_0 > 1$, Eq. (1.12) is invalid; if $M_0 = 1$, Eq. (1.12) can be satisfied only if $b_0 < \sqrt{2}a$; if $0 < M_0 < 1$, Eq. (1.12) is again invalid.

It is probably clear by now that there are essentially three solutions to the problem of combining crossing with infinitely rising trajectories. Let D_i, for $i = 1, 2, 3$, represent three, as yet unspecified, domains in the first quadrant of the real st plane, and let ΔB be the difference between the modified t-channel and s-channel background integrals, i.e.,

$$\Delta B = (B_t' - B_s'). \quad (2.14)$$

If ΔB is increasing less rapidly than $\sum R_t^{(n)}$, we obtain the asymptotic crossing relations for the Regge amplitude proposed in this paper, i.e.,

$$\sum R_s^{(n)} \to \sum R_t^{(n)} \tag{2.15}$$

as $s\to\infty$ for $t\in D_1$. If, on the other hand, $\sum R_s^{(n)}$ is increasing less rapidly than $\sum R_t^{(n)}$ for $t\in D_2$, then the modified background integrals must generate the crossed-channel Regge terms, i.e.,

$$(-\Delta B)\to \sum R_t^{(n)} \tag{2.16}$$

as $s\to\infty$ for $t\in D_2$. The third possibility is that both $\sum R_s^{(n)}$ and ΔB are increasing at essentially the same rate as $\sum R_t^{(n)}$ for $t\in D_3$, in which case

$$\sum R_s^{(n)} \to f\sum R_t^{(n)} \text{ and } (-\Delta B)\to(1-f)R_t^{(n)}, \tag{2.17}$$

where neither f nor $(1-f)$ vanish as $s\to\infty$ for $t\in D_3$. For those domains D_i in which t is not increasing ΔB may be replaced by $(-B_s')$ since in that case $|B_t'|< s^{-1/2}$ as $s\to\infty$.

A fourth possibility arises in the case that an infinite number of Regge terms can be extracted from B_s for finite values of s. Khuri[12] has investigated this case using Mandelstam's[13] modified expressions for the Regge terms and for the background integral, which we will designate as $r_s^{(n)}$ and $b_s^{(L)}$, respectively. The superscript L on $b_s^{(L)}$ refers to the verticle line, $l=-L+i\,\mathrm{Im}\,l$, $-\infty<\mathrm{Im}\,l<\infty$, in the complex l plane over which the integration in $b_s^{(L)}$ is done. Khuri assumes that the limit of $L\to\infty$ exists in which case

$$B_s\to b_s^{(\infty)}+\sum_{n=H+1}^{\infty} r_s^{(n)} \tag{2.18}$$

since $r_s^{(n)}\to R_s^{(n)}$ for $n>H$ as $s\to\infty$. In Ref. 13 Mandelstam pointed out that if the limit of $L\to\infty$ exists, $b_s^{(\infty)}$ cannot vanish. Khuri demonstrates that residues and trajectories may be chosen such that

$$\sum_{n=1}^{\infty} r_s^{(n)}\to\gamma(t)s^{\alpha(t)} \tag{2.19}$$

for large s and fixed negative t. His choice involves trajectories whose imaginary parts, $\mathrm{Im}\,\alpha_n(s)$, are independent of n and whose real parts are equally spaced, i.e., $\mathrm{Re}\,\alpha_n(s)=\alpha(s)-n$. The residue function corresponding to the leading trajectory was found to increase exponentially as $s\to\infty$. We regard this behavior as undesirable since it would be preferable to maintain some semblance of the connection between the partial widths of the hadronic resonances and the Regge residues as $s\to\infty$. Khuri assumed that as s increased, the resonances would eventually become too broad to be observable, i.e., that $\mathrm{Im}\,\alpha(s)\to\infty$ as $s\to\infty$. Thus, in his approach the exponential increase of the residues was not regarded as particularly objectionable. Also a greater similarity to the form obtained in the Veneziano model would be preferred. Nevertheless, Khuri has demonstrated that an infinite series of s-channel Regge terms can be summed for fixed t to a function that behaves as a t-channel Regge term for large s and fixed t. The infinite sum must be evaluated for $t<0$ and $s<4$ and then analytically continued to larger values of s and t. The residue functions obtained are not asymptotically self-consistent, but can be made approximately self-consistent over a certain bounded range of s values.

In this paper we restrict our attention to the first of the four possibilities, namely, Eq. (2.15). Accordingly we wish to determine for what values of t

$$\frac{1}{\Delta B}\sum_{n=1}^{K(t)} R_t^{(n)}\to\infty \tag{2.20}$$

and

$$\frac{1}{\Delta B}\sum_{n=1}^{K(s)} R_s^{(n)}\to\infty \tag{2.21}$$

as $s\to\infty$. Of course, we know from Eq. (1.6) that whenever one of the above equations is valid, the other must be valid as well. However, in practice Eqs. (2.20) and (2.21) will result in a comparison of ΔB with $R_s^{(n)}$ or $R_t^{(n)}$ since we can not evaluate the above sums over n.

III. DOMAIN OF VALIDITY AND EXISTENCE PROOFS

The consideration of the previous sections lead us to suggest that crossing applies directly to the sum of the Regge-pole amplitudes whenever one of the Mandelstam variables (s,t) is large and the other is restricted to a certain interval along the real, positive axis to be discussed below. We include R_t' with $\sum R_t^{(n)}$ so that our definition will not contain discontinuous functions of t, i.e., $\sum_1^{H(t)} R_t^{(n)}$ is a discontinuous function of t whereas $(R_t'+\sum_1^{H(t)}R_t^{(n)})$ is not. More precisely, we postulate that for any $\epsilon>0$ there exists an S such that

$$\frac{1}{\Delta(s,t)}\left|\sum_{n=1}^{K(s)} R_s^{(n)}-\left(R_t''+\sum_{n=1}^{K(t)} R_t^{(n)}\right)\right|<\epsilon \tag{3.1}$$

for all $s\geq S$ and for t in the closed interval $[t_1,t_2]$, where $\Delta(s,t)$ is an upper bound of $|\Delta B|$ on the interval $[t_1,t_2]$ and R_t'' is the sum of R_t' and the contributions from those trajectories with $K(s)<n<H(s)$. The latter are the trajectories lying above the line $\mathrm{Re}\,l=-\frac{1}{2}$ but below the leading Regge branch point $\mathrm{Re}\,\alpha^B(t)$. Equation (1.12) should be regarded as a simplified expression

of Eq. (3.1). In other words, the exact meaning of (1.12) must follow from Eq. (3.1). From a practical standpoint the inclusion of R_t'' is not important since it is negligible in comparison to $\sum R_t^{(n)}$ as $s\to\infty$. Its inclusion is mathematically convenient since one of the existence proofs to be given later is valid only if the functions in (3.1) are continuous.

One of the most difficult and also most important questions in this formulation of duality is what constitutes the correct interval in t over which (3.1) should hold. In our discussion of the consistency of crossing with infinitely rising trajectories we assumed that we could choose t to be as large as we wished by making s sufficiently large, i.e., with $s \gg t$. In that case $P_{\alpha(t)}(z_t) \approx A(t)s^{\alpha(t)}$ for large s, and the customary Regge behavior was obtained. This suggests that the interval should begin at some fixed point t_1 with an end point t_2 which increases with s but does so sufficiently slowly that $(t_2/s)\to 0$ or $s\to\infty$. We will refer to this choice for the t interval as case (A). Although we do not regard it as very likely, we also consider for completeness, the possibility that t_2 is independent of s. We will refer to this possibility as case (A) as well.

On the other hand, the proof given for the bound on $|B_s'|$ suggests that the validity of (1.12) for large s is independent of t, i.e., t may be as large or even larger than s. The implication here is that the appropriate choice for t_2 is $t_2 = s$. This is actually the largest possible value of t_2 applicable to Eq. (3.1) because those situations in which t is greater than s are covered by Eq. (3.17) below. Note also that $t=s$ is an "identity line" in the st plane in the sense that Eqs. (3.1) and (3.17) are trivially satisfied there. As we move away from this line, we expect it to become increasingly more difficult to satisfy Eqs. (3.1) and (3.17). From this viewpoint the length of the interval is actually determined by our choice of t_1. The largest interval is obtained by choosing t_1 to be independent of s, the smallest by choosing t_1 to be proportional to s. These are the other two cases to which we give primary consideration. The choice $t_2 = s$ applies to both of them, and we refer to them as case (B) and case (C), respectively.

In cases (A) and (B) it must be assumed that $|\Delta B|$ is bounded by s^{N_0} for some N_0. Then t_1 will be a fixed number. A reasonable value for t_1 in these two cases would be the larger of the numbers $4\mu^2$ and t_0, where $\mathrm{Re}\,\alpha_L(t_0) = N_0$. We assume for simplicity that $t_1 \neq 4\mu^2$ and instead that $\mathrm{Re}\,\alpha_L(t) = N_0$ for both cases (A) and (B).

In the case (C) t_1 is proportional to s, and we set $t_1 = s/r_1$ where r_1 is some positive number greater than unity. If $|B_s'|$ is bounded by a fixed power of s, then Eq. (3.1) must be satisfied for values of t less than s/r_1. Hence, we expect $[s/r_1, s]$ to be the correct interval only if the least upper bound for $c_l(s)$ is an exponential of s. Note that the ratio (t/s) never approaches zero as $s\to\infty$. In fact case (C) corresponds to the limit in which $s\to\infty$ for z_s fixed and greater than unity. (z_t is also fixed in this case.) There are cases intermediate to cases (B) and (C) which we mention briefly in Sec. IV along with the possibility that the interval extends from fixed t_1 to $t_2 = s/r_2$. In other words, we consider every possible type of interval along the positive, real t axis.

To summarize, there are three different intervals $[t_1, t_2]$ along the positive t axis to which we give our primary consideration, namely the following ones.

(1) Case (A): t_1 fixed with $(t_2/s)\to 0$ as $s\to\infty$ (t_2 may or may not depend on s),

(2) Case (B): t_1 fixed and $t_2 = s$,

(3) Case (C): $t_1 = s/r_1$ and $t_2 = s$ with r_1 fixed.

Later we will divide case (A) into two categories, depending on whether the trajectories satisfy Eqs. (2.1) and (3.13).

We now present formulas for the Regge terms that will be needed in the remainder of the paper. First we consider the case in which $(t/s)\to 0$ as $s\to\infty$. Then for both fixed and increasing values of t, $R_t^{(L)}$ is given by the usual expression which we put into a form more suitable for later use, namely

$$R_t^{(L)} \to 2i\sqrt{\pi}\,\eta_L(t)\left(\frac{4s}{t-4\mu^2}\right)^{\alpha_L(t)}, \tag{3.2}$$

where

$$\eta_n(t) \equiv \beta_n(t)(\alpha_n + \tfrac{1}{2})\,\frac{\Gamma(\alpha_n+\frac{1}{2})}{\Gamma(\alpha_n+1)}\left(\frac{e^{-i\pi\alpha_n}+\sigma_n}{-2i\sin\pi\alpha_n}\right). \tag{3.3}$$

Note that as $t\to\infty$

$$\eta_n(t) \to \beta_n(t)\sqrt{\alpha_n}\left(\frac{e^{-i\pi\alpha_n}+\sigma_n}{-2i\sin\pi\alpha_n}\right). \tag{3.4}$$

We will refer to $\eta_n(t)$ as the nth t-channel expansion coefficient. For $R_s^{(n)}$ we need expansions of the Legendre functions for large α and z approaching ± 1. From Eq. (A15) of the Appendix we find that as $s\to\infty$ with $(t/s)\to 0$

$$R_s^{(n)} \to i\sqrt{\pi}\left(\frac{s}{t}\right)^{1/4}\eta_n(s)\exp\left[2\,\frac{\alpha_n(s)}{\sqrt{s}}\sqrt{t}\right]. \tag{3.5}$$

Now we consider cases for which the ratio (t/s) does not vanish as $s\to\infty$. It never vanishes in case (C) and need not in case (B). From Eq.

(A14) we see that as $s \to \infty$

$$R_s^{(n)} \to i\,(2\pi)^{1/2}(z_s^{\,2}-1)^{-1/4}\eta_n(s) \times \exp\{[\alpha_n(s)+\tfrac{1}{2}]x_s\} \qquad (3.6)$$

and

$$R_t^{(n)} \to i\,(2\pi)^{1/2}[(z_t^{\,2}-1)]^{-1/4}\eta_n(t) \times \exp\{[\alpha_n(t)+\tfrac{1}{2})]x_t\}, \qquad (3.7)$$

where

$$x_s \equiv \ln[z_s+(z_s^{\,2}-1)^{1/2}] \qquad (3.8)$$

and x_t is defined by Eq. (3.8) with s and t interchanged. Eqs. (3.6) and (3.7) are valid for large, positive s, large $\alpha(s)$, and any real, positive value of t, subject to the condition that $|\alpha(s)\sqrt{t}/\sqrt{s}| \to \infty$ as s increases.

We now discuss the function $\Delta(s,t)$. As mentioned previously, we neglect Regge-cut amplitudes in the above domains. Thus Eq. (1.6) can be written as

$$\left(R_s' + \sum R_s^{(n)}\right) - \left(R_t' + \sum R_t^{(n)}\right) \to \Delta B. \qquad (3.9)$$

This suggests that we choose $\Delta(s,t)$ such that it is increasing slightly more rapidly than $|\Delta B|$ for $t \in [t_1, t_2]$. Then, if we divide both sides of (3.9) by $\Delta(s,t)$ we can make the right-hand side as small as we wish by choosing s sufficiently large. However, we do not want to choose $\Delta(s,t)$ so large that it is also increasing more rapidly than $\sum R_s^{(n)}$ and $\sum R_t^{(n)}$, i.e., there should exist some M such that as $s \to \infty$ for $t \in [t_1, t_2]$,

$$\frac{1}{\Delta(s,t)}\left|\sum_n R_s^{(n)}\right| > M \qquad (3.10)$$

and

$$\frac{1}{\Delta(s,t)}\left|\sum_n R_t^{(n)}\right| > M \qquad (3.11)$$

We will implement Eqs. (3.10) and (3.11) by choosing $\Delta(s,t)$ such that it is increasing no more rapidly than each $R_t^{(n)}$. Such a choice should be possible if Eqs. (2.20) and (2.21) are satisfied.

In cases (A) and (B) it is necessary that $|\Delta B|$ be bounded as a power of s, say s^{N_0}, for t in the vicinity of t_1 where $N_0 = \mathrm{Re}\alpha_L(t_1)$. Otherwise, ΔB (actually B_s') would be increasing more rapidly than $R_t^{(L)}(s, t_1)$. Thus the dominant s dependence of $\Delta(s,t)$ should be $s^{N_0}(\ln s)^\delta$, where δ is sufficiently large that $\Delta(s,t)$ is increasing more rapidly than ΔB. Such a choice will obviously satisfy Eq. (3.11) for $t > t_1$. For later purposes it is convenient, although not necessary, to include the factor $[s(z_s^{\,2}-1)]^{-1/4}$ in the definition of $\Delta(s,t)$. Our final choice for cases (A) and (B) is

$$\Delta_{AB}(s,t) \equiv (\ln s)^\delta [s(z_s^{\,2}-1)]^{-1/4} s^{\mathrm{Re}\,\alpha_L(t_1)}. \qquad (3.12)$$

We take notice of the special case in which the real parts of the trajectories have the behavior indicated earlier in Sec. II, see Eq. (2.1), i.e., $\mathrm{Re}\alpha_L(s) \to a\sqrt{s}\,\ln(s)$ as $s \to \infty$, and in addition satisfy inequality

$$\mathrm{Re}\alpha_L(t) \geq [\mathrm{Re}\alpha_L(t_1) + 2a(\sqrt{t} - \sqrt{t_1})] \qquad (3.13)$$

for $t \in [t_1, t_2]$. Then $\mathrm{Re}\alpha_L(t_1)$ in Eq. (3.12) may be replaced by $[\mathrm{Re}\alpha_L(t_1) + 2a[\sqrt{t} - \sqrt{t_1})]$ with the knowledge that Eq. (3.11) will still be satisfied. Thus we define

$$\Delta_A'(s,t) = (\ln s)^\delta t^{-1/4}(s)^{\mathrm{Re}\,\alpha_L(t_1) - 2a\sqrt{t_1}} \times \exp[x_s \mathrm{Re}\alpha_L(s)]. \qquad (3.14)$$

For future reference we will designate this special case in which the trajectories satisfy Eqs. (2.1) and (3.13) and t is in the domain of case (A) as case (A'). We may also wish to consider the domain of case (A) and explicitly exclude trajectories satisfying Eqs. (2.1) and (3.13). We will refer to this situation as case (A_0).

If the crossing relations are to be valid only in the domain of case (C), ΔB must be increasing faster than any power of s. Hence $\Delta(s,t)$ must be increasing exponentially with s. We choose the exponential in $\Delta(s,t)$ such that $\Delta(s,t)$ is increasing no more rapidly than each $R_s^{(n)}$ and $R_t^{(n)}$ as $s \to \infty$ for $t \in [t_1, t_2]$ in accordance with Eqs. (3.10) and (3.11). The asymptotic dependence of both the s- and t-channel residue functions is important in such a choice. Referring to Eq. (3.6) we define for case (C)

$$\Delta_C(s,t) \equiv s^\delta \exp[x_s \mathrm{Re}\alpha_L(s) - \theta(s)], \qquad (3.15)$$

where $\theta(s)$ is connected with the exponential decrease of the residues and is defined by Eqs. (4.6), and (3.4). In conclusion $\Delta(s,t)$ should be chosen to be $\Delta_A'(s,t)$, $\Delta_{AB}(s,t)$, or $\Delta_C(s,t)$ according to the case being considered.

It is only in cases (A) and (B) that fixed values of t are encountered and the sum over the t-channel trajectories can be replaced by the contribution of the leading trajectory as $s \to \infty$. We may also neglect R_s'' and R_t'' in this limit. Then the sum over the s-channel Regge terms must exhibit crossed-channel Regge behavior, i.e., Eq. (3.9) simplifies to

$$\sum_{n=1}^{K(s)} R_s^{(n)} = R_t^{(L)} + O(s^{N_0}) \qquad (3.16)$$

or, in terms of the expansion coefficients,

$$\frac{1}{2}\sum_{n=1}^{K(s)} s^{1/4}\eta_n(s)\exp\left[\frac{2\alpha_n(s)\sqrt{t}}{\sqrt{s}}\right] \sim t^{1/4}\eta_L(t)\left(\frac{4s}{t-4\mu^2}\right)^{\alpha_L(t)}.$$

This is essentially Eq. (11) of Ref. 4. We also expect this equation to be valid for increasing values of t as long as $(t/s)\to 0$ or $s\to\infty$; however, it is not valid in case (C).

We mention in passing that crossing symmetry and Eq. (3.1) also require that for any $\epsilon>0$ there exists a T such that

$$\frac{1}{\Delta(t,s)}\left|\sum_{n=1}^{K(t)} R_t^{(n)} - \left(R_s'' + \sum_{n=1}^{K(s)} R_s^{(n)}\right)\right| < \epsilon \qquad (3.17)$$

for all $t \geq T$ and for s in the closed interval $[s_1, s_2]$ where $s_1 = t_1$ and $s_2 = t_2$; $\Delta(t,s)$ is equal to $\Delta(s,t)$ with $s \leftrightarrow t$.

At this point we give several existence proofs for Eq. (3.1). It is convenient to regard the trajectories of both channels and the t-channel expansion coefficients as given. Then Eq. (3.1) can be viewed as an equation from which the s-channel expansion coefficients $\eta_n(s)$ are to be determined. The question before us is this: Can we prove the existence of a set of coefficients $\{\eta_n(s)\}$ satisfying Eq. (3.1) without having to make any assumptions regarding the specific form of $R_t'' + \sum R_t^{(n)}$? The answer is that we can for certain types of trajectories specified below.

Our first proof applies only to case (C) and is for trajectories that are asymptotically parallel, i.e., those for which

$$\begin{aligned} &\mathrm{Re}\alpha_n(s) \to \mathrm{Re}\alpha_L(s) - nb + \cdots, \\ &\text{for } n = 1, 2, 3, \ldots, \text{ and} \qquad (3.18) \\ &\mathrm{Im}\alpha_n(s) \to c_0(s) + \cdots \end{aligned}$$

as $s\to\infty$ where b is independent of n and s. In this case we multiply both sides of Eq. (3.1) by $|[i(2\pi)^{1/2}]^{-1}\exp(-\frac{1}{2}x_s - ic_0x_s)|$ and absorb these factors into the ϵ on the right-hand side of the equation. Then Eq. (3.1) can be replaced by the equation

$$\left|\sum_{n=1}^{K(s)} c_n(s)y^n - T_s(y)\right| < \epsilon \qquad (3.19)$$

for $y \in [y_1, y_2]$, where y_1, y_2 are finite, nonvanishing numbers, $c_n = \eta_n e^{-\theta}$, $y \equiv e^{-bx_s}$, and

$$T_s(y) \equiv \frac{(-i)}{(2\pi)^{1/2}\Delta_c(s,t)}\exp(-\tfrac{1}{2}x_s - ic_0x_s) \times \left(R_t'' + \sum_{n=1}^{K(t)} R_t^{(n)}\right). \qquad (3.20)$$

We emphasize that (3.19) is not a power-series expansion of $T_s(y)$ about the point $y=0$ (which corresponds to $t=\infty$) since we do not expect $T_s(y)$ to be analytic in a neighborhood of this point. However, we can be certain that there exists a set of s-channel expansion coefficients which satisfy Eq. (3.19) for any given s. This conclusion follows from the Weierstrass approximation theorem which reads as follows: Let the function $f(y)$ be continuous on the finite closed interval $[y_1, y_2]$. For any $\epsilon>0$ there exists a positive integer N and a corresponding polynomial $P_N(y)$ of the Nth degree such that $|P_N(y) - f(y)| < \epsilon$. This theorem may be applied to (3.19) since $T_s(y)$ is a continuous function of y; it was for this reason that R_t'' was included in (3.1).

This proves that for asymptotically parallel trajectories a set of s-channel expansion coefficients, $\{\eta_n(s)\}$, satisfying (3.19) can be determined (at least in principle) independently of the choice made for the t-channel expansion coefficients—other than that they be continuous functions of t. Naturally the set of coefficients $\{\eta_n(s)\}$ obtained will depend on the choice made for the set of t-channel coefficients $\{\eta_n(t)\}$, and *a priori* the two sets need not be the same. In Sec. IV we consider the question of self-consistency, i.e., whether Eq. (3.1) admits of solutions $\eta_n(s)$ which are functionally identical to the corresponding t-channel coefficients $\eta_n(t)$, for large values of their arguments.

Our second proof also applies only to case (C) and is for trajectories whose real parts are asymptotically degenerate and whose imaginary parts are asymptotically parallel, i.e., as $s\to\infty$

$$\mathrm{Re}\alpha_n(s) \to \mathrm{Re}\alpha_L(s) - n^2 b'(s) + \cdots \qquad (3.21)$$

and

$$\mathrm{Im}\alpha_n(s) \to \frac{2\pi}{L}[n + C(s)] + c_0 + \cdots \qquad (3.22)$$

for $n = 0, \pm1, \pm2, \ldots, \pm K$, where L is the length of the interval in the x_s variable, i.e, $L \equiv x_2 - x_1$, with

$$x_2 \equiv x_s(s, t_2), \quad x_1 \equiv x_s(s, t_1). \qquad (3.23)$$

In all the cases which we consider, $L \leq 1.763$. Also $b'(s)$ is some function which vanishes rapidly as $s\to\infty$, c_0 is a constant, and $C(s)$ is some function such that $C(s) \geq K(s)$.

Notice that we have changed our labeling system to allow n to be negative and chosen the real part of the trajectories to be asymptotically independent of the sign of n. If $\mathrm{Re}\alpha_{n_1}(s) > \mathrm{Re}\alpha^B(s)$ for some positive n_1, then $\mathrm{Re}\alpha_{-n_1}(s) > \mathrm{Re}\alpha^B(s)$ as well; thus the sum over n must extend from $n = -K$ to $n = K$. For trajectories such as these Eq. (3.1) can be rewritten as

$$\left|\sum_{n=-K(s)}^{K(s)} c_n(s)\exp\left(\frac{2\pi i n x_s}{L}\right) - F_s(x_s)\right| < \epsilon \qquad (3.24)$$

for $x_s \in [x_2, x_1]$ where $c_n = \eta_n e^{-\theta}$ and $F_s(x_s)$ is similar in form to $T_s(y)$.

As before, certain factors of order unity have been absorbed into the ϵ on the right-hand side of Eq. (3.24). It is obvious that the above series is a Fourier series of period L and that (3.24) is the statement that $F_s(x_s)$ can be approximated to an aribitary degree of accuracy by such a Fourier series. Since $F_s(x_s)$ is not expected to be periodic in x_s, it is critical that the L in Eq. (3.22) be the exact length of the interval over which Eq. (3.1) is assumed to be valid (in terms of the variable x_s). The presence of R_t'' is not important in this case. As is well known, a function need not be continuous to have a Fourier series; the Fourier series of any function of bounded variation will converge to that function except at the points where the function is discontinuous. If R_t'' is omitted from (3.1), $F_s(x_s)$ will still be of bounded variation for any finite s. This proves the existence of a set of s-channel coefficients satisfying (3.24). They will naturally be given by the usual formula

$$\eta_n(s) = \frac{e^{-\theta}}{L}\int_{x_1}^{x_2} F_s(x_s)\exp\left(-\frac{2\pi i n x_s}{L}\right)dx_s . \qquad (3.25)$$

In attempting to extend our proof to cases (A) and (B) two problems are encountered. The first is that the length of the intervals, $y_2 - y_1$ and $x_2 - x_1$, vanish as $s\to\infty$. This can be circumvented by choosing slightly different forms for the trajectories. The second is that in converting Eq. (3.1) into a form analogous to Eq. (3.19) or (3.24) it is necessary to multiply both sides of (3.1) by $\exp[-2\,\mathrm{Re}\alpha_L(s)\sqrt{t}/\sqrt{s}]$. This causes no problem in case (C) because Δ_C^{-1} also contains this exponential. However, Δ_{AB}^{-1} does not. Thus ϵ in Eqs. (3.19) and (3.24) is replaced by $\epsilon' = \epsilon\exp[-2\,\mathrm{Re}\alpha_L(s)\sqrt{t}/\sqrt{s}]$ for cases (A) and (B). Application of Weierstrass's theorem to the resulting expression does not prove the existence of solutions to Eq. (3.1) because a small value for ϵ' does not imply a small value for ϵ. The same problem is present in the application of the Fourier theorems. Therefore, we are not able to provide an existence proof for Eq. (3.1) in cases (A) and (B) by the use of the Weierstrass or Fourier theorems with the exception of the special case discussed below.

We now consider case (A'), i.e., trajectories satisfying Eqs. (2.1) and (3.13). Then $\Delta(s,t)$ is given by Eq. (3.14), and it is necessary to multiply both sides of Eq. (3.1) by only $(1/i\sqrt{\pi})$ to put it into the form

$$\left|\sum_n c_n(s)\exp\left\{\frac{2}{\sqrt{s}}[\alpha_n(s) - \mathrm{Re}\alpha_L(s)]\sqrt{t}\right\} - \frac{1}{i\sqrt{\pi}}\left(R_t' + \sum_n R_t^{(n)}\right)\right| < \epsilon, \qquad (3.26)$$

where

$$c_n(s) \equiv \eta_n(s) s^{-N_0 - 2a\sqrt{t_1}}(\ln s)^{-\delta}. \qquad (3.27)$$

It is easy to see that if the real parts of the trajectories are given by Eq. (3.21) and the imaginary parts by Eq. (3.22) but with L replaced by $2(\sqrt{t_2} - \sqrt{t_1})/\sqrt{s}$, then Eq. (3.26) can be written as a Fourier series in the variable $(t/t_2)^{1/2}$. Therefore, we can prove the existence of s-channel coefficients satisfying Eq. (3.26), and hence (3.1), for case (A) when the trajectories are of the indicated form.

The first proof, using Weierstrass's theorem, cannot be extended to case (A'). The real parts of the trajectories needed for such a proof would behave as

$$\mathrm{Re}\alpha_n(s) \to a\sqrt{s}\,\ln s - n\lambda(s/t_2)^{1/2} + \cdots . \qquad (3.28)$$

The only trajectories that can participate in the sum are those lying above the leading Regge branch point which behaves according to Eq. (1.10) as

$$\mathrm{Re}\alpha^B(s) \to a\sqrt{s}\,\ln s - a\sqrt{s}\,\ln N^2 - \frac{\lambda}{\sqrt{t_2}}\sqrt{s} + \cdots \qquad (3.29)$$

for trajectories like those in (3.28). Comparing (3.28) and (3.29) we see that $\mathrm{Re}\alpha_n(s) > \mathrm{Re}\alpha^B(s)$ only for

$$n < 1 + \frac{2a}{\lambda}\sqrt{t_2}\,\ln N_B. \qquad (3.30)$$

Therefore only a finite number of trajectories will lie above the leading Regge branch point for large s, and it would be impossible to satisfy Eq. (3.1) for arbitrarily small ϵ. This argument does not apply to the previous proof using the Fourier theorem [assuming that $b'(s)\to 0$ faster than $s^{-1/2}$.]

On the basis of the existence proofs provided in this section we expect that the number of terms contributing to Eq. (3.1) must increase without bound as $s\to\infty$ if the equation is to be satisfied to an arbitrary degree of accuracy. Even in cases (A_0) and (B), for which existence proofs were not supplied, it can be shown that no single term can dominate the left-hand side of Eq. (3.16). It is obvious that the leading trajectory $\alpha_L(s)$ cannot bootstrap itself, i.e., the equation

$$\tfrac{1}{2}s^{1/4}\eta_L(s)\exp\left[\frac{2\alpha_L(s)\sqrt{t}}{\sqrt{s}}\right]\sim t^{1/4}\eta_L(t)\times\left(\frac{4s}{t-4\mu^2}\right)^{\alpha_L(t)} \tag{3.31}$$

for large s and large t, with $s\gg t$, has no solution. Also, several authors[14] have found that a single trajectory cannot bootstrap itself in the context of finite energy sum rules. Furthermore, no reciprocal bootstrap is possible in which the left-hand side is dominated asymptotically by a single term corresponding to a trajectory $\alpha_d(t)$ ($\alpha_d\neq\alpha_L$), which then generates the leading trajectory $\alpha_L(t)$. This can be proven as follows. Assume the contrary. Then

$$\tfrac{1}{2}s^{1/4}\eta_d(s)\exp\left[\frac{2\alpha_d(s)\sqrt{t}}{\sqrt{s}}\right]\sim t^{1/4}\eta_L(t)\times\left(\frac{4s}{t-4\mu^2}\right)^{\alpha_L(t)}. \tag{3.32}$$

The most general form for $\alpha_d(s)$ consistent with this equation is $\alpha_d(s)=(b_1\ln s+b_0)\sqrt{s}$. When this is substituted into Eq. (3.32), all the remaining functions are determined to within four arbitrary constants, b_0, b_1, c, and ν to be $\alpha_L(t)=\frac{1}{2}b_1\sqrt{t}+\nu$, $\eta_d(s)\to cs^\nu$, and

$$\eta_L(t)=\exp\{\tfrac{1}{2}b_0\sqrt{t}+(\nu+\tfrac{1}{2}b_1\sqrt{t})\ln[\tfrac{1}{4}(t-4\mu^2)]\}. \tag{3.33}$$

By crossing symmetry $\eta_L(s)$ is given by Eq. (3.33) with $s\leftrightarrow t$, which shows that the left-hand side of Eq. (3.16) cannot be dominated by the term $\eta_d(s)\exp[2\alpha_d(s)\sqrt{t}/\sqrt{s}]$ because $\eta_L(s)\gg\eta_d(s)$. We conclude that many terms must be present on the left-hand side of Eq. (3.1) for any given s and that their contributions must be of comparable importance; since $K(s)$ is the number of terms, $K(s)\to\infty$ as $s\to\infty$. In order that the terms be of comparable asymptotic importance it is necessary that

$$\left|\frac{\eta_n(s)\exp[\alpha_n(s)x_s]}{\eta_k(s)\exp[\alpha_k(s)x_s]}\right|\leq M_{nk} \tag{3.34}$$

or

$$\left|\frac{\eta_n(s)}{\eta_k(s)}\right|\exp\{x_s\operatorname{Re}[\alpha_n(s)-\alpha_k(s)]\}\leq M_{nk}. \tag{3.35}$$

Since the inequality must be satisfied for a range of t values, it must hold separately for the exponential and the ration of the η's. It follows that the ratio of any two expansion coefficients must be bounded for large s, i.e.,

$$\left|\frac{\eta_n(s)}{\eta_k(s)}\right|< M_{nk}. \tag{3.36}$$

For cases (B) and (C) the maximum value of t is s, which implies from Eq. (3.35) that the asymptotic spacing of the real parts of the trajectories is bounded, i.e.,

$$\operatorname{Re}[\alpha_n(s)-\alpha_k(s)]< M_{nk}. \tag{3.37}$$

Notice that this is consistent with the forms assumed in both of the existence proofs given earlier. For case (A), the maximum value of t is not s, and in fact $t/s\to0$ as $s\to\infty$. Thus Eq. (3.35) implies only that

$$\frac{1}{\sqrt{s}}\operatorname{Re}[\alpha_n(s)-\alpha_k(s)]< M_{nk}/\sqrt{t_2}, \tag{3.38}$$

where the right-hand side of (3.38) vanishes as $s\to\infty$ when t_2 is increasing with s.

In conclusion, we do not claim that Eq. (3.1) requires trajectories which behave asymptotically as one of the three types for which existence proofs have been given. Other types of trajectories may be capable of generating, through Eq. (3.1), quite general classes of functions. Rather we have established that the set of solutions to Eq. (3.1) is definitely not the null set. The sum of the s-channel Regge amplitudes is indeed capable of reproducing, to any desired degree of accuracy, the behavior exhibited by the sum of the t-channel Regge amplitudes in the domain of case (C) for either asymptotically parallel or asymptotically degenerate trajectories. This is also true in case (A') for asymptotically degenerate trajectories. At this point it does not appear very likely that the crossing relations can be satisfied for cases (A_0) and (B). Furthermore, we will show later that neither case (A_0), (A'), nor (B) can be generalized to the scattering of unequal mass particles. Thus we will give primary consideration in the remainder of the paper to case (C).

IV. SELF-CONSISTENT DETERMINATION OF THE ASYMPTOTIC RESIDUE FUNCTION

The main purpose of this section is to examine to what extent Eq. (3.1) is an asymptotic bootstrap equation for the expansion coefficients and, if so, whether self-consistent solutions can be obtained. In so doing, we will obtain the asymptotic form of the expansion coefficients and hence of the residue functions.

We begin with some convenient definitions. If we let

$$\alpha(s)\equiv\operatorname{Re}\alpha_L(s) \tag{4.1}$$

and

$$b_n(s) \equiv \mathrm{Re}[\alpha_L(s) - \alpha_n(s)], \quad (4.2)$$

then the real part of each trajectory may be written as

$$\mathrm{Re}\alpha_n(s) = \alpha(s) - b_n(s), \quad (4.3)$$

where $b_n(s)$ is positive by definition and is bounded for large s and fixed n by virtue of Eq. (3.37). We define s-channel basis functions as follows:

$$\phi_n(s, x_s) \equiv \exp\{[\alpha_n(s) - \alpha_L(s)]x_s\}$$
$$= \exp\{[-b_n(s) + i\,\mathrm{Im}\alpha_n(s)]x_s\}, \quad (4.4)$$

where x_s is defined by Eq. (3.8) and restricted to an interval of length $L \equiv (x_2 - x_1)$, see Eq. (3.23). We make no particular assumption about the form of the trajectories as we did for the existence proofs but we do assume that the basis functions are linearly independent. Without this assumption, the postulate that the direct-channel Regge terms can be expanded in terms of the crossed-channel Regge terms would be empty. However, we do not assume that the basis functions form a complete set.

For case (C) Eq. (3.1) can be written in terms of the basis functions as follows:

$$\left|\sum_{n=1}^{K(s)} \tilde{\eta}_n(s)\phi_n(s, x_s) - f_s(x_s)\right| < \epsilon \quad (4.5)$$

for s sufficiently large and $t \in [t_1, t_2]$, where $\tilde{\eta}_n(s)$ is the reduced expansion coefficient defined to be a function increasing no faster than a power of s and satisfying the relation

$$\eta_n(s) = \tilde{\eta}_n(s)e^{-\theta(s)}. \quad (4.6)$$

Note that $\theta(s)$ must be independent of n owing to Eq. (3.36). The factor of $(s)^\delta$ is not present because we have set $\delta = 0$ for simplicity. The function $f_s(x_s)$ is defined as follows:

$$f_s(x_s) \equiv Ae^{\theta(s) - \Psi(s,t)} \sum_{n=1}^{K(t)} \eta_n(t)\phi_n(t, x_t), \quad (4.7)$$

where

$$A(s, t) \equiv \left(\frac{z_s^2 - 1}{z_t^2 - 1}\right)^{1/4} e^{(x_t - x_s)/2} \quad (4.8)$$

and

$$\Psi(s, t) \equiv \alpha(s)x_s - \alpha(t)x_t. \quad (4.9)$$

Also, the t-channel basis function $\phi_n(t, x_t)$ is defined by Eq. (4.4) with $s \leftrightarrow t$. Equations (4.5) and (4.6) also apply to case (A') but θ should be replaced by $[2a\sqrt{t_1} - \alpha(t_1)]\ln s$. For case ($A_0$) and ($B$) the expression analogous to (4.5) is

$$\left|\sum_{n=1}^{K(s)} \tilde{\eta}_n(s)e^{\alpha_n(s)x_s} - e^{\alpha(s)x_s}f_s(x_s)\right| < \epsilon, \quad (4.10)$$

where $f_s(x_s)$ is defined by Eq. (4.7) but with θ replaced by $-\alpha(t_1)\ln s$. The same replacement applies to Eq. (4.6). These replacements appropriate for cases (A) and (B) apply throughout the remainder of this section but will not be mentioned again. The most important difference between Eqs. (4.5) and (4.10) is that $|\phi_n|$ is bounded as $s \to \infty$ whereas $|e^{\alpha_n(s)x_s}|$ is not.

It will be instructive to determine for which of the various domains we can rephrase the problem as one in a linear vector space or function space. For this purpose we define f and ϕ_n to be vectors represented for a fixed value of s by the functions $f_s(x_s)$ and $\phi_n(s, x_s)$, respectively. Let the scalar product of any two vectors f and g be

$$(f, g) \equiv \frac{1}{L}\int_{x_1}^{x_2} f^*(x_s)g(x_s)dx_s, \quad (4.11)$$

where x_2, x_1 are defined by Eq. (3.23). The square root of the scalar product of a vector with itself is called the norm of f and is denoted by $\|f\|$. We make the space a metric space by defining the distance $\rho(f, g)$ between two vectors, f and g, to be the norm of the difference vector $f - g$, i.e.,

$$\rho^2(f, g) \equiv \|f - g\|^2 \equiv \frac{1}{L}\int_{x_1}^{x_2} |f(x_s) - g(x_s)|^2 dx_s. \quad (4.12)$$

These are the standard definitions. The function $g(x_s)$ representing a given vector g must be square-integrable but not continuous. In this space Eq. (4.5) is replaced by

$$\rho\left(\sum \tilde{\eta}_n \phi_n, f\right) < \epsilon. \quad (4.13)$$

We wish to choose the set of numbers $\{\tilde{\eta}_n\}$ (for a given s) such that the distance between the vectors, f and $\sum \tilde{\eta}_n \phi_n$, is minimized. It is simpler to minimize the square of the distance which we expand as follows:

$$\rho^2\left(\sum \tilde{\eta}_n \phi_n, f\right) = (f, f) - \sum_n \tilde{\eta}_n^*(\phi_n, f) - \sum_n \tilde{\eta}_n(f, \phi_n) + \sum_{n,l} \tilde{\eta}_n \tilde{\eta}_l^*(\phi_l, \phi_n). \quad (4.14)$$

Written in this way ρ^2 is easily differentiated, and setting the derivative equal to zero, we have

$$\frac{d\rho^2}{d\tilde{\eta}_m^*} = -(\phi_m, f) + \sum_n \tilde{\eta}_n(\phi_m, \phi_n) = 0$$

or

$$\sum_{n=1}^{K} (\phi_m, \phi_n)\tilde{\eta}_n = (\phi_m, f), \quad (4.15)$$

where $\tilde{\eta}_n$ and $\tilde{\eta}_n^*$ are considered as independent variables. In (4.15) we have a set of K algebraic equations for the reduced expansion coefficients $\tilde{\eta}_n$, and their solutions give the optimal choice for the coefficients.

Let Φ be the $K\times K$ matrix having matrix elements $\Phi_{mn}\equiv(\phi_m, \phi_n)$. Solutions to (4.15) will exist since the $\phi_n(s, x_s)$ are assumed linearly independent. Letting Φ^{-1} be the inverse of Φ, we see that the solutions to Eq. (4.15) are given by

$$\tilde{\eta}_n = \sum_{m=1}^{K} \Phi_{nm}{}^{-1}(\phi_m, f), \tag{4.16}$$

where $\Phi_{nm}{}^{-1}$ are the matrix elements of the matrix Φ^{-1}. The remainder of this section will be devoted primarily to the extraction of the asymptotic behavior of $\eta_n(s)$ from this equation.

For a given s the matrix elements of Φ can be explicitly evaluated in terms of the trajectories, i.e.,

$$\begin{aligned}(\Phi_m, \Phi_n) &= \frac{1}{L}\int_{x_1}^{x_2} \phi_m^*(s, x_s)\phi_n(s, x_s)dx_s \\ &= \frac{1}{L\lambda_{nm}}\left(e^{x_2\lambda_{nm}} - e^{x_1\lambda_{nm}}\right),\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\lambda_{nm}(s) &\equiv \alpha_n(s) + \alpha_m^*(s) - 2\alpha(s) \\ &= -[b_n(s) + b_m(s)] + i\,\mathrm{Im}[\alpha_n(s) - \alpha_m(s)].\end{aligned} \tag{4.18}$$

Since $(\phi_m, \phi_n) = (\phi_n, \phi_m)^*$, it is seen that Φ is a Hermitian matrix. For cases (B) and (C), $\mathrm{Re}\lambda_{nm}(s)$ is bounded for large s and fixed n, m by virtue of Eq. (3.37). Also, the definition of $\alpha(s)$ insures that $\mathrm{Re}\lambda_{nm}$ is negative. Hence $|(\phi_m, \phi_n)|$ is bounded for large s and fixed n, m for all three cases. As either n or m increases for a given s we expect $\mathrm{Re}\lambda_{nm}\to-\infty$; thus, (ϕ_m, ϕ_n) probably vanishes as n or $m\to\infty$. Since the matrix elements of Φ are bounded, those of Φ^{-1} must also be bounded.

Referring back to cases (A_0) and (B), we attempt to express Eq. (4.10) as a relationship in this vector space. For this purpose we define e_n and g to be vectors represented for any given s by the functions $e^{x_s\alpha_n}$ and $e^{x_s\alpha(s)}f_s(x_s)$, respectively. Then for any finite value of s Eq. (4.10) is replaced by

$$\rho\left(\sum\tilde{\eta}_n e_n, g\right) < \epsilon. \tag{4.19}$$

Although the vectors e_n are well defined for any finite value of s, we are considering an infinite sequence of distances, $\rho_s(\sum\tilde{\eta}_n e_n, g)$, and vectors, $e_n^{(s)}$, each member of the sequence corresponding to a given value of s. For any given n the limit as $s\to\infty$ of the sequence of vectors $e_n^{(s)}$ is not a vector in this vector space, i.e., it does not exist. Therefore, we cannot take the limit of the sequence of distances $\{\rho_s\}$ in Eq. (4.19), and Eq. (4.10) cannot be viewed as defining a relationship in this vector space in the limit of infinite s. Naturally this failure is connected with our inability in Sec. III to provide an existence proof for those cases to which Eq. (4.10) applies. Since we cannot recast Eq. (4.10) into an equation in the metric space which we have defined, it is convenient to write Eq. (4.10) in the form

$$\sum_{n=1}^{K(s)} \tilde{\eta}_n(s)\phi_n(s, x_s) - f_s(x_s) < \epsilon\exp[-x_s\alpha(s) + i\phi_0], \tag{4.20}$$

where ϕ_0 is the phase of the expression on the left-hand side of the above inequality. If we multiply both sides of this inequality by $L^{-1}\phi_m^* dx_s$ and integrate from x_1 to x_2 and then invert the resulting expression in order to solve for $\tilde{\eta}_n(s)$, we obtain an inequality analogous to Eq. (4.16), namely

$$\tilde{\eta}_n(s) - \sum_{m=1}^{K(s)} \Phi_{nm}{}^{-1}(\phi_m, f_s) < \delta_n, \tag{4.21}$$

where

$$\delta_n \equiv \epsilon\sum_{m=1}^{K} \Phi_{nm}{}^{-1}\frac{1}{L}\int_{x_1}^{x_2} \phi_m^*(s, x_s) \times \exp[-x_s\alpha(s) + i\phi_0]\,dx_s$$

and δ_n can clearly be made as small as is desired. Recall that the difficulty in these cases is not an inability to satisfy Eq. (4.21) with a small value of δ but to do so for small ϵ so that the solution to (4.21) will also be a solution to Eq. (3.1).

We now turn our attention to the matrix elements of f_s, i.e., (ϕ_m, f). First we investigate the asymptotic behavior of $e^{-\Psi}$, where Ψ is defined by Eq. (4.9). It is convenient to express the asymptotic behavior of $\alpha(s)$ in the form shown in Eq. (1.9), i.e., $\alpha(s)\to as^p(\ln s)^R$ as $s\to\infty$, where p and R are two as yet unspecified positive numbers. We expect p to either exceed $\frac{1}{2}$ or if $p=\frac{1}{2}$, for $R>0$ on the basis of the discussion of Sec. I. Consider first those values of t which are as large or almost as large as s. This will occur for cases (B) and (C). Accordingly we set $t = s/r$ where $1\leqslant r<\infty$. It follows that $z_s\to 1+2/r$ and $z_t\to 1+2r$ as $s\to\infty$. Therefore (4.9) has the asymptotic form

$$\Psi(s, t)\to ag(r, p)s^p(\ln s)^R + \cdots, \tag{4.22}$$

where

$$\begin{aligned}g(r, p) \equiv \ln&\left[1 + \frac{2}{r} + \frac{2}{r}(1+r)^{1/2}\right] \\ &- r^p\ln[1 + 2r + 2(r + r^2)^{1/2}]\end{aligned} \tag{4.23}$$

and Eq. (1.9) has been used; $g(r,p)$ will not necessarily be positive for arbitrary p and for all values of r in the interval $1\leq r<\infty$. For instance as $r\to\infty$ (which can occur when $s\to\infty$)

$$g(r,p)\to\frac{2}{\sqrt{r}}-r^{-p}\ln(4r)+\cdots \tag{4.24}$$

and this will be positive only if $p>\frac{1}{2}$. However, a more restrictive condition pertains if we consider the behavior of $g(r,p)$ near $r=1$. It is obvious that $g(1,p)=0$, and it is a simple matter to show that its derivative at $r=1$ is given by

$$\left[\frac{dg(r,p)}{dr}\right]_{r=1}=p\ln(3+2\sqrt{2})-\sqrt{2}. \tag{4.25}$$

To insure that $g(r,p)$ is positive for r slightly greater than 1, we must require that

$$p>\frac{\sqrt{2}}{\ln(3+2\sqrt{2})}\approx 0.802. \tag{4.26}$$

A more detailed consideration of the function $g(r,p)$ reveals that the above condition on p, i.e., (4.26), is the least restrictive condition on p sufficient to make $g(r,p)$ positive for all values of r in the range $1\leq r<\infty$. When p satisfies (4.26), $g(r,p)$ has a single maximum in the interval $1\leq r<\infty$ and, as indicated earlier, vanishes at both end points of the interval. We will assume in the remainder of the paper that p satisfies (4.26)—except in case (A'). For $p=1$ the maximum occurs at $r_0=6.68$ at which point the function has the value $g_0=g(r,1)=0.253$. For $p>1$ the maximum occurs below r_0, and for $0.802<p<1$ the maximum occurs beyond r_0.

Consider now those values of t in the integrand which are increasing with s but for which $t/s\to 0$ as $s\to\infty$, i.e., we let $t=s^{\lambda}/r$ with $0<\lambda<1$ and $r\geq 1$. Then we have

$$\Psi(s,t)\to a\left[\frac{2}{\sqrt{r}}s^{p+\lambda/2-1/2}-\lambda^{R}\left(\frac{s^{\lambda p}}{r^{p}}\right)\ln(4rs^{1-\lambda})\right](\ln s)^{R}. \tag{4.27}$$

In the above expression both terms are increasing as a power of s. To compare them we note that for $p>\frac{1}{2}$ (and recalling that $0<\lambda<1$)

$$\tfrac{1}{2}\lambda(p-\tfrac{1}{2})<\tfrac{1}{2}(p-\tfrac{1}{2})$$

or (4.28)

$$\lambda p<p+\tfrac{1}{2}\lambda-\tfrac{1}{2},$$

which shows that the first term in (4.27) is increasing more rapidly than the second, in which case Ψ is asymptotically positive and $e^{-\Psi}$ is decreasing exponentially for $p>\frac{1}{2}$. However, if $p=\frac{1}{2}$ and $R>0$, $\Psi(s,t)$ is negative, i.e.,

$$\Psi\to-\frac{a\lambda^{R}(1-\lambda)}{\sqrt{r}}s^{\lambda/2}(\ln s)^{R+1}. \tag{4.29}$$

This is relevant to case (A'). In this case $e^{-\Psi}$ is an increasing exponential when evaluated at $t=s^{\lambda}/r$. This will lead to an exponentially increasing $\eta_n(s)$ which we regard as undesirable. This implies that in case (A') t_2 should be independent of s.

Finally, we consider the behavior of $e^{-\Psi}$ for large s and fixed t. Then

$$\Psi(s,t)\to 2\left(\frac{t}{s}\right)^{1/2}as^{p}(\ln s)^{R}-\alpha(t)\ln\left(\frac{4s}{t-4\mu^{2}}\right)$$
$$\to -2a\sqrt{t}\,s^{p-1/2}(\ln s)^{R} \tag{4.30}$$

and $e^{-\Psi}$ is again decreasing exponentially as $s\to\infty$ for $p>\frac{1}{2}$.

In summary, we have found that for $p>0.802$ $e^{-\Psi}$ vanishes exponentially as some power of s for all values of t in the interval $4\mu^2\leq t\leq s$, except at $t=s$ where $\Psi=0$. Further, the smaller the value of t, the larger the value of $e^{-\Psi}$ for t in the interval $[t_1,t_m]$, where

$$t_m=\frac{s}{r_m(p)} \tag{4.31}$$

and $r_m(p)$ is the value of r at which $g(r,p)$ has its maximum. However, for $t>t_m$ [or $r<r_m(p)$], $g(r,p)$ begins to decrease and consequently larger values of $e^{-\Psi}$ accrue for larger values of t until $e^{-\Psi}=1$ is reached at $t=s$. For $p=\frac{1}{2}$ and $R>0$, $e^{-\Psi}$ increases more rapidly than a power of s as $s\to\infty$ when $t=s^{\lambda}/r$. This concludes our discussion of the asymptotic behavior of Ψ.

It is convenient to define $\eta(s)$ as the expansion coefficient corresponding to the leading trajectory and to let $B_n(s)$ be the ratio of the nth expansion coefficient to the leading one, i.e.,

$$\eta(s)\equiv\eta_L(s) \tag{4.32}$$

and

$$B_n(s)\equiv\frac{\eta_n(s)}{\eta(s)}. \tag{4.33}$$

Then each expansion coefficient can be written as

$$\eta_n(s)\equiv\eta(s)B_n(s), \tag{4.34}$$

where for large s $|B_n(s)|$ is both bounded and nonvanishing by virtue of Eq. (3.36). In other words, we expect $|B_n(s)|$ to either approach a nonzero constant as $s\to\infty$ or to oscillate indefinitely between two positive constants. Using Eqs. (4.7), (4.11), and (4.33), we can express the matrix elements (ϕ_m,f) as

$$(\phi_m, f) = \frac{1}{L} e^{\theta(s)} \int_{x_1}^{x_2} e^{-\Psi(s,t)} \eta(t) A(s,t) \phi_m^*(s, x_s) \times \sum_{n=1}^{K(t)} B_n(t) \phi_n(t, x_t) dx_s . \tag{4.35}$$

We can now rewrite Eq. (4.16) as an asymptotic integral equation for $\eta(s)$. Substituting (4.35) into (4.16), we have

$$\eta(s) = \int_{t_1}^{t_2} H(s,t) e^{-\Psi(s,t)} \eta(t) dt , \tag{4.36}$$

where

$$L(st+t^2)^{1/2} H(s,t) \equiv \frac{A(s,t)}{B_k(s)} \left[\sum_{n=1}^{K(t)} B_n(t) \phi_n(t, x_t) \right] \times \sum_{m=1}^{K(s)} \Phi_{km}{}^{-1}(s) \phi_m^*(s, x_s) \tag{4.37}$$

We have written the kernel of the integral equation as $He^{-\Psi}$ because $e^{-\Psi}$ is an exponentially varying function—except for trajectories with $\text{Re}\alpha_n(s) \lesssim \sqrt{s}\, \ln s$—whereas $H(s,t)$ does not vary that rapidly.

We now prove that $L(st+t^2)^{1/2}H(s,t)$ is both bounded and nonvanishing as s and/or t increase. Let $G(t, x_t)$ be the first sum in (4.37), i.e.,

$$G(t, x_t) \equiv \sum_{n=1}^{K(t)} B_n(t) \phi_n(t, x_t). \tag{4.38}$$

Multiply both sides by $\phi_l^*(t, x_t) dx_t / L$ and integrate from $x_a \equiv x_t(t, s_1)$ to $x_b \equiv x_t(t, s_2)$, where $s_1 = s_2/r_1$. Then

$$G_l(t) \equiv (\phi_l, G)_t = \sum_{n=1}^{K(t)} (\phi_l, \phi_n)_t B_n(t), \tag{4.39}$$

where we define for any two functions of t and x_t

$$(F, G)_t \equiv \frac{1}{L} \int_{x_a}^{x_b} F^*(t, x_t) G(t, x_t) dx_t . \tag{4.40}$$

The above integral would be merely the t-channel analog of the s-channel scalar product, as defined by Eq. (4.11), if the upper limit s_2 were set equal to t. However, we allow t to be any fixed value between s_1 and s_2. Let $\tilde{\Phi}$ be the matrix with matrix elements $\tilde{\Phi}_m \equiv (\phi_l, \phi_m)_t$ and G, the matrix with matrix elements G_l. Then

$$G = \tilde{\Phi} B. \tag{4.41}$$

Crossing symmetry implies that the $\phi_n(t, x_t)$ be linearly independent functions of x_t since we have assumed that the $\phi_n(s, x_s)$ are linearly independent functions of x_s. Thus the inverse of $\tilde{\Phi}$ exists, and Eq. (4.41) can be inverted to yield

$$B = \tilde{\Phi}^{-1} G. \tag{4.42}$$

Note that if $G(t, x_t)$ either vanishes or increases without bound as $t \to \infty$, then so must each $G_l(t)$, the elements of the matrix G. But if $G \to 0$ or $G \to \infty$ as $t \to \infty$, so will the matrix B. This, however, is not allowed since by Eq. (3.36) each $B_n(t)$ is both bounded and nonvanishing as $t \to \infty$. We conclude that $G(t, x_t)$ is also bounded and nonvanishing for large t. Similarly we can show that the second sum (over m) in Eq. (4.37) is bounded and nonvanishing as $s \to \infty$. Therefore, there must exist two positive numbers m_1 and m_2 such that

$$m_1 < | L(st+t^2)^{1/2} H(s,t) | < m_2 \tag{4.43}$$

for large s and large t.

This establishes that the asymptotic behavior of the integrand in Eq. (4.36) is controlled by the product $\eta(t) e^{-\Psi}$, or, more accurately, by the function

$$\exp(-[\theta(t) + \Psi(s,t)]) \tag{4.44}$$

since $\eta(t) = \tilde{\eta}_n(t) e^{-\theta(t)} / B_n(t)$ and neither $\tilde{\eta}_n$ nor B_n vary exponentially as $t \to \infty$. Let us restrict our attention to case (C), the case of primary interest. It is easy to see that the expansion coefficients must display an exponential behavior as $t \to \infty$, in particular, that $|\theta(t)| / \Psi(s,t) \gtrsim \epsilon$ for some $\epsilon > 0$ as $s \to \infty$ for $t \in [t_1, t_2]$. We prove this by assuming the contrary in which case $|\theta(t)| / \Psi(s,t) \to 0$, i.e., $\theta(t)$ is negligible compared to $\Psi(s,t)$. Since $\Psi(s,t)$ is positive and increasing ($p \gtrsim 0.805$) for all t except $t = s$ where $\Psi = 0$, (4.44) assumes its maximum value at $t = s$, and thus the integral in Eq. (4.36) will be dominated by its contribution at $t = s$. This is unacceptable because Eq. (3.1), or equivalently (4.5), is simply an identity at $t = s$. No information is contained in any of these equations at the point $t = s$. Thus it is inconceivable that $\eta(s)$ could be determined solely from the contribution of the integral for $t = s$. Thus, our assumption must be incorrect and our original statement valid. Next we prove that θ is positive. Again we assume the opposite, $\theta < 0$. Then (4.4) becomes $\exp[-\Psi(s,t) + |\theta(t)|]$, where $|\theta(t)|$ is an increasing function of t as was shown above. Since Ψ assumes its smallest value at $t = s$ and $\theta(t)$, its largest value, the dominant contribution will again occur at $t = s$ which of course is unacceptable. We conclude that $\theta > 0$.

We can avoid the unreasonable conclusion that the point $t = s$ provides the only relevant contribution to the integral as $s \to \infty$ when $\theta > 0$. We do this by showing that there is exactly one other point, call it t', whose contribution to the integrand can be made comparable in importance to that of the upper limit ($t_2 = s$) by an appropriate choice of the function $\theta(t)$. For this purpose we will assume the following asymptotic form for

$\theta(t)$,

$$\theta(t) \to c(\ln t)^h t^q + \cdots \tag{4.45}$$

as $t \to \infty$ where c, h, q are undetermined constants. These constants will be determined by the asymptotic self-consistency condition

$$\theta(t') + \Psi(s, t') \simeq \theta(s) \tag{4.46}$$

as $s \to \infty$ for some t', where

$$[\theta(t') + \Psi(s, t')] \leq [\theta(t) + \Psi(s, t)] \tag{4.47}$$

for all $t \in [t_1, t_2]$. Equation (4.46) is a bootstrap requirement only in case (C) since only then will t' be increasing with s, i.e., $t' = t'(s)$.

It is instructive to consider case (B) briefly. The above remarks are applicable to case (B) but we will see that no bootstrap mechanism is operative. The reason is that t_1, being independent of s, is the necessary choice for t' in Eq. (4.46). Note that $\theta(t_1)$ is a fixed number and $\Psi(s, t_1) \to -2a\sqrt{t_1}(\ln s)^R s^{p-1/2}$, i.e., they both assume their minimum values at t_1 for $t \in [t_1, t_2]$. Thus

$$\theta(s) \to -2a\sqrt{t_1}(\ln s)^R s^{p-1/2} \tag{4.48}$$

and the expansion coefficients and the residue functions decrease exponentially with the energy $(\sqrt{s})$ for asymptotically linear trajectories $(p = 1)$. Contrary to our expectations in Ref. 4, Eq. (3.1) does not function as a bootstrap equation over the domain of case (B). We mention that Eq. (4.36) must be obtained from (4.10) rather than (4.5) for case (B).

We return to case (C). Here it is not obvious that t' should equal t_1; all that can be said *a priori* is that t' must be determined from Eqs. (4.46) and (4.47). Recall that for $p \geq 0.802$ there are points in the interval (t_m, s) for which $g(r, p) < g(r_1, p)$. At these points $e^{-\Psi}$ exceeds its asymptotic value at t_1 and if $e^{-\Psi}$ were the only factor to consider, we would conclude that $t' \in [t_m, s]$. However, it is the function (4.44) that determines the magnitude of the integral as $s \to \infty$, and the largest value of $e^{-\theta(t)}$ will naturally occur at the smallest value of t, i.e., t_1. Without knowing the asymptotic form of $\eta(t)$, it is impossible to know which of the two competing functions, $e^{-\theta}$ or $e^{-\Psi}$, is exerting the controlling influence. To determine this and hence the value of t', we must solve Eq. (4.46). Thus we set $t' \equiv s/r'$ and substitute into (4.46). Using (4.22) and (4.45), we obtain the equation

$$c(\ln s)^h s^q \to c\left[\ln\left(\frac{s}{r'}\right)\right]^h \left(\frac{s}{r'}\right)^q + ag(r', p)(\ln s)^R s^q . \tag{4.49}$$

The only self-consistent solution to (4.49) requires that $h = R = 0$, $q = p$, and $c = c_p(r')$, where

$$c_p(r') \equiv \frac{a(r')^p g(r', p)}{(r')^p - 1} \tag{4.50}$$

and r' is to be determined from Eq. (4.47). Actually solutions for which $h = R \neq 0$ could be obtained by inclusion of additional, lower-order terms in the asymptotic expansion of $\theta(s)$. If we define the function

$$\gamma(r, p) \equiv \frac{r^p g(r, p)}{r^p - 1}, \tag{4.51}$$

then r' must be chosen such that $\gamma(r', p) < \gamma(r, p)$ for $1 < r \leq r_1$. For this reason it is important to determine the properties of the function $\gamma(r, p)$. It is easily shown that

$$\gamma(1, p) = \ln(3 + 2\sqrt{2}) - \frac{\sqrt{2}}{p}, \tag{4.52}$$

which is a positive number by virtue of Eq. (4.26). The first derivative of $\gamma(r, p)$ with respect to r is zero at $r = 1$, and the second derivative at $r = 1$ is found to be

$$\left|\frac{d^2\gamma}{dr^2}\right|_{r=1} = \frac{-1}{3\sqrt{2p}}(p - p_1)(p - p_2), \tag{4.53}$$

where

$$p_1 \equiv \tfrac{1}{8}(3 - \sqrt{17}) \approx 0.14$$

and

$$p_2 \equiv \tfrac{1}{8}(3 + \sqrt{17}) \approx 0.89. \tag{4.54}$$

Thus $\gamma(r, p)$ has a relative minima at $r = 1$ if $p_1 < p < p_2$ and a relative maxima at $r = 1$ if $p > p_2$. In the former case $\gamma(r, p)$ must increase as r is increased beyond the point $r = 1$. Since we are only interested in those cases for which $p \gtrsim 0.802$, $\gamma(r, p) \to 0$ (from above) as $r \to \infty$. Thus, $\gamma(r, p)$ must attain some maximum value and then begin to decrease as r continues to increase. We have found that it has only one maxima for $1 \leq r < \infty$ when $0.802 < p < p_2$. In the other case (with $p > p_2$), $\gamma(r, p)$ is a monotonically decreasing function of r for $1 \leq r < \infty$. We conclude that if $0.802 < p < p_2$, r' must be some point on the far side of the maxima of $\gamma(r, p)$. For example, if $p = 0.85$, the maxima of $\gamma(r, p)$ is near $r = 10$, and it does not decrease to its value at $r = 1$ until approximately $r = 38$; thus r_1 must exceed 38 or else $r' = 1$, which is unacceptable. On the other hand, if $p > p_2$ the appropriate choice for r' is clearly r_1 (the largest possible value), in which case

$$c_p = c_p(r_1) = \frac{ar_1^p g(r_1, p)}{r_1^p - 1} . \tag{4.55}$$

This implies that for asymptotically linear trajectories the expansion coefficients and the residue functions will decrease exponentially as $e^{-c_1 s}$, where c_1 is given by (4.55) with $p=1$. Since the maximum value of $\gamma(r,1)$ for $1 \leq r < \infty$ occurs at $r=1$ and $\gamma(1,1) \approx 0.349$ by Eq. (4.52), we have the inequality

$$0 < \left(\frac{c_1}{a}\right) < 0.349. \tag{4.56}$$

This is the type of behavior displayed by the residues in the Veneziano model, although the constant in that model corresponding to (c_1/a) is 0.38 and, therefore, exceeds any of the values possible in the above case. At this point it is evident that in case (C) a new possibility is emerging, namely, that the asymptotic expansion coefficient $\eta(s)$—and hence the asymptotic Regge residue function as well—bootstraps itself at infinity, unencumbered by background integrals, Regge cuts, fixed poles, or the like. It does this through the integral equation (4.36) and the bootstrap conditions (4.46) and (4.47). The solution, to first order, is given by (4.56) above.

A few remarks will be made regarding case (A). The integral in Eq. (4.36) is clearly dominated by its contribution at the lower limit in case (A_0). Using Eq. (4.30), we see that as $s \to \infty$

$$\eta_n(s) \approx \eta_L(t_1)\left[\frac{F_n(s,t_1)}{(t_2/t_1)^{1/2}-1}\right] s^{\alpha(t_1)} \times \exp[-2a\sqrt{t_1}(\ln s)^R s^{p-1/2}] + \cdots, \tag{4.57}$$

where $|F_n(s,t_1)|$ is bounded above and below by virtue of Eq. (4.43). Then in case (A_0) the residues have essentially the same behavior as in case (B). In passing we mention that if the interval in case (A_0) is extended to the point $t_2 = s/r_2$ for some $r_2 > 1$, the resulting asymptotic behavior is obviously unchanged. In case (A'), $p=\frac{1}{2}$, $R=1$ so

$$\eta_n(s) \approx \eta_L(t_1) F_n'(s,t_A) s^{\alpha(t_A) - 2a\sqrt{t_A}} + \cdots, \tag{4.58}$$

where t_A is the value of t at which the function $\alpha(t) - 2a\sqrt{t}$ has a maximum over the interval $[t_1, t_2]$. We expect t_A to be t_2; $F_n'(s,t_A)$ is bounded above and below as $s \to \infty$ — see Eq. (4.43).

For the sake of completeness, we comment briefly on those cases intermediate to cases (B) and (C). These are the cases in which $t_2 = s$ and t_1 increases with s, but does so sufficently slowly that $(t_1/s) \to 0$ as $s \to \infty$. For these cases we let $t_1 = s^\lambda / r_1$ for some λ between zero and 1 and for $1 < r_1 < \infty$. Equations (4.46) and (4.47) are valid in these cases. Then for a given λ, Eq. (4.46) has the solution

$$\theta(s) \to \exp\left(-\frac{2a}{\sqrt{r_1}} s^{p-(1-\lambda)/2} \sum_{n=1}^{\infty} r_1^{\mu_n} s^{\nu_n}\right) \tag{4.59}$$

$$\to \exp\left(-\frac{2a}{\sqrt{r_1}} s^{p-(1-\lambda)/2}\right), \tag{4.60}$$

where

$$\nu_n \equiv -(1-\lambda^{n-1})[p-(1-\lambda)/2] \tag{4.61}$$

and

$$\mu_n \equiv -\left(\frac{1-\lambda^{n-1}}{1-\lambda}\right)[p-(1-\lambda)/2]. \tag{4.62}$$

In obtaining (4.59) we found it useful to replace (4.45) by the more general form

$$\theta(s) \to \exp\left(-\sum_{n=1} c_n s^{q_n}\right), \tag{4.63}$$

but the final result is the same as would have been obtained by using (4.45) and considering only the two leading terms. Notice that for t near t_1 with $t = s^\lambda / r$, $r \leq r_1$,

$$|R_s^{(n)}(s,t)| \approx s^{-\lambda/4} |\tilde{\eta}(s)| \times \exp\left[2a\left(\frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r_1}}\right) s^{p-(1-\lambda)/2}\right] \tag{4.64}$$

so that $R_s^{(n)}$ is not increasing exponentially at $t = t_1$. Possible logarithmic factors were omitted for simplicity in obtaining Eq. (4.59)–(4.62). Also as $s \to \infty$ for $t = s^\lambda / r$

$$|R_t^{(n)}(s,t)| \approx s^{(1-\lambda)/4} \tilde{\eta}(s^\lambda / r_1) \times \exp\left(2a\left(\frac{s^\lambda}{r_1}\right)^{p-1/2} \left\{\sqrt{s} - \frac{1}{\sqrt{r_1}}\left[\left(\frac{s}{r}\right)^{1/2}\right]^{\lambda^2}\right\}\right) \tag{4.65}$$

which is increasing exponentially even at $t = t_1$. Since this is not compatible with the behavior of $R_s^{(n)}$ at $t = t_1$, we do not regard this as a viable alternative.

We conclude this section with a few remarks about the behavior of the Regge terms and hence of the modified background integral for case (C). In Sec. II we indicated that the modified background integral must increase exponentially if the correct domain for the crossing relations is that of case (C). This is necessary because, contrary to cases (A) and (B), $R_s^{(n)}$ increases exponentially as $s \to \infty$ for $t = t_1$. From Eqs. (3.6), (4.45), (4.50), and (4.51)

$$|R_s^{(n)}(s,t)| \approx |\eta_n(s)| \exp[x_s \mathrm{Re}\,\alpha_n(s)] \approx |\tilde{\eta}(s) B_n(s)| \exp\{a[x_s - \gamma(r_1,p)] s^p\}. \tag{4.66}$$

We now show that for all $r > 1$

$$x_s(s, s/r) > \gamma(r,p). \tag{4.67}$$

Proof:

$$\begin{aligned}(x_s-\gamma) &= x_s - \left(\frac{r^p}{r^p-1}\right)g \\ &= x_s - \left(\frac{r^p}{r^p-1}\right)(x_s - r^{-p}x_t) \\ &= \left[1-\left(\frac{r^p}{r^p-1}\right)\right]x_s + \left(\frac{1}{r^p-1}\right)x_t \\ &= \left(\frac{1}{r^p-1}\right)(x_t - x_s) \\ &> 0,\end{aligned} \tag{4.68}$$

Q.E.D. In the above we have used the fact that $x_t > x_s$ for all $r > 1$. It follows that $R_s^{(n)}(s, t_1)$ is an exponentially increasing function of s, independently of the value chosen for r_1. We might add that $R_t^{(n)}(s, t_1)$ also increases exponentially as $s \to \infty$. For cases (A) and (B), $R_s^{(n)}(s, t)$ increased exponentially as $s \to \infty$ for $t > t_1$ but not for $t = t_1$; an exception to this occurs for trajectories of the type satisfying Eq. (2.1) in which event $R_s^{(n)}$ never increases faster than a power of s. Returning to case (C) we mention that for $p > p_2$ ($p_2 = 0.89$), $\gamma(r, p)$ is a monotonically decreasing function of r as is x_s. The maximum value of $x_s - \gamma(r, p)$ occurs at $r = 1$ and equals $\sqrt{2}/p$ [see Eq. (4.52)]. Therefore, $R_s^{(n)}$ cannot possibly increase faster than $e^{\sqrt{2}as^p/p}$ as $s \to \infty$. It follows that if the proposed crossing relations are to be valid, the modified background integrals, in particular ΔB, must not be increasing as rapidly as $e^{\sqrt{2}as^p/p}$.

V. GENERALIZATION TO UNEQUAL MASSES

It is not the purpose of this paper to consider in detail the application of the proposed crossing relations to the scattering of particles of unequal masses. However, we outline the form such a generalization should take in order to show that the generalized crossing relations do not possess nontrivial solutions in all the domains which we have considered.

Since the t and u channels are not identical, Eq. (1.2) is replaced by the following pair of equations:

$$\sum_{n=1}^{K(s)} R_s^{(n)} \sim \sum_{n=1}^{K(t)} R_t^{(n)} \tag{5.1}$$

as $s \to \infty$ for $t \in [t_1, t_2]$, and

$$\sum_{n=1}^{K(s)} R_s^{(n)} \sim \sum_{n=1}^{K(u)} R_u^{(n)} \tag{5.2}$$

as $s \to \infty$ for $u \in [u_1, u_2]$, where the more precise statements of these two relationships would be similar to Eq. (3.1). In addition there are two similar equations involving the limit as $t \to \infty$ and two involving the limit of infinite u. The requirement that the trajectories and residues satisfy (5.1) and (5.2) simultaneously clearly places a stringent restriction on their possible forms.

The analysis presented in the previous sections is applicable to Eq. (5.1) with a few obvious modifications to take into account the lack of crossing symmetry. The equations relating the s- and u-channel parameters [resulting from Eq. (5.2)] will be identical in form to those relating the s- and t-channel parameters except $\eta_n(s)$ will be replaced by $\sigma_n\eta_n(s)$, and $\eta_n(t)$ will be replaced by $\sigma_n\eta_n(u)$; also, any kinematical factors present must be appropriate for the corresponding channel.

As an example consider the reaction $\pi^- + \bar{K}^0 \to \pi^0 + K^-$ and assume case (A_0) or (B). Then the asymptotic behavior of the expansion coefficients resulting from Eq. (5.1) will be similar to Eq. (4.57), i.e.,

$$\tilde{\eta}_n(s) \sim \eta_\rho(t_1)\left(\frac{s}{2q_t p_t}\right)_{t_1}^{\alpha_\rho(t_1)} \exp\left[-2\frac{\alpha(s)}{\sqrt{s}}\sqrt{t_1}\right]F_n(s, t_1), \tag{5.3}$$

where the subscript t_1 indicates that the c.m. momenta q_t of the $\pi\pi$ system and p_t of the $K\bar{K}$ system are to be evaluated at $t = t_1$; α_ρ is the ρ trajectory. The analogous expression resulting from Eq. (5.2) is

$$\sigma_n\tilde{\eta}_n(s) \sim \sigma_{K^*}\tilde{\eta}_{K^*}(u_1)\left(\frac{s}{2q_u}\right)_{u_1}^{\alpha_{K^*}(u_1)} \times \exp\left[-2\frac{\alpha(s)}{\sqrt{s}}\sqrt{u_1}\right]F_n(s, u_1), \tag{5.4}$$

where the u-channel c.m. momentum q_u is to be evaluated at $u = u_1$ and α_{K^*} is the K^*(892 MeV) trajectory. These two equations can easily be shown to be inconsistent. The most important observation in this regard is that the function $F_n(s, x)$ appearing in (5.3) is essentially the same function appearing in (5.4), except in the former case $x = t_1$ and in the latter $x = u_1$. In other words, this function is completely independent of the crossed channel except for the value assumed by its second argument. Furthermore, we have shown that $F_n(s, x)$ is bounded and can vanish no faster than $s^{-1/2}$ [see Eq. (4.43)]. It follows that the two exponentials $\exp[-2\alpha(s)\sqrt{t_1}/\sqrt{s}]$ and $\exp[-2\alpha(s)\sqrt{u_1}/\sqrt{s}]$ will be the same (as they must be) only if $u_1 = t_1$. Thus $F_n(s, t_1) = F_n(s, u_1)$, and the n dependence of the right-hand sides of (5.3) and (5.4) is identical, but the n dependence of the left-hand sides of (5.3) and (5.4) differ by the signature factor σ_n. Therefore, the two equations are inconsistent. We also note the inconsistency arising from $u_1 = t_1$ and the fact that the ρ and K^* trajectories are not expected to intersect, in which case the s dependence predicted by Eq. (5.3) is not the same as that predicted by Eq. (5.4).

If we attempt to duplicate the above arguments for case (A') we find again two expressions for $\eta_n(s)$ which differ by the signature factor σ_n resulting in an inconsistency.

In the above proofs presented for cases (A) and (B) the assumption was made that the function $F_n(s,x)$ appearing in the expression for $\eta_n(s)$ was identical to the one in the expression for $\sigma_n\eta_n(s)$. This can be proven rigorously for case (A_0) if we assume a Lipshitz condition on $\alpha(t)$ in the variable $\sqrt{t}$ for $\sqrt{t}$ near $\sqrt{t_1}$ or near $\sqrt{t_A}$ in case (A'). However, in case (B) additional, though not overly restrictive, assumptions are needed to ensure the equality of the two functions.

For case (C) no inconsistencies have been detected in Eqs. (5.1) and (5.2). Of course the proven consistency of the crossing relations in the domain of case (C) and generalized for external particles of unequal mass must await an existence proof or further developments indicative of their consistency. All that can be said at this point is that the approach which we have considered cannot be used to arrive at an obvious inconsistency in case (C). It fails because the expressions for $\tilde{\eta}_n(s)$ were not sufficiently simple to allow the necessary conclusions to be made. All that we have been able to obtain in this paper is an expression for $\theta(s)$. Furthermore, we can be certain that no kinematical inconsistencies reflecting a mass difference will be encountered in case (C). When s is large and t is not, t-channel mass differences will be important. However, when both s and t are large, as is always true in case (C), the kinematics of all channels are identical and mass differences are of no importance.

VI. SUMMARY OF RESULTS, CONCLUSIONS, AND ASSUMPTIONS

In this paper we have presented a general investigation of the postulate that infinitely rising trajectories combine with the crossing principle to provide a new type of relationship between the Regge parameters of the direct and crossed channels. Our main concern has been the determination of the type of domain in the real st plane in which such crossing relations might exist as $s\to\infty$. We have assumed that Regge poles rise above the leading Regge branch point at sufficiently large and positive values of their arguments and that the background integrals are increasing more rapidly than the Regge-cut amplitudes. Since the crossing relations involve trajectories of both the s and t channels, we have restricted our attention to the first quadrant of the real st plane. The result is an expansion problem: an expansion of the direct-channel Regge amplitudes in terms of the crossed-channel amplitudes, and vice versa. The basis functions entering the expansions are naturally assumed to be linearly independent. The only other assumption used is that the residue functions not increase exponentially. In cases (A_0), (B), and (C) this was found to be equivalent to requiring that the real parts of the trajectories increase faster than s^p where $p=0.802$. In case (A') the question of the exponential increase of the residue functions did not occur since the rise of the trajectories was only sufficient to induce a power behavior for the residues. Also the assumption of the Lipshitz condition mentioned in case (A) below can certainly be relinquished without changing our conclusion about its extension to unequal-mass scattering. A comparison of the results obtained for the four cases of primary interest is given in outline form below. In each case it is understood that s is increasing without bound and t is confined to a given interval along the positive, real axis.

Case (A'): $t\in[t_1,t_2]$ with t_1 and t_2 fixed and the real parts of trajectories must satisfy Eqs. (2.1) and(3.13).

Results:

(1) An existence proof can be given but only if the trajectories obey the additional condition

$$\operatorname{Im}\alpha_n(s)\to\frac{\pi\sqrt{s}}{\sqrt{t_2}-\sqrt{t_1}}[n+C(s)], \tag{6.1}$$

where $C(s)$ is a function of the order of magnitude of $K(s)$.

(2) The problem can be phrased as a relationship in a linear vector space.

(3) The residues behave as a power of s for large s.

(4) $|B_s'|<s^N$ for some fixed N, where B_s' is the modified background integral.

(5) The generalization to unequal-mass scattering is inconsistent for all two-body reactions if $\alpha(t)$ satisfies a Lipshitz condition in the variable $\sqrt{t}$ for $\sqrt{t}$ near $\sqrt{t_A}$.

Case (A_0): $t\in[t_1,t_2]$ with t_1 fixed and $(t_2/s)\to 0$ as $s\to\infty$, where t_2 may or may not depend on s and trajectories do not satisfy Eq. (2.1).

Results:

(1) An existence proof could not be supplied.

(2) The problem could not be stated as a relationship in a linear vector space.

(3) The residues decrease as $\exp[-2a\sqrt{t_1}s^p(\ln s)^R]$.

(4) $|B_s'|<s^N$ for some N.

(5) Essentially the same as for case (A').

Case (B): $t\in[t_1,s]$ with t_1 fixed.

Results:

(1)–(4) Same as for case (A_0).

(5) The generalization to unequal-mass scattering fails only if more restrictive assumptions are used.

(6) The real parts of the trajectories must be either asymptotically parallel or asymptotically degenerate.

Case (*C*): $t \in [s/r_1, s]$ with r_1 fixed and $r_1 > 1$. Results:

(1) An existence proof was provided for two types of trajectories, namely, (a) asymptotically parallel trajectories, i.e., those for which

$$\mathrm{Re}\alpha_n(s) \to \mathrm{Re}\alpha_L(s) - nb \tag{6.2}$$

and

$$\mathrm{Im}\alpha_n(s) \to c_0(s), \tag{6.3}$$

and (b) trajectories with asymptotically degenerate real parts and asymptotically equal spacing for their imaginary parts, i.e.,

$$\mathrm{Re}\alpha_n(s) \to \mathrm{Re}\alpha_L(s) \tag{6.4}$$

and

$$\mathrm{Im}\alpha_n(s) \to f(s) + \frac{2\pi n}{L}, \tag{6.5}$$

where $n = 0, \pm1, \pm2, \ldots$ and $f(s)$ is of the order of magnitude of $K(s)$.

(2) The problem can be formulated in a linear vector space.

(3) If we assume that $p \gtrsim 0.802$, the residues decrease as $\exp[-a\gamma(r_1, p)s^p]$, where

$$\gamma(r_1, p) \equiv (1 - r_1^{-p})^{-1} \left\{ \ln\left[1 + \frac{2}{r_1} + \frac{2}{r_1}(1 + r_1)^{1/2}\right] - r_1^{-p} \ln[1 + 2r_1 + 2(r_1 + r_1^2)^{1/2}] \right\}, \tag{6.6}$$

the maximum value of $\gamma(r_1, p)$ for $p = r_1 = 1$ being 0.348.

(4) $|B_s'| \leq e^{cs^p}$, where $c < \sqrt{2}\, a/p$.

(5) No obvious inconsistencies are found in the generalization to unequal-mass scattering.

(6) The real parts of the trajectories must be either asymptotically parallel or asymptotically degenerate.

Also in all four cases the expansion coefficients must be of comparable magnitude as $s \to \infty$, and it is only in case (*C*) that the crossing relations actually provide a bootstrap mechanism for the Regge parameters.

On the basis of the above results, we have come to the following conclusions:

1. Nontrivial solutions do not exist even in equal-mass scattering for cases (A_o) and (B).

2. Nontrivial solutions for case (A') exist in equal-mass scattering but not in unequal-mass scattering. In the former, the solutions are not required to be selfconsistent because the crossing relations relate the s- and t-channel Regge parameters at completely different values of their arguments.

3. Nontrivial solutions exist for case (C) in both equal-mass and unequal-mass scattering, although in the latter our conclusion is speculative in the sense that a plausibility argument, and not an existence proof, has been given. In the equal-mass case self-consistent values for the leading terms in the asymptotic expansion of $\mathrm{Re}\alpha_L(s)$ and $\eta_L(s)$ have been obtained. We suggest that the asymptotic solutions are self-consistent in general for equal-mass cases. Their self-consistency for unequal masses is yet to be investigated.

It is clear that the appropriate domain for the crossing relations for the Regge amplitudes is the domain of case (C). This is not the conclusion we expected. It involves the limit as $s \to \infty$ for fixed z_s; we expected the limit as $s \to \infty$ for either fixed t or such that the ratio $(t/s) \to 0$. In the latter case, the resulting crossing relations are reminiscent of the FESR in that an analytically continued sum of resonance amplitudes, as defined by Eq. (1.2), must behave as $s^{\alpha(t)}$ for large s, i.e., $\sum R_s^{(n)} \approx s^{\alpha_L(t)}$. This cannot occur in case (C). The sum over the resonance amplitudes, hidden in the series $\sum R_s^{(n)}$, is Regge-behaved only in that $\sum R_s^{(n)} \to \sum R_t^{(n)}$, and in case ($C$) $R_t^{(n)}$ does not reduce to $A_n(t)s^{\alpha_n(t)}$. Instead we find that asymptotic local duality as defined in this paper bears a closer resemblence to the zero-width duality of the Veneziano model.

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APPENDIX A

In this appendix we derive some useful expression for $P_\alpha(z)$ and $P_\alpha(-z)$ as $\alpha \to 1$ both for fixed z and as $z \to 1$. From Eq. 3.2(26) of Bateman[5] we have

$$P_\alpha(z) \to \frac{1}{(2\pi)^{1/2}(z^2-1)^{1/4}} \left[\frac{\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-\alpha)} G(u) e^{-(\alpha+1/2)x} + \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} F(u) e^{(\alpha+1/2)x}\right], \tag{A1}$$

where

$$F(u) \equiv \sum_{n=0}^{\infty} \frac{[(\frac{1}{2})_n]^2 u^n}{n!(\frac{1}{2}-\alpha)_n},$$

$$G(u) \equiv \sum_{n=0}^{\infty} \frac{[(\frac{1}{2})_n]^2 u^n}{n!(\alpha+\frac{3}{2})_n},$$

and

$$u \equiv \frac{-z + (z^2-1)^{1/2}}{2(z^2-1)^{1/2}} = \frac{-1}{[z + (z^2-1)^{1/2}]^2}. \tag{A2}$$

Also x is defined by Eq. (2.20) with $z \leftrightarrow z_s$, $x \leftrightarrow x_s$. Both series, F and G, converge only for $|u|<1$, i.e., for $|z|>3/2\sqrt{2}$. However, they can be analytically continued to smaller values of z as we will now show. First, we consider $F(u)$ and write it as a contour integral in the complex ν plane:

$$F(u)=\frac{\Gamma(\frac{1}{2}-\alpha)}{2\pi i}\oint_c \frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\frac{(-u)^\nu}{\Gamma(\frac{1}{2}-\alpha+\nu)}\frac{d\nu}{\sin\pi\nu} + \sum_{n=0}^{N}\frac{[(\frac{1}{2})_n]^2u^n}{n!(\frac{1}{2}-\alpha)_n}, \tag{A3}$$

where the contour c encloses the real ν axis beginning at $\mathrm{Re}\nu=n+\epsilon$ and going out to infinity. We can open up the contour and discard the contribution along the infinite semicircle since the integrand behaves for large $|\nu|$ as

$$\nu^{-1/2+\alpha}e^{\nu\ln(-u)}e^{-\pi|\mathrm{Im}\nu|} \tag{A4}$$

and at this point $|u|<1$ and $\arg(-u)<\pi$. In this way we obtain the expression

$$F(u)=-\frac{\Gamma(\frac{1}{2}-\alpha)}{2\pi i}\int_{L-i\infty}^{L+i\infty}\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\frac{(-u)^\nu}{\Gamma(\frac{1}{2}-\alpha+\nu)}\frac{d\nu}{\sin\pi\nu} + \sum_{0}^{N}\frac{(\frac{1}{2})_n{}^2u^n}{n!(\frac{1}{2}-\alpha)_n}, \tag{A5}$$

where $L=N+\epsilon$. The integral in (A5) clearly exists for $|u|>1$ which corresponds to $z<3/2\sqrt{2}$. Thus (A5) is the desired analytic continuation. From Binet's first expression for $\ln\Gamma(z)$[5] one can show that for $\mathrm{Re}z>0$

$$|\Gamma(z)| \leqslant \left|\left(\frac{2\pi}{z}\right)^{1/2}e^{-z}z^z\right|e^{k/x}|\Gamma(z)| \leqslant \left|\left(\frac{2\pi}{z}\right)^{1/2}e^{-z}z^z\right|e^{k/z_1}, \tag{A6}$$

where $z_1\equiv\mathrm{Re}z$, and for $\mathrm{Re}(z+a)>0$, $\mathrm{Re}(z+b)>0$

$$\left|\frac{\Gamma(z+a)}{\Gamma(z+b)}\right| \leqslant M|z^{a-b}|\exp\left(\frac{k}{z_1+a}+\frac{k}{z_1+b}\right), \tag{A7}$$

where k, M are constants. If we denote the integral in (A5) by f, we have

$$f=-\frac{\Gamma(\frac{1}{2}-\alpha)}{2\pi i}\int_{L-i\infty}^{L+i\infty}\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\frac{(-u)^\nu}{\sin\pi\nu}\frac{d\nu}{\Gamma(\frac{1}{2}-\alpha+\nu)} \tag{A8}$$

or, using certain properties of the γ functions,

$$f=\frac{-1}{2i}\int_{L-i\infty}^{L+i\infty}d\nu\,\frac{\cos[\pi(\nu-\alpha)]}{\cos\pi\alpha\sin\pi\nu}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} \times\frac{\Gamma(\alpha-\nu+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}(-u)^\nu\Gamma(\nu+\tfrac{1}{2}). \tag{A9}$$

We can use Eqs. (A6) and (A7) to obtain an upper bound for $|f|$ as follows. Let $\nu_2=\mathrm{Im}\nu$. Then

$$\begin{aligned}|f| &\leqslant M\int_{-\infty}^{\infty}d\nu_2|\cos\pi\nu+\tan\pi\alpha|\,\frac{(-u)^L}{(\nu_2{}^2+L^2)^{1/2}}\left|\left(\frac{2\pi}{\nu+\frac{1}{2}}\right)^{1/2}e^{-\nu-1/2}(\nu+\tfrac{1}{2})^{\nu+1/2}\alpha^{-\nu}\right|e^{K(L+1/2)}\\ &\leqslant M\left(\frac{-u}{|\alpha|}\right)^L\int_{-\infty}^{\infty}\frac{d\nu_2}{\nu_2{}^2+L^2}\exp[\nu_2\psi-\nu_2\arg(\nu+\tfrac{1}{2})+(L+\tfrac{1}{2})\ln|\nu+\tfrac{1}{2}|]\\ &\leqslant M\left(\frac{-u}{|\alpha|}\right)^L\int_{-\infty}^{\infty}d\nu_2\,\frac{[\nu_2{}^2+(L+\frac{1}{2})^2]^{1+L/2}}{[\nu_2{}^2+L^2]}\exp(-\nu_2[\arg(\nu+\tfrac{1}{2})-\psi(s)])\\ &\leqslant M\left(\frac{-u}{|\alpha|}\right)^L,\end{aligned} \tag{A10}$$

where $\psi(s)$ is the argument of $\alpha(s)$ and we have assumed that $\psi(\infty)$ is less than $\frac{1}{2}\pi$. This is necessary for the integral to converge since $\arg(\nu+\frac{1}{2})\to\pm\frac{1}{2}\pi$ as $\nu_2\to\pm\infty$. The proof can be modified to include the case $\arg(\alpha)>\frac{1}{2}\pi$ but it is cumbersome and will not be done here. From Eq. (A10) we see that $|f|$ can be made arbitrarily small by choosing L sufficiently large as long as

$$\left(\frac{-u}{|\alpha|}\right)\to 0. \tag{A11}$$

In particular $|f|$ can be made smaller than the first few terms of the finite series in (A5) whenever (A11) is satisfied. Equation (A11) is clearly satisfied for fixed z, $z\neq 1$, i.e., for fixed, finite values of u. Now consider the case of infinite u. Suppose that $(s/t)\to\infty$, in which case $z\to 1$. Then one can easily show that

$$u\to-\frac{1}{4}\left(\frac{s}{t}\right)^{1/2}\left[1-2\left(\frac{t}{s}\right)^{1/2}+\cdots\right] \tag{A12}$$

and Eq. (A11) becomes

$$\left(\frac{|\alpha|\sqrt{t}}{\sqrt{s}}\right)\to\infty. \tag{A13}$$

In the formulas in this paper (A13) is always satisfied since we consider only those trajectories which increase faster than $\sqrt{s}$.

$G(u)$ can likewise be analytically continued to $z<3/2\sqrt{2}$ and bounded in the limit of infinite s

and $\alpha(s)$. We conclude that as $s\to\infty$ and $\mathrm{Re}\alpha(s)\to\infty$

$$P_\alpha(z)\to\frac{1}{(2\pi)^{1/2}(z^2-1)^{1/4}}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}e^{(\alpha+1/2)x}+\cdots \tag{A14}$$

whenever Eq. (A11) or (A13) is satisfied. It is understood in the above that s, t, and z are real and that t, s, and $\mathrm{Re}\alpha(s)$ are positive.

Notice that if $(t/s)\to 0$ as $s\to\infty$, Eq. (A14) can be written as follows:

$$P_{\alpha(s)}(z)\to\frac{1}{2}\left(\frac{\sqrt{s}}{\pi\alpha(s)\sqrt{t}}\right)^{1/2}\exp\left\{2[\alpha(s)+\tfrac{1}{2}]\frac{\sqrt{t}}{\sqrt{s}}\right\}, \tag{A15}$$

assuming of course that $\mathrm{Re}\alpha(s)\to\infty$ as $s\to\infty$.

Next we consider $P_\alpha(-z)$ and use Eq. 3.3.1(10) of Ref. 5 to obtain

$$P_\alpha(-z)=e^{-i\pi\alpha}P_\alpha(z)-\frac{2}{\pi}\sin(\pi\alpha)\,Q_\alpha(z) \tag{A16}$$

for $\mathrm{Im}z>0$. From Eq. 3.2(44) of Ref. 5

$$\begin{aligned}\frac{2}{\pi}\sin\pi\alpha\,Q_\alpha(z)&=\left(\frac{2}{\pi}\right)^{1/2}\frac{\sin(\pi\alpha)}{(z^2-1)^{1/4}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\\&\quad\times G(u)e^{-(\alpha+1/2)x}\\&\quad+\cdots\\&=\frac{2\cos(\pi\alpha)}{(2\pi)^{1/2}(z^2-1)^{1/4}}\frac{\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-\alpha)}\\&\quad\times G(u)e^{-(\alpha+1/2)x}\\&\quad+\cdots,\end{aligned} \tag{A17}$$

where some properties of the γ functions have been used. It is clear from Eqs. (A16) and (A17) that if $\mathrm{Re}\alpha\to\infty$, the second term in Eq. (A16) is negligible and

$$P_\alpha(-z)\to e^{-i\pi\alpha}P_\alpha(z). \tag{A18}$$

Suppose we replace the index α with l and consider the limit in which $l_2\to\pm\infty$ with $l_1=-\frac{1}{2}$, where $l=l_1+il_2$. Then the second term in (A16) is not necessarily negligible. Substituting Eq. (A17) into Eq. (A16), we obtain after some simplification

$$\begin{aligned}P_l(-z)&=\frac{1}{(2\pi)^{1/2}(z^2-1)^{1/4}}\\&\quad\times\left[-e^{i\pi l}\frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-l)}G(u)e^{-(l+1/2)x}\right.\\&\quad\left.+e^{-i\pi l}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)}F(u)e^{(l+1/2)x}\right].\end{aligned} \tag{A19}$$

Then as $l_2\to+\infty$ the second term in (A19) is dominant and we obtain

$$\frac{P_l(-z)}{\sin\pi l}\to\frac{(-2i)e^{-i\pi/4}e^{il_2x}}{(2\pi l_2)^{1/2}(z^2-1)^{1/4}}+\cdots. \tag{A20}$$

As $l_2\to-\infty$ the first term in (A19) is dominant and gives

$$\frac{P_l(-z)}{\sin\pi l}\to\frac{(-2i)e^{-i\pi/4}e^{-il_2x}}{[2\pi(-l_2)]^{1/2}(z^2-1)^{1/4}}+\cdots. \tag{A21}$$

The validity of the above formulas for z near 1 depends on the analytic continuation of $F(u)$ and $G(u)$ and is therefore subject to the conditions indicated in Eqs. (A11) and (A13).

Next we consider the case in which z is approaching unity as $|l_2|\to\infty$ but much more slowly than before, i.e., we assume that

$$\lim_{|l_2|\to\infty}\left(\frac{|l_2|\sqrt{t}}{\sqrt{s}}\right)=0. \tag{A22}$$

For this case we use Eq. (A16) and an equation just below Eq. 3.6.1(11) of Ref. 5. Combining them we have

$$\frac{P_l(-z)}{\sin\pi l}=\left\{\frac{e^{-i\pi}}{\sin\pi l}-\frac{1}{\pi}\ln\left(\frac{z+1}{z-1}\right)+\frac{2}{\pi}[\gamma+\psi(l+1)]\right\}P_l(z)+\frac{1}{\pi^2}\sin\pi l\sum_{n=1}^{\infty}\frac{\Gamma(n-l)\Gamma(n+l+1)}{(n!)^2}\sigma(n)\left(\frac{1-z}{2}\right)^n, \tag{A23}$$

where $\sigma(n)=\gamma+\psi(n+1)$. One can show that

$$\sin\pi l\,\Gamma(n-l)\Gamma(n+l+1)\left(\frac{1-z}{2}\right)^n=(-1)^n\frac{\Gamma(l+n+1)}{\Gamma(l+1-n)}\left(\frac{1-z}{2}\right)^n\to(-1)^n\left[l_2^2\left(\frac{1-z}{2}\right)\right]^n=\left(\frac{l_2^2t}{s}\right)^n\to\left(\frac{l_2\sqrt{t}}{\sqrt{s}}\right)^{2n}. \tag{A24}$$

Thus each term in the infinite series in (A23) vanishes as $l_2\to\infty$ when (A22) is satisfied. From Eq. 3.2(3) of Ref. 5 one can show that in this case

$$\lim_{|l_2|\to\infty}P_l(z)=1. \tag{A25}$$

Thus as $|l_2|\to\infty$

$$\begin{aligned}\frac{P_l(-z)}{\sin\pi l}&\to\frac{1}{\pi}\ln(z-1)+\frac{2}{\pi}\ln(l+1)+\cdots\\&\to\frac{1}{\pi}\ln\left(\frac{2tl^2}{s}\right)+\cdots\end{aligned} \tag{A26}$$

when Eq. (A22) is satisfied.

APPENDIX B

In this appendix we obtain an estimate for the modified background integral B_s' using the bound for $A_l(s)$, Eq. (2.11), and also Eq. (2.12). For this we must consider the behavior of the Legendre functions as $l_2\to\pm\infty$. We need consider only $P_l(-z)$ since $P_l(z)/\sin\pi l\to 0$ in this limit. We find that the behavior of $P_l(-z)$ for large l_2 depends on the magnitude of the quantity $l\sqrt{t}/\sqrt{s}$. If $l\sqrt{t}/\sqrt{s}\to 0$ we see from Eq. (A26) of the preceding Appendix that as $l_2\to\pm\infty$

$$\frac{P_l(-z_s)}{\sin\pi l} \to \frac{1}{\pi}\ln\left(\frac{2tl^2}{s}\right) + M_l(s), \tag{B1}$$

where $|M_l(s)|$ is bounded. Robin[15] has considered the case in which $l|\sqrt{t}/\sqrt{s}|$ = constant as l_2 and s increase. Using his results, we find that as $l_2 \to \pm\infty$

$$\frac{P_l(-z_s)}{\sin\pi l} \to \frac{(-2i)}{(2\pi\omega)^{1/2}} e^{\omega}[1+O(t/s)], \tag{B2}$$

where

$$\omega \equiv (2l+1)\left(\frac{1-z_s}{2}\right)^{1/2} \to 2l\left(\frac{t}{s}\right)^{1/2}. \tag{B3}$$

A similar expression can be obtained when $l_2 \to -\infty$. It is clear that (B2) is not useful when $(t/s) \to \infty$. Then we must use Eqs. (A20) and (A21), which indicate that

$$\frac{P_l(-z_s)}{\sin\pi l} \to \frac{(-2i)\exp(i|l_2|x_s)}{(2\pi)^{1/2}|l_2|(z_s^{\,2}-1)^{1/4}} + \cdots. \tag{B4}$$

Notice that (B4) reduces to (B2) when $l_2 \to +\infty$ and $t/s \to 0$.

In estimating the behavior of B_s' as $s \to \infty$, we ignore that portion of the integral involving negative l_2 for simplicity. We write B_s' as the sum of three integrals: the first from zero to l_0, the second from l_0 to $(s/t)^{1/2}$, and the third from $(s/t)^{1/2}$ to ∞, where l_0 is an arbitrarily large but fixed number. Let the first integral be B_1, the second be B_2, and the third be B_3. Thus $B_s' = B_1 + B_2 + B_3$, where $|B_1|$ is bounded by $c(s)$ by virtue of (2.11) and (2.12),

$$B_2 \to -\frac{i}{\pi}\int_{l_0}^{s/t} c_l(s)\ln\left(\frac{2t}{s}l_2^{\,2}\right) l_2^{\,1/2-t}\,dl_2 + \cdots \tag{B5}$$

and

$$B_3 \to \frac{-2}{(2\pi)^{1/2}(z_s^{\,2}-1)^{1/4}}\int_{s/t}^{\infty} c_l(s) l_2^{\,-\epsilon} e^{il_2x_s}\,dl_2 + \cdots. \tag{B6}$$

Notice that we have used (B1) for all values of l_2 in Eq. (B5) although (B2) is the more appropriate expression for l_2 near the upper limit, $(s/t)^{1/2}$. It should be apparent that this error cannot alter our conclusions regarding the bound on $|B_s'|$ by more than some power of s. Using (2.12) and (B5), we have

$$|B_2| \le c(s)\int_{l_0}^{(s/t)^{1/2}} l_2\ln\left(\frac{2tl_2^{\,2}}{s}\right)dl_2 \le M\frac{s}{t}c(s). \tag{B7}$$

It is not convenient to obtain an upper bound for $|B_3|$. Instead we obtain an estimate of its behavior for large s by replacing $c_l(s)$ with $c(s)$. If the lower limit is replaced by zero, the integral can be evaluated to yield the result

$$|B_3| \le \left|\frac{2c(s)}{(2\pi)^{1/2}(z_s^{\,2}-1)^{1/4}}\int_0^{\infty} l_2^{\,-\epsilon} e^{il_2x_s}\,dl_2\right| = \frac{2\Gamma(1-\epsilon)}{(2\pi)^{1/2}(z_s^{\,2}-1)^{1/4}}c(s)x_s^{\,-1+\epsilon}.$$

Based on Eqs. (2.11) and (2.12) we conclude that a good estimate for a bound on $|B_s'|$ is provided by the following:

$$|B_s'| < M\frac{s}{t}c(s) \tag{B8}$$

for large s and some M. This is the bound we wished to obtain for B_s'. It's magnitude is essentially the same as the least upper bound for $A_l(s)$, that is, $c(s)$.

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