Multiperipheral models: A self-consistent field approach*

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A multiperipheral amplitude with elementary particle propagators is written in a form containing only pairwise finite-range rapidity-dependent terms, and transverse-momentum terms (in the central region). A potential is defined that represents, on the average, the effect on a particle of all the other particles. This potential confines each particle to a certain region of rapidity (with respect to nearby particles on the chain), and transverse momentum. The potential is shown in turn to result from the averaged effects of all the other particles, each confined by the same potential. It is argued that the approximation becomes exact for high densities (large coupling constants); and transverse momentum and multiplicity distributions, as well as Regge exponents, are computed in this limit for three cases: I. rapidity order corresponding to chain order; II. no rapidity restrictions but the same order on both sides of the multiperipheral ladder; III. no restrictions (all crossed rungs). The results are seen to be quite different in each case.

I. INTRODUCTION

We consider a new approximation for obtaining inclusive single-particle distributions, Regge exponents, and multiplicity distributions in a multiperipheral theory, where single particles are emitted at the vertices and elementary pions are exchanged. The model is essentially a ϕ^3 theory except that the emitted and exchanged particles may have different mass.

The complete solution for the above quantities involves taking the t-channel factorizable N-particle amplitude, summing over all permutations of the particles for a given set of rapidities and transverse momenta, squaring the sum, and integrating over all (or all but one) transverse momenta and ordered rapidities. As with most such problems, we do not have the exact solution, and various approximations are used. We first mention a few methods that have been developed, and then describe ours.

It is well known that multiperipheral models, at least in their original form with pion propagators,[1] lead to multiparticle distributions that resemble those of a classical fluid of particles interacting with short-range forces, where longitudinal and transverse dimensions correspond to rapidity and transverse momentum. Cambell and Chang[2] have shown that in a ϕ^3 theory, if transverse momenta are integrated over, the resulting N-particle probability function satisfies a cluster-decomposition property. Lee[3] has also reduced the function (in the standard ladder approximation) to that of a one-dimensional gas interacting with short-range forces, which he then explicitly computes in a certain approximation.

The above properties have led to approximation schemes for solving for various quantities, such as the Regge exponent α and the multiplicity distributions. Since in the low-density, or weak-coupling limit each particle is far away from the others, on the average, we can have an approximation[2] where each particle is only weakly affected by the others and, to first order, only interacts with at most one other particle. This is the first term of the cluster expansion, familiar in statistical mechanics, and by considering more particles interacting we can develop an expansion in either the coupling constant or the density. In another scheme, as a first approximation, we can consider only interactions between nearest neighbors in rapidity, since other particles will be mostly beyond the range of the force if the density is low. The probability function then factorizes into products of functions of neighboring rapidity differences.[3] We can then consider next-nearest-neighbor interactions, and so on, and also develop an expansion for various quantities with this method.

It is interesting to compare the above schemes with the well-known integral-equation method.[1] With this method (assuming the equations can be solved), all of the "interactions" can be included, but only for a fixed type of diagram, which corresponds to the iteration of the kernel. Thus, the first approximation would involve only the single-rung kernel, which would be a good approximation if densities are not too high. Further terms could then be added in the form of crossed-rung diagrams.[4] All these methods are exact only in the low-density limit, and are useful for not-too-high densities, when few correction terms are required.[5]

In this paper we complement the above set by providing a technique that becomes exact in the *high*-density limit. In this limit each particle is strongly influenced by the others, there are many particles within the region of interaction of each, and many high-order crossed rungs (interference terms) are important. Our method, therefore, is radically different from the others, and may prove to be an important alternative.[6] Since the mathematics in our approximation is simple, the high-density limit may be a starting point for an expansion (perhaps in inverse powers of density or coupling constant) that can be used at intermediate densities, even if the amplitude itself is not realistic at high densities. Also, the technique may prove to be useful for other, less simple theories.

We have also three other reasons for studying this limit:

(i) The multiperipheral amplitude is the simplest field-theoretic production model that we have, and the results of such a theory should be known, not only in the low-density limit.

(ii) Since experimentally measured large-transverse-momentum ($\vec{p}_\perp$) inclusive distributions seem to fall only as a power of $\vec{p}_\perp{}^2$, there is the chance that a simple model that contains only power damping in momentum transfers is correct if the produced particles are ρ's, for all events where rapidity gaps are not too large (so that Regge effects can be neglected). If we then look at events with higher-than-average multiplicity, at available energies, we see that energy-momentum conservation "compresses" the particles into a small region of phase space, and densities there are quite high. Thus, with some simple conservation constraints and resonance-interference effects[7] (which we neglect here), our technique may be directly useful in comparing theory with experiment.

(iii) It may be that coupling constants *are* quite large. In that case absorption would be important and the diagrams that we consider would be internal elements of the theory (the terms with exactly two vertical propagators), as in eikonal models.[8]

The approach that we will use is similar to a self-consistent field technique: The N-particle amplitude is represented by a product of single-particle terms. With one of the momenta fixed, the terms in the actual amplitude which depend on the other momenta are replaced by their average values, assuming the above factorized form. The resulting function of the remaining momentum is then equated to the term assumed in the factorized form. In general, an integral equation results that can be solved for this function. When the terms that were replaced by their averages depend on a large number of momenta, but only weakly on each, and are well behaved, the statistical fluctuations that were ignored in computing the averages become small and the method should give good results.

There is an interesting analogy to a classical N-body problem where particles interact through weak, long-range potentials[9]: Each particle moves in a field that is a function of the positions of the other particles. When there are many particles in the range of interaction the statistical fluctuations are small and the field can be replaced by its average value at each point. The N-particle probability function then becomes a product of single-particle functions.

The above technique will be used to calculate the partial cross sections σ_N, the Regge exponent α, and the transverse-momentum distribution $\rho(\vec{p}_\perp)$ in the limit when $Y \sim \ln s \to \infty$ *first* (so that leading-particle or end effects can be ignored) and then the density N/Y becomes large (or alternatively the coupling constant becomes large).

Three cases will be treated:

Case I. Rapidity order corresponding to chain order;

Case II. ladder approximation, no rapidity order but the order of momenta on the left side of the diagram corresponding to that on the right side (no crossed diagrams);

Case III. no restrictions (all crossed diagrams).

It will be shown that cases I and II do not give the correct results at high density and that, if the *only* assumption is our multiperipheral form for the amplitude, case III gives the correct results. The three cases, however, will be treated in a unified way, each requiring successively more complications.

Our main results will include an analytic limiting form for $\rho(\vec{p}_\perp)$ and an integer ν and constant c such that

$$\sigma_N \to (cg^2)^N Y^{\nu N}/(\nu N)! \quad (N/Y \to \infty)$$

or

$$\alpha \to (cg^2)^{1/\nu} \quad (g^2 \to \infty),$$

where g is the coupling constant. We believe that these results are new for the case with all crossed diagrams.

Suranyi[10] has also studied the multiperipheral amplitude in the high-density limit, but with a more general dependence on the momentum transfers. Although his approach was more analytical than ours (we rely heavily on the classical fluid analogy) he used similar variables and in some cases gets similar results. In particular, he treats the ladder model [case (ii)] in detail and gets the same results for $\rho(\vec{p}_\perp)$ and the behavior of σ_N as we do. He then briefly treats crossed diagrams and incorrectly concludes that they do not modify the momentum distribution (our reasons for disagreeing with him are given in the footnotes). He does, however, arrive at the correct value of ν in this case (this quantity is not sensitive to the momentum dependence and will be derived in a simplified one-dimensional model in Sec. IV).

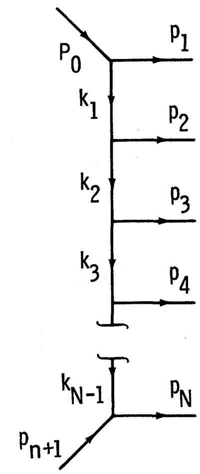

FIG. 1. Multiperipheral amplitude.

II. THE MODEL

Denoting momentum transfers k_i, $i=1,\ldots,N-1$, momenta of the produced particles p_i, $i=1,\ldots,N$ and incoming momenta p_0, p_{N+1} (see Fig. 1), the N-particle amplitude is

$$A_N(p_0, p_1, \ldots, p_{N+1}) = \prod_{1}^{N-1} (\mu_i^2 - k_i^2)^{-1}. \tag{1}$$

We label the masses of the produced particles m, the incoming ones M, and the exchanged ones μ, and we define rapidities by the equation

$$p_i = (m_{\perp i}\cosh y_i, m_{\perp i}\sinh y_i, \vec{p}_i), \tag{2}$$

where

$$m_{\perp i} = (m^2 + \vec{p}_i^{\,2})^{1/2}, \quad i=1,\ldots,N$$

and $\vec{p}_i$ is the transverse momentum.

We can write A_N in a more suggestive form by using energy-momentum conservation. Define

$$\sigma_i = m_{\perp i}/m, \quad i=1,\ldots,N$$
$$\sigma_i = -M/m, \quad i=0, N+1 \tag{3}$$
$$p_i^{\pm} = p_{i0} \pm p_{i\parallel} = m\sigma_i e^{\pm y_i}.$$

Then since

$$k_i = p_0 - \sum_{1}^{i} p_j = -\left(p_{N+1} - \sum_{i+1}^{N} p_j\right) \tag{4}$$

and

$$k_i^2 = k_i^+ k_i^- - \vec{k}_i^{\,2},$$

$$k_i^2 = -\left(p_0^+ - \sum_{1}^{i} p_j^+\right)\left(p_{N+1}^- - \sum_{i+1}^{N} p_j^-\right) - \vec{k}_i^{\,2}$$

or

$$-k_i^2 = m^2 \sum_{j\le i}\sum_{k>i} \sigma_j\sigma_k e^{-(y_k - y_j)} + \vec{k}_i^{\,2}. \tag{5}$$

We also have the energy-momentum constraints

$$\sum_{0}^{N+1} \sigma_i e^{\pm y_i} = \sum_{1}^{N} \vec{p}_i = 0. \tag{6}$$

It can be seen that, as $Y = \ln(s/m^2) \to \infty$, the constraints in y each separately affect only the particles near each end of phase space. Looking at one end, we have

$$\sum_{1}^{N} \sigma_i e^{-y_i} = \frac{M}{m} e^{Y/2}$$

or, with δy_i measuring the distance from the end at $-Y/2$ to the ith particle,

$$\sum_{1}^{N} \sigma_i e^{-\delta y_i} = M/m. \tag{7}$$

Thus, particles far from the end will have large δy_i and can be neglected in the sum. We will integrate over the rapidities of the end particles in any quantities that we consider, and if a particle is a large enough but finite distance from the walls, as $Y \to \infty$ these constraints can be neglected.[11] Also, since $\vec{p}_i^{\,2}$ will turn out to be damped by the amplitude, the constraint in $\vec{p}_i$ will have a vanishing effect on each particle as $N \to \infty$, and can be neglected. These approximations are convenient in obtaining certain asymptotic limits, but probably are important at finite energies. Our amplitude is then

$$A_N(\{y_i\}, \{\vec{p}_i\}) = m^{-2N} \prod_{1}^{N-1} (\tau_i + r_i)^{-1}, \tag{8}$$

where

$$\tau_i = \sum_{j\le i}\sum_{K>i} \sigma_j \sigma_K e^{-(y_K - y_j)},$$

$$r_i = (\mu^2 + \vec{k}_i^{\,2})/m^2.$$

We will need the N-particle cross section, σ_N for fixed N/Y as $Y \to \infty$ in three cases:

Case I. Rapidity order corresponding to chain order;

Case II. no rapidity restrictions, but the order of momenta in A_N corresponding to A_N^* [this is the standard Amati-Fubini-Stanghellini (AFS) ladder approximation];

Case III. no restrictions.

Then, with the notation of Ref. 12,

$$\sigma_N^{\rm I} = \frac{c}{s}(g^2/16\pi^3)^N \int \prod dy_i\, d\vec{p}_i\, \theta(y_{i+1} - y_i) \times |A_N(\{y_i\}, \{\vec{p}_i\})|^2, \tag{9}$$

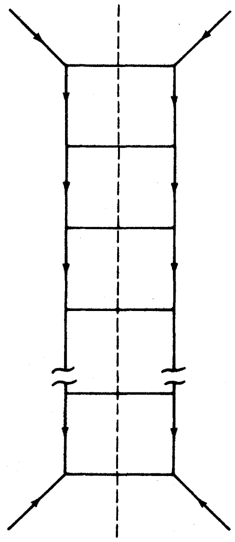

FIG. 2. Terms computed in cases I and II.

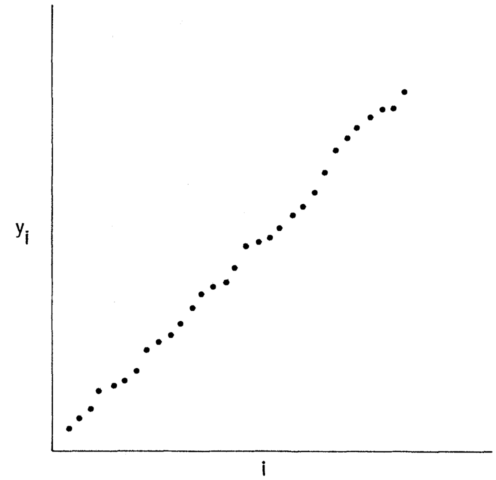

FIG. 3. An allowed set of y_i's in case I: $\rho = \rho_{\mathrm{I}}$.

$$\sigma_N^{\mathrm{II}} = \frac{c}{s}(g^2/16\pi^3)^N \int \prod dy_i d\vec{p}_i |A_N(\{y_i\},\{\vec{p}_i\})|^2, \quad (10)$$

$$\sigma_N^{\mathrm{III}} = \frac{c}{s}(g^2/16\pi^3)^N \sum_p \int \prod dy_i d\vec{p}_i A_N(\{y_i\},\{\vec{p}_i\}) \times A_N^*(\{y_i'\},\{\vec{p}_i'\}), \quad (11)$$

where c is an irrelevant constant, we ignore energy-momentum conservation but restrict rapidities to the interval $(-Y/2, +Y/2)$, and $\sum_p$ denotes a sum over all permutations that take the set $\{y_i'\},\{\vec{p}_i'\}$ into $\{y_i\},\{\vec{p}_i\}$. The transverse-momentum distributions are similarly computed by keeping one of the $\vec{p}_i$ fixed.

III. THE APPROXIMATION

A. Case I: Rapidity order corresponding to chain order

This is the simplest case in our approximation (see Figs. 2 and 3).

1. Case I: 1 + 1 dimensions

It is interesting to first ignore transverse momentum, as in a theory with 1 + 1 dimensions. Then, in this case (1 + 1 dimensions),

$$Q_N \equiv |A_N|^2 = \prod_1^{N-1} (\tau_i + r)^{-2}, \quad (12)$$

where

$$r = \mu^2/m^2$$

and

$$\tau_i = \sum_{j \le i} \sum_{k > i} e^{-(y_k - y_j)}. \quad (12')$$

We have

$$\sigma_N^{\mathrm{I}} = \sigma_0 e^{-Y} \lambda_1^N \int \prod dy_i\, \theta(y_{i+1} - y_i) \prod (\tau_i + r)^{-2}, \quad (13)$$

where

$$\lambda_1 = g^2/16\pi^3 m^4$$

and σ_0 is an irrelevant constant of dimension M^{-2}. Since $y_k > y_j$ for $k > j$, Q_N is of short-range form in y: If $|y_k - y_j| \gg 1$ for any two particles k, j, then Q_N becomes independent of this quantity.[13]

We write

$$\tau_i = \tau_i^+ \tau_i^-, \quad (14)$$

where

$$\tau_i^+ = \sum_{k > i} e^{-(y_k - y_i)},$$

$$\tau_i^- = \sum_{j < i} e^{-(y_i - y_j)}.$$

If $\rho \equiv N/Y$ becomes large, each $\tau_i^\pm$ will be $O(\rho)$, since the number of particles in its "range" is ρ. The important simplification that happens then is that, since the statistical fluctuation in the number of particles in this range is only $O(\rho^{1/2})$ (assuming correlations can be neglected, which will be verified later), then the $\tau_i^\pm$ can be replaced by their mean values:

$$\langle\tau^+\rangle=\langle\tau^- -1\rangle=\sum_{k=i+1}^{N}\frac{\int\prod dy_j\Theta(y_{j+1}-y_j)e^{-(y_k-y_i)}}{\int\prod dy_j\Theta(y_{j+1}-y_j)}=\rho. \tag{15}$$

Alternatively, by fixing the positions of the particles at their mean values,

$$y_i=ai,\quad a=Y/N=\rho^{-1}, \tag{16}$$

we get

$$\langle\tau^-\rangle\approx\langle\tau^+\rangle=\sum_{i+1}^{N}e^{-(k-i)a}=e^{-a}(1-e^{-a(N-i)})(1-e^{-a})^{-1}.$$

Then, since we let $Y\to\infty$ and $N\to\infty$ *first*, for most values of i (those not near N) we can ignore the factor $e^{-a(N-i)}$. Then, we let $a=Y/N$ become small and

$$\langle\tau^-\rangle\approx\langle\tau^+\rangle\underset{a\to 0}{\longrightarrow}a^{-1}+O(1)=\rho+O(1). \tag{17}$$

Then, for most of N-particle phase space, we have

$$\tau_i\approx\langle\tau\rangle=\rho^2 \tag{18}$$

and, since $r\ll\rho^2$

$$Q_N=\langle\tau\rangle^{-2N},$$
$$\sigma_N^{\mathrm{I}}=\sigma_0e^{-Y}\lambda_1{}^N\langle\tau\rangle^{-2N}\int\prod dy_i\,\theta(y_{i+1}-y_i) \tag{19}$$

or

$$\sigma_N^{\mathrm{I}}=\sigma_0e^{-Y}\lambda_1{}^N\rho^{-4N}Y^N/N!. \tag{20}$$

Using Stirling's approximation,[14]

$$\sigma_N^{\mathrm{I}}=\sigma_0e^{-Y}(\lambda_1e)^N(Y/N)^{5N} \tag{21}$$

and

$$\sigma_N^{\mathrm{I}}=\sigma_0e^{-Y}(\alpha_1Y)^{5N}/(5N)!, \tag{22}$$

where

$$\alpha_1=5e^{-4/5}(g^2/16\pi^3m^4)^{1/5}.$$

Defining the partition function, or normalized cross section

$$\Omega(\lambda_1,Y)=\sum_{N=0}^{\infty}\sigma_N/\sigma_0 \tag{23}$$

we have

$$\lim_{Y\to\infty}\ln\Omega(\lambda,Y)=(\alpha_1-1)Y, \tag{24}$$

which can be verified by the method of steepest descent, or by realizing that we have a multiplicity distribution that is Poisson in steps of five. The mean multiplicity is

$$\langle N\rangle=\lambda_1\frac{\partial}{\partial\lambda_1}\ln\Omega(\lambda_1,Y) \tag{25}$$

and the mean density is $\alpha_1/5$. Going back to Eqs. (12) and (12′), we can fix all positions but one at ai, compute the change in Q_N caused by moving one particle a distance $\sim a$ (which is the order that it moves, with respect to the surrounding particles, in the phase-space integration), and verify that it is insignificant, so that the free-particle approximation is justified. The function $Q_N(\{y_i\})$ in $1+1$ dimensions thus becomes similar to that for a uniform system of classical particles interacting with weak forces, with range much longer than the interparticle distance. The approximation we have used is therefore essentially a mean-field one, familiar in statistical mechanics.[9]

2. *Case I: 3+1 dimensions*

When we include transverse momentum, the problem is no longer trivial, but the mathematics is still simple. We write

$$Q_N(\{y_i\},\{\vec{q}_i\})=\prod_1^{N-1}(\tau_i+r_i)^{-2}, \tag{26}$$

where

$$\tau_i=\sum_{j\le i}\sum_{k>i}\sigma_j\sigma_ke^{-(y_k-y_j)},$$
$$r_i=(\mu/m)^2+\vec{K}_i{}^2,$$
$$\vec{K}_i=\sum_1^i\vec{q}_j, \tag{26′}$$
$$\sigma_i=(1+\vec{q}_i{}^2)^{1/2},$$

and

$$\vec{q}_i=\vec{p}_i/m.$$

With this notation

$$\sigma_N^{\mathrm{I}}=\sigma_0e^{-Y}\lambda^N\int\prod dy_i\,d\vec{q}_i\Theta(y_{i+1}-y_i)\times Q_N(\{y_i\},\{\vec{q}_i\}), \tag{27}$$

where

$$\lambda=g^2/16\pi^3m^2.$$

Again, the y dependence is short-range,[15] but the $\vec{q}$ dependence appears in "charges" $\sigma_i\sigma_j$ multiplying the $e^{-(y_k-y_j)}$ terms, and in the r_i terms. We no longer have a uniform-particle distribution, merely bounded by walls, and instead of a mean-field-type approximation we have to use a self-consistent-type theory to describe the (damped) transverse-momentum dependence.

We first simplify the denominators of (26). It is argued *a posteriori* in the Appendix that, for large ρ, the $\vec{K}_i^2$ terms are $O(\rho)$, and we have $r_i \ll \tau_i$ for most of phase space. Also, since it will turn out that the probabilities of the $\vec{q}_i$, and hence of the σ_i, are independent the arguments of the previous section can be repeated and the τ_i remain close to a mean, $\langle\tau\rangle$. Thus, for the important region of integration Q_N stays close to $\langle\tau\rangle^{-2N}$. The phase-space integration does not just give us a $Y^N/N!$ term, however, since we have a transverse-momentum integration to do. We accordingly look for a form

$$Q_N^A = \langle\tau\rangle^{-2N} \prod_1^N h(\vec{q}_i^{\,2}) \tag{28}$$

to represent Q_N.

We ignore the y dependence (which again can be seen to be insignificant), substitute a factor $Y^N/N!$ for the y integrations, and set $y_i = ai$. Then,

$$\tau_i = \sum_{j\le i}\sum_{k>i} \sigma_j \sigma_k e^{-(k-j)a}.$$

We consider the effect of varying one of the σ_j, say for $j=l$, on Q_N.

As seen in (26′), the term in τ_i that depends on σ_l is

$$\delta\tau_i(\sigma_l) = \sigma_l \sum_{j\le i} \sigma_j e^{-(l-j)a} \tag{29}$$

if $l > i$, and

$$\delta\tau_i(\sigma_l) = \sigma_l \sum_{k>i} \sigma_k e^{-(k-l)a} \tag{30}$$

if $l \le i$. If $|i-l|a$ is $O(1)$, there will be a number of terms in (29) or (30), each of which will be $O(1)$ [assuming each σ_j is $O(1)$]. The number of these terms will be $O(\rho)$, and hence $\delta\tau_i(\sigma_l)$, the part of τ_i that depends on σ_l, is $O(\rho)$. Since τ_i is $O(\rho^2)$, as seen in the $(1+1)$-dimensional case, each τ_i^{-2} for which $|i-l|a$ is $O(1)$ will only have a small dependence on σ_l. There will, however, be $O(\rho)$ values of i where τ_i will be perturbed by $O(\rho)$ and the total effect will be to damp σ_l exponentially, even though there is only power damping of the τ_i's.

We write

$$\tau_i \approx \langle\tau\rangle + \delta\tau_i(\sigma_l) - \langle\delta\tau_i(\sigma_l)\rangle$$

and to lowest order in $(\delta\tau_i/\tau_i)$

$$\tau_i^{-2} \approx \langle\tau\rangle^{-2} \exp\left[-2\frac{\delta\tau_i(\sigma_l)}{\langle\tau\rangle} + 2\frac{\langle\delta\tau_i(\sigma_l)\rangle}{\langle\tau\rangle}\right]. \tag{31}$$

Then, as a function of σ_l,

$$\overline{Q}_N(\sigma_l) = \langle\tau\rangle^{-2N} \exp\left[-\frac{2}{\langle\tau\rangle}\left(\sum_i \delta\tau_i(\sigma_l) - \sum_i \langle\delta\tau_i(\sigma_l)\rangle\right)\right]. \tag{32}$$

From (29) and (30),

$$\sum_i \delta\tau_i(\sigma_l) = \sigma_l \sum_j |l-j|\sigma_j e^{-|l-j|a}. \tag{33}$$

Since the sum contains a large number of terms, we replace it by its average,

$$\left\langle \sum_j |l-j|\sigma_j e^{-|l-j|a}\right\rangle = \langle\sigma\rangle \sum_k |k| e^{-|k|a}$$

$$= 2\langle\sigma\rangle\rho^2 + O(\rho) \tag{34}$$

since the σ_j are by hypothesis independent. We also have

$$\langle\tau\rangle = \sum_{j\le i}\sum_{k>i} e^{-(k-j)a} \langle\sigma_j\sigma_k\rangle$$

$$= \langle\sigma\rangle^2\rho^2 + O(\rho). \tag{35}$$

Substituting (34) and (35) into (32),

$$\overline{Q}_N(\sigma_l) = (\rho\langle\sigma\rangle)^{-4N} \exp[-4\sigma_l/\langle\sigma\rangle + 4].$$

If the σ_l's are independent, we should get the same result for each, and (28) is

$$Q_N^A(\{\vec{q}_j\}) = (\rho\langle\sigma\rangle)^{-4N} \exp\left(-\frac{4}{\langle\sigma\rangle}\sum_j \sigma_j + 4N\right). \tag{36}$$

Since the average $\langle\sigma\rangle$ is defined with the above probability function, we have the self-consistency condition

$$\langle\sigma\rangle = \int d\vec{q}\,\sigma e^{-4\sigma/\langle\sigma\rangle} \Big/ \int d\vec{q}\, e^{-4\sigma/\langle\sigma\rangle}. \tag{37}$$

We see that the only quantity that has to be determined self-consistently is the strength of the "field" which damps the σ_j's. The technique that we have used, however, usually results in self-consistency conditions for the *functional form* of the field, as well as the strength: If we take an N-particle probability function with arbitrary pairwise "potentials" $v(\sigma_j, \sigma_k)$ we would get, in general, a nonlinear integral equation for the function $h(\sigma)$ in Eq. (28). In particular, if we had $v(\sigma_j, \sigma_k)$ instead of $\sigma_j\sigma_k$ in (26′), we would get

$$h(\sigma) = \exp\left(-\frac{4\int dq'\, v(\sigma,\sigma')h(\sigma')}{\langle v\rangle \int dq'\, h(\sigma')} + 4\right), \tag{38}$$

where

$$\langle v\rangle = \frac{\int d\sigma_1 d\sigma_2 v(\sigma_1,\sigma_2)h(\sigma_1)h(\sigma_2)}{[\int dq' h(\sigma')]^2}.$$

It is only the fact that $v(\sigma,\sigma')$ is a separable potential in our case $(\sigma\sigma')$ that enables us to immediately determine the functional form of $h(\sigma)$. It is also this fact, of course, which would enable us to write a linear integral equation, whose solution would give the exact exponent and inclusive distribution in this case.

From the form of Eqs. (36) and (37), we can see that $h(\sigma)$ attains a limiting form as ρ becomes large.

Solving (37), we have

$$q\,dq = \sigma\,d\sigma \tag{39}$$

and

$$\langle\sigma\rangle \int_1^\infty d\sigma\,\sigma e^{-4\sigma/\langle\sigma\rangle} = \int_1^\infty d\sigma\,\sigma^2 e^{-4\sigma/\langle\sigma\rangle}. \tag{40}$$

Substituting

$$\beta = 4/\langle\sigma\rangle$$

and doing the integrations,

$$\frac{4}{\beta}e^{-\beta}\left(\frac{1}{\beta}+\frac{1}{\beta^2}\right) = -\frac{\partial}{\partial\beta}e^{-\beta}\left(\frac{1}{\beta}+\frac{1}{\beta^2}\right) \tag{41}$$

or

$$\beta^2 - 2\beta - 2 = 0. \tag{42}$$

Solving this simple self-consistency equation,

$$\beta = \beta_0 = 1+\sqrt{3}. \tag{43}$$

Our first result is that the (normalized) single-particle distribution approaches a limiting form:

$$\rho(\vec{q}) = h(\vec{q})/h_0 = e^{-\beta_0\sigma}/\pi e^{-\beta_0}, \tag{44}$$

where

$$h(\vec{q}) = e^{-\beta_0\sigma+4}, \quad h_0 = \pi e^{-\beta_0+4}. \tag{44'}$$

Then, from (27),

$$\sigma_N^{\mathrm{I}} = \sigma_0 e^{-Y}\lambda^N (Y^N/N!)(\rho\langle\sigma\rangle)^{-4N} h_0^N. \tag{45}$$

Using (44′), and Stirling's approximation,

$$\sigma_N^{\mathrm{I}} = \sigma_0 e^{-Y}\lambda^N (Y/N)^{5N} e^{5N}(\beta_0/4)^{4N}(\pi e^{-\beta_0})^N$$

or

$$\sigma_N^{\mathrm{I}} = \sigma_0 e^{-Y}(\alpha^{\mathrm{I}}Y)^{5N}/(5N)!, \tag{46}$$

where

$$\alpha^{\mathrm{I}} = 5\left[\left(\frac{\beta_0}{4}\right)^4 e^{-\beta_0}\right]^{1/5}(g^2/16\pi^2 m^2)^{1/5}. \tag{46'}$$

We then see that the exponent is

$$\alpha \to \alpha^{\mathrm{I}}$$

and the mean density

$$\rho = \alpha^{\mathrm{I}}/5, \tag{47}$$

where the multiplicity distribution is again Poisson in steps of five.

B. Case II: Ladder approximation-no rapidity order (see Figs. 2 and 4)

The exponent and inclusive distributions can also be derived, in this case, from the solutions of the Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) integral equation.[1] This equation is very difficult to solve, in general, and even if it were solved exactly we would still only have an approximate solution to the general problem, because we have ignored interference terms.

By constraining rapidities to have the same order as the chain order, we can define a modified amplitude [with $\theta(y_{i+1}-y_i)$ terms] that results in our case I. Also, case III corresponds, of course, to the exact amplitude. There is no simple set of approximations, however, that we can make in the amplitude in order to arrive at case II, and from our point of view its solution does not give us as much insight into the kind of multiparticle distributions that result from a certain kind of amplitude.

There are, however, many advantages to this case, since properties of the solutions can be obtained from the integral equation. From an analytical point of view, for instance, this case can give us a $2\to 2$ amplitude, through unitarity, that can be continued in s and t. Contact can then be made with the Bethe-Salpeter approximation for

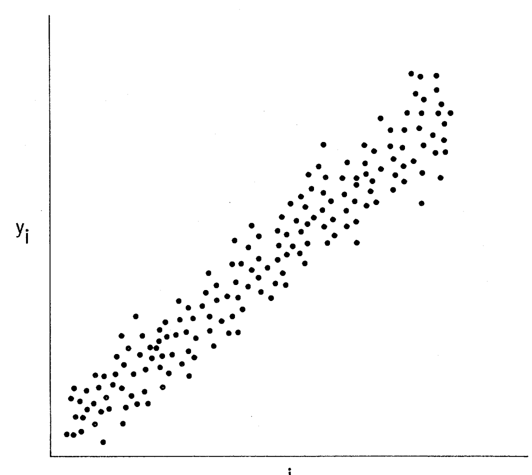

FIG. 4. An allowed set of y_i's in cases II and III: $\rho = 5\rho_{\mathrm{I}}$.

dealing with bound states. Also, in this case, the exponent has been computed in the high-density limit in the integral-equation formulation,[16] and our technique can be checked.

The probability function Q_N is given by Eq. (26), as in case I. Here, however, there are no y_i restrictions and the correspondence between Q_N and the probability function of a classical gas of particles is not as direct as in case I. We can, however, relabel the particles by their rapidity order and sum over all chain orderings (configurations) for a given set $\{y_i\},\{\vec{q}_i\}$. Then, it can be seen that if the distance $y_k - y_j$, $k>j$ is large, the configurations split into two sets. In one, $k'<j'$, where k' (j') is the chain position of particle k (j) and the Q_N's for these configurations vanish. In the other set $k'>j'$, and the rapidity order for these two particles is the same as the chain order. This latter set of Q_N do not vanish but, as in case I, become independent of $(y_k - y_j)$. Thus, the sum of all configurations has a short-range form in y

The N-particle cross section is now (labeling chain order)

$$\sigma_N^{II} = \sigma_0 e^{-Y}\lambda^N \int \prod dy_i d\vec{q}_i Q_N(\{y_i\},\{\vec{q}_i\}) \tag{48}$$

and we have to take account of overlap of the particles ($y_j > y_k$, but $k<j$). Instead of using a mean-field type of approximation and ignoring the y_i dependence, it will turn out that we also need a self-consistent field type of approximation for this variable. Separating out the dependence of τ_i on σ_l, y_l, as in the last section, we have

$$\delta\tau_i(\sigma_l, y_l) = \sigma_l e^{y_l}\sum_{k>i}\sigma_k e^{-y_k} \tag{49}$$

if $l \le i$ and

$$\delta\tau_i(\sigma_l, y_l) = \sigma_l e^{-y_l}\sum_{j\le i}\sigma_j e^{y_j} \tag{50}$$

if $l>i$. It will turn out that $\delta\tau_i$ will be $O(\rho)$ and τ_i will be $O(\rho^2)$, as in the last section. Then, the variation of τ_i^{-2} with σ_l, y_l is

$$\tau_i^{-2} \approx \langle\tau\rangle^{-2}\exp(-2\delta\tau_i/\langle\tau\rangle + 2\langle\delta\tau_i\rangle/\langle\tau\rangle). \tag{31}$$

For a given set $\{y_i\},\{\vec{q}_i\}$, we can then get the dependence of Q_N on y_l and σ_l:

$$\overline{Q}_N(\sigma_l, y_l) = \langle\tau\rangle^{-2N}\exp\left(-\frac{2\sigma_l}{\langle\tau\rangle}(e^{+y_l}v_l^+ + e^{-y_l}v_l^-) + \frac{2}{\langle\tau\rangle}\sum\langle\delta\tau_i\rangle\right), \tag{51}$$

where

$$v_l^+ = \sum_{i\ge l}\sum_{k>i}\sigma_k e^{-y_k} = \sum_{k>l}(k-l)\sigma_k e^{-y_k},$$

$$v_l^- = \sum_{i<l}\sum_{j\le i}\sigma_j e^{y_j} = \sum_{j<l}(l-j)\sigma_j e^{y_j}.$$

We write

$$e^{y_l}v_l^+ + e^{-y_l}v_l^- = 2(v_l^+v_l^-)^{1/2}\cosh(y_l - y_l^0), \tag{52}$$

$$y_l^0 = \tfrac{1}{2}\ln(v_l^-/v_l^+),$$

and[17]

$$\overline{Q}_N(\sigma_l, y_l) = \langle\tau\rangle^{-2N}\exp[-\beta_l\sigma_l\cosh(y_l - y_l^0) + \langle\beta_l\sigma_l\cosh(y_l - y_l^0)\rangle], \tag{53}$$

$$\beta_l = 4(v_l^+v_l^-)^{1/2}/\langle\tau\rangle.$$

Since β_l, y_l depend on the coordinates of many nearby particles, we have the picture of each particle being confined in a "potential well" in σ_l, y_l formed by a large number of nearby particles.

We write

$$v_l^+v_l^- = \sum_{j<l}\sum_{k>l}(l-j)(k-l)\sigma_j\sigma_k e^{-(y_k-y_j)}, \tag{54}$$

$$\tau_l = \sum_{j\le l}\sum_{k>l}\sigma_j\sigma_k e^{-(y_k-y_j)}, \tag{55}$$

and assume that these quantities do not vary too much. Replacing them by their averages, $\langle v_l^+v_l^-\rangle$, $\langle\tau\rangle$, we have the independent-particle approximation

$$Q_N^A = \langle\tau\rangle^{-2N}\prod h_l(\vec{q}_l, y_l), \tag{56}$$

where

$$h_l(\vec{q}_l, y_l) = \exp[-\beta\sigma_l\cosh(y_l - y_l^0) + \beta\langle\sigma_l\cosh(y_l - y_l^0)\rangle], \tag{56'}$$

$$\beta = 4(\langle v_l^+v_l^-\rangle)^{1/2}/\langle\tau\rangle.$$

The average of a function is given by

$$\langle U(\vec{q}_l, y_l)\rangle = \int d\vec{q}_l dy_l h_l(\vec{q}_l, y_l)U(\vec{q}_l, y_l)/h_0, \tag{57}$$

$$h_0 = \int d\vec{q}_l dy_l h_l(\vec{q}_l, y_l). \tag{57'}$$

Rewriting

$$v_l^+v_l^- = \sum_{j<l}(l-j)\sigma_j e^{-(y_l^0-y_j)}\sum_{k>l}(k-l)\sigma_k e^{-(y_k-y_l^0)}, \tag{58}$$

$$\tau_l = \sum_{j\le l}\sigma_j e^{-(y_l^0-y_j)}\sum_{k>l}\sigma_k e^{-(y_k-y_l^0)}. \tag{59}$$

We see that we need the averages

$$\langle\sigma_k e^{\pm(y_k-y_l^0)}\rangle = \frac{1}{h_0}\int d\vec{q}_k dy_k\sigma_k\, e^{\pm(y_k-y_l^0)}e^{-\beta\sigma_k\cosh(y_k-y_k^0)+c}.$$

Defining

$$b = \langle \sigma_k \cosh(y_k - y_k^0) \rangle = \langle \sigma_k e^{\pm(y_k - y_k^0)} \rangle, \tag{60}$$

we have

$$\langle \sigma_k e^{\pm(y_k - y_l^0)} \rangle = b e^{\pm(y_k^0 - y_l^0)},$$

$$c = \beta b,$$

$$b = \frac{1}{h_0} \int d\vec{q}\, dx\, \sigma(\vec{q}) \cosh x\, e^{-\beta\sigma \cosh x + c}. \tag{61}$$

Then,

$$\langle v_l^+ v_l^- \rangle = b^2 \sum_{k>l} (k-l) e^{-(y_k^0 - y_l^0)} \sum_{j<l} (l-j) e^{-(y_l^0 - y_j^0)} \tag{62}$$

and

$$\langle \tau_l \rangle = b^2 \sum_{k>l} e^{-(y_k^0 - y_l^0)} \sum_{j<l} e^{-(y_l^0 - y_j^0)}. \tag{62'}$$

Also, for a given set of y_j^0, we have the average

$$\langle y_l^0 \rangle = \tfrac{1}{2} \ln\Big(\sum_{j<l} (l-j) e^{y_j^0} \Big) - \tfrac{1}{2} \ln\Big(\sum_{k>l} (k-l) e^{-y_k^0} \Big). \tag{63}$$

If we replace each y_j^0 with its mean value, we have

$$y_j^0 \to \langle y_j^0 \rangle = aj, \tag{64}$$

which can be checked by substitution in (63).

The main point is that y_l^0, as given by (63), does not vary too much from its mean value as we integrate over the y_j's. This small variation does not affect any of the integrals that we do, and we can set each y_j^0 at its mean value. Of course, this approximation is only used locally, for regions in y of $O(1)$, as in Eq. (61): The picture of particles confined to regions centered at aj is not correct over large distances and there is no long-range order. Figure 4 describes a configuration in y_i. The approximation is that each particle "feels" a force from a uniform fluid of nearby particles. With this substitution we have

$$\langle v_l^+ v_l^- \rangle = b^2 \rho^4, \tag{65}$$

$$\langle \tau_l \rangle = b^2 \rho^2, \tag{66}$$

and

$$\beta = 4/b, \quad b = \langle \sigma \cosh x \rangle. \tag{67}$$

We then have the self-consistency condition,[18] using (61),

$$4/\beta_0 = \frac{\int d\vec{q}\, dx\, \sigma(\vec{q}) \cosh x\, e^{-\beta_0 \sigma \cosh x + 4}}{\int d\vec{q}\, dx\, e^{-\beta_0 \sigma \cosh x + 4}}, \tag{68}$$

which can be rewritten

$$\frac{4}{\beta} h_0(\beta) \Big|_{\beta_0} = -\frac{\partial}{\partial \beta} h_0(\beta) \Big|_{\beta_0}. \tag{69}$$

Doing the $\vec{q}$ integration in (57′), with $q\, dq = \sigma\, d\sigma$,

$$h_0(\beta) = 4\pi e^4 \int_0^\infty dx\, e^{-\beta \cosh x} (1/\beta \cosh x + 1/\beta^2 \cosh^2 x). \tag{70}$$

Forming $\beta^2 h_0(\beta)$ and differentiating,

$$\frac{\partial}{\partial \beta} \beta^2 h_0(\beta) = -4\pi e^4 \beta \int_0^\infty dx\, e^{-\beta \cosh x} = -4\pi e^4 \beta K_0(\beta). \tag{71}$$

But

$$\frac{\partial}{\partial \beta} \beta K_1(\beta) = -\beta K_0(\beta),$$

where $K_0(\beta)$ and $K_1(\beta)$ are modified Bessel functions.[19]

Since both $\beta h_0(\beta)$ and $K_1(\beta) \to 0$ as $\beta \to \infty$,

$$h_0(\beta) = 4\pi e^4 K_1(\beta)/\beta. \tag{72}$$

The self-consistency condition then requires that at $\beta = \beta_0$

$$\frac{\partial}{\partial \beta} \beta^3 K_1(\beta) \Big|_{\beta_0} = 0 \tag{73}$$

or

$$2K_1(\beta_0) = \beta_0 K_0(\beta_0), \quad h_0(\beta_0) = 2\pi e^4 K_0(\beta_0). \tag{74}$$

This is precisely the condition that Chang and Rosner[16] obtained. There, however, β was related to the radius of a classical orbit in the t channel, obtained as a large coupling solution of the Bethe-Salpeter equation, and the condition was that the angular momentum be maximized.

Our first result is that the (normalized) single-particle distribution approaches a limiting form. Since

$$h(\vec{q}) = \int_{-\infty}^{+\infty} dx\, h(\vec{q}, x) = e^4 \int_{-\infty}^{+\infty} dx\, e^{-\beta_0 \sigma \cosh x} = 2e^4 K_0(\beta_0 \sigma), \tag{75}$$

the transverse-momentum distribution is,[20] using (74),

$$\rho(\vec{q}) = K_0(\beta_0 \sigma(\vec{q}))/\pi K_0(\beta_0). \tag{76}$$

Solving (48), with (56), (57′), and (66),

$$\sigma_N^{\rm II} = \sigma_0 e^{-y} \lambda^N (\rho b)^{-4N} h_0{}^N. \tag{77}$$

Using (67) and (74)

$$\sigma_N^{\rm II} = \sigma_0 e^{-Y} \lambda^N (Y/N)^{4N} (\beta_0/4)^{4N} [2\pi e^4 K_0(\beta_0)]^N \tag{78}$$

or

$$\sigma_N^{II} = \sigma_0 e^{-Y}(\alpha^{II}Y)^{4N}/(4N)!,$$

where

$$\alpha^{II} = [2\beta_0{}^4 K_0(\beta_0)]^{1/4}\left(\frac{g^2}{16\pi^2 m^2}\right)^{1/4}. \tag{79}$$

The exponent is then

$$\alpha \to \alpha^{II}, \tag{80}$$

which agrees with Ref. 16 and the mean density

$$\rho = \alpha^{II}/4, \tag{81}$$

where the multiplicity distribution is now Poisson in steps of four.

C. Case III: No restrictions (see Figs. 4 and 5)

We start with a probability function similar to (26), but with an amplitude without correlations, instead of A_N^* multiplying $A_N(\{\vec{q}_i\}, \{y_i\})$. Then,

$$Q_N^L(\{\vec{q}_i\}, \{y_i\}) = \prod_1^{N-1} (\tau_i + r_i)^{-1} \prod_1^N f(\vec{q}_i). \tag{82}$$

The idea is that if Q_N^L can be approximated by a function

$$Q_N^A(\{\vec{q}_i\}) = \prod F(\vec{q}_i) \tag{83}$$

without correlations, after all (left-hand) permutations are summed over, then we can make the identification

$$F(\vec{q}) = f^2(\vec{q}), \tag{84}$$

where we use $\prod f(\vec{q})$ to represent $A_N^*(\{\vec{q}_i\}, \{y_i\})$, with all (right-hand) permutations summed over. Proceeding as in the last section,

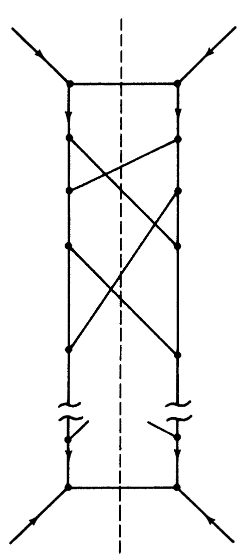

FIG. 5. One of the terms computed in case III.

$$Q_N^A = \langle\tau\rangle^{-N} \prod f(\vec{q}_l) h_l(\vec{q}_l, y_l), \tag{85}$$

where

$$h_l(\vec{q}_l, y_l) = \exp[-\beta\sigma_l \cosh(y_l - y_l^0) + c], \tag{85$'$}$$

$$\beta = 2(\langle v_l^+ v_l^- \rangle)^{1/2}/\langle\tau\rangle,$$

$$c = \beta\langle\sigma_l \cosh(y_l - y_l^0)\rangle.$$

We have a factor of 2 instead of 4 for β, since τ_i^{-1}, instead of τ_i^{-2}, was expanded. The average $\langle\sigma_l \cosh(y_l - y_l^0)\rangle$ is now

$$b = \frac{1}{h_0'} \int d\vec{q}_l dy_l \sigma_l \cosh(y_l - y_l^0) f(\vec{q}_l) h_l(\vec{q}_l, y_l), \tag{86}$$

where

$$h_0' = \int d\vec{q}\, dy f(\vec{q}) h_l(\vec{q}, y). \tag{86$'$}$$

To accomplish the summation over permutations, we first integrate over all y_i without restrictions, where, as in the last section, i labels the chain order. Since we get an uncorrelated probability function then, it does not matter if the labels on the $\vec{q}_i$ refer to chain or y order, and the function would be the same if we had summed over all chain permutations first and then integrated over ordered y_i's $(y_{i+1} > y_i)$. If further, we assume that for each permutation the amplitude does not change very much if each y_i is changed by $O(a)$, as in Sec. III A, then the sum over all permutations without the (ordered) y_i integrations is just the above function divided by $Y^N/N!$. We then have the product approximation for the complete function

$$Q_N = A_N A_N^* \tag{87}$$

summed over both left and right permutations for ordered y_i, but not integrated over y_i;

$$Q_N^A = \langle\tau\rangle^{-N} \frac{N!}{Y^N} \prod f(\vec{q}_l) h_1(\vec{q}_l), \tag{88}$$

where

$$h_1(\vec{q}_l) = \int dy_l h_l'(\vec{q}_l, y_l). \tag{89}$$

Using Stirling's approximation, we then have

$$f(\vec{q}) = \langle\tau\rangle^{-1}(N/Ye) h_1(\vec{q}), \tag{90}$$

$$Q_N^A = \prod [f(\vec{q})]^2$$
$$= \langle\tau\rangle^{-2N}(N/Ye)^{2N} \prod [h_1(\vec{q}_l)]^2. \tag{91}$$

We then have

$$b = \frac{1}{h_0} \int d\vec{q}\, \sigma(\vec{q}) \int dx \cosh x\, e^{-\beta\sigma \cosh x + c} \times \int dx' e^{-\beta\sigma \cosh x' + c}, \tag{92}$$

$$h_0 = \int d\vec{q} \int dx \int dx' e^{-\beta\sigma(\cosh x + \cosh x') + 2c}, \tag{93}$$

where we redefine

$$h_0 = \int d\vec{q} \int dx h_1(\vec{q}) h_l(\vec{q}, x)$$

$$= \int d\vec{q} [h_1(\vec{q})]^2. \tag{94}$$

Also, as in the last section,

$$\begin{aligned} &\beta = 2/b, \\ &c = \beta b = 2, \\ &\langle\tau\rangle = b^2\rho^2. \end{aligned} \tag{95}$$

We have

$$\frac{2}{\beta} h_0(\beta)\bigg|_{\beta_0} = -\frac{1}{2}\frac{\partial}{\partial\beta} h_0(\beta)\bigg|_{\beta_0} \tag{96}$$

because of symmetry in x and x' in (92), and we get exactly the same form

$$\frac{4}{\beta} h_0(\beta)\bigg|_{\beta_0} = -\frac{\partial}{\partial\beta} h_0(\beta) \tag{69}$$

or

$$\frac{\partial}{\partial\beta}[\beta^4 h_0(\beta)]\bigg|_{\beta_0} = 0 \tag{97}$$

for the self-consistency condition. The function $h_0(\beta)$ is different, though. Doing the $\vec{q}$ integration we have, with $c(x, x') = \cosh x + \cosh x'$,

$$\beta^2 h_0(\beta) = 2\pi e^4 \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' e^{-\beta c(x,x')} \times [\beta c(x, x')^{-1} + c(x, x')^{-2}] \tag{98}$$

and we can see that

$$\frac{\partial}{\partial\beta}\beta^2 h_0(\beta) = -8\pi e^4 \beta K_0^2(\beta). \tag{99}$$

It finally turns out that

$$h_0(\beta) = 4\pi e^4[K_1^2(\beta) - K_0^2(\beta)]. \tag{100}$$

The self-consistency relation (99) can then be written in a simpler form:

$$K_1(\beta_0) = \sqrt{2}\, K_0(\beta_0) \tag{101}$$

and

$$h_0(\beta_0) = 4\pi e^4 K_0^2(\beta_0). \tag{102}$$

The normalized transverse-momentum distribution is now[21]

$$\rho(\vec{q}) = h_1^2(\vec{q})/h_0, \tag{103}$$

where from (85′), (89), and (95)

$$h_1(\vec{q}) = 2e^2 K_0(\beta_0\sigma(\vec{q})) \tag{104}$$

and

$$\rho(\vec{q}) = K_0^2(\beta_0\sigma(\vec{q}))/\pi K_0^2(\beta_0). \tag{105}$$

Using (94) and (91), doing the $\vec{q}$ integrations, and introducing a factor $Y^N/N!$ for the y integrations, we have the N-particle total cross section:

$$\sigma_N^{\text{III}} = \sigma_0 e^{-Y} \lambda^N \langle\rho b\rangle^{-4N} (N/Ye)^N h_0^{\,N}.$$

Using (95) and (102),

$$\sigma_N^{\text{III}} = \sigma_0 e^{-Y} \lambda^N (Y/N)^{3N} (\beta_0/2)^{4N} [4\pi e^3 K_0^2(\beta_0)]^N$$

or

$$\sigma_N^{\text{III}} = \sigma_0 e^{-Y} (\alpha^{\text{III}} Y)^{3N}/(3N)!, \tag{106}$$

where

$$\alpha^{\text{III}} = 3\left[\frac{\beta_0^4}{4} K_0^2(\beta_0)\right]^{1/3} \left(\frac{g^2}{16\pi^2 m^2}\right)^{1/3}. \tag{107}$$

Also

$$\alpha \to \alpha^{\text{III}}, \tag{108}$$

which is the Regge exponent, the mean density

$$\rho = \alpha^{\text{III}}/3, \tag{109}$$

and the multiplicity distribution is Poisson in steps of three.

IV. DISCUSSION

We first rewrite the formulas that we have obtained for 3+1 dimensions. For all three cases we have the normalized transverse-momentum distribution:

$$\rho(\vec{q}) = \frac{1}{\pi}\frac{H(\gamma\sigma)}{H(\gamma)}, \tag{110}$$

where

$$\sigma = (1 + \vec{q}^{\,2})^{1/2},$$

$$H(z) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx'\, F(x, x') \times \exp\left(-\frac{z}{2}(\cosh x + \cosh x')\right).$$

The condition determining γ can be written

$$\int d\vec{q}\, \rho(\vec{q}) = 1 \tag{111}$$

and the exponent

$$\alpha \to \nu\left[\left(\frac{\gamma}{4}\right)^4 H(\gamma)\right]^{1/\nu} \lambda_g^{\,1/\nu} \tag{112}$$

where

$$\lambda_g = g^2/16\pi^2 m^2.$$

The multiplicity distribution can be determined from (112), and

$$\rho \to \alpha/\nu, \tag{113}$$

and the distribution is Poisson in steps of ν.

The quantities ν, $F(x,x')$, and the resulting $H(z)$ are

case I.

$$\nu = 5,$$
$$F(x,x') = \delta(x)\delta(x'),$$
$$H(z) = e^{-z}.$$

case II.

$$\nu = 4,$$
$$F(x,x') = \delta(x-x'), \tag{114}$$
$$H(z) = 2K_0(z).$$

case III.

$$\nu = 3,$$
$$F(x,x') = 1,$$
$$H(z) = 4K_0^2(z/2).$$

It is easy to understand the form of $F(x,x')$ and α in the three cases: Labeling the particles by their y order, we can define an "internal" variable x (x') for each particle such that x_i/a (x_i'/a) is the difference between the y order and the left (right) chain order. Introducing a factor that corrects for overcounting, in the high-density limit we can label the permutations of the indices in the chains with the set of numbers $\{x_i\}$, $\{x_i'\}$. Then, in case I we set $x_i = x_i' = 0$; in case II $x_i = x_i'$ and integrate over each x_i independently; and in case III integrate over each x_i and x_i' independently. Since each x_i turns out to be $O(1)$, we get a factor a^{-N} in σ_N for case II and an a^{-2N} in case III when converting the sum over permutations to integrals over the x_i. Then, we have

case I.

$$\sigma_N \sim t^{-2N} Y^N/N! \sim a^{5N}, \quad \alpha \sim \lambda_g^{1/5},$$

case II.

$$\sigma_N \sim a^{4N}, \quad \alpha \sim \lambda_g^{1/4}, \tag{115}$$

case III.

$$\sigma_N \sim a^{3N}, \quad \alpha \sim \lambda_g^{1/3}.$$

It is interesting to interpret these results in terms of a fluid "pressure" p. Regarding λ_g as a dummy variable, we can define

$$p(\lambda_g) = \alpha(\lambda_g + 1) = \frac{1}{Y}\ln[e^Y \Omega(\lambda_g, Y)],$$

where $\Omega(\lambda_g, Y)$ is defined in Eq. (23), and we have $\rho = \lambda_g(\partial/\partial\lambda_g)p$. Eliminating λ_g,

$$p = \nu\rho, \tag{116}$$

where

$$\alpha \to c\lambda_g^{1/\nu}.$$

For a noninteracting gas, which we have in our system in the low-density limit,[2]

$$p = \rho, \tag{117}$$

so that in the three cases as p increases $dp/d\rho$ changes from 1 to ν. The fact that $\nu > 1$ means that there are effectively repulsive potentials. Also, the fact that p depends linearly on ρ for large ρ means that the potential that each particle feels saturates, or reaches a limiting value, as the density increases. This is also evidenced in the limiting forms of $\rho(\vec{q})$. This "soft" behavior of the amplitude results from the power damping in the t_i's. If we instead had a form[5]

$$A^E = \prod e^{\beta t_i} \tag{118}$$

it can be seen that

$$Q_N^E = \exp\left(-\beta \sum_{i<j} (j-i)\sigma_i\sigma_j e^{-(y_j - y_i)} - \beta \sum_i \vec{K}_i^{\,2}\right) \tag{119}$$

and in the high-density limit

$$\sigma_N \sim \left(\frac{N}{Y}\right)^{dN} \exp(-\beta' N^3/Y^2), \tag{120}$$

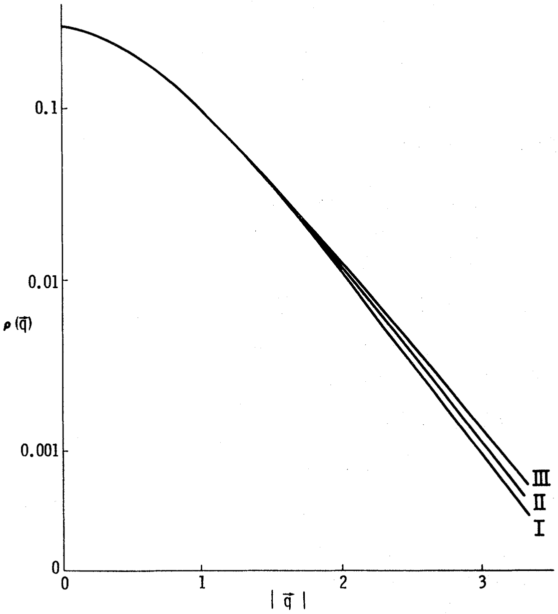

FIG. 6. Normalized transverse-momentum distributions in cases I, II, and III.

where d is an irrelevant constant. The pressure can be computed and turns out to be

$$p \to \text{const} \times \rho^3, \tag{121}$$

which means that the particles feel an ever-increasing potential. Also, there is then no limiting transverse-momentum distribution.

The normalized transverse-momentum distributions are presented in Fig. 6 for cases I, II, and III. It is seen that the results do not differ significantly at smaller values of $|\vec{q}|$. These results are meant to hold if, for fixed $\vec{q}$, ρ becomes large. For fixed but large ρ, however, as we increase $\vec{q}$ we eventually reach a point at which correlations cannot be neglected. Our approximation breaks down for these events, even though most of the distribution would be well represented if ρ is large. For instance, for sufficiently large $\vec{q}$ we would expect to recover the scaling in $x_\perp = m|\vec{q}|/\sqrt{s}$ described in Ref. 22. Perhaps the technique could still be used here, if the high $|\vec{q}|$ particle were treated separately, with the other particles moving in a self-consistent field together with a fixed external field resulting from this particle.

Also it should be mentioned that we have only presented the technique for a simple ϕ^3 theory. Before any detailed comparison with experiment can be made, energy-momentum conservation and resonance decay should be treated. These problems exist in most multiparticle calculations, and in this work we have dealt with the basic multiperipheral mechanism-independent damping in the t_i, rather than with these questions.

In conclusion, we have represented the multiperipheral amplitude with power damping in t_i by a probability function that is factorizable in the momenta of the particles. It was shown that this form becomes exact in the high-density limit, where each $\vec{q}_i$ is damped by an average "potential" which results from many nearby particles. A self-consistent form was obtained for this potential when rapidity orders were the same as chain orders, only the left and right chain orders were restricted to be equal, and in the unrestricted case (with all interference terms). It was seen that the addition of the interference terms significantly affected the exponent, but not the inclusive spectra. In all three cases the transverse-momentum distribution was found to have a limiting form as $\rho \to \infty$.

Two extensions of the theory should be looked into.

First, finite density corrections should be found, so that the technique can be applied to the multiperipheral model in more general situations.

Second, since our model, considered as a particular set of Feynman diagrams, has proved amenable to the technique, perhaps other, more general sets of diagrams will also simplify in the large-coupling limit. The crucial property here seems to be that damping in the t_i be only power-like for large t_i. If that should be the case, we would be in a better situation than classical many-body workers, who must generally deal with potentials that become infinite when particles are close, or densities are high.

APPENDIX

We argue heuristically that the $\vec{K}_i^2$ terms in the probability function (26) can be neglected. First, we assume that $\vec{K}_i^2$ is small compared to τ_i, and then calculate the mean value, which will turn out to satisfy our assumption. With these terms, the function defined in (28) becomes

$$Q_N^A(\{\vec{q}_i\}) = \langle\tau\rangle^{-2N} \prod h(\vec{q}_i) \prod e^{-2\vec{K}_i^2/\langle\tau\rangle}. \tag{122}$$

The terms $h(\vec{q})$ provide damping in $|\vec{q}_i|$, and $\vec{K}_i$ varies in random-walk fashion, and would increase indefinitely as i increases, were it not for the last term. The question then is does the last term affect any of the integrals? Instead of using the form (44′) for $h(\vec{q})$, we use a Gaussian. Then, we have the form

$$Q_N^K(\{\vec{q}_i\}) = \left(\frac{b}{\pi}\right)^N \prod e^{-b\vec{q}_i^2} \prod e^{-(2/\langle\tau\rangle)\vec{K}_i^2}, \tag{123}$$

where we ignore the $\langle\tau\rangle^{-2N}$ term, and introduce a ρ-independent constant b which is $O(1)$. Without the last term, we would have

$$\sigma_N^K = \int \prod d\vec{q}_i Q_N = 1.$$

Defining

$$\sigma_N(\vec{K}) = \int \prod d\vec{q}_i Q_N^K(\{\vec{q}_i\})\, \delta\left(\vec{K} - \sum_1^N \vec{q}_i\right), \tag{124}$$

we have the relation

$$\sigma_{N+1}(\vec{K}) = \frac{b}{\pi} e^{-(2/\langle\tau\rangle)\vec{K}^2} \int d\vec{q}\, e^{-b\vec{q}^2} \sigma_N(\vec{K} - \vec{q}). \tag{125}$$

Since

$$\sigma_2(\vec{K}) = \frac{b}{\pi} \exp\left[-\left(b + \frac{2}{\langle\tau\rangle}\right)\vec{K}^2\right], \tag{126}$$

we take a Gaussian form for $\sigma_N(\vec{K})$;

$$\sigma_N(\vec{K}) = \lambda_N e^{-b_N \vec{K}^2}. \tag{127}$$

Using (125) and (127) and completing squares,

$$\sigma_{N+1}(\vec{K}) = \frac{b}{b + b_N} \lambda_N \exp\left[-\left(\frac{b b_N}{b + b_N} + \frac{2}{\langle\tau\rangle}\right)\vec{K}^2\right], \tag{128}$$

or

$$b_{N+1}=\frac{bb_N}{b+b_N}+\frac{2}{\langle\tau\rangle} \qquad (129)$$

and

$$\lambda_{N+1}=\frac{b}{b+b_N}\lambda_N.$$

Since

$$\lambda_1=b/\pi,$$

$$b_1=b+2/\langle\tau\rangle,$$

we see that, as $N\to\infty$

$$b_{N+1}\to b_N\equiv B, \qquad (130)$$

where

$$B\ll b.$$

We then have

$$B=B(1+B/b)^{-1}+2/\langle\tau\rangle$$

or

$$B\approx\left(\frac{2b}{\langle\tau\rangle}\right)^{1/2} \qquad (131)$$

and

$$\lambda_{N+1}\to\lambda_N(1-B/b)^N.$$

We then have

$$\sigma_N(\vec{K})\to\left[1-\left(\frac{2}{b\langle\tau\rangle}\right)^{1/2}\right]^N\exp\left[-\left(\frac{2b}{\langle\tau\rangle}\right)^{1/2}\vec{K}^2\right]. \qquad (132)$$

Thus, integrating over $\vec{K}$, we have now

$$\sigma_N(\vec{K})\to\left[1-\left(\frac{2}{b\langle\tau\rangle}\right)^{1/2}\right]^N$$

$$\to\exp\left[-\left(\frac{2}{b\langle\tau\rangle}\right)^{1/2}N\right] \qquad (133)$$

instead of 1, and

$$\langle\vec{K}^2\rangle\sim O\left(\left(\frac{\langle\tau\rangle}{2b}\right)^{1/2}\right)\ll\langle\tau\rangle. \qquad (134)$$

As $\langle\tau\rangle$ becomes large, we conclude that the error made in neglecting $\vec{K}_i^{\,2}$ becomes negligible.

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[1]D. Amati, S. Fubini, and A. Stanghellini [Nuovo Cimento 26, 896 (1962)] discussed the original model; K. Wilson, [Cornell Report No. CLNS-131, 1970 (unpublished)] and more recently R. C. Arnold [Argonne Report No. ANL/HEP 7317, 1973 (unpublished)] discussed the analogy with a classical gas.

[2]D. K. Campbell and S.-J. Chang, Phys. Rev. D 4, 3658 (1971).

[3]T. D. Lee [Phys. Rev. D 6, 3617 (1972)] defines particle coordinates that are related to momenta of the exchanged particles, rather than the produced ones. See also D. K. Campbell, Phys. Rev. D 6, 2658 (1972).

[4]D. Snider and D. Tow [Phys. Rev. D 3, 996 (1971)] and H. W. Wyld [Phys. Rev. D 3, 3090 (1971)] computed the effects of the first crossed rung.

[5]R. F. Amann and P. M. Shah [Phys. Lett. 42B, 353 (1972)], using a similar model, numerically computed the effects of all relevant crossed terms in specific cases, and concluded that many interference terms were (collectively) important, even at moderate densities. Snider and Tow (see Ref. 4), however, concluded that the effect of the first crossed rung on α was small, when the coupling was chosen such that $\alpha\sim 1$.

[6]S.-J. Chang and T.-M. Yan [Phys. Rev. D 7, 3698 (1973)] discussed the ϕ^3 model in the standard ladder approximation. They used a gas-analog point of view, as we do, and obtained bounds for the exponent in the large-coupling limit.

[7]J. Steinhoff [Nucl. Phys. B55, 132 (1973)] has developed a technique for evaluating interference terms resulting from independent emission of pion pairs in the amplitude. It will be seen that the multiperipheral amplitude with ρ mesons results in just such a situation at high density.

[8]H. Cheng and T. T. Wu, Phys. Rev. D 5, 3192 (1972).

[9]M. Kac, G. E. Uhlenbeck, and P. C. Hemmer [J. Math. Phys. 4, 216 (1963), see especially Sec. V] discuss a gas model with weak long-range potentials; R. C. Arnold and J. Steinhoff, Phys. Lett. 45B, 141 (1973) use a similar form to compute certain Pomeron-cut effects.

[10]P. Suranyi, Nuovo Cimento 6A, 473 (1971); Phys. Lett. 35B, 167 (1971). The author was not aware of Suranyi's work until after completion of this paper.

[11]We could rigorously justify this assumption in our case, but since most of the other arguments will be heuristic, we will just use this simple reasoning.

[12]H. W. Wyld, Ref. 4.

[13]Since the y dependence of Q_N occurs only in separate terms containing the differences y_k-y_j, we have in mind an analogy with a classical gas with two-body potentials, even though Q_N is not of the form $\exp[\sum v(y_k-y_j)]$. As in the classical gas, we can ignore a term containing y_k-y_j if $|y_k-y_j|\gg 1$, and this is our definition of "short-range form in y."

[14]We ignore factors N^δ, which will not influence any of the quantities that we compute.

[15]The terms depending on y satisfy the criteria of footnote 13. The $\vec{K}_i^{\,2}$ terms, however, may depend on $\vec{q}_j$ and $\vec{q}_k$, even if $|y_k-y_j|$ is large. These terms have no y dependence, and in addition, are shown in the Appendix not to be important in the high-ρ limit.

[16]S.-J. Chang and J. L. Rosner, Phys. Rev. D 8, 450 (1973).

[17]This form is the same as Suranyi (see Ref. 10) obtained using a steepest-descent technique—see his Eqs. (17) and (18). Also, in his Appendix A he showed that each τ_i remains close to the mean value $\langle\tau\rangle$.

[18]The same relation can be obtained from Suranyi's Eqs. (12), (15), and (18), which express similar self-consistency criteria.

[19]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series, No. 55 (U. S. G. P. O., Washington, D. C., 1964).

[20]See Suranyi's Eq. (2).

[21]Suranyi considered the effects of changing the rapidity order, or crossing the rungs (see p. 481). He correctly derived the maximum degree of crossing (the order of the number k such that the amplitude becomes small when one rung crosses more than k other rungs). For our amplitude this becomes $O(\rho)$. He then computes the terms that appear in the amplitude which affect the momentum dependence of a nearby particle j when the rapidity order of a single particle is changed [by $O(\rho)$]. These terms do not by themselves result in a significant change in the momentum dependence of particle j, and Suranyi concluded that the over-all momentum dependence of each particle is not affected by crossing. There are, however, $O(\rho)$ *other* particles which can be exchanged at the same time, each of which will have a similar effect. The additional terms that then appear in the amplitude will be $O(\rho)$ times that resulting from the single exchange and the net effect will be a significantly different momentum dependence for particle j. Thus, in general, when all crossed (interference) terms are considered, the momentum dependence will not be the same as in the uncrossed case. Suranyi subsequently approximated these other terms and obtained the correct power dependence of σ_N on ρ. This power is not sensitive to the momentum dependence, as will be seen in Sec. IV, but the self-consistence condition (101) and the detailed form for σ_N(107) are sensitive to the momentum dependence.

[22]D. Amati, L. Caneschi, and M. Testa, Phys. Lett 43B, 186 (1973).