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Semiclassical dynamics of the "SLAC bag"*

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It is shown that the classical thin-shell states of the "SLAC bag" model may be described, in a strong-coupling limit, by the motions of a surface of constant surface tension upon which free quark fields propagate. Classical static and nonstatic solutions in two and three space dimensions are discussed. It is found that such objects are very easily deformed. The implications for excited state levels in and the quantum mechanics of a model of hadrons based on such states are discussed.

I. INTRODUCTION

Recently, Bardeen, Chanowitz, Drell, Weinstein, and Yan (BCDWY) proposed a model of hadron structure[1] based on a strongly coupled field theory of quarks interacting with a neutral quartically self-coupled scalar field. They have argued that though the quark and scalar meson masses are large, very-low-mass bound states containing quarks will form. In the semiclassical field theory these bound states correspond to extended particlelike excitations of the fields in which the energy density takes the form of a thin shell.

This paper analyzes the strong-coupling limit of the semiclassical BCDWY theory. The strong-coupling limit is defined in such a way that the quark and meson masses go to infinity, while the masses of thin-shell bound states remain finite. In this limit, the thickness of the shell goes to zero and it may be regarded as a spatially closed curved hypersurface embedded in space-time (a "bubble"). The Euler-Lagrange equations of the field theory can be re-expressed as equations relating geometric variables characterizing such a hypersurface and quark fields defined only on this surface. The resulting classical equations of bubble dynamics are Poincaré invariant and are equivalent to those derived from an action principle. This action principle is similar to those which generate the MIT bag[2] and the Nambu string.[3]

The bubble theory easily reproduces the results obtained by BCDWY for the static spherically symmetric bubble state in three dimensions and provides a very convenient starting point for a discussion of excited states of the model. We exhibit the complete solution to the static-bubble equations in two space dimensions. We find that two-dimensional static bubbles are exactly degenerate in shape—the energy of a bubble depends only on its perimeter. We analyze several approximate static solutions in three space dimensions. Our results indicate that three-dimensional bubbles, though not exactly degenerate in shape, are very easily deformed.

We obtain the exact solution to the nonstatic-bubble equation for all breathing modes of a spherically symmetric bubble in three spatial dimensions. The surface motion is quantized in the WKB approximation. In this approximation, the mass ratio of the first radially excited state and the ground state is found to be very close to that of the Roper resonance and the nucleon.

The most striking physical property of bubbles which emerges from this work is their softness to deformation. The possible implications of this property for any quantum theory of bubbles are quite important. Among these is the expectation that the thin-shell nature of the semiclassical states need not imply oscillatory form factors or the absence of scaling as would be the case for a rigid shell.

II. THE BCDWY MODEL

The BCDWY model for the binding of a single quark species is developed from the field theory defined by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\partial\sigma)^2-\lambda(\sigma^2-f^2)^2+\bar{\psi}(i\partial\!\!\!/-G\sigma)\psi,$$

whose Hamiltonian is

$$H=\int dx\left[\frac{1}{2}\left(\frac{\partial\sigma}{\partial t}\right)^2+\tfrac{1}{2}(\nabla\sigma)^2+\lambda(\sigma^2-f^2)^2+\psi^\dagger(-i\alpha\cdot\nabla+G\sigma\gamma^0)\psi\right].$$

This is a theory of a Dirac field ψ interacting, via the Yukawa coupling, with a quartically self-coupled, neutral scalar field, σ. The theory is characterized by three coupling constants: G, λ, and f.

This Lagrangian is symmetric under the discrete transformation:

$$\sigma\to-\sigma,$$
$$\psi\to\gamma_5\psi.$$

The classical potential of the σ field has symmetric minima at $\sigma=\pm f$. One therefore expects that, in the corresponding quantum theory, γ_5 reflection symmetry will be spontaneously broken and that the σ field will assume a vacuum expectation value $|\langle\sigma\rangle|=f$, which we choose, by convention, to be $+f$. In a perturbative approach one would then conclude that this theory is one of interacting quarks of mass $M_Q=Gf$ and scalar mesons of mass $M_\sigma=(8\lambda f)^{1/2}$. We consider a limit of coupling constants in which both of these masses are large.

It is easy to construct a semiclassical argument that the lowest-lying quark states need not have mass Gf. It is only in zeroth-order perturbation theory that the scalar field is not free to decrease its value from f in the neighborhood of the quark, so as to reduce the total energy. The particlelike excitations of the semiclassical BCDWY theory are formed in just this way.

The "semiclassical" field equations which we use to discuss the BCDWY theory consist of the (one-particle) Dirac equation for ψ in the presence of a classical σ field,

$$(i\partial\!\!\!/ - G\sigma)\psi=0, \tag{1}$$

where the wave function must be normalized to unit charge,

$$Q=\int dx\psi^\dagger\psi=1,$$

and of the classical equation for the σ field in the presence of a fermion source,

$$-\partial^2\sigma+4\lambda\sigma(f^2-\sigma^2)=G\bar\psi\psi. \tag{2}$$

In the "static" case, $\sigma=\sigma(\vec{x})$, $\psi=\psi(\vec{x})e^{-iEt}$ these reduce to

$$(-i\boldsymbol{\alpha}\cdot\nabla+G\sigma\gamma^0)\psi(\vec{x})=E\psi(\vec{x}),$$

$$\nabla^2\sigma+4\lambda\sigma(f^2-\sigma^2)=G\bar\psi\psi.$$

The differential equations (1) and (2) are the classical Euler-Langrange equations of the theory. The system is "semiclassical" in the sense that ψ in interpreted as if it were a single-particle Dirac wave function: It is normalized to unit charge, and negative-energy fermion states are to be given the Dirac interpretation as positive-energy antifermions. We note that the Dirac equation is one with a scalar potential, so that no Klein paradox arises—the distinction between positive- and negative-energy states is always unambiguous. Thus, the prescription by which we define a "semiclassical" theory is also unambiguous. The normalization of the fermion charge to 1 and the interpretation of negative-energy states as antiparticles arise naturally in the work of BCDWY, where the semiclassical equations are derived from the quantum field theory via an approximate variational technique.

The mechanism by which low-mass quark bound states can form is most clearly evident in the solution to the static-field equations in one dimension. Taking the representation of the γ matrices

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \alpha=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

this solution is

$$\begin{aligned}\sigma(x)&=f\tanh(2\lambda)^{1/2}f(x-x_0),\\ \psi(x)&=\frac{N}{\sqrt{2}}[\cosh(2\lambda)^{1/2}f(x-x_0)]^{-G/(2\lambda)^{1/2}}\begin{pmatrix}1\\i\end{pmatrix},\end{aligned}\tag{3}$$

where N is a normalization constant that ensures $Q=1$, and x_0 is a constant.

One finds

$$E=0,$$

$$E_{\text{total}}=\tfrac{4}{3}(2\lambda)^{1/2}f^3,$$

$$\bar\psi\psi=0.$$

In a strong-coupling limit defined by

$$G,\lambda\to\infty,$$

$$f\to0,$$

$$G\gg\lambda^{1/6},$$

$$\lambda^{1/6}f=\text{fixed},$$

this is clearly a one-quark state of much lower energy than the usual free quark.

There are several aspects of this one-dimensional solution which point toward more general features of the theory. First, because $\bar\psi\psi$ vanishes, the σ-field equation is actually independent of ψ. The above solution for σ is the well-known "kink" solution of the spontaneously broken quartic scalar theory in one dimension.[4] The dynamics of the scalar field is determined primarily by its self-coupling, rather than by its coupling to fermion sources. This will remain true in higher dimensions. The width, D, of the transition region of the the σ field is on the order of the σ Compton wavelength, which will always be small compared to $(1\text{ GeV})^{-1}$.

Perhaps the most striking feature of the solution is that the Dirac energy is small even though the Dirac wave function is very sharply peaked. Intuition based on the quantum mechanics of bosons would suggest that the energy should be comparable to the dominant Fourier components of the wave function—on the order of the bare quark mass.

This intuition need not be correct because the

Hamiltonian is linear, rather than quadratic, in the quark momentum operator. This point is fully discussed in Ref. 1.

III. THEORY OF BUBBLE STATES

In higher dimensions, as discovered by BCDWY, the low-lying bound states analogous to the one-dimensional kink have the form of finite domains within which $\sigma=-f$ and outside of which $\sigma=+f$. The transition of the σ field between these values is very sharp and takes place in a thin shell about some closed surface in space (Fig. 1). Quarks can be trapped on this domain boundary as they are on the kink. We refer to such states as "bubbles." In general, a bubble's surface may vary in time. Thus, the most natural description of a bubble is as a domain in spacetime whose boundary surface is a timelike hypertube (Fig. 2).

In this section, we discuss a general approximation scheme which affords a characterization of all bubble solutions to the Euler-Lagrange equations. Our approximate solutions become exact in the infinitely strong-coupling limit. The approximations we use in this discussion will give physical quantities to lowest order in a small parameter which we may denote schematically as "D/R." Here, D is a length on the order of the Compton wavelength of the quark or the meson and R is on the order of the smallest radius of curvature of the bubble surface. Thus, D/R is the ratio of the thickness of the shell to its size and, as we shall see, vanishes in the strong-coupling limit.

Our procedure is as follows:

(1) We assume that the desired solution to the field equations is a bubble of some undetermined shape. The σ-field equation may be solved approximately for any such configuration.

(2) We solve the Dirac equation approximately in the presence of this σ field.

(3) Finally, we derive a self-consistency condition which guarantees that the next-order corrections to this approximate solution are, in fact, small.

The net result is a reformulation of the Euler-Lagrange equations in terms of a particularly convenient set of dynamic variables which characterize the bubble surface and quark fields on it.

We begin with the assumption that the field configuration will be that of a bubble (Fig. 2). In n-dimensional Minkowski space, the bubble surface, $\sigma(x)=0$, is an $(n-1)$-dimensional hypersurface which we can parametrize by $n-1$ "internal coordinates" u^α,

$$x^\mu = R^\mu(u^\alpha), \quad \begin{array}{l}\alpha=0,\ldots,n-2\\ N=0,\ldots,n-1.\end{array}$$

Because the fields are expected to have nontrivial spacetime dependence only in a thin shell about this surface, it is convenient to adopt (non-Cartesian) coordinates (u^α, ξ) centered about it,

$$x^\mu(u^\alpha,\xi)=R^\mu(u^\alpha)+\xi n^\mu(u^\alpha),$$

where $n^\mu(u^\alpha)$ = outward unit normal at $R^\mu(u^\alpha)$.

The coordinates (u^α, ξ) are well defined only within a distance on the order of one radius of curvature away from the surface. We assume that the radii of curvature of the bubble surface are always large compared to D. This assumption has no effect whatsoever on the spectrum of low-lying excitations of the theory in the strong-coupling limit. By increasing G and λ, D may be made arbitrarily small without affecting either the spectrum or the surface geometry.[5]

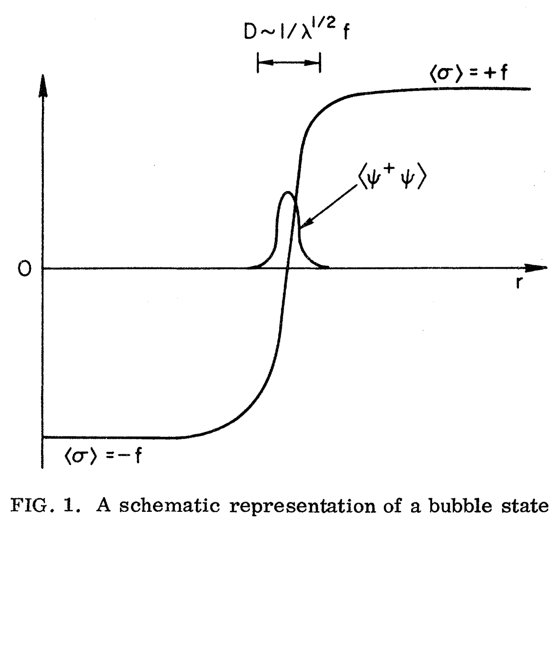

FIG. 1. A schematic representation of a bubble state.

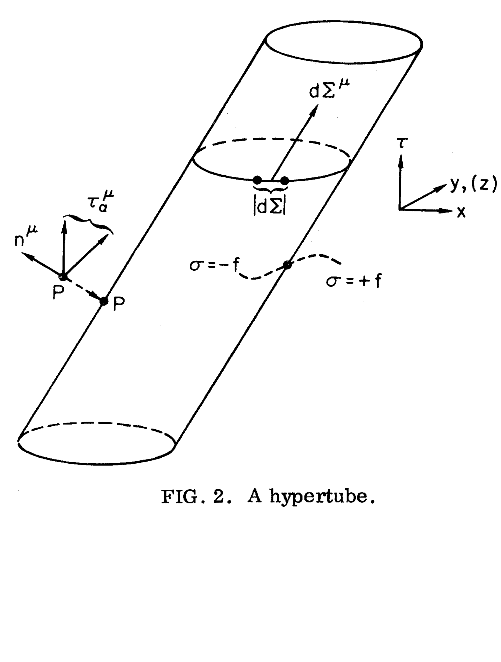

FIG. 2. A hypertube.

In the new coordinate system, we can write the gradient:

$$\partial_\mu = \partial_{\parallel\mu} - n_\mu \frac{\partial}{\partial \xi},$$

where $\partial_{\parallel\mu}$ is the "tangential" gradient which, though it depends on ξ, involves only differentiations with respect to the u^α and is tangent as a vector to the surface.

Consider the field equation for σ. Our first approximation to σ must be a function that satisfies the "largest" part of Eq. (2) near the surface. Because σ makes its transition from $-f$ to $+f$ in a distance of order D, we expect

$$\frac{\partial\sigma}{\partial\xi} \sim \frac{1}{D} f,$$

while

$$\partial_\parallel \sigma \sim \frac{1}{R} f \ll \frac{1}{D} f.$$

We also anticipate that, in analogy to the one-dimensional case, the fermion source term will be relatively unimportant in (2)—an assertion which must be verified later to ensure self-consistency. Thus, our first approximation to (2) in the neighborhood of the surface is

$$\frac{\partial^2\sigma}{\partial\xi^2} + 4\lambda\sigma(f^2 - \sigma^2) = 0.$$

This is the same as the equation for the kink of the one-dimensional theory. The solution of this equation which satisfies the boundary conditions and vanishes at $\xi = 0$ is unique:

$$\sigma(x) = \sigma(\xi) = f \tanh(2\lambda)^{1/2} f\xi. \tag{4}$$

Next, we attempt to solve the Dirac equation (1) in the presence of this σ field,

$$\left[i\partial\!\!\!/_\parallel - i n\!\!\!/ \frac{\partial}{\partial\xi} - G\sigma(\xi)\right]\Psi = 0. \tag{5}$$

We construct an approximate solution valid as $G \to \infty$, using a technique similar to one invented by Chodos[2] to derive boundary conditions for the Dirac field in the MIT bag theory. We expect that the Dirac wave functions will fall off exponentially away from the surface as $\sim e^{-Gf|\xi|}$. It is clear that such a Ψ is not an analytic function of $1/G$ as $1/G \to 0$. However, we can attempt to factor out the essential singularity in $1/G$ and then expand its coefficient in $1/G$.

We write

$$\Psi(u^\alpha, \xi) = N e^{GF(\lambda,\xi)}\left[\psi(u^\alpha, \xi) + \frac{1}{G}\psi_1(u^\alpha, \xi)\right], \tag{6}$$

where F and ψ are independent of G, ψ and ψ_1 are finite near $\xi = 0$ as $G \to \infty$, $\psi + (1/G)\psi_1$ is the beginning of an expansion of the field in powers of $1/G$. Only the properties of the first term will be important, so we use simply the ψ_1 term to represent all higher-order corrections in $1/G$.

Substituting this form in the Dirac equation (5), we have

$$0 = -G\left[\sigma(\xi) + i n\!\!\!/ \frac{dF}{d\xi}\right]\psi + \left(i\partial\!\!\!/_\parallel - i n\!\!\!/ \frac{\partial}{\partial\xi}\right)\psi - \left[\sigma(\xi) + i n\!\!\!/ \frac{dF}{d\xi}\right]\psi_1 + O\left(\frac{1}{G}\right). \tag{7}$$

This equation must be satisfied order by order in $1/G$. The equation for the coefficient of G is

$$\left[f \tanh(2\lambda)^{1/2} f\xi + i n\!\!\!/ \frac{dF}{d\xi}\right]\psi = 0.$$

To have a nontrivial solution of this matrix equation for ψ requires

$$\frac{dF}{d\xi} = \pm f \tanh(2\lambda)^{1/2} f\xi.$$

The requirement that F decrease with $|\xi|$ necessitates that we take the minus sign above. We have

$$\begin{aligned} &i n\!\!\!/ \psi = \psi, \\ &e^{GF} = [\cosh(2\lambda)^{1/2} f\xi]^{-G/(2\lambda)^{1/2}}. \end{aligned} \tag{8}$$

The equation between the terms of order unity in (7) then becomes

$$\left(i\partial\!\!\!/_\parallel - i n\!\!\!/ \frac{\partial}{\partial\xi}\right)\psi - \sigma(\xi)(1 - i n\!\!\!/)\psi_1 = 0. \tag{9}$$

Multiplying both sides by $(1 + i n\!\!\!/)$ we find

$$\frac{\partial\psi}{\partial\xi} = -k\psi, \tag{10}$$

where

$$k \equiv \tfrac{1}{2}\partial_{\parallel\mu}(n^\mu).$$

The quantity k depends on the surface geometry alone. In Sec. IV we show that k is proportional to the local mean curvature of the surface.

At $\xi = 0$, where the ψ_1 term in (9) vanishes we have

$$(i\partial_\parallel + k i n\!\!\!/)\psi(u^\alpha, 0) = 0. \tag{11}$$

This reduced Dirac equation involves only the behavior of ψ on the surface. This equation and the equation of constraint, $i n\!\!\!/ \psi = \psi$, completely characterize the quark degrees of freedom of a bubble in the strong-coupling limit.

Finally, we must show that the expressions we have obtained do constitute an approximate solution to the field equations. We will be led to a further equation of motion relating the surface geometry of the bubble and the distribution of quark energy-momentum on it. This condition is the generalization of the energy-minimization principle used by BCDWY to determine the radius of the spherically symmetric static state in Ref. 1. The detailed derivation of this condition is straightforward but technically complicated. We simply sketch the idea here and present the proof in Appendix B.

Suppose we have fields σ and Ψ in the neighborhood of a bubble surface, such that

$$\sigma(\xi) = f\tanh(2\lambda)^{1/2}f\xi,\qquad \Psi = N[\cosh(2\lambda)^{1/2}f\xi]^{-G/(2\lambda)^{1/2}}\psi(u^\alpha,\xi), \tag{12}$$

where

$$i\not{n}\psi=\psi\,.$$

$$\frac{\partial\psi}{\partial\xi}=-k\psi,$$

$$\frac{1}{N^2}\equiv\int_{-\infty}^{\infty}d\xi[\cosh(2\lambda)^{1/2}f\xi]^{-2G/(2\lambda)^{1/2}}.$$

These fields will be approximate solutions to the equations of motion in the strong-coupling limit only if further corrections to them are of order D/R. In Appendix B, these corrections are estimated as follows: The action functional is expanded quadratically about the classical fields σ and Ψ. In principle, the resulting quadratic functional can be minimized, and shifts in the fields $\delta\sigma$ and $\delta\Psi$ and the corresponding change in the action δS can be computed.

Because of the sharp gradients in the fields near the bubble surface, variations of the fields relative to this surface correspond to very-high-frequency excitations which do not enter δS to lowest order in D/R. The only variations of the fields which can cause a finite shift, δS, are those which correspond to a motion of the surface and its associated fields as a whole. Only if the action is already stationary to order D/R under such variations will the fields σ,Ψ be an approximate solution to the Euler-Lagrange equations.

That is,

$$\frac{\delta}{\delta R^\mu(u^\alpha)}\left\{\int d^4x[\bar\Psi(i\not\partial-G\sigma)\Psi+\tfrac{1}{2}(\partial\sigma)^2-\lambda(\sigma^2-f^2)^2]\right\}=0. \tag{13}$$

The Lagrangian density in (13) is very sharply peaked in the neighborhood of the bubble surface. Thus, the integral over ξ can be evaluated to lowest order in D/R.

We have

$$\begin{aligned}S[\sigma,\Psi]=&\int d^4x\{N^2[\cosh(2\lambda)^{1/2}f\xi]^{-2G/(2\lambda)^{1/2}}\bar\psi[i\not\partial_{\parallel}-i\not{n}\frac{\partial}{\partial\xi}-Gf\tanh(2\lambda)^{1/2}f\xi(1-i\not{n})]\psi\\&\quad+\tfrac{1}{2}[-n_\mu(2\lambda)^{1/2}f^2\operatorname{sech}^2(2\lambda)^{1/2}f\xi]^2-\lambda[f^2\operatorname{sech}^2(2\lambda)^{1/2}f\xi]^2\}\\&\approx\int da\,d\xi\{N^2[\cosh(2\lambda)^{1/2}f\xi]^{-2G/(2\lambda)^{1/2}}\bar\psi(i\not\partial_{\parallel}+ki\not{n})\psi-2\lambda f^4\operatorname{sech}^4[(2\lambda)^{1/2}f\xi]\}\\&\approx\int_{\text{hypertube}}da\,[\bar\psi(i\not\partial_{\parallel}+ki\not{n})\psi-C],\end{aligned}$$

where da = element of surface "area" on the hypertube and

$$C\equiv\int_{-\infty}^{\infty}d\xi\,2\lambda f^4\operatorname{sech}^4(2\lambda)^{1/2}f\xi=\tfrac{4}{3}(2\lambda)^{1/2}f^3.$$

Thus we are led to a further equation of motion in the form of a "surface action principle":

$$0=\delta\int da[\bar\psi(i\not\partial_{\parallel}+ki\not{n})\psi-C], \tag{14}$$

where the variation is to be performed over all possible bubble surfaces, $R^\mu(u^\alpha)$.

We note that the requirement that S be stationary under variations of the surface Dirac field ψ leads to the correct surface Dirac equation (11). Thus, the dynamics of bubble states in the strong-coupling limit can be completely described in terms of the geometric variables $R^\mu(u^\alpha)$, the surface Dirac field ψ, and the finite coupling C. The Dirac field obeys the constraint $i\not{n}\psi=\psi$. The equations of motion for ψ and R^μ may be derived from the surface action principle (14).

These results may be easily understood physically. In the strong-coupling limit, only a very special class of solutions to the field equations retain low energy. The requirement that their energy remain small forces these solutions to mimic, locally, the one-dimensional kink. The only remaining degrees of freedom are those which describe how such kinks are patched together continuously in spacetime $[R^\mu(u^\alpha)]$ and the quark distribution among them $[\psi(u^\alpha)]$.

IV. BUBBLE DYNAMICS

The three Eqs. (8), (11), and (14), completely characterize bubble solutions to the BCDWY field theory. In this section, we discuss some general properties of solutions to this system. The most natural language for the description of bubbles is that of the Riemannian geometry of surfaces. We will introduce some basic geometric notations and concepts in the following short discussion. The reader is referred to Appendix A and the references contained therein for further details.

The surfaces whose geometry is of interest here are spatially closed $(n-1)$-dimensional timelike hypertubes embedded in n-dimensional Minkowski space. Such a surface may be parametrized by $n-1$ "internal" coordinates $\{u^\alpha\}$.

Surface,

$$x^\mu = R^\mu(u^\alpha).$$

Our notation will be such that $\alpha, \beta, \gamma, \delta$ run from 0 to $n-2$, while $\mu, \nu, \lambda, \sigma$ run from 0 to $n-1$. The choice of internal coordinates is arbitrary. Therefore, physical quantities must be represented by tensors that are manifestly "covariant" under general coordinate transformations.

The fundamental tensors characteristic of the surface geometry are as follows:

Tangent vectors,

$$\tau^\mu_\alpha = \frac{\partial R^\mu}{\partial u^\alpha};$$

Induced metric,

$$g_{\alpha\beta} = \tau^\mu_\alpha \tau_{\beta\mu} = \tau_\alpha \cdot \tau_\beta;$$

Outward unit normal,

$$n^\mu(u^\alpha):\ n\cdot\tau_\alpha = 0,\quad n^2 = -1;$$

Coefficients of curvature,

$$h_{\alpha\beta} = -n\cdot\tau_{\alpha|\beta} = n_{|\alpha}\cdot\tau_\beta = h_{\beta\alpha},$$

where we adopt the notation

$$A_{|\alpha} = \frac{\partial A}{\partial u^\alpha}$$

for any quantity A.

The induced metric tensor $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$ are used, in the usual way, to transform between the covariant and contravariant forms of tensors. The metric is "induced" in the following sense: If V^μ is a tangent vector,

$$V^\mu = V^\alpha \tau^\mu_\alpha,$$

the length of V^μ in Minkowski space can be written in terms of its components as

$$V^\mu V_\mu = (V^\alpha \tau^\mu_\alpha)(V^\beta \tau_{\beta\mu}) = g_{\alpha\beta} V^\alpha V^\beta = V^\alpha V_\alpha.$$

The invariant element of "area" on the surface is

$$da = d^{n-1}u\sqrt{|g|},\quad g \equiv \det(g_{\alpha\beta}).$$

The n vectors $\{\tau^\mu_\alpha, n^\mu\}$ form a local "n-bein" in terms of which any Minkowski vector can be expanded:

$$(\tau^\alpha)^\mu(\tau_\alpha)^\nu - n^\mu n^\nu = \eta^{\mu\nu},$$

the Minkowski metric.

The tensor $h_{\alpha\beta}$, called the "second fundamental form," describes the local curvature of the surface. At any point, the principal values of $h^\alpha{}_\beta$ are the reciprocal radii of curvature of the surface. Along a timelike direction, the reciprocal radius of curvature is proportional to the normal acceleration of the corresponding spatial surface in its local rest frame. The quantity k is then

$$k = \tfrac{1}{2}\partial_{\parallel\mu}(n^\mu) = \tfrac{1}{2}(\tau^\alpha)_\mu \partial_\alpha n^\mu = \tfrac{1}{2}h^\alpha{}_\alpha.$$

Thus, k is proportional to the mean curvature of the surface at each point.

The flat Minkowski space induces natural laws of parallel transport along such a surface both for vectors and spinors. For a coordinate shift δu^γ, these are

Vectors,

$$\delta V^\alpha = -\left\{{}_{\beta}{}^{\alpha}{}_{\gamma}\right\} V^\beta \delta u^\gamma,$$

where the Christoffel symbol is

$$\left\{{}_{\beta}{}^{\alpha}{}_{\gamma}\right\} = \tfrac{1}{2} g^{\alpha\delta}[g_{\beta\delta|\gamma} + g_{\delta\gamma|\beta} - g_{\beta\gamma|\delta}],$$

Spinors,

$$\delta\psi = -\frac{i}{2}\sigma^{\mu\nu} n_\mu n_{\nu|\gamma}\delta u^\gamma \psi.$$

The parallel-transport law for spinors is just such that the quantity $\bar\psi\gamma^\alpha\psi$ parallel transports as a vector. There exist corresponding "covariant derivatives" of vectors and of spinors:

$$V^\alpha{}_{\parallel\beta} = V^\alpha{}_{|\beta} + \left\{{}_{\beta}{}^{\alpha}{}_{\gamma}\right\} V^\gamma,$$
$$D_\alpha \psi = \left(\partial_\alpha + \frac{i}{2}\sigma^{\mu\nu} n_\mu n_{\nu|\alpha}\right)\psi. \tag{15}$$

A little algebra gives the following relations, which will be of some use to us later:

$$\not{D} = \gamma^\mu \tau^\alpha_\mu D_\alpha = \not{\partial}_\parallel + k\not{n},$$
$$\not{D}\not{n} = -\not{n}\not{D},$$
$$\tau_{\alpha|\beta} = \left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\}\tau_\gamma + h_{\alpha\beta} n,$$
$$\tau_{\alpha\parallel\beta} = h_{\alpha\beta} n.$$
$$V^\alpha{}_{\parallel\alpha} = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}\,V^\alpha)_{|\alpha}\quad\text{for any } V^\alpha.$$

Using this notation, the bubble equations of motion can be rewritten

$$i\not{n}\psi = \psi, \tag{16}$$

$$i\not{D}\psi = 0, \tag{17}$$

$$\delta\int d^{n-1}u\sqrt{|g|}\,(\bar\psi i\not{D}\psi - C) = 0. \tag{18}$$

The Dirac equation (17) has a clear interpretation as that of a free massless fermion confined to a curved surface. The equation of constraint (16) on the Dirac field is consistent with the equation of motion (17) by virtue of the relation $\not{D}\not{n} = -\not{n}\not{D}$.

The equation of motion arising from the variation over $R^\mu(u^\alpha)$ is now straightforward to derive. For

$$R^\mu(u^\alpha) \to R^\mu(u^\alpha) + \delta R^\mu(u^\alpha)$$

after using (16) and (17), we have

$$\frac{1}{\sqrt{|g|}}\delta(\sqrt{|g|}(\bar{\psi}i\not{D}\psi - C)) = -T^{\alpha\beta}\tau_\beta \cdot \delta R_{|\alpha},$$

where

$$T^{\alpha\beta} \equiv Cg^{\alpha\beta} - \mathrm{Im}\bar{\psi}\gamma^\alpha \partial^\beta\psi.$$

We shall see presently that $T^{\alpha\beta}$ is the canonical energy-momentum tensor of the bubble.

The corresponding equation of motion is

$$0 = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}\,T^{\alpha\beta}\tau^\mu_\beta)_{|\alpha}$$

$$= T^{\alpha\beta}{}_{\|\alpha}\tau^\mu_\beta + h_{\alpha\beta}T^{\alpha\beta}n^\mu .$$

The tangential component of this equation, $T^{\alpha\beta}{}_{\|\alpha} = 0$, follows from (17). This simply reflects the fact that an infinitesimal tangential variation of $R^\mu(u^\alpha)$ is equivalent to an infinitesimal coordinate transformation—the surface itself is unchanged. The normal component of this equation provides the third equation of motion in local form:

$$h_{\alpha\beta}T^{\alpha\beta} = 0. \tag{19}$$

Accepting, for the moment, that $T^{\alpha\beta}$ is the energy-momentum tensor of the theory, this equation has a simple physical interpretation. The contraction of spatial components of $h_{\alpha\beta}$ and $T^{\alpha\beta}$ gives the net normal force density exerted on the surface due to its stresses. The orthogonal timelike component gives the rate of change of normal momentum density. Equation (19) is nothing more than Newton's second law on a relativistic hypersurface under stress.

The charge, momentum, and angular momentum of the bubble may be expressed in terms of surface fields. In the original field theory these quantities are spatial integrals of densities which are very sharply peaked at the bubble surface. As in the case of the action, the integral over the normal coordinate, ξ, can be performed explicitly, to lowest order in D/R, leaving an expression which involves only surface quantities.

An easier approach is to derive the conserved charges directly from the surface action using Noether's theorem. If the Lagrangian density is invariant under a transformation

$$R^\mu \to R^\mu + \delta R^\mu ,$$

$$\psi \to \psi + \delta\psi ,$$

then the current,

$$\delta K^\alpha = T^{\alpha\beta}\tau^\mu_\beta \delta R_\mu - \frac{i}{2}(\bar{\psi}\gamma^\alpha\delta\psi - \delta\bar{\psi}\gamma^\alpha\psi),$$

is conserved:

$$0 = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}\,\delta K^\alpha)_{|\alpha} = \delta K^\alpha{}_{\|\alpha} .$$

The following are the symmetries, currents, and conserved "charges" of the bubble:

Fermion number,

$$\begin{aligned} &\delta R^\mu = 0,\\ &\delta\psi = -i\delta\theta\psi,\\ &J^\alpha = \bar{\psi}\gamma^\alpha\psi,\\ &Q = \int d\Sigma_\alpha \sqrt{|g|}\,J^\alpha; \end{aligned} \tag{20}$$

Energy-momentum,

$$\begin{aligned} &\delta R^\mu = \text{constant},\\ &\delta\psi = 0,\\ &T^{\alpha\mu} = T^{\alpha\beta}\tau^\mu_\beta,\\ &P^\mu = \int d\Sigma_\alpha \sqrt{|g|}\,T^{\alpha\mu}; \end{aligned} \tag{21}$$

Lorentz rotations,

$$\begin{aligned} &\delta R^\mu = \delta\omega^\mu{}_\nu R^\nu ,\\ &\delta\psi = -\frac{i}{4}\delta\omega_{\mu\nu}\sigma^{\mu\nu}\psi,\\ &M^{\alpha\mu\nu} = R^\mu T^{\alpha\mu} - R^\nu T^{\alpha\tau} + \tfrac{1}{4}\bar{\psi}\{\gamma^\alpha, \sigma^{\mu\nu}\}\psi,\\ &M^{\mu\nu} = \int d\Sigma_\alpha \sqrt{|g|}\,M^{\alpha\mu\nu}. \end{aligned} \tag{22}$$

The integrals above are to be taken over any closed spacelike submanifold ("spacelike cut") of the hypertube (Fig. 2). The differential $d\Sigma_\alpha$ is the oriented element of area defined by

$$d\Sigma_{(\alpha)} \times du^{(\alpha)} = d^{n-1}u \quad (\text{no sum on } \alpha).$$

The theory we have developed is manifestly Lorentz invariant and generally covariant. Mathematically, this is a trivial consequence of the fact that all quantities are represented as tensors under Lorentz transformations and under internal coordinate transformations. We note that the spinor ψ is a spinor only in Minkowski space; it is a scalar with respect to surface coordinate transformations. One immediate consequence of Lorentz invariance is that static solutions, which have zero spatial momentum, correspond to particles of mass equal to their energy.

The conserved currents are tangential to the surface at each point. This is a physically and mathematically sensible result. If a current had a normal component, one would hardly expect that its charge could be conserved on the surface. Mathematically, only a tangential current can be integrated over a spacelike cut to produce a conserved "charge." The condition which ensures that the conserved currents are tangential is Eq. (16). This equation of constraint on the Dirac field severely restricts the possible fermionic currents that can be constructed. Essentially, we have a two-component fermion. From (16) and the relation $\{i\not n, \not\tau^{\alpha}\}=0$, we have

$$\bar\psi\not\tau^{\alpha_1}\cdots\not\tau^{\alpha_n}\psi=0 \quad \text{if } n \text{ is even,}$$

$$\bar\psi\not\tau^{\alpha_1}\cdots\not\tau^{\alpha_n}\gamma_5\psi=0 \quad \text{if } n \text{ is odd.}$$

Thus, (16) guarantees that the usual fermion current agrees with the Noether current derived above:

$$\bar\psi\gamma^{\mu}\psi=\bar\psi(\not\tau^{\alpha}\tau^{\mu}_{\alpha}-\not n n^{\mu})\psi=J^{\alpha}\tau^{\mu}_{\alpha}. \tag{23}$$

In contrast, the "axial-vector current" $\bar\psi\gamma^{\mu}\gamma_5\psi$ is purely normal:

$$\bar\psi\gamma^{\mu}\gamma_5\psi=\bar\psi(-\not n n^{\mu})\gamma_5\psi=\bar\psi(-i\gamma_5)\psi n^{\mu}. \tag{24}$$

This axial-vector current cannot be "conserved" in any sense in this theory, nor can a Lorentz- and coordinate-invariant integral over it even be defined. Every current constructed from the Dirac field can be expressed in terms of the vector and pseudoscalar currents. The expressions for the standard currents are given in Table I. We note that the scalar current vanishes, so that the Lagrangian cannot be modified in such a way as to give surface quarks an effective mass.

The generalization of the bubble equations to the case of several independent quark species is completely straightforward. Each quark field appears in the action separately,

$$S=\int du\sqrt{|g|}\left(\sum_a \bar\psi_a i\not\partial\psi_a - C\right). \tag{25}$$

Therefore, each quark field obeys the equations of motion (16) and (17), while the fermion contribution to the stress tensor in (19) is the sum over all species.

TABLE I. The Dirac currents on the bubble. The condition (16) allows all Dirac currents to be expressed in terms of the tangent vector and pseudoscalar currents.

$\bar\psi\psi=0$	$\bar\psi\gamma^{\mu}\psi\equiv J^{\alpha}\tau^{\mu}_{\alpha}$
$\bar\psi i\gamma_5\psi\equiv P$	$\bar\psi\gamma^{\mu}\gamma_5\psi=-Pn^{\mu}$
$\bar\psi\sigma^{\mu\nu}\psi=J^{\alpha}(n^{\mu}\tau^{\nu}_{\alpha}-n^{\nu}\tau^{\mu}_{\alpha})$	

In the bubble model of hadrons proposed by BCDWY, strong color gauge interactions are introduced which serve to unbind all states which are not singlets under $SU(3)_{\text{color}}$. The energy of color-singlet states remains unmodified, at least at the semiclassical level. Thus, the BCDWY scheme is equivalent, for our purposes, to a bubble theory of three independent quarks of different colors with the additional selection rule that only color-singlet bubble states are allowed.

Equations (16), (17), and (19) give a complete classical description of the dynamics of single-bubble states of the BCDWY field theory in the strong-coupling limit. These equations involve only the surface geometry and surface quark fields, and can be derived from the action principle (14) with the constraint (16). The bubble theory could have been formulated directly in terms of surface quantities, without reference to the BCDWY field theory. Such a canonical bubble theory shares many qualitative features with the Nambu string and MIT bag.

In three space dimensions, the string, bubble, and bag theories consider, respectively, one-, two-, or three-dimensional extended objects whose geometric degrees of freedom contribute to the action in proportion to the invariant "volume." Unlike the original Nambu string, the bubble theory describes an extended object upon which quarks are confined.[6] Because the embedding of the bubble surface in spacetime is nontrivial, the surface is a dynamic object carrying energy-momentum in contrast to the geometric degrees of freedom of the MIT bag.

Which, if any, of these theories may best serve to describe hadronic structure is an important question which will not be finally resolved here.

V. STATIC BUBBLE STATES

In this section and the next, we consider several exact and approximate solutions to the bubble equations (16), (17) and (19). Before examining these solutions in detail, it is important to recognize the relevance of such solutions to a model of hadron structure based on the bubble. The semiclassical theory accounts approximately for the quantum nature of the quarks. The bubble surface motion is treated entirely classically. Classical surface motion is inconsistent in principle with quantized Dirac fields. In practice, we will see that the semiclassical theory has a continuous spectrum of surface excitations and that, although the theory is Poincaré invariant, the states of the theory do not transform as irreducible "particle" representations of the Poincaré group.

The bubble model of hadrons is developed from a theory of $SU(3)_{\text{color}}$ quarks trapped in bubbles. Our goal in discussing solutions to the bubble equations is not so much to estimate hadron masses in the theory as it is to investigate and characterize the physical properties of bubbles. We will, therefore, consider, for the most part, only bubbles containing a single quark species.

This section is devoted to the analysis of static solutions to the bubble equations in two and three space dimensions. We find that all static solutions in two space dimensions can be found. In two dimensions, the energy of bubble states is independent of bubble shape. We also easily reproduce the spherical three-dimensional solution obtained in Ref. 1 by BCDWY. We consider the problem of nonspherical static bubbles in three dimensions. In such states, the quark is orbitally excited though the surface remains static. It is found that the surface is highly nonspherical even for the lowest quark orbital excitations in three dimensions. This is a reflection of a most important general property of bubbles—they are extremely "soft" to deformations.

In the case of a static surface, the geometric formalism introduced in Sec. IV simplifies considerably. Taking internal coordinates $u^0=t$, $u^1,u^2,\ldots,u^{n-2}$ we have

$$\begin{aligned}
&R^\mu(t,u^\alpha)=(t,\vec{R}(u^a)),\\
&\tau_0^\mu=(1,\vec{0}),\quad \tau_a^\mu=(0,\vec{\tau}_a),\\
&g_{\alpha\beta}=\begin{pmatrix}1 & \vec{0}\\ \vec{0} & -g_{ab}\end{pmatrix},\quad g_{ab}\equiv\vec{\tau}_a\cdot\vec{\tau}_b,\\
&n^\mu=(0,\hat{n}(u^a)),\\
&h_{\alpha\beta}=\begin{pmatrix}0 & \vec{0}\\ \vec{0} & -h_{ab}\end{pmatrix},\quad k=\tfrac{1}{2}h^a{}_a.
\end{aligned}\tag{26}$$

Here, $a,b,c,d,\ldots$ are spacelike internal coordinate indices $(1,\ldots,n-2)$, while $i,j,k,l,\ldots$ are indices in Euclidean space $(1,\ldots,n-1)$. g_{ab} is the induced metric tensor on the spatial surface $\vec{R}(u^a)$ and will be used to raise and lower indices on spatial tensors.

By virtue of (16) we can write the Dirac field in terms of a two-component spinor, χ:

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi\\ i\hat{n}\cdot\vec{\sigma}\chi\end{pmatrix}e^{-iEt},\tag{27}$$

where we have used the Dirac representation of the γ matrices. In terms of χ, the Dirac equation is

$$H\chi=E\chi,\tag{28}$$

with Hamiltonian

$$H=k-i\vec{\sigma}\cdot(\hat{n}\times\vec{\nabla}_{\parallel}).$$

The conserved currents of the theory can be written in terms of χ as follows:

$$\begin{aligned}
&J^0=\chi^\dagger\chi,\quad J^a=\chi^\dagger\vec{\sigma}\cdot(\hat{n}\times\vec{\tau}^a)\chi,\\
&T^{00}=C+E\chi^\dagger\chi,\\
&T^{0a}=\mathrm{Im}(\chi^\dagger\partial^a\chi)+\tfrac{1}{2}h^{ab}J_b,\\
&T^{a0}=-EJ^a,\\
&T^{ab}=-Cg^{ab}+\tfrac{1}{2}h^{ab}\chi^\dagger\chi+\mathrm{Im}\chi^\dagger\vec{\sigma}\cdot(\hat{n}\times\vec{\tau}^a)\partial^b\chi\,.
\end{aligned}$$

The normalization of χ is

$$Q=\int du\sqrt{|g|}\,\chi^\dagger\chi=1,$$

and the total energy is

$$U=\int du\sqrt{|g|}\,T^{00}=E+CA.$$

The requirement that the action (14) be stationary is equivalent to the condition that the total energy, U, be stationary under all variations of the spatial surface:

$$\frac{\delta}{\delta\vec{R}(u^a)}U=0$$

or

$$2Ck=\tfrac{1}{2}(h^{ab}h_{ab})\chi^\dagger\chi+h_{ab}\mathrm{Im}\chi^\dagger\vec{\sigma}\cdot(\hat{n}\times\vec{\tau}^a)\partial^b\chi.\tag{29}$$

The system of coupled equations (28) and (29) is very difficult to solve exactly or approximately in three dimensions. Before attacking the three-dimensional problem it is instructive to consider the two-dimensional case, where an exact general solution is available.

In two space dimensions, the bubble is a closed curve in the x-y plane (Fig. 3). We can choose the single parameter describing this curve as its length, l. Then

$$\begin{aligned}
&\vec{R}=\vec{R}(l),\\
&\hat{e}=\frac{d\vec{R}}{dl}=\text{unit tangent vector},\\
&\hat{n}\times\hat{e}=\hat{z}.
\end{aligned}$$

The curvature is then

$$k=\frac{1}{2}\hat{e}\cdot\frac{d\hat{n}}{dl}=\frac{1}{2}\frac{d\Phi}{dl},$$

where Φ is the angle of the normal with respect to some fixed direction in the plane (Fig. 3).

The Dirac equation is

$$\left(\frac{1}{2}\frac{d\Phi}{dl}-i\sigma_3\frac{d}{dl}\right)\chi=E\chi,$$

which may be integrated immediately to yield

$$\chi(l)=\exp\{i\sigma_3[E-\tfrac{1}{2}\Phi(l)+\tfrac{1}{2}\Phi(0)]\}\chi(0).$$

χ must be single valued, so we have

$$\chi(L)=\chi(0),\ \text{where } L=\text{total length}$$

or

$$2n\pi = EL - \tfrac{1}{2}[\Phi(L)-\Phi(0)] = EL-\pi\,,$$

where n is an integer. The Dirac energy is

$$E=\frac{2\pi m}{L}\,,\quad m\equiv n+\tfrac{1}{2}\,,$$

and the normalized Dirac wave function can be written

$$\chi=\frac{1}{\sqrt{L}}\exp\{i\sigma_3[El-\tfrac{1}{2}\Phi(l)]\}u\,,$$

where u is a fixed unit spinor.

The Dirac energy depends only on the perimeter of the bubble, L, not on its shape. There are paired positive- and negative-energy levels of the same magnitude. There is no zero-energy mode. These results can be readily understood geometrically. A one-dimensional manifold has no intrinsic curvature; from the point of view of a quark trapped on a curve, the geometry in the neighborhood of any one point is equivalent to the geometry in the neighborhood of any other point. This leads to a "translation" invariance along the curve. For spinors, this translation is realized by parallel transport, under which the spinor changes only in phase. The Dirac Hamiltonian is just the generator of such translations. Because the quark has spin $\frac{1}{2}$, transport around a closed path induces a phase factor -1, which must be compensated by the factor $e^{i\sigma_3 EL}$. Hence, the energy cannot vanish.

We interpret negative-energy quark states as positive-energy antiquarks. The local bubble energy is, then,

$$U=\frac{2\pi|m|}{L}+CL\,. \tag{30a}$$

Minimizing over L, we have

$$L=\left(\frac{2\pi|m|}{C}\right)^{1/2},\quad U=(8\pi C)^{1/2}m^{1/2}. \tag{30b}$$

It is straightforward to check that, if L is chosen to minimize U as above, Eq. (29) is satisfied at each point on the bubble surface.

The two-dimensional bubble is then extremely soft. Static bubble states occur only with perimeters fixed by the Dirac quantum number m; but bubbles of all shapes with this perimeter are degenerate classically. It is not to be expected that a fully quantized theory will have such an infinite degeneracy.[7] The reflection of the bubble's softness there lies in the large quantum fluctuations of the surface. We shall see that the three-dimensional bubble is also soft, but not so soft that all shapes are degenerate.

We note that there is one conserved quantity which does depend on the bubble shape. This is the angular momentum, J_3,

$$J_3=M^{12}=\oint dl\,(R^1T^{0(2)}-R^2T^{0(1)})\,,$$

where (1) and (2) refer to a spatial index, i.

$$T^{01}=\mathrm{Im}\left[\frac{u^\dagger}{\sqrt{L}}\,i\sigma_3(E-k)\frac{u}{\sqrt{L}}\right]+\tfrac{1}{2}(2k)u^\dagger\sigma_3 u$$

$$=\frac{E}{L}\langle\sigma_3\rangle,\ \text{where } \langle\sigma_3\rangle=u^\dagger\sigma_3 u.$$

Then

$$J_3=\frac{E}{L}\langle\sigma_3\rangle\int dl[\vec{\mathrm{R}}\times\hat{e}]_3=\frac{E}{L}\langle\sigma_3\rangle A\,,$$

where A is the total area of the bubble, and, of course, depends on its shape. Using the expression for E, we can rewrite this result:

$$J_3=|m|\,\langle\sigma_3\rangle\left[\frac{A}{\pi(L/2\pi)^2}\right]$$

or

$$J_3=(8\pi C)^{-1}U^2\langle\sigma_3\rangle\left[\frac{A}{\pi(L/2\pi)^2}\right].$$

The ratio $A/[\pi(L/2\pi)^2]$ is the ratio of the area of the bubble to the maximum area it could have, given perimeter L. The curve of maximum area with fixed perimeter is unique—a circle. Thus, the maximum possible angular momentum of a state of energy U is

$$J_{3,\max}(U^2)=\frac{1}{8\pi C}U^2. \tag{31}$$

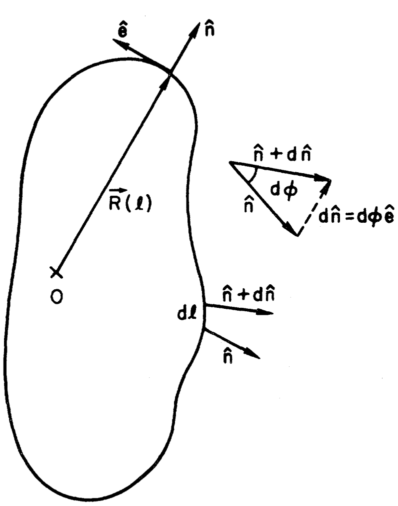

FIG. 3. The two-dimensional bubble.

Thus, the leading Regge trajectory of the two-dimensional model is nondegenerate and linear in $(\text{mass})^2$ with slope $(8\pi C)^{-1}$.

Unfortunately, the static-bubble equations in three dimensions are not so easily solved. The only known exact solution is a spherically symmetric one corresponding to the approximate solution of the field equations found by BCDWY. It is simply a very difficult technical problem to simultaneously solve the Dirac equation and satisfy the condition that the total energy by minimal under all local variations of the surface. In principle, however, one can find all solutions to the static equations as follows: (1) Solve the Dirac equation exactly on a general closed spatial surface, $\vec{R}(u^a)$. Because the surface is compact, the Dirac spectrum is discrete and the energy levels can be labeled by two discrete parameters, m_1, m_2 such that the Dirac energy is a continuous functional of the surface variables: $E_{m_1 m_2}[\vec{R}(u^a)]$. (2) Choose which levels are to be occupied by quarks or antiquarks. (3) Minimize the total-energy functional

$$U[\vec{R}(u^a)] = CA[\vec{R}(u^a)] + \sum_{\substack{\text{occupied}\\ \text{levels}}} E_{m_1 m_2}[\vec{R}(u^a)]$$

in the space of functions $\vec{R}(u^a)$.

Such a procedure is much too difficult to be carried out in practice. It suggests, however, a practical scheme for finding the energy levels approximately. Namely, we attempt to carry out the above procedure, not on a general surface, but over a class of surfaces sufficiently limited that the Dirac equation is tractable. We will choose a form for the bubble surface that depends on several real parameters, solve for the Dirac energy as a function of these parameters, then minimize the total energy over the parameters that define the surface. Because the total-energy functional is positive definite, such a variational estimate of the energy is an upper bound on the energy of the lowest bubble state with the assumed Dirac quantum numbers m_1, m_2. The accuracy of such a variational estimate depends entirely on whether the trial surfaces we consider are sufficiently "near" the true solution. This in turn depends, as a practical matter, on how well we understand the character of the distortions of the excited states of the theory.

We begin by considering the simplest possible trial surface—a sphere. We rederive the BCDWY solution of the static Euler Lagrange equations, now expressed in the geometric language of bubble theory. Let the sphere have radius R and angular coordinates be given by the usual polar angles θ, ϕ. Then we have

$$g_{ab} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad \hat{n} = \hat{r}(\theta, \phi),$$

$$h_{ab} = \frac{1}{R} g_{ab}, \quad k = \frac{1}{R},$$

$$\vec{\nabla}_{\parallel} = \frac{1}{R}\left(\hat{\theta}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{\sin\theta}\frac{\partial}{\partial\phi}\right).$$

The two-component Dirac Hamiltonian is

$$H = \frac{1}{R} - i\sigma\cdot(\hat{r}\times\vec{\nabla}_{\parallel}) = \frac{1}{R}(1 + \vec{L}\cdot\vec{\sigma}).$$

Its normalized eigenfunctions are

$$\chi = \frac{1}{R}\,\phi^{l}_{jm}(\theta, \phi),$$

where ϕ^l_{jm} is the Pauli wave function of spin j, m and orbital angular momentum l, and the corresponding Dirac energies are

$$E = \begin{cases} \dfrac{j+\frac{1}{2}}{R} & \text{if } j = l + \frac{1}{2} \\ -\dfrac{j+\frac{1}{2}}{R} & \text{if } j = l - \frac{1}{2} \end{cases}. \tag{32}$$

We interpret states with $j = l + \frac{1}{2}$ as quarks, those with $j = l - \frac{1}{2}$ as antiquarks. The total energy is

$$U = \frac{j+\frac{1}{2}}{R} + C4\pi R^2.$$

Minimizing over the parameter R, we have

$$\begin{aligned} R &= (8\pi C)^{-1/3}(j+\tfrac{1}{2})^{1/3} \equiv R_0 (j+\tfrac{1}{2})^{1/3}, \\ U &= \frac{3}{2R_0}(j+\tfrac{1}{2})^{2/3}. \end{aligned} \tag{33}$$

This gives the best approximation to the energy of single-quark states with the quantum numbers (j, m) over spherical surfaces.

The local equation for the minimization of the total energy is

$$0 = -h_{\alpha\beta}T^{\alpha\beta} = h_{ab}T^{ab} \equiv F,$$

F = outward normal force density.

For the spherical quark state (j, m),

$$F = -\frac{2C}{R} + \frac{E}{R^3}\,|\phi^l_{jm}|^2. \tag{34}$$

This vanishes locally only if $j = \frac{1}{2}$, so that $|\phi^l_{m/2}|^2 = 1/4\pi$ is independent of θ, ϕ. For $j = \frac{1}{2}$, the solution obtained by varying over spherical trial surfaces is exact. In the bubble with $j = \frac{1}{2}$, the net surface tension vanishes locally. Physically, this reflects the exact balance of the uniform surface tension C and the uniform fermi pressure due to the quark field.

For $j > \frac{1}{2}$, the surface tension and Fermi pressure balance only on the average; there is a tension-induced normal force that will tend to distort the surface from sphericity. From (34), we see that this force tends to push the surface out where the quark density is high, and allows the surface to collapse where the quark density is low (Fig.4). A particularly simple example is the case of a quark of maximal z-component angular momentum, $m=j=l+\frac{1}{2}$. The normal force density is

$$F=2\left(\frac{8\pi C}{l+1}\right)^{1/3} C\left[\frac{(l+1)(2l+1)!!}{(2l)!!}(\sin\theta)^{2l}-1\right].$$

This is a force which is axially symmetric and has a single peak in the equatorial plane. It will tend to stretch the sphere at the equator and depress it at the poles. The force densities associated with quark states with $|m|<j$ have one or more azimuthal nodes, and tend to distort the sphere to rather more complicated shapes.

The angular dependence of these force densities on the sphere suggests the shapes we should use for trial surfaces in a variational estimate of excited-state energies. We note that, because the force densities differ for spherical quark states of the same j but different m, the surfaces which actually minimize the total energy will presumably be of different shapes. Thus, it appears, the semiclassical spectrum will not necessarily consist of $(2j+1)$-fold degenerate levels corresponding to particle states of the same j but varying m. This result, though disturbing, is not terribly surprising. It is a consequence of the semiclassical treatment of the surface degrees of freedom. In a full quantum theory, the surfaces corresponding to states of the same j but different m will, because their shapes differ, have slightly different energies associated with their quantum fluctuations. This relative shift should precisely cancel the semiclassical splitting, and restore rotational invariance to the spectrum.

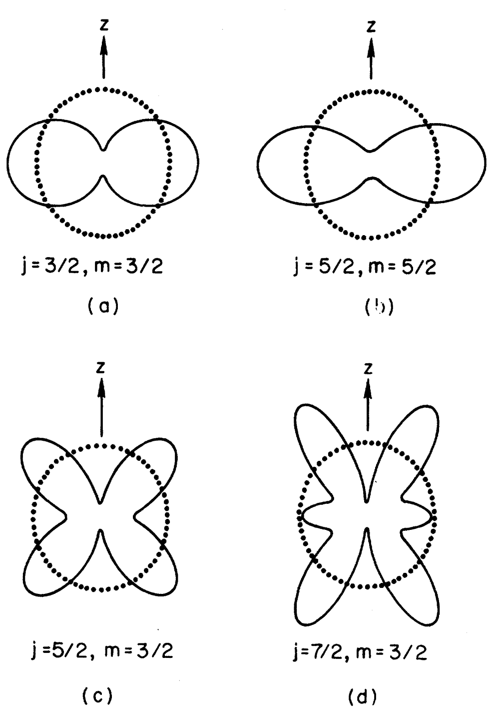

FIG. 4. The normal force density on a spherical bubble for various single-quark states. The dotted line is the zero of force.

We will sidestep this problem by considering only quark states corresponding to $|m|=j$, and interpreting the resulting energies as estimates of the energy of a multiplet of spin j. We can adduce several arguments for this interpretation. The surfaces corresponding to $|m|=j$ states are simple and smooth. Those corresponding to other values of m will be complicated and "bumpy." As a practical matter, it is extremely difficult to do the required variational calculations for surfaces of very complicated shapes. Further, because these surfaces are "bumpy," we suspect the effects of their quantum fluctuations to be relatively more important than they are for smoother surfaces. Thus, the most relatively consistent way of neglecting quantum fluctuations is to estimate the energies using states which have smooth surfaces. Finally, as we shall see, the effects of distortions of static surfaces are numerically small for the low-lying excited states. In no case will our variational estimate of the energy of single-quark bubbles be more than 10% lower than the value estimated from the sphere. Thus, whatever approximation we make, we commit no gross numerical error.

As a simple trial surface that is smooth and flattened at the poles, consider the oblate spheroid:

$$\vec{R}(\theta,\phi)=R(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ (1-d^2)^{1/2}\cos\theta),$$

where

$$0\leq d\leq 1.$$

This surface depends on two parameters: R which determines its over-all size, and d which determines shape. For $d=0$, the surface is a sphere. As d increases from zero, the spheroid becomes flatter and flatter, until at $d=1$ it is an infinitely thin "pancake." The area of the spheroid is

$$A=\frac{1}{2}\left(1+\frac{1-d^2}{2d}\ln\frac{1+d}{1-d}\right)4\pi R^2.$$

The Hamiltonian of the surface is

$$H=\frac{1}{R(1-d^2\sin^2\theta)^{1/2}}\left\{\tfrac{1}{2}(1-d^2)^{1/2}\left(1+\frac{1}{1-d^2\sin^2\theta}\right)-i\vec{\sigma}\cdot\hat{\phi}\,\frac{\partial}{\partial\theta}+i\left[\cot\theta(\cos\phi\,\sigma_1+\sin\phi\,\sigma_2)-\sigma_3(1-d^2)^{1/2}\right]\frac{\partial}{\partial\phi}\right\}.$$

Because the surface is axially symmetric, the z component of angular momentum is conserved:

$$[H,J_3]=0,\quad \text{where } J_3=-i\frac{\partial}{\partial\phi}+\tfrac{1}{2}\sigma_3 .$$

Thus, we can choose Dirac wave functions which are eigenstates of J_3.

The remaining diagonalization of the Hamiltonian must be done numerically. The level on the spheroid which corresponds to $j=m$ on the sphere is simply the lowest positive-energy state in the sector $J_3=m$. We compute the total energy, U, of a spheroidal bubble occupied by a single quark of spin m, and minimize it over R at fixed d. The ratio of this energy to the corresponding energy estimate on the sphere,

$$\rho_m(d)=\frac{U_m(d)}{(3/2R_0)(m+\frac{1}{2})^{2/3}}$$

is plotted as a function of d for $m=\frac{3}{2}$ and $m=\frac{5}{2}$ in Fig. 5.

We see immediately that, in both cases, the total energy decreases monotonically as a function of d. Indeed, these calculations show that the energy of the spheroid is lowest in the limit where it becomes a completely flattened disk. Despite the fact that such a disk has very large curvature at its edge, the Dirac energy remains small. This result is actually quite general—the static Dirac equation can be solved on surfaces with sharp edges. In the limit that an edge becomes infinitely sharp, the Dirac equation gives a boundary condition across the edge:

$$\chi(2)=\exp[-(i/2)\Delta\vec{\phi}_{21}\cdot\vec{\sigma}]\,\chi(1)\,, \tag{35}$$

where $\Delta\vec{\phi}_{21}$ is the vector rotation angle of the normal at its discontinuity across the edge between surfaces 1 and 2 (Fig. 6).

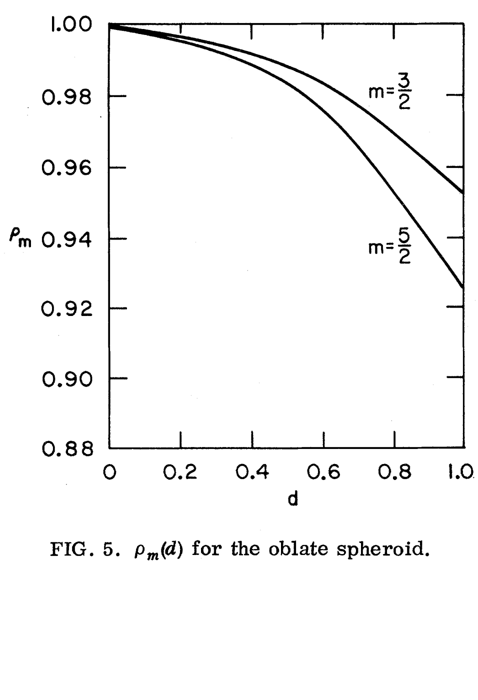

FIG. 5. $\rho_m(d)$ for the oblate spheroid.

The oblate spheroid is not an adequate trial surface. It takes into account the tendency of the surface to spread at the equator, but does not allow for sufficient depression at the poles. We note, however, that although the energy decreases uniformly as the spheroid flattens, the numerical size of the decrease is rather small. Even the completely flattened disk has energy down by less than 10% from that estimated on the sphere.

We want to find a trial surface which is both spread at the equator and dips inward at the poles. We could begin to consider surfaces that are defined by three or more parameters, but it is computationally more straightforward to continue to work with two-parameter surfaces as long as possible. A simple two-parameter surface in which the region near the poles is completely depressed is the torus (Fig. 7). This surface may be regarded as one where the poles have dipped in so far as to create a hole through the center.

We coordinatize the torus as follows:

$$\vec{R}(\theta,\phi)=b\big((\gamma+\sin\theta)\cos\phi,(\gamma+\sin\theta)\sin\phi,\cos\theta\big)\,,$$

where

$$0\le\theta\le 2\pi\,,$$

$$0\le\phi\le 2\pi\,,$$

$$\gamma\ge 1\,.$$

b is the radius of the circular vertical cross sections of the torus; γb is the radius of the torus in the x-y plane. The area of the torus is

$$A=4\pi^2\gamma b^2\,.$$

The surface Hamiltonian is

$$H=\frac{1}{b}\left(\frac{\frac{1}{2}\gamma+\sin\theta}{\gamma+\sin\theta}-i\vec{\sigma}\cdot\hat{\phi}\frac{\partial}{\partial\theta}+\frac{i}{\gamma+\sin\theta}\vec{\sigma}\cdot\hat{\theta}\frac{\partial}{\partial\phi}\right),$$

where

$$\hat{\phi}=(-\sin\phi,\ \cos\phi,0)\,,$$

$$\hat{\theta}=(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)\,.$$

As before, $J_3=-i\partial/\partial\phi+\frac{1}{2}\sigma_3$ commutes with H, and we can work in a sector of definite J_3: $J_3=m$. The state corresponding to $j=m$ on the sphere is again the lowest positive-energy state in this sector.

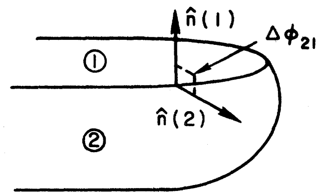

FIG. 6. Geometry at a sharp edge.

This Hamiltonian must also be diagonalized numerically. We compute the total energy, U, of single-quark states with $m=\frac{3}{2}$ and $m=\frac{5}{2}$, and minimize over b at fixed γ. The ratio

$$\rho_m(\gamma)=\frac{U_m(\gamma)}{(3/2R_0)(m+\frac{1}{2})^{2/3}}$$

is plotted versus γ in Fig. 8. The minima of the total energy in γ are given in Table II. The energy estimate of the $m=\frac{5}{2}$ state is lower than the corresponding estimate on a flattened disk and suggests that single-quark bubbles of spin $\frac{5}{2}$ and larger will have a toroidal shape. The energy estimate for the spin-$\frac{3}{2}$ bubble on the torus is larger than the estimate on the flattened disk. Presumably, the $m=\frac{3}{2}$ state is extremely depressed at the poles but remains connected.

Despite the radical differences in their shape and topology, we see that the energies of low-lying single-quark states on spheres and on toruses are not very different. We interpret this as a reflection of the "softness" of the three-dimensional bubble. This three-dimensional result is analogous to the complete shape degeneracy of the two-dimensional bubble. In order to estimate static energies more accurately, we should consider trial surfaces defined by more than two adjustable parameters. As a practical matter, as long as we are interested in only the energies of low-lying states, the computational difficulties involved in such calculations are not justified by the results we would hope to obtain. For single-quark states of spin less than $\frac{5}{2}$, we have seen that the correction to the energy due to distortions is less than 10%. For multiquark bubbles where one or more quarks remain in the lowest state, the effects are still smaller. Three quarks of spin $\frac{5}{2}$ could combine to form baryonic states of maximum spin $\frac{15}{2}$. There are not yet observed hadrons of such high spin, nor are the experimental masses of the higher resonances known to within 10%. We have neglected the effects of SU(3) breaking, which must be sizeable in the higher multiplets. Further, as we shall see in the case of the radial mode, quantum fluctuations may be expected to give corrections to the energy levels at least as large as those due to static distortions of the bubble shape.

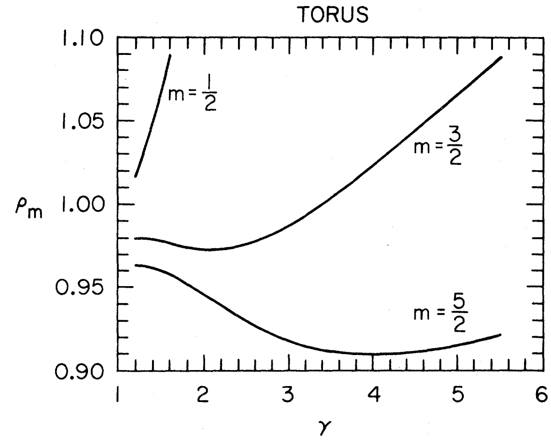

FIG. 8. $\rho_m(\gamma)$ for the torus.

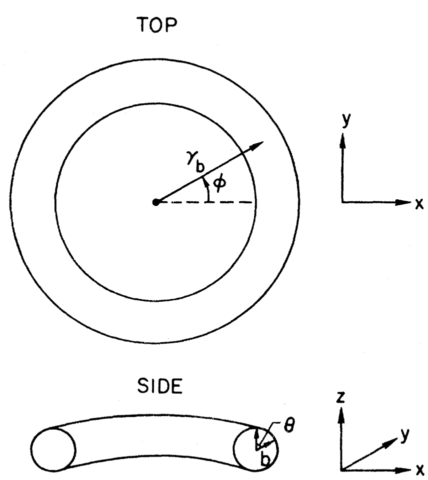

FIG. 7. Coordinates on a torus.

VI. RADIAL MODE

So far, we have not discussed either semiclassical states in which the bubble surface is nontrivially time dependent or the effects of quantum fluctuations of the surface on the spectrum. It is quite difficult to find nonstatic solutions to the bubble equations (16), (17), and (19). No general prescription for quantizing the surface motion exists.

In this section, we hope to shed some light on

TABLE II. Energies on the torus.

	γ	$\rho_m(\gamma)$
$m=\frac{3}{2}$	2.09	0.973
$m=\frac{5}{2}$	4.04	0.910

these problems by studying the one class of nonstatic bubbles in three dimensions for which an exact semiclassical solution has been found. These are spherically symmetric bubbles with a time-dependent radius—the breathing modes. We first exhibit the exact solutions of the semiclassical equations. Then we "quantize" the set of all such modes using the WKB approximation. We will see that the softness of the bubble is reflected dynamically in the large size of quantum fluctuations of its surface

We begin with the semiclassical time-dependent equations of motion. Let us assume that there is a solution of these equations whose surface is a sphere of time-dependent radius $R(t)$:

$$R^{\mu}(t,\theta,\phi)=(t,R(t)\hat{r}(\theta,\phi)).$$

Defining

$$\dot{R}=\frac{dR}{dt}=\tanh\omega(t)$$

we have

$$g_{\alpha\beta}=\begin{bmatrix}\frac{1}{\cosh^2\omega} & 0 & 0\\ 0 & -R^2 & 0\\ 0 & 0 & -R^2\sin^2\theta\end{bmatrix},$$

$$\sqrt{|g|}=\frac{R^2\sin\theta}{\cosh\omega},$$

$$n^{\mu}=(\sinh\omega,\cosh\omega\;\hat{r}),$$

$$h^{\alpha}_{\beta}=\begin{bmatrix}\cosh\omega\,\dot{\omega} & 0 & 0\\ 0 & \frac{\cosh\omega}{R} & 0\\ 0 & 0 & \frac{\cosh\omega}{R}\end{bmatrix}.$$

We take a form for the Dirac field that has $L=0$ and automatically satisfies (16):

$$\psi=\frac{1}{\sqrt{2}}(1+i\not{n})\begin{pmatrix}F(t)\\ 0\end{pmatrix},$$

where $F(t)$ is some two-component spinor. The Dirac equation becomes

$$\dot{F}(t)=-\frac{k}{\cosh^2\omega}(i+\sinh\omega)F(0),$$

whose integral is

$$F(t)=\exp\left(-\int_0^t dt\,\frac{k(i+\sinh\omega)}{\cosh^2\omega}\right)F(0)$$
$$=\frac{R(0)[\cosh\omega(0)]^{1/2}}{R(t)[\cosh\omega(t)]^{1/2}}\exp\left(-i\int_0^t dt\,\frac{k}{\cosh^2\omega}\right)F(0).$$

The normalization condition

$$1=\int d\theta\,d\phi\,\frac{R^2\sin\theta}{\cosh\omega}\,\bar{\psi}\gamma^0\psi$$

allows the wave function to be written

$$F(t)=\frac{1}{[4\pi R(t)^2\cosh\omega(t)]^{1/2}}\times\exp\left[-i\int_0^t\frac{dt}{\cosh\omega}\left(\frac{1}{R}+\frac{1}{2}\dot{\omega}\right)\right]u,\qquad(36)$$

where u is a fixed two-component unit spinor.

We see that the Dirac equation is solvable exactly for arbitrary $R(t)$. Equation (19) will determine which of these surfaces are actually allowed dynamical states. Putting the solution for ψ into Eq. (19), we have

$$0=1-R\dot{\omega}-8\pi CR^3(1+\tfrac{1}{2}R\dot{\omega}).$$

This equation can be more simply expressed in rescaled variables

$$R_0\equiv(8\pi C)^{-1/3},$$
$$t=\tau R_0,$$
$$R(t)=\rho(\tau)R_0.$$

We have

$$\frac{d\omega}{d\tau}=\frac{1-\rho^3}{\rho(1+\frac{1}{2}\rho^3)},\quad\frac{d\rho}{d\tau}\equiv\dot{\rho}=\tanh\omega.$$

This can be integrated once to give

$$\epsilon=\frac{1}{(1-\dot{\rho}^2)^{1/2}}\left(\frac{2}{3\rho}+\tfrac{1}{3}\rho^2\right),\qquad(37)$$

where ϵ is a constant. A straightforward integration of the energy density shows that the total energy is

$$U=\epsilon\,\frac{3}{2R_0}.$$

Thus, ϵ is the total energy of the radial mode measured in units of the static-ground-state energy.

If $\epsilon=1$, we recover the static solution: $\rho=1$, $\dot{\rho}=0$. For $\epsilon<1$, there are no solutions. For each $\epsilon>1$, there exists a unique solution in which $\rho(\tau)$ is periodic, with turning points determined by

$$\epsilon=\frac{2}{3\rho}+\tfrac{1}{3}\rho^2.$$

The equation for ρ is similar to that for a relativistic particle in a scalar potential

$$V(\rho)=\frac{2}{3\rho}+\tfrac{1}{3}\rho^2,$$

shown in Fig. 9.

We note that the total energy is continuous. As mentioned in Sec. V, this is an effect due to the classical treatment of the surface degrees of freedom. In order to get some idea of the level structure of the radial modes, we quantize this excitation in the WKB approximation.

We treat the equation for ρ as if it were, indeed, the equation of motion of a relativistic particle in a potential. We take the expression for the total energy (37) to be the Hamiltonian

$$H=\frac{1}{(1-\dot{\rho}^2)^{1/2}}\left(\frac{2}{3\rho}+\tfrac{1}{3}\rho^2\right).$$

The most general Lagrangian from which this H could have been derived is

$$L(\rho,\dot{\rho})=-(1-\dot{\rho}^2)^{1/2}\left(\frac{2}{3\rho}+\tfrac{1}{3}\rho^2\right)+f(\rho)\dot{\rho},$$

where $f(\rho)$ is some undetermined function. The canonical momentum conjugate to ρ is

$$P=\frac{\dot{\rho}}{(1-\dot{\rho}^2)^{1/2}}\left(\frac{2}{3\rho}+\tfrac{1}{3}\rho^2\right)+f(\rho).$$

The WKB approximation gives the discrete energy levels from the quantization condition

$$2\pi(n+\tfrac{1}{2})=\oint_{\text{orbit}}Pd\rho$$

$$=2\int_{\rho_{\min}}^{\rho_{\max}}\left[\epsilon_n{}^2-\left(\frac{2}{3\rho}+\tfrac{1}{3}\rho^2\right)^2\right]^{1/2}. \qquad (38)$$

This equation can be easily solved numerically. The first few values of ϵ_n and the corresponding turning points are given in Table III.

In the lowest state, $n=0$, we see that the effects of surface zero-point motion are very large. The radius fluctuates by a factor of 2 about its static value. The energy in the surface excitation is 60% of the static-ground-state energy. This is a quite dramatic illustration of the softness of the bubble dynamically and suggests that if fluctuations are properly accounted for, the bubble will be quite smeared out in space.

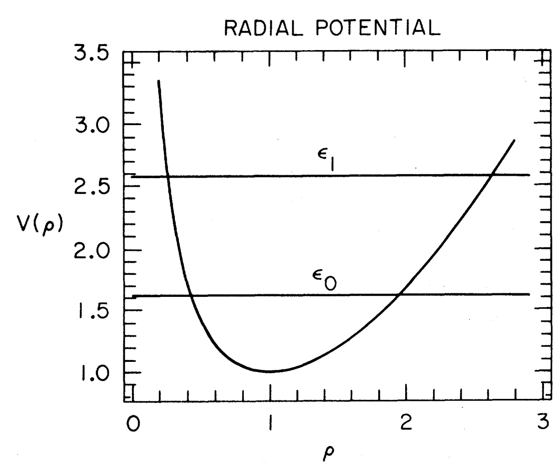

FIG. 9. The scaled radial potential.

The $n=1$ state is the lowest radial excitation of the bubble. Its energy is a factor $\epsilon_1/\epsilon_0=1.60$ higher than that of the ground state. It is easy to convince oneself that, in the case where several quarks occupy the lowest level in the bubble, all the energies of the radial mode simply rescale. Thus, the model predicts radial excitations of baryons and mesons with energies 1.6 times higher than the ground-state energies.

No radially excited meson candidates have been confirmed experimentally. There is, however, a presumed radial excitation of the nucleon—the Roper resonance—of mass 1470 MeV. We note that 1470/940 = 1.56. In the face of our inability to derive solid numerical predictions of excited-state masses, this is a pleasing bit of numerology.

VII. SUMMARY AND CONCLUSIONS

The principal result of this paper is that the low-lying bound states of the semiclassical BCDWY field theory can be completely characterized in the infinitely strong-coupling limit by the geometric theory of bubbles. The nontrivial field degrees of freedom describing such states are equivalent to the geometric variables defining the bubble surface and a set of quark fields defined on it.

The Euler-Lagrange equations of the field theory go over to the equations of bubble dynamics which can be as well derived from the surface action principle (14) with the constraint (16). The theory of bubbles is classically Poincaré invariant. The conserved charges derived from the action principle agree, in the strong-coupling limit, with those of the original field theory.

We have examined several exact and approximate solutions to the semiclassical bubble equations. The most important physical property of bubbles which emerges from this work is their softness. This property is reflected for static bubbles in the small cost in energy for large deformations of the bubble shape. Bubbles in two space dimensions are degenerate over all shapes of given perimeter. Even the lowest excited static bubbles

TABLE III. Excitation energies and turning points for the radial mode.

n	ϵ_n	$\rho_{\min}$	$\rho_{\max}$
0	1.615	0.429	1.956
1	2.577	0.261	2.641
2	3.381	0.198	3.081

in three dimensions are highly distorted from sphericity. The softness of the bubble is reflected dynamically in the large size of the estimated quantum fluctuations of the bubble surface as seen in the example of the radial mode.

In this paper, we have not seriously attempted to compute the hadronic spectrum in the bubble model and to confront experiment. The principal obstacle to such a procedure is the absence of a fully quantized theory of the bubble. The indication from the WKB quantization of the radial mode is that the inclusion of quantum fluctuations is essential to the calculation of masses as functions of the coupling C. It may be hoped that the static semiclassical spectrum gives an indication of the relative levels of excited states, but there is presently no compelling argument that such is the case.

In principle, there are two routes to a quantum theory of the bubble. One is to go back to the quantum field theory defined by the BCDWY Lagrangian and look for low-lying states in the strong-coupling limit.[8] The other is to begin with the classical bubble theory we have developed and attempt to quantize it. At this moment, neither approach has been sufficiently successful to allow for systematic calculation of the quantum spectrum of bubble states in three dimensions. Indeed, it is not immediately evident that the quantization scheme of the classical bubble theory which is free of anomalies exists, or whether, if it exists, it is equivalent to the strong-coupling limit of the quantum BCDWY field theory.

We remark that the explicit quantization of the bubble theory in two space dimensions and one time dimension is possible. In this context, some of the problems related to the more general questions of quantization can be intelligently discussed.[7]

Accepting for the moment that a quantum theory of bubbles can be constructed, the softness which we have seen is characteristic of the semiclassical theory has important implications. The softness of the bubble suggests that quantum fluctuations of the surface will be large and have the effect of smearing the sharp classical energy-momentum and charge distributions of the classical theory over a finite volume of space. One might expect that, unlike the form factor of a shell of charge, the form factors of hadronic states will be smoothly falling functions of q^2. The softness of the bubble affords a qualitative explanation of how scaling might occur in the bubble model. A quark is free to move tangentially to the bubble surface. Because the bubble surface is easily deformed, a quark is nearly free to move normal to the bubble surface by dragging that surface along with it. Thus it need not be surprising if, at large momentum transfers, a quark trapped in a bubble appears to be a free particle.

It is to be hoped that such simple intuitive pictures of hadron dynamics in this model can be supported by calculations with which one might hope to confront experiment.

ACKNOWLEDGMENTS

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APPENDIX A

The description of the bubble surface developed in Sec. IV characterizes its geometry both in terms of intrinsic geometry objects (e.g., $g_{\alpha\beta}, h_{\alpha\beta}$) existing in the local tangent spaces to the coordinate manifold $U=\{u^\alpha\}$, and in terms of the embedding of the surface in Minkowski space, Ω. In this appendix, we briefly discuss some aspects of the relation of these two points of view. The discussion is quite standard[9-11] and is included for completeness only.

Locally, the connection between intrinsic geometric quantities and Minkowski space is described by τ^μ_α and n^μ. τ^μ_α is a mixed tensor—it transforms as a vector in U and independently as a vector in Ω. n^μ is a scalar in U and a vector in Ω.

The completeness relation for the local basis τ^μ_α, n^μ of Ω,

$$n^{\mu\nu}=\tau^{\alpha\mu}\tau^{\nu}_{\alpha}-n^\mu n^\nu\,, \tag{A1}$$

allows us to transform between the intrinsic and external description of geometric objects.

As a Riemannian manifold, the surface is characterized by $g_{\alpha\beta}$. Its embedding in Ω is characterized in the intrinsic description by the coefficients of curvature, $h_{\alpha\beta}$. The Gauss-Codazzi theorem states the relation between the fundamental forms $g_{\alpha\beta}, h_{\alpha\beta}$ and the embedding of the surface in Ω:

There exists a unique (up to Lorentz transformations) surface in Ω whose induced metric is $g_{\alpha\beta}$ and whose coefficients of curvature are $h_{\alpha\beta}$ if and only if

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= h_{\alpha\gamma}h_{\beta\delta}-h_{\alpha\delta}h_{\beta\gamma}\,,\\ h_{\alpha\beta\Vert\gamma} &= h_{\alpha\gamma\Vert\beta}\,, \end{aligned} \tag{A2}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor defined from $g_{\alpha\beta}$ and its derivatives.

In principle, the bubble theory can be discussed purely in terms of tensors in U as long as we append the Gauss-Codazzi equations to the equations of motion. In this approach, the geometric variables are $g_{\alpha\beta}, h_{\alpha\beta}$ rather than $R^\mu(u^\alpha)$. Such an approach is extremely difficult in practice, as the equations are highly nonlinear.

The parallel-transport laws of spinors and vectors introduced in Sec. IV are quite natural when viewed from Minkowski space. If V^μ is a tangent vector field

$$V^\mu = (V\cdot\tau^\alpha)\tau^\mu_\alpha , \tag{A3}$$

then the natural parallel transport of V^μ is to slide V^μ along the surface while preserving its length, for

$$\begin{aligned} &u^\alpha \to u^\alpha + \delta u^\alpha ,\\ &V^\mu \to V^\mu + n^\mu(\delta n\cdot V). \end{aligned} \tag{A4}$$

Then

$$\begin{aligned} \delta V^\alpha &= \delta\tau^\alpha\cdot V + \tau^\alpha\cdot\delta V\\ &= \{{}_\beta{}^\alpha{}_\gamma\} V^\gamma , \end{aligned}$$

which is identical to the usual parallel transport on a Riemannian manifold.

The spinor fields which we consider are spinors only in Ω; they are scalars in U. The parallel-transport law we have assumed for spinors is

$$\delta\psi = -\frac{i}{2}\,\sigma^{\mu\nu} n_\mu \delta n_\nu \psi . \tag{A5}$$

Under parallel transport, a spinor field is rotated by the same Lorentz rotation suffered by a tangent vector transported along the same path.

Finally, we note the form of conservation laws in a curved space. If K^α is a vector field, K^α is locally conserved if

$$\begin{aligned} 0 = K^\alpha{}_{\|\alpha} &= K^\alpha{}_{|\alpha} + \{{}_\alpha{}^\alpha{}_\tau\} K^\tau\\ &= \frac{1}{\sqrt{|g|}}\,(\sqrt{|g|}\,K^\alpha)_{|\alpha} . \end{aligned}$$

Then the charge defined by

$$Q = \int_{\substack{\text{spacelike}\\ \text{cut}}} d\Sigma_\alpha \sqrt{|g|}\,K^\alpha$$

is conserved. A tensor field $W^{\alpha\beta}$ satisfying $W^{\alpha\beta}{}_{\|\beta} = 0$ does *not*, in general, lead to a conserved *vector* charge.

APPENDIX B

In this appendix, we present a more detailed discussion of fields σ_0 and Ψ_0, constructed in Sec. II, and show they are good approximate solutions to the Euler-Lagrange equations (1) and (2) in the strong-coupling limit. The proof proceeds along the following lines.

Solutions to the Euler-Lagrange equations are local extrema of the action functional in function space. If σ_0, Ψ_0 is a good approximate solution, there will be a true extremum of the action in the neighborhood of σ_0, Ψ_0. In this neighborhood, we may expand the action functional to second order in deviations about σ_0, Ψ_0:

$$\begin{aligned} S[\sigma_0+\delta\sigma, \Psi_0+\delta\Psi] &= S[\sigma_0,\Psi_0] + \delta S ,\\ \delta S &= \int dx\,(J\delta\sigma + \bar\eta\delta\Psi + \delta\bar\Psi\eta)\\ &\quad + \int dx\,[\tfrac12(\partial\delta\sigma)^2 - 2\lambda(3\sigma_0^2 - f^2)(\delta\sigma)^2 + \delta\bar\Psi(i\partial\!\!\!/ - G\sigma_0)\delta\Psi - G\delta\sigma(\delta\bar\Psi\Psi_0 + \bar\Psi_0\delta\Psi)] , \end{aligned} \tag{B1}$$

where

$$\begin{aligned} &\text{(a)}\ \ J \equiv -[\partial^2\sigma_0 + 4\lambda\sigma_0(\sigma_0^2 - f^2) + G\bar\Psi_0\Psi_0]\\ &\text{and}\\ &\text{(b)}\ \ \eta \equiv (i\partial\!\!\!/ - G\sigma_0)\Psi_0 . \end{aligned} \tag{B2}$$

The deviation of the true solution from σ_0, Ψ_0 may be estimated as the position of the extremum of the quadratically expanded functional (B1) if $\delta\sigma$ and $\delta\Psi$ are, indeed, small. The equations that must be satisfied by $\delta\sigma$ and $\delta\Psi$ are simply the first-order corrections to the Euler-Lagrange equations:

$$\begin{aligned} &\text{(a)}\ \ [\partial^2 + 4\lambda(3\sigma_0^2 - f^2)]\,\delta\sigma + G(\bar\Psi_0\delta\Psi + \delta\bar\Psi\Psi_0) = J ,\\ &\text{(b)}\ \ (i\partial\!\!\!/ - G\sigma_0)\delta\Psi - G\delta\sigma\Psi_0 = -\eta , \end{aligned} \tag{B3}$$

for which the shift in the action is

$$\delta S = \tfrac12 \int dx\,(J\delta\sigma + \bar\eta\delta\Psi + \delta\bar\Psi\eta) . \tag{B4}$$

If σ_0, Ψ_0 were exact solutions, J and η would vanish and, therefore, $\delta\sigma$ and $\delta\Psi$ would also vanish.

We prove the following theorem.

If $\sigma_0(x), \Psi_0(x)$ are chosen to satisfy

$$\begin{aligned} &\text{(a)}\ \ \sigma_0(x) = f\tanh(2\lambda)^{1/2} f\xi ,\\ &\text{(b)}\ \ \Psi_0(x) = N[\cosh(2\lambda)^{1/2} f\xi]^{-G/(2\lambda)^{1/2}}\,\psi_0(u^\alpha, \xi) ,\\ &\text{(c)}\ \ i n\!\!\!/\psi_0 = \psi_0 ,\\ &\text{(d)}\ \ (i\partial\!\!\!/_\parallel + k i n\!\!\!/)\psi_0 = 0 ,\\ &\text{(e)}\ \ \frac{\partial\psi_0}{\partial\xi} = -\tfrac12(\partial_\mu n^\mu)\psi_0 ,\\ &\text{(f)}\ \ h_{\alpha\beta}(Cg^{\alpha\beta} - \mathrm{Im}\bar\psi_0\gamma^\alpha\partial^\beta\psi_0)|_{\xi=0} = 0 , \end{aligned} \tag{B5}$$

for spacetime coordinates $x^\mu(u^\alpha,\xi)=R^\mu(u^\alpha)+\xi n^\mu$ about some bubble, then the corrections $\delta\sigma$, $\delta\Psi$ computed in (B3) are small in the sense that they induce a correction δS which vanishes in the strong-coupling limit.

The size of δS may be estimated by a detailed examination of Eqs. (B3) and (B4). We will classify the behavior of quantities in the strong-coupling limit by giving their power dependence in f. For example, we write

$$\lambda=(\lambda f^6)f^{-6}\sim f^{-6} \quad\text{as}\quad f\to 0\,.$$

For convenience, we take $G\sim\sqrt{\lambda}$; this choice simplifies the discussion, but is not essential.

The currents J,η computed from (B2) and (B5) are

$$\begin{aligned}&\text{(a)}\;\; J(x)=2(2\lambda)^{1/2}f^2[k+O(\xi)]\,\mathrm{sech}^2[(2\lambda)^{1/2}f\,\xi]\,,\\ &\text{(b)}\;\; \eta(x)=N[\cosh(2\lambda)^{1/2}f\,\xi]^{-G/(2\lambda)^{1/2}}\\ &\qquad\times[\xi(-\tfrac{1}{2}h^{\alpha\beta}h_{\alpha\beta}-ih^{\alpha\beta}\gamma_\alpha\partial_\beta)\psi+O(\xi^2)].\end{aligned}\tag{B6}$$

Both J and η are nonzero only in a thin shell $|\xi|\lesssim D\sim f^2$. Inside this region,

$$\begin{aligned}&\text{(a)}\;\; J\sim f\,,\\ &\text{(b)}\;\; \eta\sim 1\,.\end{aligned}\tag{B7}$$

The leading terms in J and η arise from the cross terms between normal and tangential derivatives in the field equations. If the normal direction were the only one, the solution to the equations of motion would be exact.

Because J and η cut off sharply in ξ, we need only determine $\delta\sigma$ and $\delta\Psi$ in the thin shell $|\xi|\lesssim D$ in order to estimate δS. If the integral (B4) for δS is to remain nonzero as $f\to 0$, we must have, in the thin shell,

$$\begin{aligned}&\text{(a)}\;\; \delta\sigma\gtrsim\frac{1}{f}\,,\\ &\text{(b)}\;\; \delta\Psi\gtrsim\frac{1}{f^2}\,.\end{aligned}$$

That is, in the strong-coupling limit, the leading terms in $\delta\sigma$ and $\delta\Psi$ must be of orders $1/f$ and $1/f^2$ for δS to be nonvanishing. Below we show that no such leading terms arise.

The shifts $\delta\sigma$, $\delta\Psi$ are not uniquely determined by Eqs. (B3) alone. To any given solution of (B3) a solution to the corresponding homogeneous equations [(B3) with J and η set to zero] can be added. This ambiguity corresponds to the possible translations along the surface in function space of all exact solutions to the Euler-Lagrange equations. Here we are interested not in such translations but rather in those solutions to (B3) which vanish as $J,\eta\to 0$. We may pick out such solutions by imposing the boundary condition that $\delta\sigma,\delta\Psi\to 0$ where $J,\eta\to 0$. That is, $\delta\sigma$, $\delta\Psi$ vanish rapidly for $|\xi|>D$.

Now, consider the behavior of the terms in (B3) as $f\to 0$. We note that, by the boundary conditions and the fact that J,η are of width $1/\lambda^{1/2}f$, $\partial/\partial\xi\sim\lambda^{1/2}f$ on $\delta\sigma$ or $\delta\Psi$, while $\partial/\partial u^\alpha\sim$finite.

We have for the various terms which arise in (B4a):

$$\begin{aligned}&G\bar\Psi_0\delta\Psi\sim\frac{1}{f^7}\,,\\ &\frac{\partial^2\delta\sigma}{\delta^2\xi}\sim\frac{1}{f^5}\,,\\ &\frac{\partial}{\partial u^\alpha}\frac{\partial\sigma}{\partial\xi}\sim\frac{1}{f^3}\,,\\ &4\lambda(3\sigma_0^2-f^2)\delta\sigma\sim\frac{1}{f^5}\,,\end{aligned}$$

while

$$J\sim\frac{1}{f}\,.$$

It is straightforward to show that the terms of order $1/f^7,1/f^5,1/f^3$ cancel only if the leading terms $\delta\sigma(u^\alpha,\xi)$, $\delta\Psi(u^\alpha,\xi)$ satisfy

$$\bar\Psi_0\delta\Psi=0 \quad\text{or}\quad i\not{n}\delta\Psi=\delta\Psi\,,\tag{B8}$$

$$\delta\sigma(u^\alpha,\xi)=\rho_0(u^\alpha)e^{-2k\xi}\sigma'(\xi)\,,\tag{B9}$$

where $\rho_0(u^\alpha)$ is finite in the strong-coupling limit and is determined by the details of the $1/f$ terms on both sides of (B3a).

Similarly, using (B9), the consistency of the Dirac equation as $f\to 0$ requires

$$\begin{aligned}\delta\Psi|_{\text{leading}}&=N[\cosh(2\lambda)^{1/2}f\,\xi]^{-G/(2\lambda)^{1/2}}\\ &\quad\times[-G\rho_0(u^\alpha)\sigma(\xi)\psi_0(u^\alpha)+\delta\psi_1]\,,\end{aligned}\tag{B10}$$

where $\delta\psi_1(u^\alpha,\xi)$ *and* its derivatives with respect to ξ are finite as $f\to 0$. $\delta\psi_1$ is determined by the detailed form of η.

We may now put (B9) and (B10) back into Eq. (B4) for δS, keeping only the terms which give finite contributions to δS in the strong-coupling limit:

$$\tfrac{1}{2}\int dx\,J\delta\sigma\approx\tfrac{1}{2}\int da\int d\xi\,4\lambda f^2k\,\mathrm{sech}^4[(2\lambda)^{1/2}f\,\xi]\,\rho_0(u^\alpha)=\tfrac{1}{2}\int da\,2Ck\rho_0(u^\alpha)\,,\tag{B11}$$

$$\begin{aligned}\tfrac{1}{2}\int da\,d\xi(\bar\eta\delta\Psi+\delta\bar\Psi\eta)&=\int da\,d\xi\,N^2[\cosh(2\lambda)^{1/2}f\,\xi]^{-2G/(2\lambda)^{1/2}}\xi\,\mathrm{Re}[\delta\bar\psi_1-G\rho_0\sigma(\xi)\bar\psi_0](-ih^{\alpha\beta}\gamma_\alpha\partial_\beta+\tfrac{1}{2}h_{\alpha\beta}h^{\alpha\beta})\psi_0\\ &=\int da\,d\xi\,N^2[\cosh(2\lambda)^{1/2}f\,\xi]^{-2G/(2\lambda)^{1/2}}\xi\,\mathrm{Im}[\delta\bar\psi_1-G\rho_0\sigma(\xi)\bar\psi_0](h^{\alpha\beta}\gamma_\alpha\partial_\beta)\psi_0\,.\end{aligned}\tag{B12}$$

Now, if $F(\xi)$ is any function with finite derivatives in ξ, we have

$$\int d\xi\, N^2[\cosh(2\lambda)^{1/2} f\,\xi]^{-2G/(2\lambda)^{1/2}}\,\xi F(\xi) \sim F(\xi)/\sqrt{\lambda}\, f\,.$$

Thus, the $\delta\bar{\psi}_1$ term in (B12) does not give a finite contribution to δS. Further, we have

$$\begin{aligned}\int d\xi\, N^2[\cosh(2\lambda)^{1/2} f\,\xi]^{-2G/(2\lambda)^{1/2}}\,\xi G\sigma F(\xi) &= \int d\xi\,\xi F(\xi)\left(-\frac{1}{2}\,\frac{\partial}{\partial\xi}\right)\{N^2[\cosh(2\lambda)^{1/2} f\,\xi]^{-2G/(2\lambda)^{1/2}}\}\\ &\approx \int d\xi\,\tfrac{1}{2}F(0)N^2[\cosh(2\lambda)^{1/2} f\,\xi]^{-2G/(2\lambda)^{1/2}} = \tfrac{1}{2}F(0)\,.\end{aligned} \tag{B13}$$

Combining (B12) and (B13), we have

$$\tfrac{1}{2}\int dx\,(\bar{\eta}\delta\Psi + \delta\bar{\Psi}\eta) \approx \int da[-\tfrac{1}{2}\rho_0(u^\alpha)h_{\alpha\beta}\,\mathrm{Im}\bar{\psi}_0\gamma^\alpha\partial^\beta\psi_0]\,.$$

So the finite contribution to δS is

$$\begin{aligned}\delta S &= \tfrac{1}{2}\int da\,\rho_0(u^\alpha)h_{\alpha\beta}(Cg^{\alpha\beta} - \mathrm{Im}\bar{\psi}\gamma^\alpha\partial^\beta\psi)\\ &= \int da\,\rho_0(u^\alpha)h_{\alpha\beta}T^{\alpha\beta}\\ &= 0\,.\end{aligned} \tag{B14}$$

Thus, corrections lead to no shift which remains finite in the strong-coupling limit. Q.E.D.

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[2]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[3]Y. Nambu, in Lectures at the Copenhagen Summer Symposium, 1970 (unpublished).

[4]See, for example, the discussion in R. F. Dashen, B. Hasslacher, and Andre Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974).

[5]A similar statement could not be made in the quantum theory. It is possible that surface fluctuations always involve local radii of curvature smaller than any finite D. Also, the classical picture of bubble scattering by fusion and fission involves zero radii of curvature.

[6]In three spacetime dimensions, the bubble becomes a closed string. Our action is similar to one of those discussed by L. N. Chang and F. Mansouri, Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, 1974 (unpublished).

[7]In fact, the full time-dependent bubble theory can be exactly solved classically and then quantized. R. Giles and S.-H. H. Tye, following paper, Phys. Rev. D 13, 1690 (1976).

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[11]See also the discussion in R. J. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).