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Aspects of quark confinement

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Various aspects of quark confinement are considered. First we show in the context of potential theory that steep confinement potentials lead to mass formulas which are similar to those obtained for confining boundary constraints imposed on the free quark motion. We then consider the field-theoretic model of a scalar quark field which is subjected to a strong self-coupling in a finite domain of a three-dimensional lattice space. We show that the strong self-coupling prevents the quark field from spreading easily, but that by itself it does not suffice to confine quarks. A confinement constraint or equivalent potential is still required. We also show that the mass of a single quark grows with increasing self-coupling and becomes infinite in the continuum limit (thus necessitating the use of the lattice).

I. INTRODUCTION

Quark confinement has attracted considerable interest recently. The reason is that despite numerous successes of the quark model all attempts to isolate and observe single quarks have failed so far. The suggestion made by Johnson,[1] that the elementary constituents of hadrons possibly never appear singly and instead are confined permanently to bounded regions of space, has therefore aroused a large number of investigations and the construction of sophisticated model theories. All these investigations (an incomplete list is given in Refs. 1 to 8) attempt to understand the quark-confining mechanism while preserving the known properties of hadrons.

A particularly ambitious model of quark confinement has been developed by Wilson.[6] This model involves an Abelian gauge field coupled to a spinor quark field. The theory is quantized on a discrete four-dimensional space-time lattice. Using the Feynman path-integral method, Wilson then finds quark confinement for the case of strong coupling in which the theory is invariant under local gauge transformations.

One of the main advantages of the formulation chosen by Wilson[6] is stated to be its easy generalization to non-Abelian gauge theories. However, even in its Abelian form the theory is already sufficiently sophisticated in order to make it difficult to understand various basic aspects such as those posed by the questions: Has the lattice only mathematical significance or is it related to the confinement? What is the source of the confinement mechanism and why does it arise only in the strong-coupling domain? Does strong coupling (e.g., self-coupling) by itself suffice to entrap quarks? These questions (and others) prompted us to consider a purely scalar self-coupled field theory on a three-dimensional spatial lattice. This model, based on the work of Schiff,[9] exhibits general features similar to those of various confinement theories, but makes an understanding of our questions more transparent.

In Sec. II we begin with some simple considerations in potential theory in order to demonstrate the equivalence between quark-confining potentials and boundary constraints imposed on the free-quark motion.

In Sec. III we consider quark confinement in the lattice space. The model is defined by the usual Hamiltonian density for a self-coupled neutral scalar field integrated over a finite volume V. The energy of the confinement is an additive constant. We then introduce the lattice which makes the theory violently noncovariant, and investigate the lattice Hamiltonian and its expectation value for states of single quarks and quark-antiquark pairs. Finally we summarize our conclusions.

II. QUARK CONFINEMENT IN THE CONTINUUM SPACE

A. Quark-confining potentials

For a better understanding of our lattice-space arguments we begin with a brief look at potential scattering as defined, for instance, by the nonrelativistic Schrödinger equation

$$\frac{d^2\psi}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}-\frac{2\mu}{\hbar^2}V(r)\right]\psi=0\,, \tag{2.1}$$

where $k^2=2\mu E/\hbar^2$. Here $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass of two quarks, r their separation, and ψ is the radial wave function defined such that the solution of the Schrödinger equation is

$$\Psi=(1/r)\psi(r)P_l^m(\cos\theta)e^{im\varphi}.$$

We now assume that the unscreened quark-quark (or, more precisely, quark-antiquark) interaction is given by the power potential

$$V(r)=G^2r^s\,,$$
$$G^2\equiv g^2/r_0^{\,s}\,,\quad g^2>0,\quad r_0\neq 0,\quad s\geqslant 1\,. \tag{2.2}$$

Here r_0 is an important part of the coupling constant. For confinement of one quark in the neighborhood of the other we require[8] $s\geqslant 1$. Solving (2.1) for the potential (2.2) one finds the Regge trajectories or, better, the energy–angular-momentum relationship[8]

$$l\equiv\alpha_n(E)$$
$$=-2n-\tfrac{3}{2}+\frac{1}{\pi^{1/2}}\,\frac{\Gamma(1+1/s)}{\Gamma(\frac{3}{2}+1/s)}\,kr_0\left(\frac{E}{g^2}\right)^{1/s}+O\left(\frac{1}{n}\right), \tag{2.3}$$

where $n=0,1,2,\ldots$. This result gives the complete set of eigenvalues associated with the power potential (2.2). In the following we will refer to formulas of the type (2.3) as mass formulas because their values of E, i.e. E_n, determine the masses M_n of the excited states, i.e.,

$$M_n=m_1+m_2+E_n$$

(ignoring an arbitrary additive constant in V). From (2.3) we deduce that power potentials of the type (2.2) lead to Regge trajectories which rise with increasing energy. Interesting cases are the linear potential ($s=1$) which yields the fastest rise (proportional to $E^{3/2}$), the harmonic oscillator ($s=2$) and the limiting case $s\to\infty$ which yields the trajectory with the slowest-rising behavior:

$$s\to\infty:\;\; l\equiv\alpha_n(E)=-2n-\tfrac{3}{2}+\frac{2kr_0}{\pi}+O\left(\frac{1}{n}\right). \tag{2.4}$$

This last and seemingly academic example $s\to\infty$ corresponds, in fact, to an infinitely steep potential well, as Fig. 1 demonstrates.

Considering relativistic kinematics for quarks of the same mass m, we have the so-called relativistic Schrödinger equation

$$(\vec{p}^{\,2}+m^2)\Psi=\tfrac{1}{4}(E-V)^2\Psi\,,$$

i.e.,

$$\vec{\nabla}^2\Psi+\frac{1}{\hbar^2}\left[\tfrac{1}{4}(E-V)^2-m^2\right]\Psi=0\,, \tag{2.5}$$

where $\vec{p}$ is the momentum of one of the quarks in the center-of-mass frame, and E the total energy of the pair. It is clear that—assuming the potential is again given by (2.2)—in this case the calculation of the discrete eigenvalues is more complicated. We shall not go into details but will return to a discussion of Eq. (2.5) at a later stage.

B. Quark-confining constraints

We consider again the motion of one quark relative to another as described by the Schrödinger equation. But this time we do not assume a specific form of the potential. Instead we argue that the effect of the quark-confining force can be described equally by constraints on the wave function of the free-particle equation. This is an alternative way of formulating the same dynamics. Of course, in practice it is an extremely complicated mathematical problem to establish a one-to-one correspondence between a potential and a corresponding boundary condition. A well-known exception is the potential of the form of a δ function: In this case it is easy to construct the equivalent constraints. Thus, the examples that we consider are difficult to relate to specific equivalent potentials. However, we can use some loose arguments in order to make the connection plausible.

We consider the free Schrödinger equation, i.e.,

$$\vec{\nabla}^2\Psi+k^2\Psi=0\,,$$
$$\frac{d^2\psi}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}\right]\psi=0\,, \tag{2.6}$$

where again $k^2=2\mu E/\hbar^2$. Suitable constraints of confinement of one quark near the other are those which exclude their scattering and hence require the vanishing of the outgoing particle current at or beyond some finite nonzero quark-quark separation r_0, i.e.,

$$\psi(r_0)=0 \tag{2.7}$$

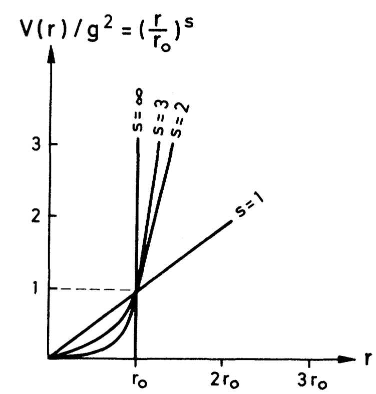

FIG. 1. The relative behavior of power potentials. The parameter r_0 represents something like the radius of a hadron. Thus, assuming a suitable energy (i.e., mass) dependence of r_0, the relation (2.4) corresponds to a linearly rising Regge trajectory.

or

$$\left(\frac{d\psi}{dr}\right)_{r_0}=0. \tag{2.8}$$

We consider first the case (2.7). The solution $\psi(r)$ of the radial Schrödinger equation (2.6), which is regular at the origin, is

$$\psi(r)=\frac{1}{r^{1/2}}J_{l+1/2}(kr). \tag{2.9}$$

The constraint (2.7) therefore requires

$$J_{l+1/2}(kr_0)=0. \tag{2.10}$$

The nth zero of this Bessel function is given by

$$kr_0=(n+\tfrac{1}{2}l)\pi-\frac{4(l+\frac{1}{2})^2-1}{8\pi(n+\frac{1}{2}l)}-\cdots \tag{2.11}$$

for $n \gg l+\frac{1}{2}$. Solving this equation for l we obtain

$$l=-2n+\frac{2kr_0}{\pi}+\frac{(-2n+2kr_0/\pi+\frac{3}{2})(-2n+2kr_0/\pi-\frac{1}{2})}{\pi kr_0}+\cdots \tag{2.12}$$

for $(n \gg l+\frac{1}{2})$. We observe that this mass formula is similar to (2.4), which we obtained for the infinitely high potential well. Thus we can argue that the effect of the quark-confining constraint (2.7) is roughly equivalent to that of the infinitely high potential well. Of course, we do not expect this correspondence to be unique. In fact, the constraint (2.8) also leads to a mass formula of the type (2.4). For the solution (2.9) the constraint (2.8) yields the relation

$$\begin{aligned} l&=kr_0\frac{J_{l+3/2}(kr_0)}{J_{l+1/2}(kr_0)}\\ &=kr_0\tan(kr_0-\tfrac{1}{2}l\pi-\tfrac{1}{2}\pi)+O((kr_0)^0). \end{aligned} \tag{2.13}$$

Extracting l, we obtain

$$\begin{aligned} l&=-1+\frac{2kr_0}{\pi}-\frac{2}{\pi}\tan^{-1}\left(\frac{l}{kr_0}\right)\\ &=-2n-1-\frac{4}{\pi^2}+\frac{2kr_0}{\pi}+O(1/n) \end{aligned} \tag{2.14}$$

for $|l/kr_0| \ll 1$, where we have used

$\tan^{-1}x = n\pi+\arctan x$,

$0 \le \arctan x \le \frac{1}{2}\pi$ for $x>0$,

and

$-\frac{1}{2}\pi \le \arctan x < 0$ for $x<0$.

Thus the constraints (2.7) and (2.8) which are much stronger than the usual bound-state boundary condition at infinity include the effect of quark-confining potentials. In fact, the confinement range r_0 is intimately related to the coupling strength of the corresponding potential, as (2.2) indicates. In the nonrelativistic case discussed so far k is proportional to $E^{1/2}$; however, in the relativistic case k is proportional to $(E^2-m^2)^{1/2}$.

Finally, we consider the free-field Hamiltonian H of a real (i.e., neutral) scalar field ϕ of mass m which is confined, by a boundary condition

$$\phi(r,\theta,\varphi) \text{ or } \frac{\partial\phi(r,\theta,\varphi)}{\partial r}=0 \text{ at } r=r_0, \tag{2.15}$$

to a finite region V of the infinite continuum space, i.e.,

$$H=\int_V d\vec{r}\,[\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\vec{\nabla}\phi)^2+\tfrac{1}{2}m^2\phi^2+B]. \tag{2.16}$$

Here $\Pi(\vec{r})=\partial\mathcal{L}/\partial\dot{\phi}$, $\dot{\phi}=\partial\phi/\partial t$, is the momentum canonical to $\phi(\vec{r})$, and B is the constant energy associated with the constraint per unit area of the surface enclosing the confinement domain. Also, we set $\hbar=c=1$. The classical equation of motion

$$(\partial_\mu\partial^\mu+m^2)\phi=0$$

is solved by expanding the field ϕ in terms of a complete set of solutions $\phi_i(x)$ of the equation

$$(\partial_\mu\partial^\mu+m^2)\phi_i(x)=0. \tag{2.17a}$$

The expansion is

$$\phi(x)=\sum_i \sigma_i\phi_i(x) \tag{2.17b}$$

where σ_i are constant operator coefficients.

The spherical-harmonic solutions

$$\phi_i(x)=e^{\pm iEt}\,\psi_l(r)P_l^m(\cos\theta)e^{im\varphi}, \tag{2.18}$$

having $E^2=\vec{p}^{\,2}+m^2$, $\vec{p}=-i\hbar\vec{\nabla}$ describe particles in eigenstates of angular momentum. It is clear that Eq. (2.17) is identical to (2.5) for $V=0$. Thus, imposing the constraints (2.15) on each of the wave functions $\phi_i(x)$ we again expect solutions and mass formulas of the type discussed above.[10]

III. QUARK CONFINEMENT IN THE LATTICE SPACE

A. Lattice Hamiltonian

Our starting point is the continuum field Hamiltonian[2,9]

$$H=\int_V\left[\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\vec{\nabla}\phi)^2+\tfrac{1}{2}m^2\phi^2+\frac{1}{n}\,\alpha^2\phi^n+B\right]d\vec{r}, \tag{3.1}$$

where ϕ is again a real scalar quark field, m is the field rest mass (or bare mass of the quark, which may be taken to be zero), V is a finite volume, and α ($\alpha^2>0$) is a parameter of the nonlinear self-coupling of the scalar field. We choose the Hamiltonian (3.1) because it represents the simplest model which enables us to investigate the essential *physical* aspects of quark confinement without the enormous complications of spin and several fields. It will be seen that in this model of confined scalar quarks it is neither the strong self-coupling of the field nor the exchange of $n-2$ quarks which provides the confinement gluon force but a constraint of energy BV which restricts the quarks to the region V of space. We shall need the canonical commutation relation

$$[\phi(\vec{r},t),\Pi(\vec{r}',t)]=i\delta(\vec{r}-\vec{r}')\,. \tag{3.2}$$

It is well known that the scalar theory is renormalizable in the case $n=4$.

We now consider the ordinary three-dimensional space to be a cubic lattice of $Z^3=(L/a)^3$ points.[11] Here a is the lattice constant (i.e., the distance between adjacent lattice points) and L is the length Za of one edge of the world cube. It should be noted that this lattice is different from that used by Wilson[6] (who uses a four-dimensional Euclidean lattice in which an imaginary linear lattice is allocated to the variable of time). The concept of a lattice space offers a convenient or even natural way to incorporate the cutoff necessary in calculations for nonrenormalizable interactions.

In the lattice space a field is defined only at the lattice points $\vec{n}=\vec{r}/a$ (counted from, say, the center of the world cube) where it is characterized by quantum-mechanical operators $\phi(\vec{n})$, $\Pi(\vec{n})$. A convenient way to define[9] $\phi(\vec{n})$ is by means of the average (in some sense) of the continuum field over a distance of length a about the lattice point, i.e. we define the lattice field by the correspondence

$$\frac{1}{a}\,\phi(\vec{n})\rightarrow\phi(\vec{r}_n)\equiv\int_{K_a}f(\vec{r}-\vec{r}_n)\phi(\vec{r})\,d\vec{r}\,, \tag{3.3}$$

and similarly

$$\frac{1}{a^2}\,\Pi(\vec{n})\rightarrow\Pi(\vec{r}_n)\equiv\int_{K_a}f(\vec{r}-\vec{r}_n)\Pi(\vec{r})\,d\vec{r}\,, \tag{3.4}$$

where $f(\vec{r}-\vec{r}_n)$ is an averaging function, and K_a implies integration over a sphere of radius a about the point with position vector $\vec{r}_n$. The factors $1/a$ are inserted for convenience.

Next, in order to be able to use momentum-space representations, we require the reciprocal lattice. Thus, if b is the lattice constant in the reciprocal space, and if $b\vec{h}$ is the lattice vector corresponding to the momentum vector $\vec{p}$ in the continuum, we define the Fourier transform $\tilde{\phi}(\vec{h})$ of $\phi(\vec{n})$ by

$$\phi(\vec{n})=\frac{1}{Z^3}\sum_{\vec{h}}\tilde{\phi}(\vec{h})\exp\left(\frac{2\pi i\vec{h}\cdot\vec{n}}{Z}\right), \tag{3.5}$$

so that

$$\tilde{\phi}(\vec{h})=\sum_{\vec{n}}\phi(\vec{n})\exp\left(-\frac{2\pi i\vec{h}\cdot\vec{n}}{Z}\right).$$

Here

$$\vec{p}=b\vec{h}\,, \tag{3.6}$$

and b is related to a by

$$b=\frac{2\pi}{Za}\,. \tag{3.7}$$

In this way we will be working in terms of the four-vectors

$$x^\mu=(t,a\vec{n}),\quad p^\mu=(\omega,b\vec{h})\,, \tag{3.8}$$

where $\omega=(\vec{p}^{\,2}+m^2)^{1/2}$. The rules for transcribing formulas from the continuum (coordinates $\vec{r},\vec{p}$) to the lattice (coordinates $\vec{n},\vec{h}$) and vice versa are

$$\begin{aligned}
&\phi(\vec{r})\leftrightarrow\frac{1}{a}\,\phi(\vec{n})\,,\\
&\Pi(\vec{r})\leftrightarrow\frac{1}{a^2}\,\Pi(\vec{n})\,,\\
&\delta(\vec{r}-\vec{r}')\leftrightarrow\frac{1}{a^3}\,\delta_{\vec{n}\vec{n}'}\,,\\
&f(\vec{r}-\vec{r}')\leftrightarrow\frac{1}{a^3}f(\vec{n}-\vec{n}')\,,\\
&\int d\vec{r}\leftrightarrow a^3\sum_{\vec{n}}\,,\\
&\tilde{\phi}(\vec{p})\leftrightarrow a^2\tilde{\phi}(\vec{h})\,,\\
&\delta(\vec{p}-\vec{p}')\leftrightarrow\frac{1}{b^3}\,\delta_{\vec{h}\vec{h}'}\,,\\
&\tilde{f}(\vec{p}-\vec{p}')\leftrightarrow\frac{1}{b^3}\tilde{f}(\vec{h}-\vec{h}')\,,\\
&\int d\vec{p}\leftrightarrow b^3\sum_{\vec{h}}\,,
\end{aligned} \tag{3.9}$$

where $\tilde{f}$ is the Fourier transform of f. Of course, the correspondences (3.9) are not unique. Terms can always be included in the lattice formulation which vanish in the limit $a\rightarrow 0$.

If the averaging function f is taken to be the δ function, we have

$$\begin{aligned}f(\vec{r}_n-\vec{r}_{n'})&=\delta(\vec{r}_n-\vec{r}_{n'})\\&=\frac{1}{(2\pi)^3}\int d\vec{p}\exp[i\vec{p}\cdot(\vec{r}_n-\vec{r}_{n'})]\,,\end{aligned} \tag{3.10}$$

and so [using (3.9)]

$$f(\vec{n}-\vec{n}')=\delta_{\vec{n}\vec{n}'}$$

$$=\frac{1}{Z^3}\sum_{\vec{h}}\exp\left(\frac{2\pi i\vec{h}\cdot(\vec{n}-\vec{n}')}{Z}\right). \quad (3.11)$$

Using (3.2) and (3.7), it is readily seen that the canonical commutation relation of the lattice field is

$$[\phi(\vec{n},t),\Pi(\vec{n}',t)]=i\delta_{\vec{n}\vec{n}'}. \quad (3.12)$$

The last point we have to deal with in order to be able to transcribe the Hamiltonian (3.1) into the lattice-space formulation is the gradient term $\int_V(\vec{\nabla}\phi)^2d\vec{r}$. Using Green's theorem we may write this term in the form

$$\int_V(\vec{\nabla}\phi)^2d\vec{r}=-\int_V(\phi\vec{\nabla}\cdot\vec{\nabla}\phi)d\vec{r}+\int_{S_V}\phi\vec{\nabla}\phi\cdot\vec{n}_S\,dS, \quad (3.13)$$

where $\vec{n}_S$ is a unit vector along the normal to the element dS of the surface S_V enclosing the volume V. In the following we will be concerned with a volume V which is finite and nonzero even in the continuum limit $a\to 0$.

For permanent confinement of quarks we now impose the constraint

$$\int_{S_V}\phi\vec{\nabla}\phi\cdot\vec{n}_S\,dS=0. \quad (3.14)$$

This relation is satisfied if ϕ or $\vec{\nabla}\phi$, i.e., the corresponding wave function [see (2.17)], is zero everywhere on the surface S_V—this is, in fact, the constraint equivalent to (2.7) and (2.8) in our earlier considerations. Of course, in the case of strong self-coupling of the scalar quark field, the surface integral is small compared with the integral over the self-coupling. But without the constraint (3.14) (imposed with energy BV) the field would gradually radiate beyond V and finally be distributed over the whole of space. Thus the strong (but noninfinite) self-coupling of the field does not by itself entrap the quarks permanently.

Using the rules (3.9) we can transcribe (3.13) into the lattice-space formulation:

$$\int_V(\vec{\nabla}\phi)^2d\vec{r}\to\frac{1}{2\pi aZ^5}\sum_{\vec{h}\vec{h}'}\tilde{f}_V(\vec{h}-\vec{h}')\tilde{\phi}(\vec{h}')\vec{h}^2\tilde{\phi}(\vec{h}), \quad (3.15)$$

where $\tilde{f}_V(\vec{h}-\vec{h}')$ is derived from

$$\tilde{f}_V(\vec{p}-\vec{p}')=\int_V d\vec{r}\,e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}$$

$$=\prod_{j=x,y,z}\frac{\sin[j(p_j-p_j')]}{\pi(p_j-p_j')} \quad (3.16)$$

and $V=(2x)(2y)(2z)$. In the limit $x,y,z\to\infty$, $\tilde{f}_V(\vec{p}-\vec{p}')$ becomes $\delta(\vec{p}-\vec{p}')$. Using the transforms (3.5) we may rewrite (3.15)

$$\int_V(\vec{\nabla}\phi)^2d\vec{r}\to\frac{1}{2\pi aZ^2}\sum_{\vec{n}\vec{n}'}A_{\vec{n}\vec{n}'}\phi(\vec{n})\phi(\vec{n}'), \quad (3.17)$$

where

$$A_{\vec{n}\vec{n}'}=\frac{1}{Z^3}\sum_{\vec{h}\vec{h}'}\tilde{f}_V(\vec{h}-\vec{h}')\vec{h}^2\exp\left[\frac{2\pi i}{Z}(\vec{h}'\cdot\vec{n}'-\vec{h}\cdot\vec{n})\right]. \quad (3.18)$$

It is clear that the terms of (3.17) represent the kinetic energy of the field at the points of the lattice. Again the lattice representation (3.17) is not unique but convenient for our purposes.

The Hamiltonian (3.1) can now be written down in the lattice formulation. We have

$$H=H_0+H'$$

where

$$H_0=\sum_{\vec{n}\in V}\frac{1}{a}\left[\tfrac{1}{2}\Pi^2(\vec{n})+\frac{1}{2}\left(m^2a^2+\frac{A_{\vec{n}\vec{n}}}{2\pi Z^2}\right)\phi^2(\vec{n})+\frac{\alpha^2}{na^{n-4}}\phi^n(\vec{n})\right]+BV \quad (3.19)$$

and

$$H'=\frac{1}{2\pi aZ^2}\sum_{\substack{\vec{n}\vec{n}'\\ \vec{n}\neq\vec{n}'\in V}}A_{\vec{n}\vec{n}'}\phi(\vec{n})\phi(\vec{n}'). \quad (3.20)$$

From (2.17) we know that the field ϕ is associated with a wave function which is (conveniently) represented by the same letter ϕ. Physically, H' represents the energy of propagation of the scalar field through the lattice. Our condition of confinement requires the wave function ϕ or its gradient to be zero on the surface of V [see (3.14)]. Now we observe that the lattice Hamiltonian H_0 is separable and describes a set of uncoupled oscillators, one at each lattice point of the volume V. This decoupling was achieved with the strong self-coupling of the field. The term H' then couples these oscillators together, in general [i.e., without the constraint (3.14)] both inside and outside V. We shall consider H' as a perturbation on H_0. It is plausible to expect this perturbation procedure to be valid for sufficiently large coupling constants α, at least for $n=4$, because then the expectation value of H_0 is larger than that of H'. Thus, inside V the field at each lattice point behaves like a free self-coupled anharmonic oscillator.

Next we adopt

$$\Pi(\vec{n})=-i\frac{\partial}{\partial\phi(\vec{n})} \quad (3.21)$$

as a convenient representation of the canonical momentum, which is compatible with (3.12). The Schrödinger equation may now be set up and separated as follows:

$$H\Psi = E\Psi\,,$$

$$H_0\Psi_0 = E_0\Psi_0\,,$$

where

$$\Psi_0 = \prod_{\vec{n}\in V} u_{\vec{n}}\,, \quad E_0 = \sum_{\vec{n}\in V} \mathcal{E}_{\vec{n}}\,, \tag{3.22}$$

and the $\mathcal{E}_{\vec{n}}$ are given by

$$\frac{1}{a}\left[-\frac{\partial^2}{\partial\phi^2(\vec{n})} + \frac{1}{2}\left(m^2a^2 + \frac{A_{\vec{n}\vec{n}}}{2\pi Z^2}\right)\phi^2(\vec{n}) + \frac{\alpha^2}{na^{n-4}}\,\phi^n(\vec{n})\right]u_{\vec{n}} = (\mathcal{E}_{\vec{n}} - BV)u_{\vec{n}}\,, \tag{3.23}$$

where we have used the orthonormality of the wave functions $u_{\vec{n}}$. The energies $\mathcal{E}_{\vec{n}}$ are seen to be the eigenvalues for the one-dimensional motion of a particle in a potential of the type of an anharmonic oscillator. Physically Eq. (3.23) describes the independent excitations of the self-coupled quark field at each point $\vec{n}$ of the region V of the lattice space.

B. Isolated quarks

In confinement theories the mass of a quark can be a concept of very dubious meaning. Hence, before we can talk about it, we have to say what we mean. The most immediate way to define the mass of an isolated quark assuming that one exists somewhere is to identify it with the total energy of the quark in its rest frame. This seemingly precise definition becomes vague if we require the quark to be imprisoned in a certain bounded region of space or equivalently held fixed at a specified point in space. This is due to the fact that the confining potential V may include an arbitrary additive constant so that the total energy E in the relation

$$(E-V)^2 = p^2 + m^2$$

has no well-defined zero point. This is also physically plausible. Because if a quark is held fixed at (or near) a point, it is difficult to distinguish between the mass of the quark and the energy of confinement. More important, there is probably no need for the distinction. Also the difficulty can presumably be removed if vacuum-polarization effects which in Abelian theories screen the bare interactions are taken into account and scattering becomes possible. At present it is not clear to us how this difficulty (which is essentially a problem of renormalization for large coupling[12]) can be overcome.

In the following we take the world lattice with the quark field $\phi(\vec{n})$ confined at the point $\vec{n}$ as our picture of an isolated and confined single quark. We use Eq. (3.23) to describe the various states of this system. Thereby we ignore the terms in $A_{\vec{n}\vec{n}'}$, i.e., H'.

As noted above, this approximation is legitimate only if the self-coupling of the field is sufficiently strong, i.e., α^2, the coupling constant, sufficiently large. We could also consider non-self-interacting quarks which are confined within a volume V. However, in that case the Hamiltonian could not be approximated by a countable number of independent oscillators. Phrased in a different way: The strong self-coupling of the quark field is responsible for its approximate localizability (which in turn implies particlelike solutions) owing to the fact that the energy associated with the propagation of the field through the lattice (H') is much smaller in magnitude than the energy associated with the self-coupling of the field (the term proportional to α^2 in H_0). This picture of confinement looks similar to that described by Wilson,[6] who also does not obtain quark confinement in the weak-coupling domain.

In the Hamiltonian [e.g., (3.19)], m describes the bare mass of the quark field. The dressed or physical mass of the quark is its mass which *would* be observed if it (i.e., the single, isolated quark) *could* be observed. This is simply the ground state of the single-particle expectation value of the Hamiltonian, i.e.,

$$E_1 = \frac{\langle 0|\,\phi^{(+)}(\vec{n})|H|\,\phi^{(-)}(\vec{n})|0\rangle}{\langle 0|\,\phi^{(+)}(\vec{n})|\,\phi^{(-)}(\vec{n})|0\rangle}\,, \tag{3.24}$$

where $\phi^{(-)}(\vec{n})$ is the configuration-space representation of the creation operator of a single scalar quark at the lattice point $\vec{n}$, i.e., $\phi(\vec{n}) = \phi^{(+)}(\vec{n}) + \phi^{(-)}(\vec{n})$. We write

$$u_{\vec{n}}^{(i)} \equiv \langle \vec{n}, i|\,\phi^{(-)}(\vec{n})|0\rangle$$

for the configuration-space wave function, and $\mathcal{E}_{\vec{n}}^{(i)}$ for the corresponding energy E_1, the superscript i enumerating the excited states ($i=0$ for the ground state). For the sake of generality we consider first the volume V enclosing N_V lattice points at each of which the quark field is defined. We ignore the propagation of the quark field through the lattice which, as we have seen, is a plausible approximation in the strong-coupling domain. Thus, neglecting H' and using the orthonormality of the wave functions, we have Eq. (3.23), i.e.,

$$\frac{\partial^2 u^{(i)}}{\partial \phi^2}+\left[a(\mathcal{E}^{(i)}-BV)-\frac{1}{2}\left(m^2a^2+\frac{A_{\vec{n}\vec{n}}}{2\pi Z^2}\right)\phi^2-\frac{\alpha^2}{na^{n-4}}\phi^n\right]u^{(i)}=0 \tag{3.25}$$

at each lattice point $\vec{n}\in V$. This equation describes the state of the self-coupled quark field system at each point $\vec{n}\in V$ as the value of the quark wave function at that point changes as a result of the strong self-coupling of the field at that point. For our present purposes it suffices to obtain an approximate form of the eigenenergies $\mathcal{E}^{(i)}$. For sufficiently large values of α^2, we therefore neglect the term in ϕ^2. The approximate form of the eigenvalues can be obtained with the help of the WKB method. Thus, using (2.3) we find

$$\mathcal{E}^{(i)}\simeq BV+\frac{1}{a}\left[\frac{n}{2}\;\frac{(4i+3)\Gamma(\frac{3}{2}+1/n)}{\Gamma(1/n)}\right]^{2n/(n+2)}\times\left(\frac{\alpha^2}{na^{n-4}}\right)^{2/(n+2)}. \tag{3.26}$$

Strictly speaking, this approximation assumes that i is large. However, the WKB method is known to yield surprisingly accurate approximations so we shall assume that its dependence on α and a is also that for the ground state $i=0$. Hence for $n=4$ we have

$$\mathcal{E}^{(0)}-BV\sim\frac{\alpha^{2/3}}{a}. \tag{3.27}$$

The total ground-state energy of the system of volume V enclosing N_V lattice points is therefore

$$\mathcal{E}^{(0)}\simeq N_V(BV+C\alpha^{2/3}/a),$$

where C is a number.

Next we consider a single isolated quark enclosed in the volume element a^3 (i.e., a bag containing only one quark). Then

$$\mathcal{E}^{(0)}\simeq Ba^3+C\alpha^{2/3}/a\equiv m' \tag{3.28}$$

is the energy of this quark. We assume that B, the confinement energy per unit volume, is a constant independent of a. Hence $\mathcal{E}^{(0)}$, i.e., the physical mass of the quark, is large and becomes infinite in the continuum limit $a\to 0$ for α large but finite. This result is physically plausible. It means that the quark is heavy because it is static or vice versa. This conclusion is similar to, though not identical with, that arrived at by Wilson.[6] In Wilson's theory the mass of the isolated or static quark is stated to be infinite and a consequence of gauge invariance irrespective of the lattice. Since the local gauge invariance used by Wilson is broken spontaneously for weak coupling, his conclusion also seems to require strong coupling.

C. Quark-antiquark system

In the simple model we are considering here quarks and antiquarks are indistinguishable since ϕ is a neutral scalar field. We now consider a state consisting of a quark at the lattice point n_1 and an antiquark at the lattice point n_2. We are interested in the ground-state energy of this pair, i.e., (using our earlier notation) in the quantity

$$E_2=\frac{\langle 0|\phi^{(+)}(\vec{n}_1)\phi^{(+)}(\vec{n}_2)|H|\phi^{(-)}(\vec{n}_1)\phi^{(-)}(\vec{n}_2)|0\rangle}{\langle 0|\phi^{(+)}(\vec{n}_1)\phi^{(+)}(\vec{n}_2)|\phi^{(-)}(\vec{n}_1)\phi^{(-)}(\vec{n}_2)|0\rangle}. \tag{3.29}$$

V is a volume enclosing the points n_1 and n_2. If these points are far apart, the minimal volume is of the order $a^3(n_1-n_2)$ where $n_1-n_2=|\vec{n}_1-\vec{n}_2|$. Ignoring H' as before, Eq. (3.29) yields with the help of (3.28)

$$E_2=2m'+2BV,\quad m'=C\alpha^{2/3}/a. \tag{3.30}$$

Here $2BV$ is the energy required in order to impose the constraint (3.14), i.e., to imprison the quarks in the volume V. Without a specific field-theoretic ansatz for B it is not possible to extract the equivalent potential. However, it is clear that this equivalent potential is of the type

$$V(\vec{r})\simeq g^2|\vec{r}|^{p+1},\quad p\geq 0.$$

IV. CONCLUSIONS

Starting from the nonrelativistic Schrödinger equation, we showed in Sec. II that an infinitely steep quark-confining potential corresponding to a hard bag leads to mass formulas which are similar to those obtained by subjecting the free-quark motion to boundary constraints. Analogous results were obtained for relativistic equations.

In Sec. III we considered quark confinement in the lattice space, and arrived at the following conclusions for our Abelian theory:

(a) the strong local self-coupling of the field prevents the quark field from spreading easily but does not by itself imprison the quarks;

(b) the physical mass of a single, isolated quark increases with the self-coupling;

(c) the physical mass of a single, isolated quark is proportional to the reciprocal of the lattice constant of the spatial lattice and thus becomes infinite in the continuum limit (in ϕ^4 theory this is one of the finite number of renormalizable infinities);

(d) the lattice is essential mathematically in order to define an otherwise uncountably infinite number of states of infinite energy;

(e) the confinement requires the imposition of a confinement constraint.

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