Quarks at the ends of the string

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We develop a technique for attaching quark quantum numbers to world lines joined by relativistic strings. We are able to describe spin-0 and spin-$\frac{1}{2}$ U(n)-symmetric quarks attached to world lines. One spin-$\frac{1}{2}$ theory based on the Dirac equation yields a classical particle with helical motion, interpretable as *Zitterbewegung*. Another spin-$\frac{1}{2}$ model has no helical motion, but yields an algebra resembling that of supersymmetry. Motivated by duality diagrams and some general properties of quark-gluon models, we then construct quark-string models of mesons and baryons. The analysis of the meson model with unequal quark masses implies a stringlike spectrum with broken trajectory intercepts. A simple baryon model suggests a dynamical reason for diquark configurations in the lowest states. Physical weak and electromagnetic currents for the quark-string system follow from a minimal-coupling scheme as in gauge field theories.

I. INTRODUCTION

The quark model has provided a successful framework for understanding many properties of elementary particles and their interactions, such as classification and spectroscopy, deep-inelastic scattering from nucleons, and possibly e^+e^- annihilation into hadrons. The details of the strong interactions between quarks, and consequently between elementary particles, are presently unknown. However, some general features of the interaction such as Regge behavior and approximately linear trajectories have emerged. There are at present two philosophical approaches to the strong interactions, the dual resonance models[1] and the quark-gluon color gauge theories.[2]

The dual resonance model provides a reasonably successful approximation to the required properties of the strong-interaction S matrix even in the "Born approximation." Mandelstam[3] has shown a direct connection between dual models and interacting strings,[4] thus providing an appealing physical picture for the dynamics underlying the dual models.

In the interacting-string picture a meson is represented by a string with two free ends. During interaction, two such strings (mesons) join ends to form a single continuous string with two free ends; this object is interpretable as a resonance which can in turn split (decay) into two or more strings.

As shown in Fig. 1, the spacetime paths of the string ends thus form a realization of the duality diagrams which were used originally[5] to keep track of the quark-model quantum numbers of the intermediate states in dual scattering amplitudes.

This picture strongly suggests that the quark and antiquark inside the meson are bound together by a string. Then we are led to the interpretation that in the interacting-string model the propagation of the string is dictated by the string action of Nambu, while the interaction between strings is a *local interaction between quarks on different strings*. Two end quarks annihilate during the formation of a larger string, and a pair of quarks is created when the string splits. Thus the meson interacts only through its "valence" quarks. We note that in Mandelstam's model the interior of the string does not interact. Although the local quark interaction is sufficient to reproduce the dual model amplitudes, it is not clear *a priori* that the string does not interact also in other ways.

Another popular approach to the strong interactions of elementary particles is the non-Abelian

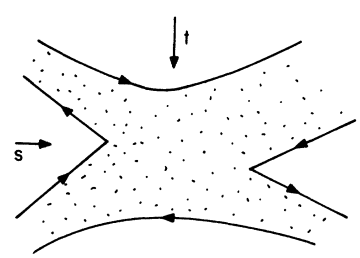

FIG. 1. Duality diagram for the s-t contribution to meson-meson scattering. The dotted area represents the surface swept out by the string.

gauge theory of SU(3)-colored gluons and quarks. Calculations in this framework are still at an elementary stage, but there are several indications in the literature that some form of the string model may emerge from such theories: Nielsen and Olesen[6] have argued that vortexlike classical solutions of field theories may be identified with dual strings, and Wilson[7] and Kogut and Susskind[8] have suggested a form of string in their lattice formalism for gauge theories. Both proposals imply that strings are made of color glue. If this picture is correct, then color singlet mesons can be made only if the string terminates on color "monopoles—that is, the quarks must be attached to the ends of the string. The end points would therefore carry all observable quantum numbers such as spin, charge, isospin, etc., while the body of the string carries none. In addition, 't Hooft[9] has proposed a two-dimensional color gauge-theory model for mesons. The model which will be presented here yields results very close to those of 't Hooft, thus establishing a further connection between the string picture and color gauge theory. Other pictures may also emerge from the color quark-gluon model, but the color model does not seem to be inconsistent with the idea of placing quarks on the ends of strings.

With these motivations in mind, we will propose here models for mesons as strings with quarks at the ends and models for baryons as strings with three quarks. The new feature in our approach is the introduction of quark spin and internal-symmetry degrees of freedom in the string formalism. Our new variables are *not* related to the Neveu-Schwarz or Bardakci-Halpern variables[10] previously introduced in the string formulation, but rather they directly correspond to the usual spin and internal symmetry of the quark fields. Our formulation makes a close connection between the standard phenomenological quark model and the string model.

The physical picture that emerges is appealing: The string action produces a relativistic potential which binds the quarks together. Furthermore, a two-dimensional analysis indicates that the potential energy of the system depends *linearly* on the separation between the quarks; the quarks are thus trapped in a manner reminiscent of the proposals of 't Hooft,[9] Wilson,[7] and Kogut and Susskind.[8]

Our formulation has nontrivial implications that follow from the introduction of quark internal-symmetry variables. First, the internal symmetry is broken by the *unequal masses* of the quarks. This then leads to a spectrum of Regge trajectories with nondegenerate intercepts. These trajectories curve at low energies but are asymptotically essentially linear. The amount of curvature increases with the masses of the quarks determining the quantum numbers of the trajectory. Second, *weak and electromagnetic interactions can be coupled directly to the quarks* following the same prescription as unified gauge field theories. This then leads to the definition of the *physical currents* in the string formalism. Weak and electromagnetic interactions couple only to the quarks, not to the string, just as in the quark-gluon model where the colored gluons do not possess weak and electromagnetic interactions. The string, just like the gluons, is the medium of strong interactions between the quarks. Weak and electromagnetic interactions can be treated perturbatively as in the standard field-theory approach.

We remark that there are two very different ways of regarding our model. On the one hand, the quarks and the color gluons interacting with them could be considered as the fundamental basis for strong interactions. Then our picture would be a phenomenological approximation to the stringlike vortices of, say, Nielsen and Olesen.[6] On the other hand, one might believe that some more sophisticated version of the interacting-string model will give the exact solution to the strong-interaction problem. If this were the case, then the known connection between the zero-slope limit of dual models and non-Abelian gauge theories would suggest a different viewpoint: Vector-gluon field theories of strong binding would be phenomenological approximations to the richer structure of a stringlike theory. From this second point of view our model is an attempt to incorporate the quark quantum numbers into the picture.

We should point out that the observable quantum numbers of the quarks could conceivably arise from topological properties of more complex geometrical models, e.g., "membrane" or "jelly" models. For example, the left-twisted and right-twisted Möbius strips could be associated with two different values of a quantum number. Such connections between topology and internal quantum numbers also occur in field theory.[11] Thus our model with quarks on the ends of the string could be an *approximation* to a theory based on a geometrical structure more complicated than the string. In such a theory purely topological quark quantum numbers might appear in some limit to be joined by an extended stringlike structure.

In the end, it might even happen that a perfect quark-gluon model and a perfect geometrical model were "dual" to one another in the sense that both would give equivalent descriptions of physical processes.

The present paper deals mainly with the basic principles of our general formalism. We will discuss various simple examples to develop intuition, but will leave for later work a number of difficult problems presented by the most realistic models. We begin in Sec. II by discussing a new approach[12] to the incorporation of field-theoretic degrees of freedom into point particles lying on a world line. We develop models for spin-0 and spin-$\frac{1}{2}$ particles carrying internal symmetry. These then form the basis of our technique for attaching point quarks to the string. In Sec. III we summarize what is known about the relativistic string with massive ends,[13] since some cases of our model reduce effectively to this one. Much of our intuition is based upon our knowledge of the string with massive ends. Section IV deals with our essential problem—that of building mesons by replacing the ends of the string with massive quarklike point particles of the type discussed in Sec. II. We also suggest a model of baryons with three quarks. In Sec. V, we generalize the field-theoretical minimal-coupling principle to couple external electromagnetic and non-Abelian gauge fields to our point quarks. We are then able to define the physical currents of our model in a natural way. Suggestions for future investigations and a summary of the current work are contained in the final section. An appendix is devoted to a general technique for restricting fields to a subspace of the physical spacetime.

II. POINT PARTICLES WITH INTERNAL SYMMETRY AND SPIN

In order to describe quarks as point particles following world lines attached to the ends of the string, we must find a way of attaching spin and internal-symmetry indices to a world line. Since it is clear how to describe conventional fields possessing these extra indices, we will accomplish our goal by starting with conventional fields and restricting them to a world line. We begin for simplicity with a free spinless U(n)-symmetric quark. Next, we treat the more realistic case of a spin-$\frac{1}{2}$ U(n)-symmetric quark. Here two models are considered: the first, following directly from the Dirac equation, possesses a classical *Zitterbewegung* while the other does not. The quantum theory of the second model leads to canonical qunatization rules reminiscent of supersymmetry.

A. Spinless particle with U(n) symmetry

The standard τ-reparametrization invariant action for a free, spinless, relativistic point particle is

$$S=-\int_{\tau_1}^{\tau_2} d\tau\,\mu(-x_\tau^{\,2})^{1/2}, \tag{2.1}$$

where μ is the mass of the particle and $x_\tau^\mu=\partial x^\mu/\partial\tau$. Our metric is such that $x^2=-x_0^{\,2}+\vec{x}^2$. The canonical momentum is

$$p^\mu=\mu x_\tau^\mu/(-x_\tau^{\,2})^{1/2}$$

and obeys the constraint $p^2+\mu^2=0$. Minimizing the action, one finds that p^μ is a constant of motion. We may thus solve the equations of motion for x^μ in the form

$$x^\mu=q^\mu+p^\mu s(\tau)/(-p^2)^{1/2}. \tag{2.2}$$

Here the τ-reparametrization invariant function $s(\tau)$ may depend also on the integration constants q^μ and p^μ. Choosing a gauge, for example, the proper-time gauge $x^0=p^0\tau/(-p^2)^{1/2}$, fixes the form of $s(\tau)$ and q^μ. In general, we may write

$$s(\tau)=\int^\tau dt(-x_t^{\,2})^{1/2}. \tag{2.3}$$

We thus have a correct classical description of the motion of the particle, but are able to say nothing about its internal symmetries. In order to describe a particle which carries internal-symmetry indices, it is clear that we need more variables in addition to the position $x^\mu(\tau)$. We begin by introducing functions $\phi_\alpha(\tau)$, $\alpha=1,\ldots,n$, which form a basis for the spinor representation of U(n). As described in the Introduction this U(n) symmetry refers only to observable symmetries of the quark at the end of the string rather than hidden color symmetry. The ϕ_α are also taken to be scalars under Lorentz transformations and τ reparametrizations. The simplest action for a massive point particle which is invariant under U(n) symmetry, Poincaré transformations, and τ reparametrizations is

$$S=\int_{\tau_1}^{\tau_2} d\tau\, L_0(\phi_\alpha(\tau),\partial_\tau\phi_\alpha(\tau)),$$

where the point-particle Lagrangian is

$$L_0=\sum_\alpha\left(\frac{\partial_\tau\phi_\alpha^\dagger\partial_\tau\phi_\alpha}{(-x_\tau^{\,2})^{1/2}}-m^2(-x_\tau^{\,2})^{1/2}\phi_\alpha^\dagger\phi_\alpha\right). \tag{2.4}$$

Hereafter, sums over α will be implicit.

This Lagrangian is closely related to the standard field-theoretic description of a free spinless particle with internal symmetry.[14] To see this, consider the spacetime Lagrangian density for a free U(n) Klein-Gordon particle:

$$\mathcal{L}=-\partial_\mu\phi^\dagger(x)\partial^\mu\phi(x)-m^2\phi^\dagger(x)\phi(x). \tag{2.5}$$

To restrict the field to a world line, we require that x^μ be replaced by $x^\mu(\tau)$, where τ parametrizes the world line. The Cartesian Minkowski metric $\eta^{\mu\nu}=\text{diag}(-1,1,1,1)$ can be decomposed at any point on the world line into a complete set of vectors consisting of the timelike tangent to the world-line, $x_\tau^\mu = \partial x^\mu/\partial\tau$, and all of the spacelike normals $n_i^\mu(\tau)$. Thus we have

$$\eta^{\mu\nu} = \frac{x_\tau^\mu x_\tau^\nu}{x_\tau^2} + n_i^\mu n_i^\nu , \tag{2.6a}$$

where a sum over the index i is implied. Note that

$$\begin{aligned} x_\tau^\mu \eta_{\mu\nu} x_\tau^\nu &= x_\tau^2 < 0, \\ n_i^\mu \eta_{\mu\nu} x_\tau^\nu &= 0, \\ n_i^\mu \eta_{\mu\nu} n_j^\nu &= \delta_{ij} . \end{aligned} \tag{2.6b}$$

The metric $\eta^{\mu\nu}$ is τ-reparametrization invariant and raises or lowers indices in Minkowski space as usual. Using Eq. (2.6a) we can write

$$\eta^{\mu\nu}\partial_\mu \phi^\dagger(x)\partial_\nu\phi(x) = \frac{x_\tau^\mu \partial_\mu \phi^\dagger x_\tau^\nu \partial_\nu\phi}{x_\tau^2} + n_i^\mu \partial_\mu\phi^\dagger n_i^\nu \partial_\nu\phi. \tag{2.7}$$

If the field $\phi(x)$ is not to leave the world line, we cannot allow any nonvanishing normal derivatives. Thus we take

$$n_i^\mu \partial_\mu \phi(x) = 0. \tag{2.8}$$

Furthermore, we note that by the chain rule of differentiation

$$x_\tau^\mu \partial_\mu \phi(x(\tau)) = \partial_\tau \phi(\tau). \tag{2.9}$$

Therefore, we may now consider ϕ to be effectively a function of τ, and substitute Eqs. (2.7), (2.8), and (2.9) into Eq. (2.5). Multiplying by $(-x_\tau^2)^{1/2}$ (which effectively is a Jacobian), we obtain Eq. (2.4). The general technique for restricting an arbitrary field theory to a subspace of arbitrary dimension such as a world sheet instead of a world line is discussed in the Appendix.

The equations of motion are obtained by varying the action in the standard way with respect to both $\phi_\alpha(\tau)$ and $x^\mu(\tau)$. We find the canonical momenta

$$\frac{\partial L_0}{\partial(\partial_\tau \phi_\alpha^\dagger)} = \frac{\partial_\tau\phi_\alpha}{(-x_\tau^2)^{1/2}} \equiv \Pi_\alpha, \tag{2.10}$$

$$\frac{\partial L_0}{\partial(\partial_\tau x_\mu)} = (\Pi^\dagger\Pi + m^2\phi^\dagger\phi)\frac{x_\tau^\mu}{(-x_\tau^2)^{1/2}} \equiv p^\mu, \tag{2.11}$$

and the Euler equations

$$\partial_\tau \Pi_\alpha + (-x_\tau^2)^{1/2} m^2 \phi_\alpha = 0, \tag{2.12}$$

$$\partial_\tau p^\mu = 0. \tag{2.13}$$

The constants of motion of this system are the total momentum p^μ, the Lorentz-transformation generators

$$M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau),$$

and (2.14)

$$\mu = \Pi^\dagger(\tau)\Pi(\tau) + m^2\phi^\dagger(\tau)\phi(\tau).$$

We also identify the following constraint, which results from τ-reparametrization invariance:

$$p^2 + \mu^2 = 0. \tag{2.15}$$

For arbitrary $x^\mu(\tau)$, we can solve Eq. (2.12) for $\phi_\alpha(x(\tau))$ using the parameter $s(\tau)$ defined by Eq. (2.3). The result is

$$(2m)^{1/2}\phi_\alpha(\tau) = a_\alpha e^{-ims} + b_\alpha^* e^{ims}, \tag{2.16}$$

where a_α, b_α^* are dimensionless complex constants. Replacing (2.16) in (2.14), we find that the constraint takes the form

$$\mu = (-p^2)^{1/2} = m(a_\alpha^* a_\alpha + b_\alpha^* b_\alpha). \tag{2.17}$$

Furthermore, using (2.11) and (2.13) we may solve for $x^\mu(\tau)$:

$$x^\mu(\tau) = q^\mu + p^\mu s(\tau)/(-p^2)^{1/2}. \tag{2.18}$$

This is the same as Eq. (2.2).

Finally, we note that if the U(n) symmetry is *broken* by assigning different masses to the components of ϕ_α,

$$m = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \end{bmatrix} . \tag{2.19}$$

Equation (2.17) becomes

$$\mu = (-p^2)^{1/2} = a^* m a + b^* m b. \tag{2.20}$$

It is straightforward to quantize the theory in a given gauge such as $x^0 = \tau p^0/(-p^2)^{1/2}$ by assuming standard commutation rules for $\vec{p}$ and $\vec{q}$ and taking

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta} = [b_\alpha, b_\beta^\dagger] \tag{2.21}$$

and all other commutators zero. This is then consistent with the canonical commutation rules $[\phi_\alpha, \Pi_\beta^\dagger] = i\delta_{\alpha\beta}$ etc.

The *interpretation* of the solution (2.18) and (2.20) is that the center of mass $\vec{x}$ moves like a free particle, while the effective mass of the system, $(-p^2)^{1/2}$, is equal to the sum of the *number of quanta at the point* $\vec{x}$ times the appropriate free mass m_i. The more quanta we put at the point $\vec{x}$, the heavier the system becomes. The effective mass of the system also depends on the *kind* of quanta we put in if the U(n) symmetry is broken as in Eq. (2.19). This is a satisfactory description of free scalar particles with (broken) internal symmetry.

Although we have restricted ourselves to a U(n) multiplet for the purposes of illustration, it is clear that the treatment can be extended to any representation of any internal-symmetry group. It is also clear that our approach could be generalized to interacting theories such as ϕ^4 etc., which would change the solution (2.16) as well as the spectrum of (2.20). We will not attempt to treat these matters here.

B. Dirac particles with internal symmetry

Since it is believed that free quarks would obey the Dirac equation, we now proceed to derive the Lagrangian for a Dirac particle restricted to a world line using the methods of the previous subsection. We begin with the spacetime Lagrangian density for U(n) quarks,

$$\mathcal{L}=\sum_{\alpha,\beta}\left[-\tfrac{1}{2}\bar{\psi}_\alpha(x)\overleftrightarrow{\partial\!\!\!/}\psi_\alpha(x)-\bar{\psi}_\alpha(x)m_{\alpha\beta}\psi_\beta(x)\right]. \tag{2.22}$$

Hereafter sums over the indices α will be implicit. Recall that the indices α refer only to observable symmetries, not to color. Our γ-matrix conventions are, e.g., those of Weinberg.[15]

We now restrict $x^\mu(\tau)$ to a world line parametrized by τ and forbid $\psi(x(\tau))$ to leave the world line by imposing the condition

$$n_i^\mu(\tau)\partial_\mu\psi(x(\tau))=0. \tag{2.23}$$

Then when we replace the metric $\eta^{\mu\nu}$ in Eq. (2.22) by the expression (2.6a), we find the following Lagrangian for a classical pointlike Dirac particle:

$$L_0=\frac{x_\tau^\mu}{2(-x_\tau^2)^{1/2}}\bar{\psi}(\tau)\gamma_\mu\overleftrightarrow{\partial}_\tau\psi(\tau)-(-x_\tau^2)^{1/2}\bar{\psi}(\tau)m\psi(\tau). \tag{2.24}$$

The canonical momenta may now be identified as

$$\begin{aligned}p^\mu&=\frac{x_\tau^\mu}{(-x_\tau^2)^{1/2}}\left(-\frac{x_\tau^\nu}{2x_\tau^2}\bar{\psi}\gamma_\nu\overleftrightarrow{\partial}_\tau\psi+\bar{\psi}m\psi\right)+\frac{1}{2(-x_\tau^2)^{1/2}}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}_\tau\psi,\\ \chi&=\frac{\partial L_0}{\partial(\partial_\tau\bar{\psi})}=-\frac{x\!\!\!/_\tau}{2(-x_\tau^2)^{1/2}}\psi;\\ \bar{\chi}&=\frac{\partial L_0}{\partial(\partial_\tau\psi)}=\bar{\psi}\frac{x\!\!\!/_\tau}{2(-x_\tau^2)^{1/2}}.\end{aligned} \tag{2.25}$$

To simplify our expressions, we now define

$$\begin{aligned}\mathring{x}^\mu&=\frac{dx^\mu}{ds}=\frac{x_\tau^\mu}{(-x_\tau^2)^{1/2}}\quad(\mathring{x}^2=-1),\\ \mathring{\psi}&=\frac{d\psi}{ds}=\frac{\psi_\tau}{(-x_\tau^2)^{1/2}},\end{aligned} \tag{2.26}$$

where $s(\tau)$ is the parameter (2.3) defined earlier. The Euler equations then become

$$\begin{aligned}&\mathring{p}^\mu=0,\\ &\mathring{\psi}-(\tfrac{1}{2}\mathring{x\!\!\!/}\mathring{\mathring{x\!\!\!/}}-m\mathring{x\!\!\!/})\psi=0,\\ &\mathring{\bar{\psi}}+\bar{\psi}(\tfrac{1}{2}\mathring{x\!\!\!/}\mathring{\mathring{x\!\!\!/}}-m\mathring{x\!\!\!/})=0.\end{aligned} \tag{2.27}$$

We may thus derive a number of constants of motion, including the total momentum p^μ and

$$\begin{aligned}&\bar{\psi}\lambda_\alpha\psi,\\ &\bar{\psi}\lambda_\alpha\mathring{x\!\!\!/}\psi,\end{aligned} \tag{2.28}$$

where λ_α is any U(n) matrix which commutes with the mass matrix,

$$[\lambda_\alpha,m]=0.$$

It is convenient to define the constant mass parameter

$$\mu=\bar{\psi}m\psi=-p\cdot\mathring{x}. \tag{2.29}$$

Integrating this equation we find

$$p\cdot x=-\mu s+d,$$

where d is a constant.

The generators of the Lorentz group are also constants of the motion. To derive an expression for them, we note that x^μ, ψ, and $\bar{\psi}$ transform under Lorentz transformations as

$$\begin{aligned}\delta x^\mu&=\omega^{\mu\nu}x_\nu,\\ \delta\psi_a&=\frac{i}{4}\omega_{\mu\nu}\sigma_{ab}^{\mu\nu}\psi_b,\\ \delta\bar{\psi}_a&=-\frac{i}{4}\omega_{\mu\nu}\bar{\psi}_b\sigma_{ba}^{\mu\nu},\end{aligned}$$

where

$$\sigma^{\mu\nu}=\frac{1}{2i}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu). \tag{2.30}$$

Noether's theorem then implies that

$$M^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu+S^{\mu\nu} \tag{2.31}$$

is the constant generator of Lorentz transformations and the spin matrix is defined by

$$S^{\mu\nu}=\frac{i}{2}\bar{\psi}\sigma^{\mu\nu}\chi-\frac{i}{2}\bar{\chi}\sigma^{\mu\nu}\psi$$
$$=\frac{1}{4i}\bar{\psi}(\sigma^{\mu\nu}\dot{\not x}+\dot{\not x}\sigma^{\mu\nu})\psi. \quad (2.32)$$

We note the total momentum can be written as

$$p^\mu=\mu\dot{x}^\mu+S^{\mu\nu}\ddot{x}_\nu. \quad (2.33)$$

From Eq. (2.32), we see that

$$S^{\mu\nu}\dot{x}_\nu=0, \quad (2.34)$$

where we have used the fact that

$$\sigma^{\mu\nu}\gamma^\lambda+\gamma^\lambda\sigma^{\mu\nu}=\sigma^{\lambda\mu}\gamma^\nu+\gamma^\nu\sigma^{\lambda\mu}$$
$$=\sigma^{\nu\lambda}\gamma^\mu+\gamma^\mu\sigma^{\nu\lambda}.$$

Equations (2.31), (2.33), and (2.34) are recognizable as the basis of Frenkel's theory of spinning relativistic particles.[16] However, our theory differs substantially from that of Frenkel because our μ is a dynamical variable defined by (2.29). We also have additional equations of motion (2.27) which determine ψ, and hence the properties of μ. The variables $\psi_\alpha(\tau)$ are absent in Frenkel's theory.[17]

We also find the following relation among our variables:

$$\tfrac{1}{2}p^2S^{\mu\nu}S_{\mu\nu}+p_\mu S^{\mu\nu}S_{\nu\lambda}p^\lambda=-\tfrac{1}{2}(S^{\mu\nu}S_{\mu\nu})(\bar{\psi}m\psi)^2. \quad (2.35)$$

Defining the Pauli-Lubanski vector

$$W^\mu=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}M_{\nu\lambda}p_\sigma\equiv M^{*\mu\nu}p_\nu, \quad (2.36)$$

we find that Eq. (2.35) may also be written

$$W^2=\tfrac{1}{2}(S^{\mu\nu}S_{\mu\nu})(\bar{\psi}m\psi)^2. \quad (2.37)$$

We next find it convenient to separate $x^\mu(s)$ into two parts,

$$x^\mu(s)=q^\mu(s)+r^\mu(s), \quad (2.38)$$

where

$$r^\mu(s)=-S^{\mu\nu}p_\nu/p^2, \quad (2.39)$$

so that

$$p\cdot q=p\cdot x,\quad p\cdot r=0. \quad (2.40)$$

Examining $M^{\mu\nu}$, we find

$$q^\mu=(M^{\mu\nu}p_\nu+p^\mu p\cdot x)/p^2, \quad (2.41a)$$
$$\dot{q}^\mu=p^\mu p\cdot\dot{x}/p^2=-\mu p^\mu/p^2. \quad (2.41b)$$

Now we define

$$R^{\mu\nu}=M^{\mu\nu}-q^\mu p^\nu+q^\nu p^\mu$$
$$=r^\mu p^\nu-r^\nu p^\mu+S^{\mu\nu}, \quad (2.42)$$

where

$$R^{\mu\nu}p_\nu=0 \quad (2.43)$$

and

$$R^{\mu\nu}R_{\mu\nu}=-2W^2/p^2=S^{\mu\nu}S_{\mu\nu}-2r^2p^2. \quad (2.44)$$

An analysis of the Poisson brackets of the elements of $R^{\mu\nu}$ indicates that they generate a little group which leaves the momentum p^μ invariant. Thus the constraint equation (2.44) relates the ratio of the two Casimir operators of the Poincaré group W^2 and p^2 to the Casimir operator $(R^{\mu\nu})^2$ of the little group. Using Eqs. (2.34) and (2.39), we find

$$0=p_\mu S^{\mu\nu}\dot{x}_\nu/p^2=r\cdot\dot{x}. \quad (2.45)$$

We may now find an equation of motion for r^μ by examining

$$0=S^{\mu\nu}\dot{x}_\nu=(R^{\mu\nu}-r^\mu p^\nu+r^\nu p^\mu)\dot{x}_\nu$$
$$=R^{\mu\nu}\dot{x}_\nu+\mu r^\mu, \quad (2.46)$$

where we have used Eqs. (2.29) and (2.45). Since Eqs. (2.41b) and (2.43) imply $R^{\mu\nu}\dot{q}_\nu=0$, we finally obtain

$$R^{\mu\nu}\dot{r}_\nu+\mu r^\mu=0. \quad (2.47)$$

Examining the expression

$$S^{\mu\nu}S_{\nu\lambda}p^\lambda=S^{\mu\nu}S_{\nu\lambda}S^{\lambda\sigma}\ddot{x}_\sigma$$
$$=-\tfrac{1}{2}(S^{\alpha\beta}S_{\alpha\beta})S^{\mu\nu}\ddot{x}_\nu$$
$$=-\tfrac{1}{2}(S^{\alpha\beta}S_{\alpha\beta})(p^\mu-\mu\dot{x}^\mu) \quad (2.48)$$

which follows from Eqs. (2.33) and (2.34), we may use Eq. (2.39) to express $\dot{x}^\mu$ as

$$\mu\dot{x}^\mu=p^\mu-S^{\mu\nu}r_\nu p^2/(\tfrac{1}{2}S^{\alpha\beta}S_{\alpha\beta}). \quad (2.49)$$

Since $p\cdot\dot{x}=-\mu$ is a constant, Eq. (2.41b) implies $\ddot{q}^\mu=0$. Thus if we differentiate Eq. (2.49), we find

$$\mu\ddot{x}^\mu=\mu\ddot{r}^\mu=-S^{\mu\nu}\dot{r}_\nu p^2/(\tfrac{1}{2}S^{\alpha\beta}S_{\alpha\beta}),$$

where we used (2.40), (2.42), and (2.45). But Eq. (2.47) indicates $r\cdot\dot{r}=0$ and Eq. (2.40) makes $p\cdot\dot{r}=0$. Thus from Eq. (2.42),

$$S^{\mu\nu}\dot{r}_\nu=R^{\mu\nu}\dot{r}_\nu.$$

This information combined with Eq. (2.47) allows us to write

$$\ddot{r}^\mu+\gamma^2r^\mu=0, \quad (2.50)$$

where

$$\gamma^2=-2p^2/(S^{\alpha\beta}S_{\alpha\beta})$$
$$=-p^2\mu^2/W^2$$
$$=2\mu^2/(R^{\alpha\beta}R_{\alpha\beta}). \quad (2.51)$$

The variable $r^\mu(s)$ thus executes *harmonic motion* with angular frequency γ, so

$$r^\mu=(a^\mu e^{i\gamma s}+a^{*\mu}e^{-i\gamma s})/(-2p^2)^{1/2}. \quad (2.52)$$

From the definition (2.36) for W^μ and the expression (2.34) for r^μ, we see that

$$\begin{aligned} p\cdot r &= 0, \\ W\cdot r &= -p_\mu S^{*\mu\nu} S_{\nu\lambda} p^\lambda / p^2 \\ &= -\tfrac{1}{4} S^{\mu\nu} S^*_{\mu\nu} \\ &= 0, \end{aligned} \tag{2.53}$$

where $S^{\mu\nu}S^*_{\mu\nu}=0$ follows from $S^{\mu\nu}\dot{x}_\nu=0$. Thus r^μ moves in a plane perpendicular to both p^μ and W^μ. From Eqs. (2.47) and (2.53), we see that a^μ and W^μ are eigenvectors of $R^{\mu\nu}$:

$$\begin{aligned} iR^{\mu\nu}a_\nu + a^\mu(\tfrac{1}{2}R^{\alpha\beta}R_{\alpha\beta})^{1/2} &= 0, \\ iR^{\mu\nu}W_\nu &= 0. \end{aligned} \tag{2.54}$$

Furthermore,

$$a^2=0, \quad p\cdot a=0, \quad W\cdot a=0. \tag{2.55}$$

Here a^μ, $a^{*\mu}$, and W^μ are analogous to the $m=+1$, $m=-1$, and $m=0$ components of a spin-one vector. p^μ is invariant under rotations by $R^{\mu\nu}$, so $R^{\mu\nu}$ generates a little group of p^μ. This becomes clear from an analysis of the Poisson brackets of $R^{\mu\nu}$. Thus we may finally write

$$\begin{aligned} x^\mu(s) &= [M^{\mu\nu}p_\nu + p^\mu(d-\mu s)]/p^2 \\ &\quad + (a^\mu e^{i\gamma s} + a^{*\mu}e^{-i\gamma s})/(-2p^2)^{1/2}, \end{aligned} \tag{2.56}$$

so $x^\mu(s)$ consists of periodic circular motion in a plane superimposed upon a pure translation. The over-all helical motion is identifiable as the effective classical *Zitterbewegung* resulting from the quantum-mechanical interference of positive- and negative-frequency components in the Dirac equation. The *Zitterbewegung* is a familiar consequence of attempting to localize a Dirac particle, as may be seen explicitly from an appropriate wave-packet construction.[18]

Now we turn to the solution of the equations of motion (2.27) for ψ. Equation (2.54) can be used to show that

$$(e^{\theta R})^{\mu\nu}a_\nu = a^\mu \exp[i\theta(\tfrac{1}{2}R^{\alpha\beta}R_{\alpha\beta})^{1/2}]. \tag{2.57}$$

Next we define the Lorentz transformation in the space of Dirac matrices as

$$U(\theta) = \exp[\tfrac{1}{4}\theta\sigma_{\mu\nu}R^{\mu\nu}/(\tfrac{1}{2}R^{\alpha\beta}R_{\alpha\beta})^{1/2}]. \tag{2.58}$$

Thus

$$\begin{aligned} U(\theta)\not{a}U^\dagger(\theta) &= e^{i\theta}\not{a}, \\ U(\theta)\not{p}U^\dagger(\theta) &= \not{p}. \end{aligned} \tag{2.59}$$

We can therefore write

$$\begin{aligned} &-\tfrac{1}{2}\not{x}(s)\ddot{\not{x}}(s) + m\dot{\not{x}}(s) \\ &\quad = U(\gamma s)[-\tfrac{1}{2}\not{x}(0)\ddot{\not{x}}(0) + m\dot{\not{x}}(0)]U^\dagger(\gamma s). \end{aligned} \tag{2.60}$$

The equation of motion for ψ can now be written

$$\dot{\tilde{\psi}} + M\tilde{\psi} = 0,$$

where the s-independent matrix M is given by

$$M = \tfrac{1}{4}\sigma^{\mu\nu}R_{\mu\nu} + [-\tfrac{1}{2}\not{x}(0)\ddot{\not{x}}(0) + m\dot{\not{x}}(0)]$$

and (2.61)

$$\tilde{\psi}(s) = U^\dagger(\gamma s)\psi(s).$$

This equation can now be solved directly by quadratures to give

$$\psi(s) = U(\gamma s)e^{-isM}\lambda, \tag{2.62}$$

where λ is an s-independent spinor.

The solution of our classical spin-$\frac{1}{2}$ Dirac particle problem is now complete. The quantum theory, however, is nontrivial and will be deferred to a later investigation.

C. Supersymmetric spin-$\frac{1}{2}$ particle without *Zitterbewegung*

The spin-$\frac{1}{2}$ Dirac particle discussed in the previous section possessed a classical *Zitterbewegung* with the result that the particle's velocity did not vanish in the frame where $\vec{p}=0$, as seen from Eqs. (2.49) and (2.50). One might therefore ask if there exists a spin-$\frac{1}{2}$ particle Lagrangian which gives the particle's momentum *proportional* to its velocity; such a particle would correspond more closely to the traditional picture of a positive-energy classical spinning particle.

A sufficient condition to make a particle's momentum and velocity proportional is the constraint

$$S^{\mu\nu}p_\nu = 0, \tag{2.63}$$

where $S^{\mu\nu}$ is the spin part of the Lorentz-group generator.[19] This condition guarantees that the particle's spin degrees of freedom consist only of *spatial* rotations in the rest frame. That p^μ is parallel to x^μ can be seen directly from Eq. (2.63) by writing

$$\begin{aligned} M^{\mu\nu}p_\nu &= (x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu})p_\nu \\ &= x^\mu p^2 - p^\mu x\cdot p, \end{aligned}$$

and taking a derivative to yield

$$p^\mu = \dot{x}^\mu(p^2/p\cdot\dot{x}).$$

One way to ensure that $S^{\mu\nu}p_\nu=0$ is to search for a Lagrangian implying that

$$S^{\mu\nu} = \frac{1}{4i}\bar{\psi}\left(\sigma^{\mu\nu}\frac{\not{p}}{m} + \frac{\not{p}}{m}\sigma^{\mu\nu}\right)\psi \tag{2.64}$$

instead of Eq. (2.32). This can be achieved if our canonical momenta conjugate to $\bar{\psi}$ and ψ take the form

$$\chi = -\frac{\not{p}}{2m}\psi, \quad \bar{\chi} = \bar{\psi}\frac{\not{p}}{2m} \tag{2.65}$$

instead of Eq. (2.25). We have constructed a Lagrangian with all the required invariance properties; it is given by the expression

$$L_0 = -\tfrac{1}{2}[(\bar{\psi}\gamma^\mu m\psi)^2]^{1/2}\left[-\left(x^\mu_\tau - \bar{\psi}\frac{1}{2m}\overleftrightarrow{\partial}_\tau\gamma^\mu\psi\right)^2\right]^{1/2}$$
$$-\tfrac{1}{2}(i\bar{\psi}\gamma_\mu m\psi)\left(x^\mu_\tau - \bar{\psi}\frac{1}{2m}\overleftrightarrow{\partial}_\tau\gamma^\mu\psi\right). \tag{2.66}$$

The canonical momentum conjugate to x^μ is

$$p^\mu = +\tfrac{1}{2}\frac{(-V^2)^{1/2}}{(-T^2)^{1/2}}T^\mu - \tfrac{1}{2}V^\mu, \tag{2.67}$$

where

$$V^\mu = i\bar{\psi}\gamma^\mu m\psi,$$
$$T^\mu = x^\mu_\tau - \bar{\psi}\frac{1}{2m}\overleftrightarrow{\partial}_\tau\gamma^\mu\psi. \tag{2.68}$$

The equations of motion are

$$\partial_\tau p^\mu = 0,$$
$$i\partial_\tau\psi - m^2\frac{(-T^2)^{1/2}}{(-V^2)^{1/2}}\psi = 0, \tag{2.69}$$
$$-i\partial_\tau\bar{\psi} - m^2\frac{(-T^2)^{1/2}}{(-V^2)^{1/2}}\bar{\psi} = 0.$$

We see that

$$\mu^2 \equiv i(\bar{\chi}m^2\psi + \bar{\psi}m^2\chi) = i\bar{\psi}\not{p}m\psi \tag{2.70}$$

is a constant of the motion.

The constraint associated with τ reparametrization can now be written

$$p^2 + \mu^2 = 0. \tag{2.71}$$

After some calculations using the equations of motion and the constraints, we find first that

$$T^\mu + x^\mu_\tau - V^\mu(-T^2)^{1/2}/(-V^2)^{1/2} = 0, \tag{2.72}$$

so that Eq. (2.67) may be written

$$p^\mu = x^\mu_\tau(-V^2)^{1/2}/2(-T^2)^{1/2}.$$

Squaring this equation, we finally obtain

$$\tfrac{1}{2}(-V^2)^{1/2}/(-T^2)^{1/2} = (-p^2)^{1/2}/(-x_\tau{}^2)^{1/2}$$
$$= \mu/(-x_\tau{}^2)^{1/2},$$

so

$$p^\mu = \mu x^\mu_\tau/(-x_\tau{}^2)^{1/2}. \tag{2.73}$$

The equations for ψ can now also be simplified:

$$i\partial_\tau\psi - \frac{m^2(-x_\tau{}^2)^{1/2}}{2\mu}\psi = 0,$$
$$-i\partial_\tau\bar{\psi} - (-x_\tau{}^2)^{1/2}\bar{\psi}\frac{m^2}{2\mu} = 0 \tag{2.74}$$

We now find the explicit solutions for $\psi(s)$ in the form

$$\psi(s) = \exp\left(-\frac{im^2}{2\mu}s\right)\lambda, \tag{2.75}$$

where s is the parameter defined by Eq. (2.3), and λ is a constant spinor. The coordinate $x^\mu(s)$ consists of a pure translation,

$$x^\mu = q^\mu + \frac{p^\mu}{(-p^2)^{1/2}}s. \tag{2.76}$$

The quantum theory of this system is straightforward. We find in the $x^0 = p^0\tau/\mu$ gauge that the canonical commutation rules are satisfied provided

$$[q^i, p^j] = i\delta^{ij}, \tag{2.77a}$$

$$\{\lambda^\alpha_a, \bar{\lambda}^\beta_b\} = i\frac{2m_{\alpha\beta}\not{p}_{ab}}{p^2}. \tag{2.77b}$$

Furthermore, p^μ commutes with λ^α_a and $\bar{\lambda}^\beta_b$, but q^i does not. However, both x^i and p^i commute with ψ and χ in accordance with the canonical commutation rules.

We note the similarity of Eq. (2.77b) to the supersymmetry commutation relations.[20] This is why we call ψ the "supersymmetric" quark. Here, however, the λ^α_a are simply canonical variables. We remark that the $\not{p}$ on the right-hand side of Eq. (2.77b) is essential to obtain only *positive-norm* states. This is best seen in the rest frame ($\vec{p} = 0$), where we calculate the norm to be

$$\langle 0|\lambda^\alpha_a\lambda^{\beta\dagger}_b|0\rangle = +\frac{2m_{\alpha\beta}}{p^0 p_0}p_0(\gamma^0\gamma_0)_{ab}$$
$$= \frac{2m_{\alpha\beta}}{p^0}\delta_{ab} > 0.$$

(Remember that $\bar{\psi} = \psi^\dagger\gamma_4$, $\gamma_4 = i\gamma^0$.) A similar ghost-eliminating factor was previously discussed in connection with Chan-Paton-type spin factors in the context of dual models.[21]

III. REVIEW OF STRING WITH POINT MASSES

Before proceeding to attach quark fields to world lines connected by strings, we review the closely related problem of attaching structureless point masses.[13] For simple quarks, like those in Sec. II A and II C, the dynamics of the two systems are nearly equivalent.[22] Some of the results of this section will therefore be directly applicable to our quark models for mesons and baryons to be given in Sec. IV. In subsections A 1 and A 2 we study mesonlike cases and in Sec. III B we analyze a baryonlike system. The structureless point masses which appear in this section correspond to those of the quarks attached to the ends of the string, as will be demonstrated in the next section.

A. Two-mass case (mesons)

We first consider the action for two masses, at the points $x^\mu(0) \equiv x^\mu(\tau, \sigma=0)$ and $x^\mu(\pi) \equiv x^\mu(\tau, \sigma=\pi)$, joined by a relativistic string,

$$S = \int_{\tau_1}^{\tau_2} d\tau \left\{ -\mu_0[-x_\tau^2(0)]^{1/2} - \mu_\pi[-x_\tau^2(\pi)]^{1/2} - \gamma \int_0^\pi d\sigma(-g)^{1/2} \right\}. \quad (3.1)$$

Here

$$x_\sigma^\mu = \partial x^\mu(\sigma, \tau)/\partial\sigma,$$

$$x_\tau^\mu = \partial x^\mu(\sigma, \tau)/\partial\tau,$$

and

$$-\gamma(-g)^{1/2} \equiv -\gamma[(x_\tau \cdot x_\sigma)^2 - x_\tau^2 x_\sigma^2]^{1/2}$$

is the Nambu Lagrangian density for a free relativistic string. The string contribution to the action is effectively a relativistic potential exerting a force on the two mass points. This will become clearer below.

The action (3.1) has been extensively analyzed in Ref. 13, so we will give here only an outline of the main results. The equations of motion are

$$\partial_\tau K^\mu(\sigma, \tau) + \partial_\sigma N^\mu(\sigma, \tau) = 0, \quad 0 < \sigma < \pi \quad (3.2a)$$

$$\left.\begin{aligned} &\partial_\tau p^\mu(0) - \gamma N^\mu(\sigma=0, \tau) = 0, \quad \sigma = 0 \\ &\partial_\tau p^\mu(\pi) + \gamma N^\mu(\sigma=\pi, \tau) = 0, \quad \sigma = \pi \end{aligned}\right. \quad (3.2b)$$

where

$$\left.\begin{aligned} K^\mu &= (x_\tau^\mu x_\sigma^2 - x_\sigma^\mu x_\sigma \cdot x_\tau)/(-g)^{1/2}, \\ N^\mu &= (x_\sigma^\mu x_\tau^2 - x_\tau^\mu x_\sigma \cdot x_\tau)/(-g)^{1/2}, \end{aligned}\right. \quad (3.2c)$$

$$\left.\begin{aligned} p^\mu(0) &= \mu_0 x_\tau^\mu(0)/[-x_\tau^2(0)]^{1/2}, \\ p^\mu(\pi) &= \mu_\pi x_\tau^\mu(\pi)/[-x_\tau^2(\pi)]^{1/2}. \end{aligned}\right. \quad (3.2d)$$

1. Timelike gauge

We examine the simplest possible *longitudinal* motions of this system, which occur when the string lies always along a single line, thus we are effectively in 2 spacetime dimensions. We may then choose a timelike gauge such that

$$x^0(\tau, \sigma) = \tau \quad (3.3a)$$

and x_σ is independent of σ,

$$x(\tau, \sigma) = x(\tau, 0) + \frac{\sigma}{\pi}[x(\tau, \pi) - x(\tau, 0)]. \quad (3.3b)$$

All other components of $x^\mu(\tau, \sigma)$ are assumed to vanish. The Hamiltonian in this gauge is

$$H = [p^2(0) + \mu_0^2]^{1/2} + [p^2(\pi) + \mu_\pi^2]^{1/2} + \gamma|x(\pi) - x(0)|. \quad (3.4)$$

In the center-of-mass system where $p(0) + p(\pi) = 0$, H reduces to the invariant mass

$$M = (k^2 + \mu_0^2)^{1/2} + (k^2 + \mu_\pi^2)^{1/2} + \gamma|r|, \quad (3.5)$$

where k and r are canonically conjugate relative variables.

The periodic classical motions and the Bohr-Sommerfeld approximation to the quantum theory follow from the analysis of the k-r phase-space diagram for fixed M in Fig. 2.

The action variable J is

$$J = \oint k\,dr = -\oint r\,dk,$$

and may be computed to be

$$\gamma J = \Delta^{1/2} - 2\mu_0^2 \ln\left(\frac{1}{2M\mu_0}(M^2 + \mu_0^2 - \mu_\pi^2 + \Delta^{1/2})\right) - 2\mu_\pi^2 \ln\left(\frac{1}{2M\mu_\pi}(M^2 + \mu_\pi^2 - \mu_0^2 + \Delta^{1/2})\right), \quad (3.6)$$

where

$$\Delta(M, \mu_0, \mu_\pi) = [M^2 - (\mu_0 + \mu_\pi)^2][M^2 - (\mu_0 - \mu_\pi)^2]. \quad (3.7)$$

The Bohr-Sommerfeld quantization rule is then

$$J = 2\pi\hbar(n + \text{const}); \quad n = 0, 1, 2, \ldots. \quad (3.8)$$

The exact quantum spectrum of this equation can be found by solving the nonlocal Schrödinger equation

$$0 = \int_{-\infty}^{\infty} \gamma\,dk' G(k, k')\psi(k') + [(k^2 + \mu_0^2)^{1/2} + (k^2 + \mu_\pi^2)^{1/2} - M]\psi(k), \quad (3.9)$$

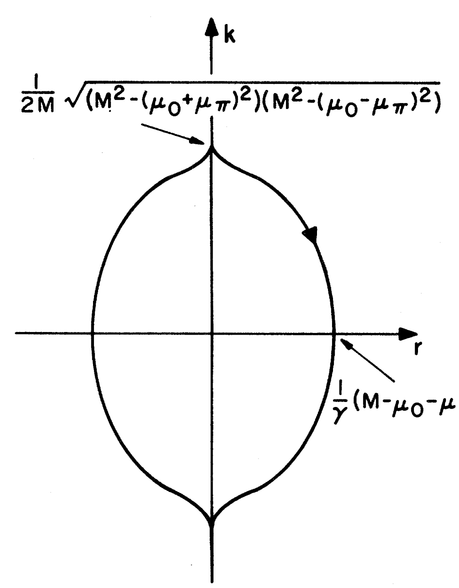

FIG. 2. Phase-space diagram following from Eq. (3.5).

where

$$G(k, k') = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_0^\infty dr\, r\cos[r(k-k')]e^{-\epsilon r} = \mathrm{P}\left(\frac{1}{\pi(k-k')^2}\right). \quad (3.10)$$

and P denotes the principal value. Exact solutions of this equation are not known.

2. *Lightlike gauge*

We now repeat the analysis of the previous subsection in the lightlike gauge

$$\begin{aligned} x^+ &\equiv x^0 + x = \tau, \\ x^- &\equiv x^0 - x = x^-(0) + \frac{\sigma}{\pi}[x^-(\pi) - x^-(0)]. \end{aligned} \quad (3.11)$$

The Hamiltonian in this gauge is

$$\begin{aligned} p^- &\equiv p^0 - p \\ &= \frac{\mu_0^2}{2p^+(0)} + \frac{\mu_\pi^2}{2p^+(\pi)} + \gamma|x^-(\pi) - x^-(0)|, \end{aligned} \quad (3.12)$$

and the total (+) momentum is $[p^+(0), p^+(\pi) > 0]$

$$p^+ = p^+(0) + p^+(\pi). \quad (3.13)$$

We note that *the quantum theory is Lorentz covariant* for this 2-dimensional system. The boost operator M^{+-} is derived from the Lagrangian via Noether's theorem and in the lightlike gauge the operators are ordered to ensure Hermiticity. We obtain

$$\begin{aligned} M^{+-} = \tau p^- &- \tfrac{1}{2}[x^-(0)p^+(0) + p^+(0)x^-(0)] \\ &- \tfrac{1}{2}[x^-(\pi)p^+(\pi) + p^+(\pi)x^-(\pi)], \end{aligned}$$

and it satisfies

$$[M^{+-}, p^\pm] = \pm i p^\pm,$$

where p^+, p^- are given in Eqs. (3.12) and (3.13). We now make the canonical transformation

$$\begin{aligned} \kappa &= [p^+(\pi) - p^+(0)]/2p^+, \\ \rho &= p^+[x^-(\pi) - x^-(0)], \end{aligned} \quad (3.14)$$

so that the invariant mass-squared can be written

$$M^2 = 2p^+p^- = \frac{\mu_0^2}{\frac{1}{2}-\kappa} + \frac{\mu_\pi^2}{\frac{1}{2}+\kappa} + 2\gamma|\rho|, \quad |\kappa| < \tfrac{1}{2}. \quad (3.15)$$

The Bohr-Sommerfeld quantization procedure yields the same result as before, Eqs. (3.6)–(3.8), while the exact quantum spectrum follows from the Schrödinger equation

$$2\gamma\int_{-1/2}^{+1/2} d\kappa' G(\kappa, \kappa')\psi(\kappa') + \left(\frac{\mu_0^2}{\frac{1}{2}-\kappa} + \frac{\mu_\pi^2}{\frac{1}{2}+\kappa} - M^2\right)\psi(\kappa) = 0, \quad (3.16)$$

where $G(\kappa, \kappa')$ is given in Eq.(3.10). 't Hooft[9] has derived this equation using a color gauge theory in two spacetime dimensions. We find it remarkable that such similar results arise from such different origins. Exact solutions of this equation are unknown.

B. Three masses (baryons)

Now we join three masses, at the points $x^\mu(0) = x^\mu(\sigma=0)$, $x^\mu(1) = x^\mu(\sigma=\sigma_1)$, $x^\mu(\pi) = x^\mu(\sigma=\pi)$, with two strings. We take the action to be

$$S = \int_{\tau_1}^{\tau_2} d\tau\left\{-\mu_0[-x_\tau^2(0)]^{1/2} - \mu_1[-x_\tau^2(1)]^{1/2} - \mu_\pi[-x_\tau^2(\pi)]^{1/2} - \gamma\int_0^{\sigma_1} d\sigma(-g)^{1/2} - \gamma\int_{\sigma_1}^{\pi} d\sigma(-g)^{1/2}\right\}. \quad (3.17)$$

We now restrict ourselves to longitudinal motions of the string lying on a straight line: We use the timelike gauge

$$x^0(\sigma, \tau) = \tau \quad (3.18)$$

and choose the σ gauge so that x_σ is independent of σ between masses,

$$\begin{aligned} 0 < \sigma < \sigma_1&: \ x(\tau, \sigma) = x(\tau, 0) + \frac{\sigma}{\sigma_1}[x(\tau, \sigma_1) - x(\tau, 0)]. \\ \sigma_1 < \sigma < \pi&: \ x(\tau, \sigma) = x(\tau, \sigma_1) + \frac{\sigma-\sigma_1}{\pi-\sigma_1}[x(\tau, \pi) - x(\tau, \sigma_1)]. \end{aligned} \quad (3.19)$$

All other components of $x^\mu(\tau, \sigma)$ are taken to vanish. The Hamiltonian then becomes

$$H = [p^2(0) + \mu_0^2]^{1/2} + [p^2(1) + \mu_1^2]^{1/2} + [p^2(\pi) + \mu_\pi^2]^{1/2} + \gamma|x(1) - x(0)| + \gamma|x(\pi) - x(1)|. \quad (3.20)$$

Going to the center-of-mass frame and choosing appropriate canonical pairs of relative coordinates, H becomes the invariant mass

$$M = [\tfrac{1}{4}(k_1 + k_2) + \mu_0^2]^{1/2} + [\tfrac{1}{4}(k_1 - k_2)^2 + \mu_\pi^2]^{1/2} + (k_2^2 + \mu_1^2)^{1/2} + \gamma|r_1 + r_2| + \gamma|r_1 - r_2|. \quad (3.21)$$

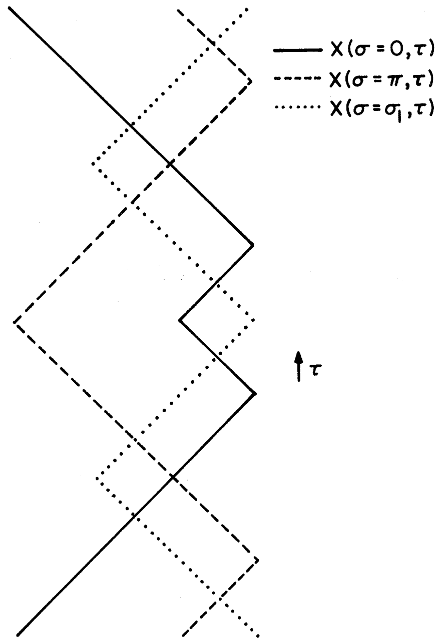

FIG. 3. A typical motion resulting from the Hamiltonian (3.21) in the zero-mass limit.

It is simpler to examine the motion of this system in the zero-mass limit. A typical motion is plotted in Fig. 3.

Analysis of the action variables for vanishing masses gives the result

$$M^2=\gamma(J_1+J_2). \tag{3.22}$$

The action variables J_1 and J_2 correspond to the normal modes of the system. When the initial conditions are such that $J_2=0$, the motion described by J_1 is plotted in Fig. 4(a). When $J_1=0$, the motion is that of Fig. 4(b). We see that J_1 describes an oscillation with a "diquark" remaining

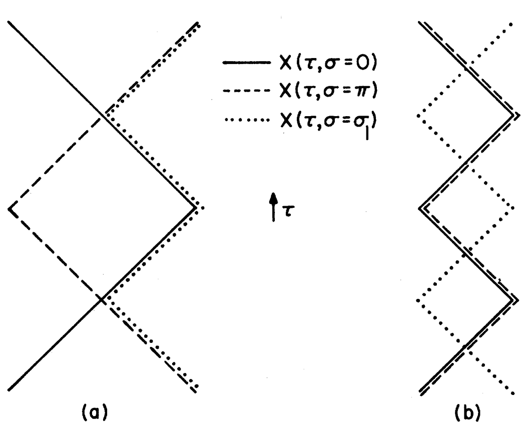

FIG. 4. (a) Pure J_1 mode, (b) pure J_2 mode, indicating diquark correlations inside low-mass baryons.

FIG. 5. Model for meson consisting at a given time of a point quark and a point antiquark connected by a string.

on one edge of the system, while J_2 gives an oscillation with a "diquark" periodically crossing the middle of the system. The lowest modes are then purely of these two types.

The experimental data on baryon spectra favor the 56 and 70 representations of SU(6). Such representations have a natural interpretation[23] in terms of diquark correlations inside the baryons. The string model discussed here then seems to give the dynamics required for these correlations to occur.

IV. QUARKS ON THE ENDS OF STRINGS

We are now ready to apply the methods of Sec. II to attach quark quantum numbers to world lines joined by the relativistic string potential. These systems constitute our proposed model for hadrons.

Mesons will be represented by a quark and an antiquark attached to opposite ends of the string, as shown in Fig. 5. Our model Lagrangian for mesons gives a σ- and τ-reparametrization invariant action with the form

$$S=\int_{\tau_1}^{\tau_2} d\tau\left[L_0(\sigma=0)+L_0(\sigma=\pi)-\gamma\int_0^{\pi}(-g)^{1/2}\right], \tag{4.1}$$

where L_0 is one of the point-quark Lagrangians examined in Sec. II and $-\gamma(-g)^{1/2}$ is the string Lagrangian density treated in Sec. III.

Our baryon model consists of three quarks connected by strings. Of three possible configurations shown in Fig. 6, the simplest is probably that of Fig. 6(a), with all quarks lying on the string. The motions exhibited in Fig. 3 apply to this case and show that each quark spends some time at the edges as well as the middle. The configuration of Fig. 6(b) is intuitively an excited state with respect to the one in Fig. 6(a), thus its low-energy spectrum is probably included in this latter case. The action corresponding to Fig. 6(a) takes the form

$$S=\int_{\tau_1}^{\tau_2} d\tau\Big[L_0(\sigma=0)+L_0(\sigma=\sigma_1)+L_0(\sigma=\pi) -\gamma\Big(\int_0^{\sigma_1}+\int_{\sigma_1}^{\pi}\Big)d\sigma(-g)^{1/2}\Big], \tag{4.2}$$

where L_0 is again any point-quark Lagrangian. The three point Lagrangians are functions of the coordinates $x^\mu(\tau,\sigma=0)$, $x^\mu(\tau,\sigma=\sigma_1)$, and $x^\mu(\tau,\sigma=\pi)$,

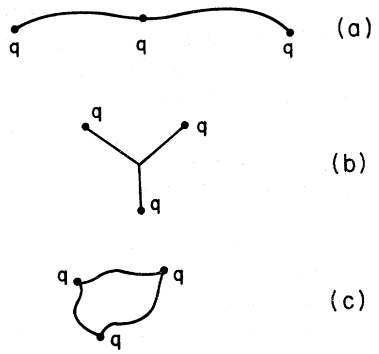

FIG. 6. Three possible configurations for the quarks on the string, giving model baryons.

respectively. The action corresponding to Fig. 6(c) will have an additional potential connecting $\sigma=0$ to $\sigma=\pi$. We will not study the other possible models for baryons in this paper.

We will impose good triality upon our systems from the outset as a phenomenological principle; only one quark will be assigned to each world line, and we will allow only strings attached to three quarks, or to one quark and one antiquark. This principle is ordinarily dictated by color-symmetry arguments[2] which play an implicit rather than an explicit role in our treatment, as described in the Introduction.

Clearly the most desirable course at this point is to analyze the quantum spectrum of each of our models for mesons and baryons using the spin-$\frac{1}{2}$ point-quark Lagrangians in four spacetime dimensions. As one might expect, this analysis becomes exceedingly complex. Since our main goal here is to introduce the basic ideas of our method, we will be content to develop a feeling for the implications by analyzing the simpler, but less realistic models. In the rest of this section, we will treat mainly the *spinless quark model* for mesons with motion restricted to a two-dimensional subspace of spacetime. The model for baryons will be outlined at the end.

A. Model for mesons

We begin our investigation of meson models by examining the action (4.1) with L_0 taken as

$$L_0(x(0)) = \partial_\tau\phi_0^\dagger\partial_\tau\phi_0[-x_\tau^{\,2}(0)]^{-1/2} - \phi_0^\dagger m^2\phi_0[-x_\tau^{\,2}(0)]^{1/2}, \tag{4.3}$$

and similarly for $L_0(x(\pi))$ with $\sigma=\pi$ quantities substituted for $\sigma=0$ quantities. Our action principle is defined by requiring the variations

$$\delta\phi_0(\tau),\quad \delta\phi_\pi(\tau),\quad \delta x^\mu(\tau,0),\quad \delta x^\mu(\tau,\pi),$$
$$\delta x^\mu(\tau,\sigma),\quad 0<\sigma<\pi$$

to be arbitrary except at $\tau=\tau_1,\tau_2$, where they vanish. We thus obtain the equations of motion

$$0=\partial_\tau K^\mu(\tau,\sigma)+\partial_\sigma N^\mu(\tau,\sigma),\quad 0<\sigma<\pi \tag{4.4}$$

$$\partial_\tau\left(\frac{x_\tau^\mu(0)}{[-x_\tau^{\,2}(0)]^{1/2}}\,(\Pi_0^\dagger\Pi_0+\phi_0^\dagger m^2\phi_0)\right)+\gamma N^\mu(\tau,\sigma=0)=0,$$
$$\partial_\tau\left(\frac{x_\tau^\mu(\pi)}{[-x_\tau^{\,2}(\pi)]^{1/2}}\,(\Pi_\pi^\dagger\Pi_\pi+\phi_\pi^\dagger m^2\phi_\pi)\right)-\gamma N^\mu(\tau,\sigma=\pi)=0, \tag{4.5}$$

where K^μ and N^μ are defined by Eq. (3.2c). Introducing the s parameter of Eq. (2.3), the ϕ equations of motion, essentially the same as Eq. (2.12), may be written

$$\frac{d^2\phi_0}{ds_0^{\,2}}+m^2\phi_0=0,\quad \sigma=0$$
$$\frac{d^2\phi_\pi}{ds_\pi^{\,2}}+m^2\phi_\pi=0,\quad \sigma=\pi. \tag{4.6}$$

Thus we see that ϕ_0 possesses the solution

$$\phi_{0\alpha}(s_0)=\left(\frac{e^{-ims_0}}{(2m)^{1/2}}\right)_{\alpha\beta}a_\beta(0)+\left(\frac{e^{ims_0}}{(2m)^{1/2}}\right)_{\alpha\beta}b_\beta^*(0), \tag{4.7}$$

with ϕ_π having a similar expression.

Equations (4.6) also imply that

$$\mu_0=\partial_{s_0}\phi_0^\dagger\partial_{s_0}\phi_0+\phi_0^\dagger m^2\phi_0,$$
$$\mu_\pi=\partial_{s_\pi}\phi_\pi^\dagger\partial_{s_\pi}\phi_\pi+\phi_\pi^\dagger m^2\phi_\pi \tag{4.8}$$

are constants of motion, so that Eqs. (4.5) may be reexpressed in *exactly* the same form as Eqs. (3.2b), (3.2d). The other constants of motion are the Poincaré-group generators p^μ and $M^{\mu\nu}$ and the U(n)-symmetry group generators

$$Q_{\alpha\beta}=\phi_{0\alpha}^\dagger\overleftrightarrow{\partial}_{s_0}\phi_{0\beta}+\phi_{\pi\alpha}^\dagger\partial_{s_\pi}\phi_{\pi\beta}, \tag{4.9}$$

which commute with the mass term in the Lagrangian L_0.

B. Longitudinal motions and their spectrum

Restricting ourselves to motion in two spacetime dimensions and choosing the timelike gauge analogous to Eqs. (3.3), we find the Hamiltonian

$$H=[p^2(0)+\mu_0^{\,2}]^{1/2}+[p^2(\pi)+\mu_\pi^{\,2}]^{1/2} +\gamma|x(\pi)-x(0)|. \tag{4.10}$$

Here $p(0),p(\pi)$ are identical in form to Eq. (3.2d), except that μ_0 and μ_π are the nontrivial canonical variables (4.8).

While the quantum theory of the variables p and x is nontrivial, as we saw in Sec. III, we may quantize the fields ϕ_0 and ϕ_π in a straightforward manner. From Eq. (4.7), we deduce that for $\sigma=0$

$$[a_\alpha(0),a_\beta^\dagger(0)]=\delta_{\alpha\beta}=[b_\alpha(0),b_\beta^\dagger(0)] \tag{4.11}$$

are acceptable commutation relations implying

$$i[\Pi_{0\alpha},\phi_{0\beta}^\dagger]=\delta_{\alpha\beta}, \tag{4.12}$$

where $\Pi_{0\alpha}$ is the canonical momentum, e.g., (2.10). Similar equations hold at $\sigma=\pi$. Thus Eq. (4.8) may be written

$$\mu_0 = a^\dagger_\alpha(0)\, m_{\alpha\beta} a_\beta(0) + b^\dagger_\alpha(0)\, m_{\alpha\beta} b_\beta(0) + \text{const}, \qquad (4.13)$$

and similarly for μ_π. It is possible to cancel all or part of the normal-ordering constant in Eq. (4.13) by adding an extra term to the Lagrangian proportional to $[-x_\tau{}^2(0)]^{1/2}$ (and $[-x_\tau{}^2(\pi)]^{1/2}$). We will set the normal-ordering constant equal to zero in the analysis which follows.

We see that our meson system has been reduced effectively to that of Sec. III, except that the mass variables μ_0, μ_π are now *operators* which take on different values depending on the different masses of the quarks in the multiplet. Symmetry breaking thus appears in a natural way, and the masses at the ends of the string are now identified *directly* with the masses of the quarks making up a given meson.

For example, the π^+ meson will be a string with $\mathcal{P}$ and $\bar{\mathfrak{N}}$ quark masses on the ends, while for a K^+ meson, the $\mathcal{P}$ and $\bar{\lambda}$ masses appear. The internal-symmetry content of mesons is described by the states $a^\dagger_\alpha(0) b^\dagger_\beta(\pi)|0\rangle$, where $a^\dagger_\alpha(0)$ creates a quark of type α at $x(0)$, while $b^\dagger_\beta(\pi)$ creates an antiquark of type β at $x(\pi)$. The particles π^+, K^+ etc. and their excitations are described by the states, e.g.,

$$|K^+\rangle = a^\dagger_{\mathcal{P}}(0) b^\dagger_\lambda(\pi)|0\rangle\,. \qquad (4.14)$$

The spectrum, e.g., of the K^+ family, is then given by

$$\begin{aligned} H_{K^+} &= \langle K^+|H|K^+\rangle \\ &= [p^2(0) + m_{\mathcal{P}}{}^2]^{1/2} + [p^2(\pi) + m_\lambda{}^2]^{1/2} \\ &\quad + \gamma|x(\pi) - x(0)|\,. \end{aligned} \qquad (4.15)$$

We see from the form of Eq. (4.15) that (a) the system becomes heavier for larger quark separations, (b) the lowest mass occurs for a shrunk string $[x(0)\cong x(\pi)]$ and corresponds to the ground state of the standard quark model where the quarks are approximately at the same spacetime point. The quark masses determine the intercept of the trajectories, which are therefore in general nondegenerate.

We can calculate approximately the quantized radial excitation spectrum of the π^+, K^+, etc. families from our longitudinal-mode Hamiltonian (4.10). This spectrum would correspond to the mass states with fixed spin in a Chew-Frautschi plot. To obtain the angular excitations one would have to include the transverse modes of the string as well, which we have not yet done. We now apply the semiclassical Bohr-Sommerfeld quantization procedure as in Sec. III and Ref. 13. Proceeding as for Eq. (4.15), we find for a meson with quark masses m_0 and m_π the effective Hamiltonian

$$\begin{aligned} H &= [p^2(0) + m_0{}^2]^{1/2} + [p^2(\pi) + m_\pi{}^2]^{1/2} \\ &\quad + \gamma|x(\pi) - x(0)|\,. \end{aligned} \qquad (4.16)$$

The resulting spectrum has the following properties:

(a) If $m_0 = m_\pi = 0$ (e.g., pion, with massless quarks)

$$2\pi\hbar\gamma(n + \text{const}) = M_n{}^2, \quad n = 0, 1, 2, \ldots\,. \qquad (4.17a)$$

(b) If $m_0 = 0$, $m_\pi = m \neq 0$ (e.g., kaon),

$$2\pi\hbar\gamma(n + \text{const}) = M_n{}^2 - m^2 - m^2 \ln(M_n{}^2/m^2)\,. \qquad (4.17b)$$

(c) If $m_0 = m_\pi = m$,

$$\begin{aligned} 2\pi\hbar\gamma(n + \text{const}) &= M_n(M_n{}^2 - 4m^2)^{1/2} \\ &\quad - 4m^2 \ln\left[\frac{M_n}{2m} + \left(\frac{M_n^2}{4m^2} - 1\right)^{1/2}\right]. \end{aligned} \qquad (4.17c)$$

(d) If $m_0 \neq m_\pi \neq 0$,

$$\begin{aligned} 2\pi\hbar\gamma(n + \text{const}) &= \Delta^{1/2}(M_n, m_0, m_\pi) - 2m_0{}^2 \ln\left(\frac{M_n{}^2 + m_0{}^2 - m_\pi{}^2}{2m_0 M_n} + \frac{\Delta^{1/2}(M_n, m_0, m_\pi)}{2m_0 M_n}\right) \\ &\quad - 2m_\pi{}^2 \ln\left(\frac{M_n{}^2 + m_\pi{}^2 - m_0{}^2}{2m_\pi M_n} + \frac{\Delta^{1/2}(M_n, m_0, m_\pi)}{2m_\pi M_n}\right), \end{aligned} \qquad (4.17d)$$

where Δ is defined in Eq. (3.7).

This shows that with massive quarks the spectrum is *not* linear. However, for mesons containing small quark masses, $m_0{}^2, m_\pi{}^2 \lesssim 0.3\ \text{GeV}^2$, the deviation from linearity is very small and the spectrum quickly becomes essentially linear. On the other hand, if the quark mass is large, such as the conjectured charmed quark with $m_c \approx 2$ GeV, then the curvature is substantial. This may be a welcomed feature if the new resonances[24] are interpreted as charmonium states.[25]

We remark that since we have included neither spin nor spin-spin interactions at this stage, our present results are not necessarily realistic.

If we assume, however, that the above formulas are applicable to the octet of observed pseudoscalar mesons, and take, e.g.,

$$|\pi^+\rangle = a_{\mathcal{P}}^\dagger(0) b_{\mathfrak{N}}^\dagger(\pi)|0\rangle\,, \quad |K^+\rangle = a_{\mathcal{P}}^\dagger(0) b_{\lambda}^\dagger(\pi)|0\rangle\,,$$
$$|\eta\rangle = \frac{1}{\sqrt{3}}\left[a_{\mathcal{P}}^\dagger(0) b_{\mathcal{P}}^\dagger(\pi) + a_{\mathfrak{N}}^\dagger(0) b_{\mathfrak{N}}^\dagger(\pi) - 2a_{\lambda}^\dagger(0) b_{\lambda}^\dagger(\pi)\right]|0\rangle\,, \tag{4.18}$$

while setting $(n+\text{const})=0$ in Eq. (4.17), we obtain the following masses for the ground states:

$$m_{\pi^+} = m_{\mathcal{P}} + m_{\mathfrak{N}}\,,$$
$$m_{K^+} = m_{\mathcal{P}} + m_{\lambda}\,, \tag{4.19}$$
$$m_{\eta} = \tfrac{1}{3}(m_{\mathcal{P}} + m_{\mathfrak{N}} + 4m_{\lambda})\,.$$

For $m_{\mathcal{P}} \approx m_{\mathfrak{N}}$, Eq. (4.19) leads to the *linear* mass formula

$$m_\pi + 3m_\eta = 4m_K\,, \tag{4.20}$$

which is in reasonable agreement with experiment. Given our crude model, we consider this result encouraging. The coefficients in Eq. (4.20) are the same as those in Gell-Mann and Okubo's *quadratic* mass formula.

We will not attempt to treat baryons in detail here. The basic procedure would be to construct baryonic states analogous to Eq. (4.18) and take matrix elements of the Hamiltonian following from an action like Eq. (4.2). In this case the Hamiltonian would take the form of Eq. (3.20). The Bohr-Sommerfeld spectrum for massless quarks would then be given by Eq. (3.22), while for massive quarks the trajectories would in general be nonlinear.

V. WEAK AND ELECTROMAGNETIC INTERACTIONS

The close connection between the present model and field theory suggests a compelling approach for dealing with weak and electromagnetic interactions. We propose to extend the procedure of Sec. II to include quark fields coupled minimally to a set of vector mesons, where the gauge group may in general be non-Abelian. The string itself will not be coupled to these vector mesons because, as discussed in the Introduction, the string is assumed to have the same properties as color glue. Provided the meson-quark coupling is small, as in the unified theories of weak and electromagnetic interactions[26] we may treat the interaction perturbatively. Thus the successes of gauge theories in their application to weak and electromagnetic interactions would be expected to persist in our model. The picture that emerges is one in which the strong interactions mediated by the string are solved in the absence of weak forces, which are then considered as small perturbations on the system.

For the purpose of illustration, let us consider the model of Sec. IV with spinless quarks. The point-quark Lagrangians $L_0(x(0))$ and $L_0(x(\pi))$ will be modified in the presence of interactions. The new form of L_0 at each point follows from examining the gauge-invariant spacetime Lagrangian density

$$\mathcal{L}(x) = -[D^\mu \phi(x)]^\dagger [D_\mu \phi(x)] - \phi^\dagger(x) m^2 \phi(x)\,, \tag{5.1}$$

where D^μ is the covariant derivative defined by

$$D^\mu \phi(x) = \partial^\mu \phi(x) - \frac{i}{2} g \lambda_\alpha A^\mu_\alpha(x)\phi(x)\,. \tag{5.2}$$

Following the same procedure as in Sec. II A, we find the modified point-particle Lagrangian

$$L_0(x) = \frac{1}{(-x_\tau{}^2)^{1/2}}\left(\phi_\tau - \tfrac{1}{2} i g \lambda_\alpha A^\mu_\alpha x_{\tau\mu}\phi\right)^\dagger\left(\phi_\tau - \tfrac{1}{2} i g \lambda_\alpha A^\mu_\alpha x_{\tau\mu}\phi\right) + \tfrac{1}{4} g^2 (-x_\tau{}^2)^{1/2} \phi^\dagger \lambda_\alpha \lambda_\beta \phi \sum_i (n_i^\mu A_{\alpha\mu} n_i^\nu A_{\beta\nu}) - (-x_\tau{}^2)^{1/2} \phi^\dagger m^2 \phi\,, \tag{5.3}$$

where $A^\mu_\alpha(x(\tau))$ may be considered as an external field. The Lagrangian (5.3) is invariant under the infinitesimal gauge transformation *restricted to the world line*:

$$\lambda \cdot \delta A^\mu(x) = i[\lambda \cdot \Lambda(x), \lambda \cdot A^\mu(x)] + \lambda \cdot \partial^\mu \Lambda(x)\,,$$
$$\delta\phi(\tau) = i\lambda \cdot \Lambda(x)\phi(\tau)\,, \tag{5.4}$$

where

$$n_i^\mu(\tau)\partial_\mu \Lambda^\alpha(x) = 0\,. \tag{5.5}$$

This latter condition is necessary to keep $\phi(\tau)$ on the same world line following the gauge transformation. The normal components of A^μ_α, namely, $n_i^\mu A_\mu^\alpha(x)$, cannot be gauge-transformed away in general.

We observe that the above procedure for coupling gauge fields to the ends of the string is quite different from that of Ademollo *et al.*,[27] who did not use field theory as a starting point.

The current that couples to the gauge fields is located only on the world lines of each quark, where $x^\mu = x^\mu(\tau, \sigma=0)$ or $x^\mu = x^\mu(\tau, \sigma=\pi)$. The local current at $x^\mu = x^\mu(0)$ is then given by

$$J^\mu_\alpha(x(0)) = \frac{1}{g}\frac{\partial L_0(x(0))}{\partial A^\alpha_\mu(x(0))} = \left(\phi_0^\dagger \frac{i\lambda^\alpha}{2}\overleftrightarrow{\partial_\tau}\phi_0 + \frac{g}{4}\phi_0^\dagger\{\lambda^\alpha, \lambda^\beta\}\phi_0 A^\nu_\beta(x(0))\partial_\tau x_\nu(0)\right)\frac{x^\mu_\tau(0)}{[-x_\tau{}^2(0)]^{1/2}} + \sum_i \left(\frac{g}{4}\phi_0^\dagger\{\lambda^\alpha, \lambda^\beta\}\phi_0 A^\nu_\beta(x(0)) n_{i\nu}(x(0))\right) n_i^\mu(x(0))\,, \tag{5.6}$$

and a similar expression gives the current at $x^\mu = x^\mu(\pi)$. The coupling scheme described here allows us in principle to calculate hadron form factors. We have not yet carried out this program.

VI. CONCLUSION

Motivated by the many parallels between the dual-string picture of hadrons and the quark-gluon field theories of hadron dynamics, we have sought a method of attaching quark quantum numbers to world lines joined by relativistic strings. We began by developing techniques for restricting classical quark fields, with any desired measureable quantum numbers, to a world line. Very simple point-particle theories resulted when we considered spinless quarks and spin-$\frac{1}{2}$ quarks without *Zitterbewegung*. The latter spin-$\frac{1}{2}$ model contains an algebra reminiscent of supersymmetry. A much more complex and interesting theory possessing *Zitterbewegung* arose when we restricted a classical Dirac field to a world line.

By attaching structureless masses to the string as in Ref. 13, we developed an intuitive picture for the dynamics of the simple longitudinal string oscillations. The string with masses on each end gave a roughly linear mass-squared spectrum as expected of a mesonlike system. In a lightlike gauge, this system is described by an integral equation found also by 't Hooft in a totally different context. A baryonlike system resulted from placing a third mass in the middle of the string; for small masses, the normal modes of this system simulate diquarks oscillating against a third single quark.

Next, we analyzed the longitudinal spectrum of a model for mesons consisting of spinless SU(3) quarks on the ends of the string. Systematic deviations from a linear spectrum were found in the Bohr-Sommerfeld approximation to the quantum theory, while symmetry breaking appeared in a natural way. The SU(3) pseudoscalar-meson masses were found to obey a formula similar to that of Gell-Mann and Okubo, but with masses replacing squared masses. Our technique for including internal symmetries therefore has nontrivial implications.

Finally, we observed that we could couple external fields to quarks on the ends of the string in a straightforward way. The essence of the technique consisted of examining the spacetime field-theoretic Lagrangian for a quark coupled to a vector gauge field and restricting the quark fields to a world line. The gauge fields were then interpretable as external field potentials and the system was invariant under a restricted class of gauge transformations. Furthermore, the point quark Lagrangian permitted us to identify clearly the physical currents.

Only the simplest aspects of our proposed models have been worked out in detail here. There are clearly many other facets which would be interesting to explore. The baryon spectrum needs to be investigated more thoroughly, as do the problems of using Dirac quarks for both mesons and baryons. Understanding the quantum mechanics of these models will surely be challenging. Many additional effects will occur when one allows arbitrary motions of the string, instead of considering only longitudinal motions as we did here. Calculating form factors for models with Dirac-like point quarks will give stringent conditions on the phenomenological validity of our proposals.

It is also possible to replace or supplement the relativistic string potential by more complicated interactions. For example, an Iwasaki-Kikkawa spinning string,[28] corresponding to the Neveu-Schwarz model, would be expected to generate spin-spin interactions between the quarks. One could also conceive of stringlike potentials that would generate more complicated spin-spin interactions, or even isospin-isospin interactions; such "strings" would then have a close phenomenological correspondence to the effects of field-theoretic quark binding.

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APPENDIX: FIELDS ON A WORLD LINE, A WORLD SHEET, ETC.

We outline here a new method to treat a local field on a space smaller than physical spacetime. We use the induced metric on the smaller subspace to generate Poincaré-invariant Lagrangians. Parametrizing the subspace by the variables τ^a, we write the spacetime position of any point in the subspace as $x^\mu(\tau^a)$. The induced metric is defined in terms of the tangents $\partial x^\mu/\partial\tau^a$ as

$$g_{ab} = \frac{\partial x^\mu}{\partial \tau^a}\,\eta_{\mu\nu}\,\frac{\partial x^\nu}{\partial \tau^b} \equiv x_a \cdot x_b, \qquad \text{(A1)}$$

where the x^μ-space metric $\eta^{\mu\nu}$ is taken to be flat, but could depend on x^μ, e.g., in polar coordinates. In Cartesian coordinates, $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. More generally, if $\eta^{\mu\nu}$ were the metric of a curved space it might be possible to include gravitation in this formalism. Writing

$$g = \det(g_{ab}), \qquad \text{(A2)}$$

we find the invariant volume element in τ^a space to be

$$(d\omega) = (d\tau)(-g)^{1/2}. \qquad \text{(A3)}$$

If we now define g^{ab} as the inverse of g_{ab}, we can examine the metric $\eta^{\mu\nu}$ written in terms of a complete set of vectors following a suggestion of Giles and Tye[12]

$$\eta^{\mu\nu}=x_a^\mu g^{ab}x_b^\nu+\sum_i n_i^\mu n_i^\nu. \tag{A4}$$

$\eta_{\mu\nu}$ becomes the inverse of $\eta^{\mu\nu}$ provided the n_i^μ are an independent set of normals to the subspace,

$$\begin{aligned} n_i^\mu\eta_{\mu\nu}n_j^\nu&=\delta_{ij},\\ n_i^\mu\eta_{\mu\nu}x_a^\nu&=0,\\ x_a^\mu\eta_{\mu\nu}x_b^\nu&=g_{ab}. \end{aligned} \tag{A5}$$

The metric g_{ab} has one timelike direction, while n_i^2 is always spacelike. We confirm that

$$\begin{aligned} \eta^{\mu\nu}\eta_{\nu\lambda}x_c^\lambda&=x_c^\mu,\\ \eta^{\mu\nu}\eta_{\nu\lambda}n_i^\lambda&=n_i^\mu. \end{aligned} \tag{A6}$$

Now let us consider a local Lagrangian field theory with fields $\phi(x)$, and restrict $\phi(x)$ to live only on the τ^a subspace. We demand that the normal derivatives out of the subspace vanish,

$$n_i^\mu\partial_\mu\phi(x(\tau^a))=0. \tag{A7}$$

Then we take ϕ to depend on τ^a through the variables $x^\mu(\tau^a)$. This gives effectively $\phi=\phi(\tau^a)$, where each point τ^a corresponds to a particular point on the subspace embedded in x^μ space.

The result is that derivatives of ϕ with respect to x_μ are replaced by

$$\begin{aligned} \partial^\mu\phi=\eta^{\mu\nu}\partial_\nu\phi&=\left(x_a^\mu g^{ab}x_b^\nu+\sum_i n_i^\mu n_i^\nu\right)\frac{\partial\phi}{\partial x_\nu}\\ &=x_a^\mu g^{ab}\frac{\partial\phi(\tau^a)}{\partial\tau^b}. \end{aligned} \tag{A8}$$

The rest of the Lagrangian remains unchanged, while the volume element gets replaced by $d^4x \to (-g)^{1/2}(d\tau)$.

In a one-dimensional timelike τ space corresponding to a point particle's world line, we find

$$\begin{aligned} g_{ab}&=g=x_\tau^\mu\eta_{\mu\nu}x_\tau^\nu\equiv x_\tau{}^2,\\ g^{ab}&=g^{-1}=1/x_\tau{}^2,\\ \eta^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}&=\frac{x_\tau^\mu}{x_\tau{}^2}\frac{\partial\phi(\tau)}{\partial\tau}. \end{aligned} \tag{A9}$$

The invariant line element is simply

$$ds=d\tau(-g)^{1/2}=d\tau(-x_\tau{}^2)^{1/2}. \tag{A10}$$

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[1]For a review and a comprehensive list of references, see S. Mandelstam, Phys. Rep. 13C, 260 (1974).

[2]For a discussion of the advantages of the colored quark model, see H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973). Confinement with non-Abelian gauge bosons is discussed by D. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[3]S. Mandelstam, Nucl. Phys. B64, 205 (1973); B69, 77 (1974).

[4]Y. Nambu, in *Symmetries and Quark Models*, proceedings of the conference at Wayne State University, 1969, edited by Ramesh Chand (Gordon and Breach, New York, 1970); H. Nielsen, in *High Energy Physics*, proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, 1970, edited by V. Shelest, (Naukova Dumka, Kiev, U. S. S. R., 1972); L. Susskind, Nuovo Cimento 69, 210 (1970); L. N. Chang and F. Mansouri, Phys. Rev. D 5, 2535 (1972); P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B61, 45 (1973).

[5]H. Harari, Phys. Rev. Lett. 22, 562 (1969); J. L. Rosner, *ibid.* 22, 689 (1969).

[6]H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 47 (1973); P. Olesen, Phys. Lett. 50B, 255 (1974).

[7]K. Wilson, Phys. Rev. D 10, 2445 (1974).

[8]J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

[9]G. 't Hooft, Nucl. Phys. B75, 461 (1974). We thank C. Rebbi for bringing this reference to our attention.

[10]A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971); K. Bardakci and M. B. Halpern, Phys. Rev. D 3, 2493 (1971).

[11]M. B. Halpern, Phys. Rev. D 12, 1684 (1975).

[12]See the Appendix and R. C. Giles, thesis, Stanford University, 1975 (unpublished); R. C. Giles and S.-H. H. Tye, this issue, Phys. Rev. D 13, 1690 (1976); L. N. Chang and F. Mansouri, in *Proceedings of the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, 1974,* edited by G. Domokos *et al.* (Johns Hopkins Univ., Baltimore, 1974).

[13]W. A. Bardeen, I. Bars, A. J. Hanson, and R. D. Peccei, Phys. Rev. D (to be published). Massive end points have been briefly considered also by A. Chodos and C. Thorn, Nucl. Phys. B72, 509 (1974).

[14]For a rudimentary attempt in this direction, see S. K. Wong, Nuovo Cimento 65A, 689 (1970). We would in general obtain a different set of equations.

[15]S. Weinberg, Phys. Rev. D 7, 1068 (1973).

[16]J. Frenkel, Z. Phys. 37, 273 (1926); J. Weyssenhoff, Acta. Phys. Pol. 9, 1 (1947).

[17]See however, F. Gürsey, Nuovo Cimento 5, 784 (1957). The spin-$\frac{1}{2}$ field in this reference satisfies a different equation than our two models. We thank F. Gürsey for pointing out to us this work.

[18]K. Huang, Am. J. Phys. 20, 479 (1952).

[19]See, e.g., A. J. Hanson and T. Regge, Ann. Phys. (N. Y.) 87, 498 (1974).

[20]J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974); A. Salam and J. Strathdee, *ibid.* B76, 477 (1974);

D. Volkov and P. Akulov, Phys. Lett. 46B, 109 (1974).

[21] I. Bars, Nucl. Phys. B31, 15 (1971).

[22] In these two cases we can show *without* using the free Euler equation $\partial_\tau p^\mu = 0$ that the interacting quark-string system reduces effectively to the system considered in Sec. III. The only change is the replacement of the parameters μ_0 and μ_π by time-independent mass operators, as in Eq. (4.13).

[23] J. D. Bjorken, private communication.

[24] J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1404 (1974); C. Bacci *et al.*, *ibid.* 33, 1408 (1974); G. S. Abrams *et al.*, *ibid.* 33, 1453 (1974).

[25] T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975).

[26] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Wiley, New York, 1969), p. 367.

[27] M. Ademollo *et al.*, Nuovo Cimento 21A, 77 (1974).

[28] Y. Iwasaki and K. Kikkawa, Phys. Rev. D 8, 441 (1973); L. N. Chang, K. Macrae, and F. Mansouri, Phys. Lett. 57B, 59 (1975).