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Noncovariance of the Coulomb-gauge Schwinger model*

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Formal proofs of the covariance of gauge theories in two-dimensional space-time are questionable in the Coulomb gauge in view of the highly singular nature of the inverse Laplacian. It is shown that such considerations do in fact destroy the covariance of the Schwinger model as well as that of the more general non-Abelian gauge theory.

I. MOTIVATION

The famous model in two-dimensional space-time of an electromagnetic field in interaction with a massless charged fermion (more commonly known as the Schwinger model[1]) has recently received considerable attention within the context of the quark-binding problem. Thus this model has continued to be useful long after the original motivation for which it was proposed (namely the connection between gauge invariance and mass) has ceased to be controversial.

Although Schwinger's original paper developed the theory in a covariant gauge, the Coulomb gauge is in many respects a more interesting framework for a discussion of its properties. Thus Brown[2] has studied the model in that gauge in considerable detail and claims to have demonstrated its covariance. However, that proof, is suspect since it requires the *ad hoc* addition of a term $-\frac{1}{8}e^2Q^2$ to the energy density, where Q is the total charge operator

$$Q=\int j^0(x,t)dx .$$

While such a modification of $T^{00}(x)$ would be acceptable if Q were a conserved operator, it has been observed by Zumino[3] that the equation

$$(-\partial^2+e^2/\pi)j^\mu=0$$

implies

$$({\partial_0}^2+e^2/\pi)Q=0,$$

so that Q cannot be constant unless it vanishes. Since Q cannot identically vanish if it is to fulfill its role as the generator of gauge transformations of the first kind, Zumino concludes "that the Coulomb gauge formulation of the theory is not truly covariant unless one is willing to restrict oneself to states of zero charge."

In view of Zumino's remarks the noncovariance of the model in the Coulomb gauge would seem to be well established. However, reference to the literature suggests that this fact has generally been overlooked or forgotten by authors of recent papers dealing with the Schwinger model. Thus, for example, Li and Willemsen[4] in their recent study of the Schwinger model generalized to a non-Abelian gauge found considerable difficulty in demonstrating covariance in the Coulomb gauge and eventually relied upon incorrect assertions such as [their Eq. (2.25)]

$$\int_{-L}^{L} dx\,\epsilon(x-x')=0$$

in order to obtain the desired commutators of the Poincaré group. Since all experience in this field suggests that results of Abelian theories should generalize to non-Abelian gauge theories, one expects the Schwinger model and its Yang-Mills extension to either stand or fall together. The results of Ref. 3 and the above remarks concerning Ref. 4 do in fact support this intuitive notion by showing that neither theory is covariant.

In the present paper we reexamine this old question in order to emphasize Zumino's observations and to display in clear fashion where the breakdown of covariance occurs. In the following section we thus review certain aspects of the Schwinger model including in particular a more careful consideration of the inversion of the Laplacian in one spatial dimension. In the concluding section the failure of covariance is displayed.

II. OPERATOR PROPERTIES

In order to fix the notation to be used here we begin by writing the Lagrangian of the model in the form

$$\mathfrak{L}=\frac{i}{2}\psi\alpha^\mu\partial_\mu\psi+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+ej^\mu A_\mu , \qquad (2.1)$$

where the current operator is formally defined by the limit

$$j^\mu=\lim_{\vec{x}\to\vec{x}'}\psi(x)\alpha^\mu q\psi(x'),$$

where q is the usual charge matrix

$$q=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

From (2.1) there follow the equations of motion

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu, \qquad (2.2)$$

$$\partial_\nu F^{\mu\nu}=ej^\mu, \qquad (2.3)$$

$$\alpha^\mu\left(\frac{1}{i}\partial_\mu-eqA_\mu\right)\psi=0, \qquad (2.4)$$

to which set one appends the Coulomb gauge condition

$$A_1=0. \qquad (2.5)$$

Using (2.5) Eqs. (2.2) and (2.3) imply, respectively,

$$F^{01}=-\partial_1 A^0$$

and

$$-\nabla^2 A^0=ej^0. \qquad (2.6)$$

By displaying A^0 in this way as a nonlocal function of the charge density one makes explicit the well-known fact that there are no dynamical degrees of freedom associated with the electromagnetic field in one dimension and that the only interaction in the model is the nonlocal coupling of the fermion field to itself.

While it is customary to invert the Laplacian in (2.6) and to write immediately

$$A^0(x)=-\frac{e}{2}\int_{-\infty}^{\infty}|x-x'|j^0(x')dx', \qquad (2.7)$$

we choose to be more circumspect in carrying out that operation in order to determine the most general solution of (2.6). To that end it is convenient to consider (2.6) in a one-dimensional box $|x|<L$ and to solve the Dirichlet boundary value problem corresponding to that domain. The appropriate Green's function is then found to have the unique symmetric form

$$G(x,x')=-\tfrac{1}{2}|x-x'|+\frac{L}{2}\left(1-\frac{xx'}{L^2}\right), \qquad (2.8)$$

which clearly has the properties

$$G(x,x')=G(x',x),$$

$$G(|x|=L,x')=0,$$

This implies for $A^0(x)$ the form

$$\begin{aligned}A^0(x)&=e\int_{-L}^{L}G(x,x')j^0(x')dx'-[(\nabla'G)A^0(x')]_{-L}^{L}\\&=e\int_{-L}^{L}G(x,x')j^0(x')dx'+\tfrac{1}{2}(A_+^0+A_-^0)\\&\quad+\frac{x}{2L}(A_+^0-A_-^0),\end{aligned} \qquad (2.9)$$

where we have used the notation $A_\pm^0\equiv A^0\ (x=\pm L)$ to denote the operator $A^0(x)$ evaluated at the boundaries of the box. In order to compare this result with (2.7) one reduces (2.9) by means of (2.8) to

$$A^0(x)=-\frac{e}{2}\int_{-L}^{L}|x-x'|j^0(x')dx'+\frac{LeQ}{2}-\frac{xe}{2L}D+\tfrac{1}{2}(A_+^0+A_-^0)+\frac{x}{2L}(A_+^0-A_-^0),$$

where we have defined the dipole operator

$$D=\int_{-L}^{L}xj^0(x)dx.$$

At this point one can ask whether there are any conditions which must be placed on the operators $A_\pm^0$ or whether these can be assigned freely. In fact because the commutator of the Hamiltonian with $\psi(x)$ must imply (2.4) and because of the growth of $|x-x'|$ for large values of its argument, one is severely constrained. Thus using

$$T^{00}=-\frac{i}{2}\psi\alpha_1\partial_1\psi+\tfrac{1}{2}(F^{01})^2,$$

$$T^{01}=-\frac{i}{2}\psi\partial_1\psi,$$

one finds that consistency requires

$$A_+^0+A_-^0=0,$$

$$A_+^0-A_-^0=eD.$$

thereby yielding

$$A^0(x)=-\frac{e}{2}\int_{-L}^{L}|x-x'|j^0(x')dx'+\tfrac{1}{2}LeQ$$

and (2.10)

$$F^{01}=\frac{e}{2}\int_{-L}^{L}\epsilon(x-x')j^0(x')dx',$$

which results differ from (2.7) in the limit $L\to\infty$ solely by the additional term $\frac{1}{2}LeQ$ in (2.10). It is important to note that this latter term is not present in Brown's solution because of the modification of T^{00} described in his paper which was alluded to earlier. However, if one insists that the equations of motion (2.2)–(2.4) be satisfied, then the forms given above for $T^{0\mu}$ are unique and lead to a conserved energy-momentum tensor. (We remind the reader that the nonconservation of Q precludes the possibility of adding a Q^2 term to T^{00} as done by Brown.)

With the above result one can now calculate all the Green's functions of the model. This is done simply by noting that (2.10) can be rewritten as

$$A^0(x)=-\frac{e}{2}\int_{-L}^{L}(|x-x'|-L)j^0(x')dx',$$

so that by replacing

$$\mathfrak{D}(k)=\frac{1}{2}\left[\frac{1}{(k+i\epsilon)^2}+\frac{1}{(k-i\epsilon)^2}\right]$$

with $\mathfrak{D}(k)+L\pi\delta(k)$ in Brown's solution (and formally taking the limit $L\to\infty$) one obtains the exact solution of the model. It will now be shown that even with this more precise treatment of the problems associated with the Laplacian in one dimension there remain inconsistencies in the theory because of the stringent requirements imposed by Lorentz invariance.

III. BREAKDOWN OF COVARIANCE

One can now proceed to demonstrate the noncovariance of the model using the results of the preceding section together with certain details of Brown's solution. A convenient framework for such a discussion is the Dirac-Schwinger covariance condition[5]

$$-i[T^{00}(x),T^{00}(x')]=-[T^{01}(x)+T^{01}(x')]\partial_1\delta(x-x'),\tag{3.1}$$

the satisfaction of which is sufficient to guarantee Poincaré invariance. However, since it is not a necessary condition one must show that any additional terms in (3.1) are incompatible with the structure relations of the Lorentz group in order to prove noncovariance.

Before dealing directly with this question it should be observed that the operators

$$P^\mu=\int T^{0\mu}dx$$

correctly generate displacements in time and space. In the case of the spatial momentum operator this is trivially demonstrated. For P^0 one has already used $[P^0,\psi]$ to infer the correct form of A^0 and thus one has only to verify that the commutator of P^0 with F^{01} is consistent with (2.3). One finds by direct calculation

$$\partial_0F^{01}=-i[F^{01},P^0]$$
$$=\frac{e}{2}\int_{-L}^{L}\epsilon(x-x')\partial_0 j^0(x')dx',$$

which becomes upon using local charge conservation

$$\partial_0F^{01}=-ej^1+\frac{e}{2}[j^1(L)+j^1(-L)].\tag{3.2}$$

Although the bracketed term in (3.2) might seem to contradict the field equations one readily verifies by reference to Brown's solution that for $L\to\infty$

$$j^1(L)+j^1(-L)=0\tag{3.3}$$

in all matrix elements. It is crucial to note that $j^1(\pm L)$ are not separately zero as such a result used in conjunction with the equation of continuity

$$\partial_\mu j^\mu=0$$

would lead to the incorrect result that

$$\partial_0 Q=0.$$

It is thus necessary only to examine the two-dimensional Poincaré group to reach a conclusion concerning the consistency of the model.[6] One thus computes the commutator (3.1) and finds additional terms on the right-hand side of the form

$$\frac{e}{2}F^{01}(x')j^1(x)\epsilon(xx')[\delta(L+x)-\delta(L-x)]-(x\leftrightarrow x'),$$

with operator symmetrization being understood. Since the commutator of P^0 with itself must necessarily vanish, a failure of covariance can occur only in $[J^{01},P^0]$. Thus one integrates the energy-density commutator to obtain

$$-i[T^{00}(x),P^0]=-\partial_1T^{01}(x)-ej^1(x)A^0(x)[\delta(L+x)-\delta(L-x)],\tag{3.4}$$

where use has been made of (3.3). If a covariant result is to be obtained it is clearly essential that the term in (3.4) proportional to e vanish upon multiplication by x and integration over that same variable. Since one cannot in fact get rid of the unwanted terms one infers the asserted failure of covariance for the model.[7]

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[5]J. Schwinger, Phys. Rev. 127, 824 (1962).
[6]Since the Coulomb gauge and axial gauge are identical in two-dimensional space-time, it is of interest to compare the results obtained here with remarks made by Schwinger [J. Schwinger, Phys. Rev. 130, 402 (1963)] concerning the axial gauge in four-dimensional space.
[7]Identical conclusions concerning the covariance of the model are obtained from explicit calculation of the Lorentz transformation properties of the fields.