Two-dimensional Yang-Mills theory: A model of quark confinement*

Curtis G. Callan, Jr., Nigel Coote, and David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 8 September 1975)

We analyze the structure of two-dimensional Yang-Mills theory as a model of quark confinement. 't Hooft's solution, in the large-N limit, is extended to investigate the consistency and the properties of the model. We construct the hadronic color singlet bound-state scattering amplitudes. We show that they are unitary, that colored states cannot be produced, and that all long-range interactions are absent. Current amplitudes are constructed, and we show that the theory is asymptotically free and the quark mass sets the scale of mass corrections. The properties of bound states of heavy quarks are discussed, and a dynamical basis for the Okubo-Zweig-Iizuka rule is suggested. We show how confinement can occur with an infrared prescription that leads to finite-mass quarks which decouple from physical states and discuss the dependence of gauge-variant amplitudes on the cutoff procedure. Higher-order effects in $1/N$ are shown not tc change the qualitative features of the model.

I. INTRODUCTION

Surely the central problem of particle physics is why quarks are not observed as physical states. One of the standard answers is that the quarks are "confined." The most popular quark model assumes that the quarks possess a hidden SU(3) symmetry, color,[1] and physical states consist solely of color singlets.[2] One of the ideas put forward to explain confinement arises in the context of asymptotically free gauge theories of the strong interactions.[3,4] These theories appear to be unique in their ability to explain why hadrons appear to consist of quarks at short distances. At the same time they contain a mechanism for the dynamical confinement of quarks. It has been suggested[3,5] that the severe infrared behavior of these gauge theories at large distances might provide the strong forces necessary for confinement. This mechanism has been called "infrared slavery."

Confirmation of this hypothesis is an extremely difficult problem for a realistic four-dimensional gauge theory. A simpler approach would be to find a model field theory, simple enough to solve, but sufficiently nontrivial to test whether confinement is a viable concept. Two-dimensional Yang-Mills theory appears to be ideal in this respect. The theory is certainly asymptotically free, since it is superrenormalizable. Furthermore, it is manifestly "infrared enslaving," even in perturbation theory, because the Coulomb potential in two dimensions increases linearly for large spatial separations. This is to be contrasted with four-dimensional gauge theories, where the conjectured[3] strong forces at large distances must arise from nonperturbative renormalization effects. If there is any hope for infrared slavery as a confinement mechanism it must be present in two dimensions. On the other hand, two-dimensional Yang-Mills theory is highly nontrivial, in contrast with two-dimensional quantum electrodynamics, which indeed confines quarks[6] but does not provide a model of hadrons.

't Hooft has proposed an expansion of SU(N) gauge theories in powers of $1/N$,[7] which is powerful enough that one can explicitly solve two-dimensional Yang-Mills theories to leading order in $1/N$.[8] In this theory 't Hooft has demonstrated that the quarks are effectively removed from the physical spectrum, whereas there exist an infinite number of quark-antiquark (color singlet) bound states with finite masses.

In this paper we shall expand in some detail on 't Hooft's solution of two-dimensional Yang-Mills theory. Our aim is twofold. First we wish to check the consistency of the model. Does it satisfy the physical requirements of a sensible theory: unitarity, analyticity, current conservation, etc.? Do the quarks, which have been removed from the physical spectrum, reveal themselves in the short-distance behavior of the theory? Is the confinement found by 't Hooft independent of how one introduces the infrared cutoff? Are the qualitative features of the model preserved in higher orders in the $1/N$ expansion?

Our answer to all of these questions is yes. This then lends credence to the conjecture that infrared slavery can produce a consistent theory in which there exist bound states of constituent quarks and gluons which themselves cannot be produced as physical states. In addition the model can serve as a laboratory to test various ideas about realistic confining theories. Of course in two space-time dimensions there are many things one cannot discuss (Regge behavior, large-angle scattering, etc.). However, one can discuss such questions as the nature of hadronic scattering amplitudes, the approach to scaling at short distances, the properties of bound states of heavy quarks ("charmonium"),[9] the meaning and validity of the Okubo-Zweig-Iizuka (OZI) rule[10] and the mechanism for dynamical chiral-symmetry breaking.

This paper is organized in the following way. In Sec. II we review 't Hooft's solution and extend it to construct the full quark-antiquark scattering amplitude. In addition we summarize the features of the model and the results of our investigation. Explicit details are provided in the following sections.

In Sec. III we discuss the multi-bound-state scattering amplitudes, prove unitarity and the absence of long-range forces between hadronic states. Section IV is directed to a discussion of the properties of vector current amplitudes, and the short-distance behavior of the theory. In Sec. V we discuss scalar and pseudoscalar densities. In Sec. VI we discuss some of the insights into the theory obtained by treating the model with another, "regular," cutoff procedure. In Sec. VII we investigate heavy quark bound states and their decay. Section VIII consists of a discussion of higher-order corrections.

II. THE STRUCTURE OF TWO-DIMENSIONAL YANG-MILLS THEORY

A. The model

The model we shall consider consists of M quarks interacting via an SU(N) color gauge group. The Lagrangian for this two-dimensional theory is

$$\mathcal{L}=\tfrac{1}{4}G_{\mu\nu i}{}^{j}G^{\mu\nu}{}_{j}{}^{i}+\bar{q}^{ai}(i\gamma^{\mu}D_{\mu}-m_{a})q_{i}^{a}\ , \tag{1}$$

where

$$G_{\mu\nu i}{}^{j}=\partial_{\mu}A_{i}{}^{j}{}_{\nu}-\partial_{\nu}A_{i}{}^{j}{}_{\mu}+g[A_{\mu},A_{\nu}]_{i}{}^{j}\ , \tag{2a}$$

$$D_{\mu}q_{i}^{a}=\partial_{\mu}q_{i}^{a}+g\tilde{A}_{i}{}^{j}{}_{\mu}q_{j}^{a}\ , \tag{2b}$$

$$\tilde{A}_{i}{}^{j}{}_{\mu}(x)=A_{i}{}^{j}{}_{\mu}(x)-\frac{1}{N}\delta_{i}{}^{j}A_{k\mu}^{k}(x)=-\tilde{A}_{j\mu}^{*i}(x)\ , \tag{2c}$$

$$i,j=1,2,\ldots,N,\quad a=1,2,\ldots,M\ . \tag{2d}$$

We differ slightly from Ref. 8 by taking the gauge group to be SU(N) instead of U(N). Thus in the above Lagrangian the U(N) singlet, A_{k}^{k} decouples and describes a free field. To leading order in $1/N$ this distinction is immaterial, but in higher orders it is important. The Abelian charge of a U(N) gauge model cannot, of course, be confined. In the real world one would set $N=3$.

Two-dimensional gauge theories are particularly simple in gauges where some component of $A_{i}{}^{j}{}_{\mu}$ is set equal to zero. The advantages of such a gauge are as follows: (1) There are no ghosts in such a linear gauge. (2) There are no nonlinear interactions between the gauge mesons. The simplest of such gauges is the "light-cone gauge," where, following Ref. 8, we set

$$A_{-}=\frac{1}{\sqrt{2}}(A_{0}-A_{1})=A^{+}=0\ . \tag{3}$$

We use the standard light-cone coordinates:

$$x^{\pm}=x_{\mp}=\frac{1}{\sqrt{2}}(x^{0}\pm x^{1}),\ a\cdot b=a_{+}b^{+}+a_{-}b^{-}=a_{+}b_{-}+a_{-}b_{+}\ ,$$
$$\gamma^{\pm}=\frac{1}{\sqrt{2}}(\gamma^{0}\pm\gamma^{1}),\quad (\gamma^{+})^{2}=(\gamma^{-})^{2}=0,\quad \{\gamma^{+},\gamma^{-}\}=2\ . \tag{4}$$

The Feynman rules are represented in Fig. 1.

Two-dimensional gauge theories are extremely infrared singular. This is compounded in the light-cone gauge where the A^{i}_{-j} propagator [suppressing SU(N) indices] is $-i/q_{-}^{2}$. One therefore must introduce an infrared cutoff. The nature of the infrared cutoff, as well as the choice of gauge, should be irrelevant in the evaluation of matrix elements of gauge-invariant operators, since these are free of infrared singularities. However, this must be checked explicitly. We shall, following 't Hooft, remove the infrared singularities by drilling a hole in q space, about $q_{-}=0$, restricting $|q_{-}|\geq\lambda$. One then must check that Green's functions of gauge-invariant operators are independent of λ as $\lambda\to 0$. We also note that both the light-cone gauge and the cutoff procedure are manifestly Lorentz-invariant, since under a boost q_{-} simply scales. On the other hand, parity invariance is not a property of this gauge and cutoff procedure. All gauge-invariant quantities, including the S matrix, should nonetheless manifest

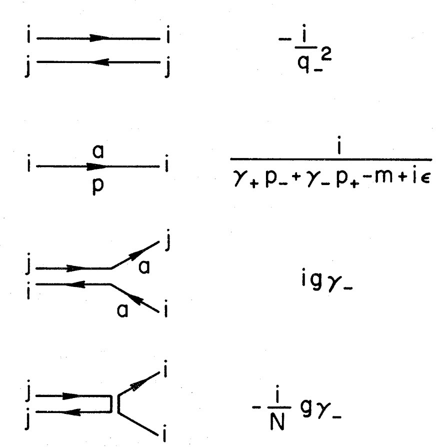

FIG. 1. Feynman rules.

parity invariance. To show this explicitly is one of the more challenging problems of this model.

The confinement mechanism is particularly transparent with the above infrared-cutoff procedure. The poles in the quark propagator are pushed to infinity as $\lambda\to 0$, thus removing colored states from the physical spectrum, whereas color singlet bound-state masses are independent of λ. On the other hand, one can define the gluon propagator with a principal-value prescription and avoid infrared infinities altogether.[11] With this procedure confinement is much less transparent. The properties of the gauge-variant sector are quite different. For example, the quark propagator has a pole at a *finite mass*. We have investigated this cutoff procedure and will show how the properties of the gauge-invariant sector remain unchanged in Sec. VI.

It is unlikely that two-dimensional Yang-Mills theories can be solved without recourse to perturbation theory. However, ordinary perturbation theory is of little value in discussing the spectrum and infrared properties of a non-Abelian gauge theory. A more useful expansion is the "large-N expansion" which will be employed in this paper. One expands Green's functions in powers of $1/N$, summing to all order in g^2N. This type of expansion, which has proved extremely useful in other field-theoretic models, has none of the obvious limitations of ordinary perturbation theory. 't Hooft has shown[7] that the Feynman diagrams of a non-Abelian gauge theory in the large-N limit exhibit a striking topological character. The dominant diagrams in this limit consist solely of planar diagrams with quarks at the edges. No fermion loops can occur at lowest order. Noting the analogy with the topological structure of the dual resonance string model he has suggested that the large-N limit of a four-dimensional gauge theory might provide a dynamical basis for the string model. The two-dimensional Yang-Mills theory, in the light-cone gauge, is especially simple, since the role of the gluons is merely to provide an instantaneous Coulomb force between the quarks and since one can, in the large-N limit, neglect pair creation. Most important, there is no vertex correction or correction to the gluon propagator in leading order. Examples of Feynman diagrams contributing to the quark propagator and the quark-antiquark scattering amplitude are illustrated in Fig. 2.

It is an important question whether nonleading orders in $1/N$ can alter the qualitative features of the model. We have investigated the next-to-leading order in $1/N$. We find enormous simplification due to the fact that the lowest-order theory already confines quarks. For example, we find

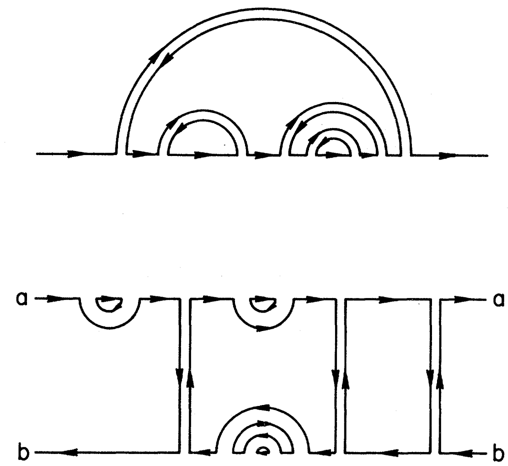

FIG. 2. Examples of leading-order graphs in the large-N limit.

that the gluon propagator is unchanged to order $1/N$. Our conclusion is that there are *no* qualitative changes due to higher-order corrections, and the $1/N$ expansion is indeed trustworthy. These corrections will be discussed in greater depth in Sec. VIII.

The quark propagator, $S(p)$, satisfies the following integral equation in the large-N limit:

$$S^{-1}(p)+i(\not{p}-m_a)=ig^2N\int\frac{d^2K}{(2\pi)^2}\,\theta(|K_-|-\lambda)\,\gamma_-\times S(p+K)\,\gamma_-(K_-)^{-2}\,. \qquad (5)$$

This equation was solved by 't Hooft,

$$S(p)=\frac{i\{p_-\gamma_++\gamma_-[p_+-(g^2N/2\pi)\,(\mathrm{sgn}p_-/\lambda-1/p_-)]+m_a\}}{2p_+p_--(g^2N/\pi)\,(|p_-|/\lambda-1)-m_a{}^2+i\epsilon}\,, \qquad (6)$$

and exhibits the infinite self-energy of the quarks (as $\lambda\to 0$) which eliminates them from the physical spectrum.

't Hooft[8] has solved the homogeneous Bethe-Salpeter equation for quark-antiquark scattering, and showed that the spectrum is discrete. We shall require the full quark-antiquark scattering amplitude, $T_{\alpha\beta,\gamma\delta}(p,p';r)$, which satisfies the following equation in the large N-limit (see Fig. 3) [SU(N) and SU(M) indices have been suppressed]:

$$T_{\alpha\beta,\gamma\delta}(p,p';r)=\frac{ig^2}{(p_--p'_-)^2}\,(\gamma_-)_{\alpha\gamma}(\gamma_-)_{\beta\delta}+ig^2N\int\frac{d^2K}{(2\pi)^2}\,\frac{(\gamma_-)_{\alpha\epsilon}(\gamma_-)_{\beta\lambda}}{(K_--p_-)^2}\,S(K)_{\epsilon\mu}\times S(K-r)_{\lambda\nu}\,T_{\mu\nu,\gamma\delta}(K,p';r)\,. \qquad (7)$$

Since $\gamma_-^2=0$, it follows that $T_{\alpha\beta,\gamma\delta}=(\gamma_-)_{\alpha\gamma}(\gamma_-)_{\beta\delta}T(p,p';r)$, and $S(K)$ can be replaced in Eq. (7) by

$$2\gamma_- S_E(p)=\gamma_- S(p)\gamma_-=2\gamma_- \frac{i}{2p_+-(g^2N/\pi)[1/p_- - \mathrm{sgn}(p_-)/\lambda]-(m^2-i\epsilon)/p_-}. \tag{8}$$

Since the interaction is instantaneous this equation can be solved by introducing

$$\phi(p_-,p'_-;r)=\int dp_+ S_E(p)\,S_E(p-r)\,T(p,p';r),$$

from which we can construct $T(p,p';r)$:

$$T(p,p';r)=\frac{ig^2}{(p_- - p'_-)^2}+\frac{ig^2N}{\pi^2}\int\frac{dk_-\phi(k_-,p';r)}{(k_- - p_-)^2}. \tag{9}$$

It then follows that $\phi(p_-,p';r)$ satisfies the equation [m_a (m_b) is the mass of the quark (antiquark)]

$$\left(\frac{m_a^2-g^2N/\pi}{p_-}+\frac{m_b^2-g^2N/\pi}{r_- - p_-}+\frac{2g^2N}{\pi\lambda}-2r_+\right)\phi(p',p_-;r)$$
$$=\theta(p_-)\,\theta(r_- - p_-)\left[\frac{\pi g^2}{(p_- - p'_-)^2}+\frac{g^2N}{\pi}\int dk_-\,\frac{\phi(k_- + p_-,p';r)}{k_-^2}\right]. \tag{10}$$

Owing to our infrared-cutoff procedure the integral in Eq. (10) is to be regarded as

$$\int dk_-\,\frac{\phi(k_- + p_-,p';r)}{k_-^2}=\frac{2}{\lambda}\,\phi(p_-,p';r)+\mathrm{P}\int dk_-\,\frac{\phi(k_- + p_-,p';r)}{k_-^2}. \tag{11}$$

Therefore, it is evident that the infrared ($1/\lambda$) singularities in Eq. (10) cancel. Furthermore, ϕ depends only on $x=p_-/r_-$, $x'=p'_-/r_-$, and r, and vanishes for x and x' outside the range 0 to 1. We can therefore rewrite (10) in the form

$$\mu^2\phi(x,x';r)=\frac{\pi^2}{Nr_-(x-x')^2}+\left(\frac{\gamma_a}{x}+\frac{\gamma_b}{1-x}\right)\phi(x,x';r)+\int_0^1\frac{[\phi(x,x';r)-\phi(y,x';r)]\,dy}{(x-y)^2}, \tag{12}$$

where

$$r^2=2r_+r_-=\left(\frac{g^2N}{\pi}\right)\mu^2,\quad m_a^2=\gamma_a\frac{g^2N}{\pi}. \tag{13}$$

't Hooft[8] has discussed the solutions of the homogeneous equation

$$H\phi_k(x)=\mu_k^2\phi_k(x)=\left(\frac{\gamma_a}{x}+\frac{\gamma_b}{1-x}\right)\phi_k(x)+\int_0^1\frac{[\phi_k(x)-\phi_k(y)]\,dy}{(x-y)^2}, \tag{14}$$

which have the following properties:

1. H is positive-definite and self-adjoint on the space of functions which vanish at $x=0$ [$x=1$] like x^{β_a} [$(1-x)^{\beta_b}$], where $\pi\beta_a\cot\beta_a\pi=1-\gamma_a$.

2. H has only a discrete spectrum. The eigenfunctions ϕ_k are complete and orthogonal:

$$\sum_k\phi_k(x)\,\phi_k(x')=\delta(x-x'),$$

$$\int_0^1\phi_n^*\phi_m\,dx=\delta_{nm}.$$

3. When $m_a=m_b=0$ the ground state has zero energy. The corresponding eigenfunction is $\phi_0(x)=1$ [$H\phi_0(x)=0$].

4. The following identity, which will prove useful later, is easily proved:

$$\mu_k^2\int_0^1 dx\,\phi_k(x)=\int_0^1 dx\,\phi_k(x)\left(\frac{\gamma_a}{x}+\frac{\gamma_b}{1-x}\right). \tag{15}$$

5. For large k (large energy) the eigenfunctions can be approximated by

$$\phi_k(x)\simeq\sqrt{2}\,\sin\pi kx,\qquad k\gg1 \tag{16}$$
$$\mu_k^2\simeq\pi^2k.$$

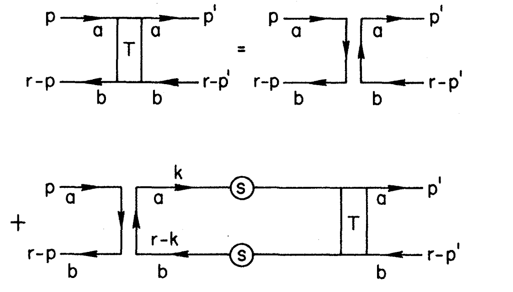

FIG. 3. Bethe-Salpeter equation for quark-quark scattering amplitude.

It is now straightforward to construct $\phi(x, x'; r)$,

$$\phi(x, x'; r) = -\sum_k \frac{\pi g^2}{(r^2 - r_k^2)} \frac{1}{r_-} \int_0^1 dy \, \frac{\phi_k(x)\,\phi_k^*(y)}{(y - x')^2} , \tag{17}$$

and the quark-antiquark scattering amplitude

$$\begin{aligned} T(x', x; r) &= \frac{ig^2}{r_-^2(x'-x)^2} - \frac{ig^2(g^2N)}{\pi r_-^2} \sum_k \frac{1}{(r^2 - r_k^2)} \int_0^1 dy \int_0^1 dy' \, \frac{\phi_k^*(y')\,\phi_k(y)}{(y'-x')^2 (y-x)^2} \\ &= \frac{ig^2}{r_-^2(x'-x)^2} - \sum_k \frac{i}{(r^2 - r_k^2)} \left\{ \phi_k^*(x') \frac{2g}{\lambda} \left(\frac{g^2 N}{\pi}\right)^{1/2} \left[\theta(x'(1-x')) + \frac{\lambda}{2|r_-|} \left(\frac{\gamma_a - 1}{x'} + \frac{\gamma_b - 1}{1-x'} - \mu_k^2 \right) \right] \right\} \\ &\quad \times \left\{ \phi_k(x) \frac{2g}{\lambda} \left(\frac{g^2 N}{\pi}\right)^{1/2} \left[\theta(x(1-x)) + \frac{\lambda}{2|r_-|} \left(\frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} - \mu_k^2 \right) \right] \right\} , \end{aligned} \tag{18}$$

where we have used the homogeneous equation (14) to define $\phi_k(x)$ for $x \geq 1$ or $x \leq 0$.

The dynamics of confinement is now clear. The infinite self-mass of the quark is canceled by the quark-antiquark interaction—producing an infinite number of color singlet bound states, whose mass squared increases linearly for large mass. There are no continuum states in the quark-antiquark amplitude—only bound states at $r^2 = r_k^2$, whose residue yields the normalized bound-state wave function

$$\Phi_K^{a,b}(x) = \frac{2g}{\lambda} \left(\frac{g^2 N}{\pi}\right)^{1/2} \phi_K(x) \left[\theta(x(1-x)) + \frac{\lambda}{2|r_-|} \left(\frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x} - \mu_K^2 \right) \right], \tag{19}$$

where x is the fraction of the total momentum (r_-) of the bound state carried by the quark.

We note that the bound-state wave function is of order $1/\lambda$ as $\lambda \to 0$. The fact that the amplitude for a bound state to decay into quarks is infinite as $\lambda \to 0$ compenstates for the vanishing quark propagator in this limit to produce *finite* bound-state amplitudes, which contain no multiquark discontinuities. However, the finite pieces of the wave function cannot be neglected, for as we shall see below they can sometimes yield the leading contribution to various scattering amplitudes.

Given this scattering amplitude one can now answer all physically interesting questions. In the rest of this section we shall pose these questions and describe the answers that we have found.

B. Hadronic scattering amplitudes

From the above results 't Hooft's conclusion that the only finite-energy states are color singlet bound states of confined quarks (which we call hadrons) is eminently reasonable. To test whether the resulting theory is physically sensible one must examine the hadronic scattering amplitudes. These of course must be finite (as $\lambda \to 0$), Lorentz-invariant, and (presumably) parity-invariant. Furthermore, they must be unitary in the subspace of physical hadronic states. A consequence of unitarity is the absence of long-range, Van der Waals type, forces between color singlets which would correspond to the exchange of gluons between color singlet states. The verifications of these properties is in principle straightforward, since with the aid of the quark propagator, Eq. (6), and the bound-state wave function, Eq. (19), all hadronic scattering amplitudes can be explicitly constructed. In practice many delicacies arise.

Consider the 3-particle vertex function, which is of order $g \sim 1/\sqrt{N}$, and arises from the diagram in Fig. 4(a). Now if we examine the bound-state wave function $\Phi_K(x)$ we see that the bound state can decay to a quark and an antiquark moving in the same direction with amplitude $\sim 1/\lambda$, whereas if either the quark or the antiquark move in the opposite direction to the bound state, the amplitude is of order λ^0. Keeping only the leading term, one would find that the 3-point vertex vanishes identically. This illustrates why one *must not* take the $\lambda \to 0$ limit until the end of a calculation. To see if the 3-point vertex has a finite limit as $\lambda \to 0$, one can simply count powers of λ. Since the quark propagator, Eq. (6), is sandwiched between γ_- matrices, it can be replaced by S_E, Eq. (7), and is of order λ. The p_+ loop momentum in Fig. 4(a) is of order $1/\lambda$, since it is dominated by poles at $p_+ \approx 1/\lambda$. Finally from the three bound-state wave functions we get a factor of $(1/\lambda)^2$ (since at least one wave function must be of order unity to conserve momentum). All in all, the factors of λ cancel producing a finite vertex function as $\lambda \to 0$, which will be real and Lorentz-invariant. This vertex will be discussed in greater detail in Sec. III.

The 2-particle scattering amplitude, which is of order $1/N$, will certainly receive a contribution

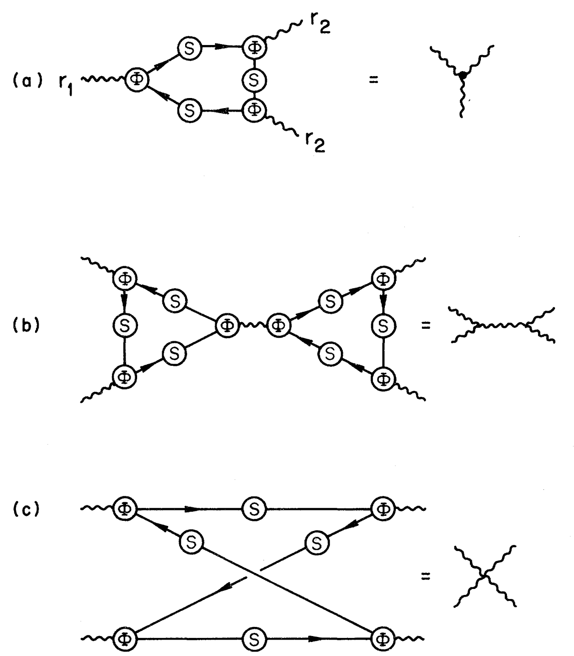

FIG. 4. (a) 3-particle vertex function. (b) Hadronic exchange contribution to 2-particle scattering amplitude. (c) Quark exchange contribution to 2-particle scattering amplitude.

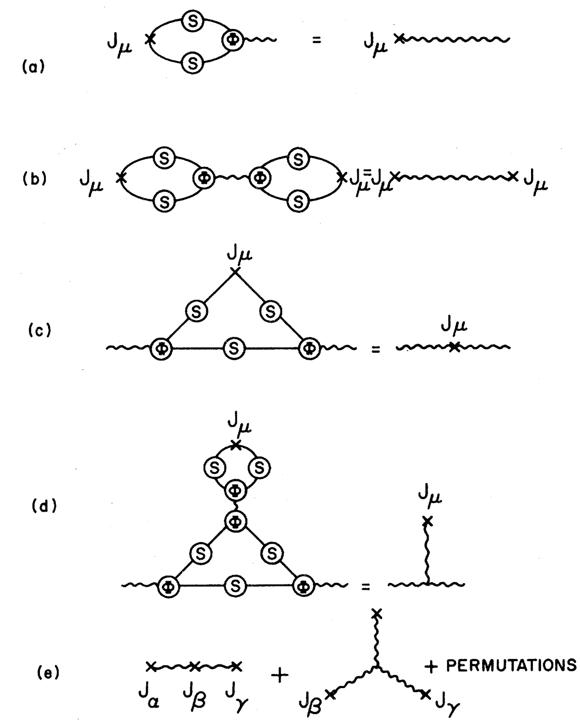

FIG. 5. Current-hadron coupling. (b) 2-point function for currents. (c), (d) Contribution to the current form factor. (e) 3-point function for currents.

from hadronic exchange, as in Fig. 4(b). However, in addition, there may be a contribution from quark exchange diagrams, such as Fig. 4(c). At first sight the latter are infinite as $\lambda \to 0$, since now all quarks and antiquarks can be moving in the same direction. Thus the powers of λ that appear in Fig. 4(c) are λ^4 from the quark propagators, $(1/\lambda)^4$ from the wave functions, and $1/\lambda$ from the loop momentum integration. This problem is discussed in Sec. III, where we show that when one adds all diagrams that contribute to this order in $1/N$, the terms of order $1/\lambda$ cancel, leaving a finite remainder, which is a real contribution to the scattering amplitude, corresponding to a 4-point interaction. We have thus verified the unitarity of the theory to first nontrivial order. One could continue in this fashion and construct an effective theory of hadrons (to lowest order in $1/N$) which would involve 3- and 4-point couplings, and in addition n-point vertices which would arise when one evaluates the $\lambda \to 0$ limit of n-point hadronic amplitudes.

While unitarity and Lorentz invariance can thus be established, the question of parity invariance, crossing, and analyticity of the resulting scattering amplitudes is much trickier. The problems involved in verifying these properties are discussed in Sec. III.

C. Current Green's functions

The Green's functions of local operators constructed out of products of quark and gluon fields can easily be constructed. Of particular importance are gauge-invariant operators which should have finite Green's functions as $\lambda \to 0$. We have examined the matrix elements of the vector currents $J_\mu^a = \bar{\psi}\gamma_\mu \lambda^a \psi$, where λ^a labels a matrix of the fundamental representation of SU(M). Our aim was first to verify that these have finite matrix elements, and that the currents which are formally conserved by virtue of the equations of motion are indeed conserved. Consider for example the matrix element of J_μ^a between the vacuum and a hadronic state, given to lowest order in $1/N$ by the diagram in Fig. 5(a). Counting powers of λ, it is easily seen that this is finite, and is equivalent to a direct coupling of the current to the hadron. It is less trivial to check the conservation of J_μ. We have verified that both the direct current-hadron coupling, as well as the hadronic form factors, given by Figs. 5(c) and 5(d), are conserved. The conservation properties of J_μ are not manifest. To establish them it proved necessary to use quite detailed properties of the bound-state wave functions. This is discussed in Sec. IV.

Since the theory is asymptotically free (superrenormalizable), the short-distance behavior of products of currents should be that of the free-quark model. On the other hand, the quarks have infinite energy and have disappeared from the space of physical states. Current amplitudes can be constructed out of effective current-hadron couplings. This theory should provide an explicit example of how a confining theory can "remember" at short-distances that hadrons are constructed out of quarks. We have examined the 2- and 3-point Green's functions of vector currents, given by Fig. 5(b) and Fig. 5(e), and verified that their high-energy behavior is indeed the same as in the free-quark model.

The 2-point function provides us with a model of e^+e^- annihilation, which to lowest order in $1/N$ is given solely by the "vector dominance" contributions of Fig. 5(b). This process illustrates the constraints on current-hadron couplings that reproduce free-quark model asymptotic behavior as well as the rate of approach to the asymptotic limit in the timelike region and the effects of heavy quark thresholds in the total annihilation cross section. The evaluation of the high-energy behavior of the current 3-point function provides a test of crossing symmetry and analyticity. The individual contributions to the 3-current vertex, illustrated in Fig. 5(e), contain various nonanalytic and non-crossing-symmetric pieces. The verification that the complete vertex reduces to the free-quark model vertex at high energies provides an indication that these nonanalytic pieces combine to yield, as expected, an analytic and crossing-symmetric result.

D. Heavy quarks and charmonium

With the recent discovery of the ψ resonance there has been much speculation that this is a bound state of a charmed quark pair.[9] In an asymptotically free gauge theory it has been argued that one can treat this particle as a nonrelativistic Coulomb-type bound state—charmonium—if the effective coupling is sufficiently small at energies of the order of the ψ mass. Furthermore, it has been argued that one can estimate the hadronic width of the ψ by analogy with the decay of positronium. Assuming that the decay proceeds via a 3-gluon state (the minimum number of gluons that can be produced when the charmed and anticharmed quarks in ψ annihilate) one can estimate the hadronic-to-leptonic branching ratio.

Two-dimensional Yang-Mills theory provides us with an ideal model to test these ideas. To this end we have examined the nature of resonances formed from heavy quarks, the effect on e^+e^- annihilation of the opening of heavy quark thresholds, and analyzed the decay modes of these resonances.

We find that the bound states of quarks whose masses (m_H) are large compared to the natural dimensional scale parameter of the theory $m_H^2 \gg g^2N/\pi$ can be well described by a nonrelativistic Schrödinger equation, with a (two-dimensional) Coulomb potential. We construct explicitly the wave functions of such bound states in this large-m_H limit. In a similar fashion we can determine the masses and wave functions of bound states of one heavy and one light quark ("charmed mesons"). It is interesting to note that the mass scale of the hadrons constructed from light quarks, i.e., the inverse slope of the linear trajectories of the bound states, is $(g^2N)\pi = (\alpha')^{-1}$ [see Eq. (16)]. Thus the charmonium picture applies when $m_H^2 \gg (\alpha')^{-1}/\pi^2$. If such factors of π^2 were to occur in four dimensions one could argue that the charmed quark mass squared would only have to be large compared to 1 GeV/$\pi \approx 300$ MeV, in order to be able to use asymptotic freedom to discuss charmonium.

The suppression of the decay amplitude for charmonium is an example of the Okubo-Zweig-Iizuka rule.[10] Since this state lies below the threshold for the production of charmed particles, the charmed quarks must annihilate. Therefore, there is no contribution to this amplitude to leading order in $1/N$ (i.e., to order $g \sim 1/\sqrt{N}$). The first nonvanishing contribution arises in next order, and proceeds through the twisted duality diagram Fig. 6. There is clearly a suppression factor of $1/N$ in this decay amplitude relative to the decay amplitude of a resonance formed from light quarks. This is a general feature of gauge theories—all amplitudes which violate the OZI rule will be suppressed by at least $1/N^2$ in rate. However, there will be additional dynamical suppression factors, which can be much more important. These dynamical effects are also responsible for the distinction between various decay modes which have the same topological suppression factors, such as $\psi' \to \psi$ + hadrons compared to $\psi' \to$ hadrons. In our model these can be calculated since for large m_H the wave functions of heavy-heavy and light-heavy resonances are known. We estimate the mass dependence of the decay amplitude for heavy resonances and find them suppressed by a factor of $m_H^{-4(2/3)}$. However, it is clear from Fig. 6, as well as from our final result, that the 3-gluon intermediate state plays no special role. Indeed the amplitude for charmonium to decay into three gluons vanishes, as it should in a confining theory. One must sum, as in Fig. 6, the amplitudes for producing an arbitrary number of gluons nonpertur-

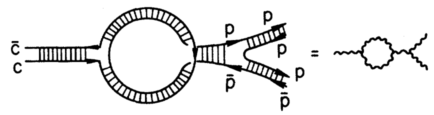

FIG. 6. Diagram for decay of charmonium.

batively in such a way as to produce physical, hadronic, intermediate states.

III. *S*-MATRIX ELEMENTS

As we noted earlier, to leading order in N^{-1} this theory yields a set of noninteracting bound states. In the next order these particles may interact with one another and we would now like to discuss the resulting scattering and decay amplitudes. The simplest amplitude we can discuss is the 3-point function responsible for the decay of a heavy meson into two lighter particles. To leading order in N^{-1}, the diagram we must compute is the simple duality diagram of Fig. 4(a). Radiative corrections of the same order in $1/N$ are redundant, since they are already contained in the bound-state wave functions. The quark propagators and the bound-state wave functions are singular as the infrared cutoff is removed, and our problem is to show that everything conspires to yield a finite S-matrix element. For the purposes of this discussion we shall take the quarks to fill the fundamental representation of the gauge group only, not worrying about a possible global symmetry.

The amplitude to be computed then is

$$A=N\int\frac{d^2l}{(2\pi)^2}\frac{\Phi_1(l,r_1-l)\Phi_2(l,r_2-l)\Phi_3(r_2,r_1-l)}{\left(l_+-\dfrac{m^2-i\epsilon}{l_-}-\dfrac{g^2N}{2\pi\lambda}\right)\left(l_+-r_{2+}-\dfrac{m^2-i\epsilon}{l_--r_{2-}}-\dfrac{g^2N}{2\pi\lambda}\right)\left(l_+-r_{1+}-\dfrac{m^2-i\epsilon}{l_--r_{1-}}-\dfrac{g^2N}{2\pi\lambda}\right)}\,.$$

Since the Φ do not depend on the + components of momentum, the l_+ integral can be evaluated immediately yielding

$$A=\frac{2iN}{(g^2N/\pi\lambda)^2}\left(\int_0^{r_{2-}}dl_--\int_{r_{2-}}^{r_{1-}}dl_-\right)\Phi_1(l_-,r_{1-}-l_-)\,\Phi_2(l_-,r_{2-}-l_-)\,\Phi_3(l_--r_{2-},r_{1-}-l_-)\,.$$

The denominators have been simplified by recognizing that $g^2/\pi\lambda$ is large compared with all external momentum components. We recall that $\Phi(r_-,s_-)$ are of order λ^{-1} so long as r_-,s_- are both positive and of order λ^0 otherwise. The kinematics is such that this condition is satisfied for only two wave functions at a time, which is just right to cancel the $O(\lambda^2)$ factor in A which came from the quark propagators. If we insert the explicit forms of Eq. (19) for the wave functions, we finally obtain

$$\begin{aligned}A&=\frac{4g^2\sqrt{N}}{\sqrt{\pi}}\left\{\int_0^{r_{2-}}dl_-\,\phi_1\!\left(\frac{l_-}{r_{1-}}\right)\phi_2\!\left(\frac{l_-}{r_{2-}}\right)\int_0^{r_{3-}}dp_-\,\frac{\phi_3(p_-/r_{3-})}{[p_--(l_--r_{2-})]^2}\right.\\&\qquad\left.-\int_{r_{2-}}^{r_{1-}}dl_-\,\phi_1\!\left(\frac{l_-}{r_{1-}}\right)\phi_3\!\left(\frac{l_--r_{2-}}{r_{3-}}\right)\int_0^{r_{2-}}dp_-\,\frac{\phi_2(p_-/r_{2-})}{(p_--l_-)^2}\right\}\\&=O\!\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

This has two important features. It is $O(1/\sqrt{N})$ and so is small in the large-N limit, and the limit $\lambda\to 0$ may be taken freely leaving a well-defined convolution of bound-state wave functions. It is remarkable that the physical amplitude, though finite, involves the "small" components of the bound-state wave functions which in turn are nonzero only when the infrared cutoff is nonzero. In other words, the walls of the spatial box giving the cutoff play a crucial role in the scattering process. It is encouraging that the delicate cancellation of divergence occurs as needed.

The next most complicated process to consider is the 4-point function, or meson-meson scattering amplitude. There are three basic duality diagrams for this process as shown in Figs. 7(a), 7(b), and 7(c). To obtain all graphs of the same order in N^{-1} it is necessary to dress the quark propagators and to exchange gluons in ladder fashion between nonadjacent quark lines (gluon exchange between adjacent quark lines is already contained in the bound-state wave functions). Thus we must make the replacement shown in Fig. 8, where the blob stands for the full quark-antiquark 4-point function. This 4-point function was computed earlier and we found it to have the structure shown graphically in Fig. 9, where the sum is taken over all the bound states of the theory. The resonance sum contribution evidently gives in its contribution to the 4-point function just tree graphs

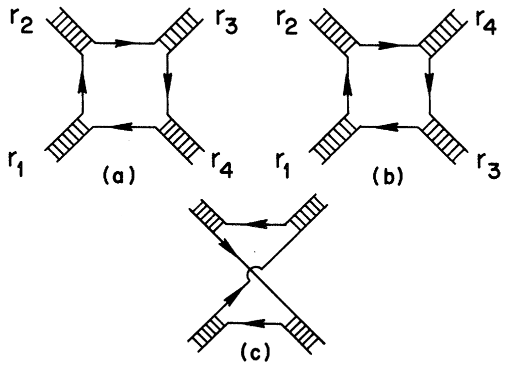

FIG. 7. Basic duality diagrams for 4-point function.

of the type shown in Fig. 4(b). The 3-point functions out of which these trees are constructed have already been shown to be infrared-finite and $O(1/\sqrt{N})$, so they lead to 4-point functions which are infrared-finite and $O(1/N)$. The disconnected piece plus Born piece of the quark 4-point function will lead to a nontrivial direct 4-meson coupling which we must discuss in detail.

Possible finite contributions are severely limited by the fact that the quark propagators which survive integration over the + component of loop momentum each contribute a factor λ^1, while bound-state wave functions contribute a factor λ^{-1} so long as the quarks carry a positive fraction of the momentum of the meson. The only possible survivors of the $\lambda\to 0$ limit are the "quark interchange" diagrams of Fig. 10. These graphs are potentially divergent since the kinematics permits all *four* bound-state wave functions to be simultaneously $O(\lambda^{-1})$ while, in effect, only *three* quark propagators survive loop integration. To avoid disaster a cancellation of divergences must take place.

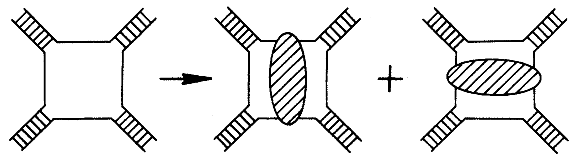

FIG. 8. Corrections to Fig. 7(a) due to latter exchange of gluons.

We will display the cancellation of divergences and not concern ourselves with the residual contact term, other than to remark that it is indeed finite. The diagram of Fig. 10 has as its most divergent contribution the integral

$$N\left(\frac{1}{\lambda\sqrt{N}}\right)^4\int\frac{d^2l}{(2\pi)^2}\,\frac{\phi_1(l,p_1-l)\,\phi_3(l,p_3-l)\,\phi_2(p_3-l,p_2-p_3+l)\,\phi_4(p_4-p_1+l,p_1-l)}{[l_+-m^2/l_-+(g^2N/\pi\lambda)\,\mathrm{sgn}(l_-)-i\epsilon]\cdots}\,,$$

where the bound-state wave functions each have implicit θ functions restricting their momentum arguments to be positive. Carrying out the l_+ integration and keeping only the leading term in λ, we have

$$N\left(\frac{2}{\lambda\sqrt{N}}\right)^4 2\left(\frac{\pi\lambda}{g^2N}\right)^3\frac{1}{2\pi}\int dl_-\,\phi_1(l_-,p_1-l_-)\cdots\phi_4(p_{4-}-p_{1-}-l_-,p_{1-}-l_-)\,.$$

Let us now look at Fig. 10(b). It is given, at least to leading order in $1/N$, by the expression

$$g^2N^2\left(\frac{2}{\lambda\sqrt{N}}\right)^4\int\frac{d^2k}{(2\pi)^2}\int\frac{d^2l}{(2\pi)^2}\,\frac{1}{k^2}\,\frac{\phi_1(l+k,p_1-l-k)\,\phi_3(l+k,p_3-l-k)\,\phi_2(p_3-l,p_2-p_3+l)\,\phi_4(p_4-p_1+l,p_-)}{D(l+k)\,D(p_1+l+k)}$$

where D represents the product of three quark propagators and where again the arguments of the ϕ_i must be positive. On doing the two + loop momentum integrations and passing to the limit of small λ we get

$$g^2N^2\left(\frac{2}{\lambda\sqrt{N}}\right)^4\left(\frac{\pi\lambda}{g^2N}\right)^4\int\frac{dk_-}{2\pi}\,\frac{1}{k_-^2}\int\frac{dl_-}{2\pi}\,\phi_1(l+k,p_1-l-k)\cdots\phi_4(p_4-p_1+l,p_1-l)\,.$$

This is $O(\lambda^{-1})$ by virtue of the singularity in the k integral. Extracting that part, we get

$$g^2N^2\left(\frac{2}{\lambda\sqrt{N}}\right)^4\left(\frac{\pi\lambda}{g^2N}\right)^4\left(\frac{1}{2\pi}\,\frac{2}{\lambda}\right)\int\frac{dl}{2\pi}\,\phi_1(l,p_1-l)\cdots\phi_4(p_4-p_1+l,p_1-l)\,.$$

This is precisely $-\frac{1}{2}$ times the divergent contribution of Fig. 10(a). The divergent part of Fig. 10(c) is the same and leads to exact cancellation of the $O(\lambda^{-1})$ terms. The $O(\lambda^0)$ remainder which we have not bothered to write out is finite and $O(1/N)$. It is, in effect, a fundamental 4-meson vertex which

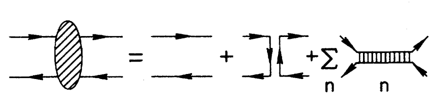

FIG. 9. Structure of blob in Fig. 8.

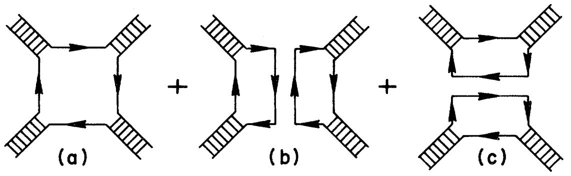

FIG. 10. Potentially divergent contributions to 4-point function.

is just as important as the "tree" graphs of Fig. 4(b) constructed out of the basic 3-point functions. The key fact is that although the final expression for the S matrix is complicated, it is free of infrared divergences due to a rather delicate cancellation. We expect this to be true for the N-point function.

We have thereby verified that the theory, to this order in $1/N$, is unitary and that no colored intermediate states appear. In particular color-singlet mesons *do not* have long-range, Van der Waals type, interactions. All interactions between hadrons are either contact interactions or are mediated by the exchange of hadrons themselves.

The formulas for S-matrix elements we have obtained, while perfectly explicit, have some disturbing features. Most important is the lack of explicit reflection invariance. This is most easily seen in the amplitude for $A \to B + C$ in the frame where A is at rest. The amplitude is a function the $-$ components of momentum of the particles. For fixed masses m_A, m_B, and m_C there are two solutions for these components, corresponding to B moving to the right or B moving to the left. By reflection invariance these two amplitudes should be equal, but in our formulas this equality is not explicit. Their equality, if true, is the consequence of a complicated convolution equality on the bound-state wave functions which we have not been able to prove. The fault lies in the use of the light-cone gauge which obviously violates explicit reflection invariance. We of course believe that if we had better control of the wave functions it would be possible to demonstrate the requisite cancellations since eccentric choices of gauge should not affect the S-matrix elements. The second trouble is that the analyticity properties one demands of satisfactory S-matrix elements are not manifest in our formulas either. Because amplitudes depend explicitly on momentum components rather than the usual invariants physically unreasonable singularities may appear. Again, we expect that the wave functions are clever enough to eliminate unphysical cuts, but we do not know enough to demonstrate this. The same problem arises for current amplitudes, but the situation there is simple enough that we can actually

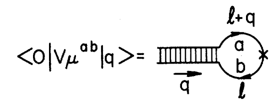

FIG. 11. Leading contribution to current-meson coupling.

analyze these questions and show that the difficulty is only apparent.

IV. CURRENT MATRIX ELEMENTS

Of the various densities one might consider, phenomenologically the most important is the current. Fortunately, it is also in this model the easiest to compute, although, as we shall see, certain important properties of current matrix elements, normally manifest, are here consequences of curious identities on the bound-state wave functions. To get an idea of what is involved we first consider the vacuum–single-meson current matrix element.

The leading contribution to this amplitude is given by the graph of Fig. 11, where, as usual, radiative corrections are already contained in the bound-state wave function. We are interested only in color-singlet densities since the physical states are themselves color singlets. The general color-singlet current is

$$V_\mu^{ab} = \sum_{i=1}^{N} \bar{\psi}_i^a \gamma_\mu \psi_i^b \;, \tag{20}$$

where $a, b = 1, \ldots, M$ index the independent SU(N) multiplets.

With our Feynman rules, the insertions of the + and $-$ components are quite different:

$$\begin{aligned} &V_-^{ab} \to N\,, \\ &V_+^{ab} \to \frac{N}{2}\,\frac{m_a m_b}{l_-(q+l)_-}\;. \end{aligned} \tag{21}$$

The + loop momentum integration involves only the two quark propagators and yields a result of order λ so that only the λ^{-1} piece of the bound-state wave function survives in the limit $\lambda \to 0$. Doing the loop integrations and removing the infrared cutoff, we find

$$\begin{aligned} \langle 0|\,V_-^{ab}\,|q\rangle &= q_-\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\phi^{ab}(x)\;, \\ \langle 0|\,V_+^{ab}\,|q\rangle &= -\frac{m_a m_b}{2q_-}\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\frac{\phi^{ab}(x)}{x(1-x)} \\ &= -q_+\frac{m_a m_b}{m^2}\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\frac{\phi^{ab}(x)}{x(1-x)}\;, \end{aligned} \tag{22}$$

where $m^2=2q_+q_-$ is the mass squared of the meson. The general form of the matrix element of V_μ is

$$\langle 0|V_\mu^{ab}|q\rangle = A\epsilon_{\mu\nu}q^\nu + Bq_\mu \ , \tag{23}$$

and the above results allow us to make the identification

$$A=\frac{1}{2}\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\phi^{ab}(x)\left[1+\frac{m_a m_b}{m^2}\,\frac{1}{x(1-x)}\right],$$

$$B=\frac{1}{2}\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\phi^{ab}\left[1-\frac{m_a m_b}{m^2}\,\frac{1}{x(1-x)}\right].$$

These expressions may be written, with the help of the identity displayed in Eq. (15), in an equivalent and more useful form as follows:

$$A=\frac{1}{2}\left(\frac{N}{\pi}\right)^{1/2}\frac{(m_a+m_b)}{m^2}\int_0^1 dx\,\phi^{ab}(x)\left(\frac{m_a}{x}+\frac{m_b}{1-x}\right),$$
$$B=\frac{1}{2}\left(\frac{N}{\pi}\right)^{1/2}\frac{(m_a-m_b)}{m^2}\int_0^1 dx\,\phi^{ab}(x)\left(\frac{m_a}{x}-\frac{m_b}{1-x}\right). \tag{24}$$

In the next section we shall identify these moments of ϕ^{ab} with the matrix elements of $\overline{\psi}\psi$ and $\overline{\psi}\gamma_5\psi$. For the moment, we simply observe that Eq. (23) makes current conservation manifest. The current V_μ^{ab} should be conserved if $m_a=m_b$. On the other hand,

$$\langle 0|\partial^\mu V_\mu^{ab}|q\rangle = iBq^2 \ ,$$

which is zero only if B vanishes. But B is explicitly proportional to $m_a=m_b$. When $m_a=m_b$ we may use the identity of Eq. (15) to cast the current matrix element in a particularly simple form:

$$\langle 0|V_\mu^{ab}|q\rangle=\left(\frac{N}{\pi}\right)^{1/2}\int_0^1 dx\,\phi^{ab}(x)\,\epsilon_{\gamma\nu}q^\nu \ . \tag{25}$$

It should be noted that current conservation is demonstrable only with the help of a nontrivial identity for the bound-state wave functions. This is a general feature of the theory: To demonstrate properties which are manifest in simple theories, we must make use of nontrivial identities involving the bound-state wave functions.

In this model the axial-vector current is just the dual of the vector current and, when $m_a=m_b$, we have as an immediate consequence of Eq. (22) that

$$\langle 0|A_\mu^{ab}|q\rangle=\left(\frac{N}{\pi}\right)^{1/2}q_\mu\int_0^1 dx\,\phi^{ab}(x)\ ,$$
$$\langle 0|\partial^\mu A_\mu^{ab}|q\rangle=\left(\frac{N}{\pi}\right)^{1/2}q^2\int_0^1 dx\,\phi^{ab}(x)\ . \tag{26}$$

One expects formally that this axial-vector current is conserved if $m_a=m_b=0$. But when $m_a=m_b=0$, the identity, Eq. (15), which we have already made use of, says that

$$q^2\int_0^1 dx\,\phi^{ab}(x)=0\ , \tag{27}$$

causing the matrix element of $\partial^\mu A_\mu^{ab}$ to vanish as expected. On the other hand, Eq. (27) means that $\int_0^1 dx\,\phi^{ab}(x)$ vanishes for all bound states of nonzero mass. As mentioned in Sec. II, there is, in the zero-quark-mass theory, a zero-mass bound state whoses wave function is $\phi_0^{ab}(x)=1$. The axial-vector current has a nonzero coupling *only* to this zero-mass state. There are indications that the zero-mass state decouples from the other bound states, thereby evading a Goldstone boson interpretation. This and other questions related to the possibility of dynamical symmetry breaking have not yet been fully explored.

One may discuss more complicated objects such as the meson form factors of the current V_μ^{ab}. Demonstrating that the current matrix element is conserved is not easy since it calls for an identity on sums of bound-state wave functions. We were able to derive the required identity but refrain from discussing the problem here in order to save space.

Much more interesting and transparent is the current 2-point function

$$M_{\mu\nu}=\int d^2x\,e^{iqx}\langle|T(V_\mu(x)\,V_\nu(0))|0\rangle\ .$$

This is the two-dimensional analog of the e^+e^- annihilation amplitude and we would like to verify for it the analog of asymptotic freedom: Namely that for large q^2, $M_{\mu\nu}$ approaches the free-quark amplitude for the same process. If we expand $M_{\mu\nu}$ in powers of g this should be automatic since the theory is, after all, superrenormalizable. However, the $1/N$ expansion is quite different since the quarks disappear from the spectrum and the question is whether the bound-state mesons can somehow, collectively, simulate the now-extinct quarks. One could easily imagine that confinement would change the short-distance behavior of the theory, if the large-energy limit and the infinite-volume limit (which produces infinite-energy states) did not commute. This is in fact the case for gauge-noninvariant operators, however, not for physical gauge-invariant Green's functions.

$M_{\mu\nu}$ in principle requires a renormalization subtraction to make it perfectly well defined. However, in this gauge the subtraction affects only M_{++}—the other current matrix elements are finite—and we may, for instance, compute M_{--} and compare it with the corresponding free-quark loop without further ado.

The graphs contributing to $M_{\mu\nu}$ in the leading N^{-1} limit are displayed in Fig. 12. The quark propagators are of course all dressed and it is easy to see in the case of M_{--} that the first two terms in Fig. 12 vanish when $\lambda\to 0$. Only the sum over resonances survives and one finds explicitly [using the results of Eqs. (18) and (22)]

$$M_{--}=i\frac{N}{\pi}(q_-)^2\sum_n\frac{[\int_0^1 dx\,\phi_n(x)]^2}{q^2-m_n^2}\,, \qquad (28)$$

where q is the four-momentum carried by the current. We will return in a moment to the question of M_{+-} and M_{++}.

It is easy to show that the free massless quark loop yields $M_{--}=Nq_-^2/q^2\pi$. However, the asymptotic behavior of M_{--} is

$$M_{--}\underset{q^2\to-\infty}{\frown}\frac{N}{\pi}\frac{(q_-^2)}{q^2}\sum_n\left[\int_0^1 dx\,\phi_n(x)\right]^2 .$$

The sum over states can be shown to be precisely unity with the help of the completeness relation for the bound-state wave functions. Thus we have the desired asymptotic freedom result

$$M_{--}\underset{q^2\to-\infty}{\sim}M_{--}(g=0,m=0)\,. \qquad (29)$$

One may ask: What is the energy scale which governs the approach to the asymptotic limit? This is equivalent to computing the $O(q^{-4})$ correction to Eq. (29), since in this superrenormalizable theory there are no logarithmic corrections:

$$\Delta M_{--}\simeq\frac{1}{q^4}\sum_n^{\Lambda}\frac{m_n^2}{\pi}\left[\int_0^1 dx\,\phi_n(x)\right]^2$$

(we must cut the sum off because it is logarithmically divergent). From the bound-state wave equation it follows that

$$\int_0^1 dy\sum_n m_n^2\,\phi_n(x)\,\phi_n(y)=\left(\frac{m_a^2}{1-x}+\frac{m_b^2}{x}\right).$$

Consequently $\Delta M_{--}/M_{--}\sim(m^2/\pi q^2)\ln q^2$, so that it is the quark mass itself which sets the scale for the approach to the asymptotic limit, even though the quark mass has nothing directly to do with the masses of the resonances which directly determine the amplitude. So, although the quarks never appear directly in any of the relevant Feynman diagrams, and although the physically relevant mass parameter is g^2N/π, the Regge slope, the asymptotic value, and rate of approach to the asymptotic limit of the 2-current amplitude is governed by noninteracting quarks. The mesons do indeed simulate the quarks and in principle allow us to measure the quark charge and mass.

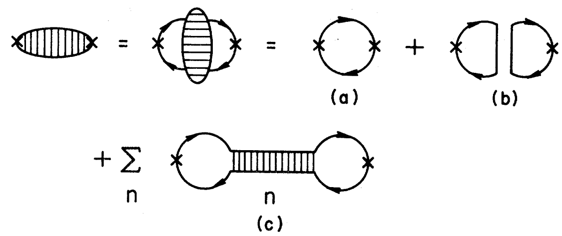

FIG. 12. Leading graphs for current 2-point function.

The discussion of other components of $M_{\mu\nu}$ is slightly more complicated due to the anomaly and the need for renormalization subtractions. In general we must have

$$M_{\mu\nu}(q)=\Pi(q^2)\,(q_{\mu\nu}q^2-q_\mu q_\nu)+C_{\mu\nu}$$

when C is an *a priori* unknown constant arising from the need for an over-all subtraction in $M_{\mu\nu}$. Our discussion of M_{--} has identified

$$\Pi=\frac{N}{\pi}\sum_n\frac{[\int_0^1 dx\,\phi_n(x)]^2}{q^2-m_n^2}\,.$$

If we evaluate M_{+-} according to our standard procedure and use the identity of Eq. (15) we easily find that

$$M_{+-}=\frac{N}{\pi}\sum_n m_n^2\frac{[\int dx\,\phi_n(x)]^2}{q^2-m_n^2}$$

$$=-\frac{N}{\pi}+q^2\frac{N}{\pi}\sum_n\frac{[\int dx\,\phi_n(x)]^2}{q^2-m_n^2}\quad .$$

This is consistent with the general form for $M_{\mu\nu}$ and our identification of Π if we set $C=-N/\pi$. That $C\neq 0$ says that there is an anomaly in the Ward indentity for the product of two vector currents. This is not particularly sinister since we are by now well acquainted with anomalies and nothing new is added to the discussion of asymptotic behavior.

Finally, we may obtain some insight into the problems of crossing symmetry and analyticity which bothered us in the case of the meson S-matrix elements (notably the 3-point function) by studying the 3-current amplitude. Consider the 3-point function of currents $V_\mu^{(ab)}$, $V_\nu^{(bc)}$, and $V_\lambda^{(ca)}$ represented graphically in Fig. 13 and denoted by $M^{abc}_{\mu\nu\lambda}$. Each term in the expansion in g is infrared-finite and has the usual crossing and analyticity properties. Furthermore, in the limit where all $q_i^2\to\infty$, because the theory is superrenormalizable, the amplitude is dominated by the free-quark loop (which falls like q^{-1}). On the other hand, in the N^{-1} expansion, the leading terms are given by graphs of the type shown in Fig. 14 where quark propagators are now dressed and sums over resonances are understood. For what should by now be quite familiar reasons the graphs of 14(a) and 14(b) vanish when the cutoff is removed and the other two are infrared-finite. In the limit $q_i^2 \to \infty$, the graph of Fig. 14(d) vanishes more rapidly than the graph of Fig. 14(c) because it involves three meson propagators instead of two. As far as asymptotic properties are concerned, then, we may focus on graphs of the type Fig. 14(c). Let us now consider a typical such contribution with kinematics as defined in Fig. 15. We need *both* bound-state wave functions to be infrared-divergent in order to overcome the factor λ^{+2} which arises from the loop integration over the quark propagators and yield a finite result. This is possible only if q_{1-} and q_{2-} are of opposite sign, so this graph will contain θ functions in the form $\theta(q_{2-})\,\theta(q_{3-})$, etc. These θ functions are nonanalytic and must somehow cancel out if the full amplitude is to have the analyticity and crossing properties it should. If we choose the $-$ component for all currents and pass to the limit $q_i^2 \to \infty$, arguments similar to those used in

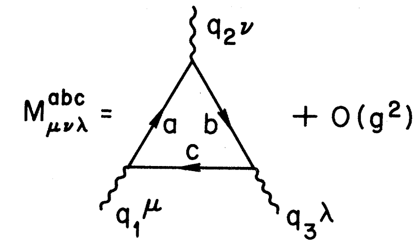

FIG. 13. 3-point function of currents.

the discussion of the current 2-point function yield the result that this graph becomes

$$M^{abc}_{---} \sim N \frac{1}{q_{1+}} \left(\frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) \times [\theta(q_{2-})\,\theta(q_{3-}) + \theta(-q_{2-})\,\theta(-q_{3-})] .$$

There are two other graphs in which the roles of momenta 1, 2, and 3 are permuted. It is an easy algebraic exercise to show that

$$\frac{1}{q_{1+}} \left(\frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) = (\text{cyclic permutations on } q_1, q_2, q_3) ,$$

so that the total amplitude is just

$$M^{abc}_{---}\big|_{\text{total}} \sim N \frac{1}{q_{1+}} \left(\frac{q_{2-}}{q_{2+}} - \frac{q_{3-}}{q_{3+}} \right) [\theta(q_{2-})\,\theta(q_{3-}) + \theta(-q_{2-})\,\theta(-q_{3-}) + \text{permutations}] . \quad (30)$$

However, the sum of θ functions is just unity and we conclude that the total amplitude does indeed have the proper analyticity and crossing properties (this amplitude is of course also equal to the free-quark amplitude, so we also verify the asymptotic freedom result as well). To show that the amplitude for finite q_i^2 has proper analyticity is horrendously difficult, and we have not done it—this calculation is meant simply to illustrate how, in a simple case, the theory manages to produce crossing-symmetric amplitudes from noncrossing-symmetric elements. We expect that it does so in general.

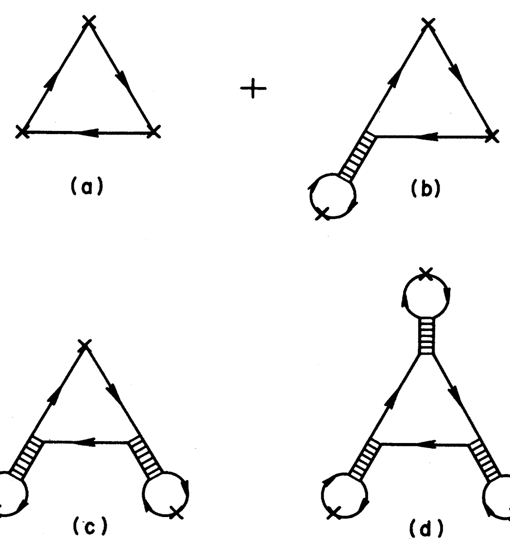

FIG. 14. Various possible structures for contributions to the 3-point function of currents.

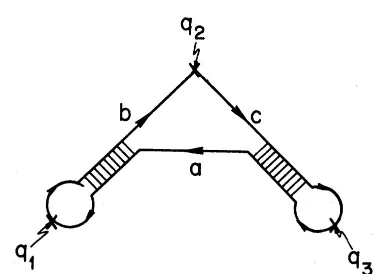

FIG. 15. Kinematics for Fig. 14(c).

V. SCALAR AND PSEUDOSCALAR DENSITIES

A number of interesting issues are raised by a study of the gauge-invariant scalar and pseudoscalar densities

$$S^{ab} = \sum_{i=1}^{N} \bar{\psi}_i^a \psi_i^b ,$$

$$P^{ab} = \sum_{i=1}^{N} \bar{\psi}_i^a \gamma_5 \psi_i^b .$$

To appreciate these questions it is convenient first to look at the vacuum–single-meson matrix elements of S and P. To leading order in $1/N$, one has to compute the graph shown in Fig. 16 where the ladder represents the bound-state wave function, the quark propagators are dressed, and the cross is either 1 or γ_5. The graph may be evaluated by techniques that should by now be familiar and we find, in the limit $\lambda \to 0$,

$$\langle 0 | S^{ab} | n \rangle = \left(\frac{N}{\pi}\right)^{1/2} \int_0^1 dx \left(\frac{m_a}{x} - \frac{m_b}{1-x}\right) \phi_n^{ab}(x) ,$$
$$\langle 0 | P^{ab} | n \rangle = \left(\frac{N}{\pi}\right)^{1/2} \int_0^1 dx \left(\frac{m_a}{x} + \frac{m_b}{1-x}\right) \phi_n^{ab}(x) . \tag{31}$$

We have previously remarked that while space-inversion invariance should be a property of the gauge-invariant sector of this theory, it is by no means manifest. Since the mass eigenstates $|n\rangle$ are nondegenerate they should have a definite parity and one should find that either $\langle 0|S^{ab}|n\rangle$ or $\langle 0|P^{ab}|n\rangle$ should vanish for every n. Inspection of Eq. (31) shows that this is far from obvious. However, it is true, as the following operator device (whose discovery we owe to G. 't Hooft) shows. Define operators K and J by

$$K\phi(x) = \mathrm{P}\int_0^1 dy \frac{\phi(y)}{y-x} ,$$

$$J\phi(x) = \int_0^1 dx\, \phi(x)$$

and let H be the mass-squared operator of this theory. It is then straightforward algebra to show that

$$[H, P] = m_1^2 \frac{1}{x} J \frac{1}{x} - m_2^2 \frac{1}{1-x} J \frac{1}{1-x} .$$

The expectation of the commutator in an H eigenstate $|n\rangle$ is of course zero. By virtue of Eq. (31) it is also equal to $(\pi/N)\langle 0|S^{ab}|n\rangle\langle 0|P^{ab}|n\rangle$. Hence one or the other of $\langle 0|S|n\rangle$ and $\langle 0|P|n\rangle$ must vanish.

Another interesting property emerges when we consider the limit $m_a, m_b \to 0$. Since the expressions

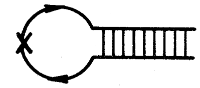

FIG. 16. Matrix elements of a density between the vacuum and a single meson.

in Eq. (31) are explicitly proportional to m_a, m_b, one might expect the matrix elements of S and P to vanish in that limit. This might even be regarded as natural since S and P create pairs of left- and right-moving quarks which, in the light-cone gauge and in the limit of zero quark mass, cannot interact. However, nothing of the sort happens: The bound-state wave functions vanish at $x=0$ [$x=1$] like $x^{Cm_a}[(1-x)^{Cm_b}]$ so that, even in the zero-mass limit, there is a finite end-point contribution to the integrals in Eq. (31). The mass singularities of the theory are just strong enough to overcome the theory's apparent asymmetry in its treatment of the left- and right-moving quarks. As a general rule, it seems to be very dangerous to draw any conclusions about the consistency, or lack thereof, of the theory by working directly in the limit of zero quark mass.

In this connection it is interesting to consider the asymptotic limit of the 2-point function of S (or P):

$$M(q^2) = \int dx\, e^{iq\cdot x} \langle 0 | T(S^{ab}(x) S^{ab}(0)) | 0 \rangle .$$

One expects, for the same reasons mentioned in the discussion of the current 2-point function, that the large-q^2 behavior of M should be governed by the free, massless theory, where

$$M(q^2) \sim \frac{N}{\pi} \ln \frac{q^2}{\mu^2} .$$

In the leading-N approximation one of course has explicitly that

$$M(q^2) = \sum_n \frac{|\langle 0|S^{ab}|n\rangle|^2}{q^2 - \mu_n^2} .$$

The large-q^2 behavior of M is clearly governed by the large-n behavior of the summand. The problem then is to evaluate $\langle 0|S^{ab}|n\rangle$ for large n. We know that $\mu_n^2 \to (g^2N/\pi)\pi^2 n$ and that $\phi_n(x) \sim \sqrt{2}\sin n\pi x$ except at the end points. Let us define

$$R_n = \int_0^1 dx \frac{m_a}{x} \phi_n(x) ,$$

$$L_n = \int_0^1 dx \frac{m_b}{1-x} \phi_n(x) .$$

Then $\langle 0|S^{ab}|n\rangle = (N/\pi)^{1/2}(R_n - L_n)$ and $\langle 0|P^{ab}|n\rangle = (N/\pi)^{1/2}(R_n + L_n)$. The theorem that $|n\rangle$ has definite parity implies that $L_n = \pm R_n$. Indeed since the bound states will alternate in parity, it is clear that as we increase n by one unit, L_n will simply change sign (we choose ϕ_n so that R_n is always positive). For large n, R_n will presumably change slowly as a function of n. On the other hand, it obviously is determined by how ϕ_n behaves near $x=0$ and so our approximate wave function is not adequate to evaluate R_n. Instead we evaluate $\langle n+1|[H,P]|n\rangle=(\mu_{n+1}{}^2-\mu_n{}^2)\langle n+1|P|n\rangle$. This can be evaluated for large n with the help of our approximate formulas for $\mu_n{}^2$ and ϕ_n (end points do not dominate) and yields $2\pi g^2N$. By virtue of the identity for $[H,P]$ it also equals $R_nR_{n+1}-L_nL_{n+1}$ which is identically $2R_nR_{n+1}\simeq 2R_n{}^2$. This shows that the leading behavior of $\langle 0|S^{ab}|n\rangle$ for large n is $2(g^2N\pi)^{1/2}$. Feeding this back into the expression for $M(q^2)$ we easily recover the "asymptotic freedom" result that $M(q^2)\sim(N/\pi)\ln q^2$.

VI. CONFINEMENT WITH A REGULAR CUTOFF

The calculations in the body of this paper have been done in the light-cone gauge (setting $A_-=0$), and with a particularly singular choice of infrared cutoff (restricting $|q_-|\geq\lambda$). The cutoff prescription that we used, following 't Hooft, gave rise to severe infrared singularities as $\lambda\to 0$. In fact it might appear that these singularities produced the confinement of the quarks, since it was because of them that the poles in the quark propagator were removed to infinity. On the other hand, one could have chosen a much less singular prescription for dealing with the infrared divergences. The choice of infrared cutoff should be independent of color, since gauge-invariant Green's functions should not be infrared singular. However, gauge-variant Green's functions could have drastically different properties with different prescriptions.

We have investigated the theory formulated with a principal-value prescription for the gluon propagator, $-i/q_-{}^2$, which we call the regular cutoff (the prescription $|q_-|\geq\lambda$ will be called the singular cutoff). It is clear that with this prescription *there are no infrared infinities at all* in the theory. How then does the confinement mechanism work?

The theory is easily constructed with the regular cutoff. One must merely eliminate all factors of $1/\lambda$ in our previous calculation of the quark propagator and scattering amplitude. Thus the quark propagator is given by Eq. (6), without the $1/\lambda$ term. This propagator then contains a pole at $q^2=m^2-g^2N/\pi$. With the regular cutoff, then, finite mass quarks exist. However, confinement can still take place if these do not appear as real intermediate states (discontinuities) in Green's functions of gauge-invariant operators or hadronic scattering amplitudes.

The quark-antiquark scattering amplitude can be constructed as before. The $1/\lambda$ terms canceled in the bound-state equation, so that Eq. (12) is unaltered. The scattering amplitude will be given by Eq. (18), where again all terms involving $1/\lambda$ are to be replaced with zero.

To see how confinement works with the regular cutoff let us examine the current 2-point function. To leading order in $1/N$ the diagrams in Fig. 12 contribute. Unlike the case of the singular cutoff, the first two diagrams *do not vanish*. In fact they contain discontinuities corresponding to the quark-antiquark threshold. However, the resonance term [Fig. 12(c)] is altered, due to the change of $\Phi_n(x)$. It contributes to $M^{ab}_{--}(q)$ the term

$$\frac{iq_-{}^2}{\pi}\left(\frac{g^2N}{\pi}\right)^2\sum_n\frac{N}{q^2-m_n{}^2}\int_0^1dx\int_0^1dx'\,\frac{\phi_n(x)\,\phi_n(x')\left(\dfrac{\gamma_a-1}{x}+\dfrac{\gamma_b-1}{1-x}-\mu_n{}^2\right)\left(\dfrac{\gamma_a-1}{x'}+\dfrac{\gamma_b-1}{1-x'}-\mu_n{}^2\right)}{\left[q^2-\dfrac{g^2N}{\pi}\left(\dfrac{\gamma_a-1}{x}+\dfrac{\gamma_b-1}{1-x}\right)\right]\left[q^2-\dfrac{g^2N}{\pi}\left(\dfrac{\gamma_a-1}{x'}+\dfrac{\gamma_b-1}{1-x'}\right)\right]}\ ,$$

where we have performed the + momentum fermion loop integrations. This can be rewritten, using the completeness of the wave function $\phi_n(x)$, as well as the relation

$$\sum_n\mu_n{}^2\,\phi_n(x)\,\phi_n(y)=\left(\frac{\gamma_a-1}{x}+\frac{\gamma_b-1}{1-x}\right)\delta(x-y)-\mathrm{P}\,\frac{1}{(x-y)^2}\ , \tag{33}$$

as a sum of three terms:

$$\frac{iq_-{}^2}{\pi}\sum_n\frac{N}{(q^2-m_n{}^2)}\int_0^1dx\int_0^1dx'\,\phi_n(x)\,\phi_n(x')\ ,$$

which is the result derived previously, Eq. (28), with the singular cutoff;

$$\frac{-iq_-{}^2}{\pi}\int_0^1dx\,\frac{N}{q^2-(g^2N/\pi)\left[(\gamma_a-1)/x+(\gamma_b-1)/(1-x)\right]}\ ,$$

which cancels the contribution of Fig. 12(a); and

$$\frac{+q_-^2}{\pi}\left(\frac{g^2N}{\pi}\right)$$
$$\times \mathrm{P}\int_0^1 dx\int_0^1 dx' \frac{N}{(x-x')^2\{q^2-(g^2N/\pi)[(\gamma_a-1)/x+(\gamma_b-1)/(1-x)]\}\{q^2-(g^2N/\pi)[(\gamma_a-1)/x'+(\gamma_b-1)/(1-x')]\}},$$

which cancels the contribution of Fig. 12(b). The final result is, as expected, unchanged. Quark continuum states do not appear due to the above cancellations.

On the other hand, gauge-variant Green's functions are totally altered with the regular cutoff procedure. Consider the 2-point function of a colored operator, say $\langle 0|T\{\bar\psi_i\psi^j, \bar\psi_j\psi^i\}|0\rangle$. With the regular cutoff this amplitude is nonvanishing, contains quark-antiquark discontinuities, and exhibits at high energies the free-field structure one would expect from asymptotic freedom. On the other hand, if we work with the singular cutoff, the amplitude vanishes as $\lambda\to 0$. This is to be expected since with this cutoff there do not exist any finite-energy colored states. This result contradicts the asymptotic theorem that one can prove using the asymptotic freedom of the model. This occurs because the high-energy and the zero-cutoff limits do not commute. In fact the above amplitude contains integrals of the form

$$q^2\int\frac{dx}{q^2x(1-x)-g^2N/\pi\lambda},$$

which approach their free-field theory value when $q^2 \gg g^2N/\pi\lambda$, but which vanish when $\lambda\to 0$. Thus the existence of infinite-energy states with the singular cutoff can alter the short-distance structure of the theory for gauge-variant Green's function. Gauge-invariant Green's functions, as we have seen in Sec. IV, are unaffected by the infinite-energy states.

Our conclusion is that confinement can occur even if infrared slavery does not produce infinite energies for the colored states. All that is required is that these states decouple from the color singlets. On the other hand, it is clearly advantageous to work with the singular cutoff. Confinement is then manifest, and calculations are much simpler due to the vanishing of many gauge-variant Green's functions.

VII. CHARMONIUM

Recent experimental discoveries (of the ψ and its partners) have made the question of the dependence on quark mass of various bound-state properties of considerable topical interest. In particular, it has been suggested that mesons constructed out of charmed quarks many times more massive than the "familiar" $\mathcal{P}$, $\mathfrak{N}$, λ quarks would have anomalously small amplitudes to decay into low-mass mesons (the OZI rule) and might also have particularly simple mass formulas by virtue of a rather heuristic asymptotic freedom argument. The model we have been discussing is particularly well suited to a study of these questions—we have only to give the appropriate structure to the mass matrix $m^2_{ab}=m_a^2\delta_{ab}$ and study various bound-state wave functions and decay amplitudes behave as one of the mass eigenvalues is allowed to become large.

To simplify matters, we shall let there be one heavy quark, c, and one light quark, $\mathfrak{N}$. We shall come back shortly to a more precise definition of what we mean by "heavy" in this context. We shall be interested in the decay of low-lying $c\bar c$ states into multiple $\mathfrak{N}\bar{\mathfrak{N}}$ states [we concern ourselves with low-lying $c\bar c$ states so that the direct (and rapid) decay into charmed mesons, $(c\bar c)\to(c\bar{\mathfrak{N}})+(c\bar{\mathfrak{N}})$, is energetically forbidden]. Since the c quarks must annihilate, this process cannot occur to leading order in $1/N$. We have already displayed in Fig. 6 the generic graph for $(c\bar c)\to(\mathfrak{N}\bar{\mathfrak{N}})+(\mathfrak{N}\bar{\mathfrak{N}})'$ and remarked that it is $O(N^{-3/2})$ as opposed to $O(N^{-1/2})$ for a normal 3-body decay. Therefore, there is a topological suppression of $O(N^{-2})$ (in real life this might be a factor of 10) of decays in which somewhere a quark must annihilate. To explain the supersuppression of the ψ decay one needs more than a factor of 10 and we want to investigate whether such suppression might come from the mass dependence of the diagram. To see whether this is so we must first evaluate the $(cc)\to(c\bar{\mathfrak{N}})+(\bar c\mathfrak{N})$ and $(\bar c\mathfrak{N})+(c\bar{\mathfrak{N}})\to(\mathfrak{N}\bar{\mathfrak{N}})$ vertices, which are themselves determined by the $c\bar c$, $c\bar{\mathfrak{N}}$, and $\mathfrak{N}\bar{\mathfrak{N}}$ wave functions. Our first task is therefore to find out how the $c\bar c$ and $c\bar{\mathfrak{N}}$ wave functions behave for large m_c.

The best we can do is to give a rough variational calculation of the desired wave functions, based on the energy integral of Eq. (14), which we choose to reexpress in dimensionless form

$$\mu^2=(\mu|H|\mu)=\int_0^1 dx\,|\phi_\mu|^2\left(\frac{\beta_a}{x}+\frac{\beta_b}{1-x}\right)+\tfrac{1}{2}\int_0^1 dx\,dy\,\frac{|\phi_\mu(x)-\phi_\mu(y)|^2}{(x-y)^2},\quad \mu^2=\frac{\pi M_{ab}^2}{g^2N},\quad \beta_{a,b}=\frac{\pi m_{a,b}^2}{g^2N}.$$

Here, ϕ_μ is the trial wave function for a state constructed out of a and b quarks and M_{ab} is the resulting mass. To find the ground state in the a, b sector, we must minimize μ^2, of course. A convenient normalized trial function, incorporating the boundary condition that ϕ vanish at $x=0, 1$, is

$$\phi(x) = x^r(1-x)^s \left[\frac{\Gamma(2r+2s+2)}{\Gamma(2r+1)\,\Gamma(2s+1)}\right]^{1/2} .$$

The kinetic-energy part of μ^2 can be evaluated exactly and one can make adequate approximations, in the cases of interest to us, to the potential-energy integral.

For the $c\bar{c}$ wave functions one finds that $r=s=\beta^{2/3}/\pi^{1/3}$ is an approximate solution to the variational problem so long as β is large and that an accurate approximation to the trial wave function is

$$\phi_{c\bar{c}}(x) = \frac{c}{\pi}\, e^{(c/2)(x-1/2)^2} ,$$

$$c = \frac{8\beta^{2/3}}{\pi^{1/3}} .$$

Furthermore, the approximate solution for μ^2 is

$$\mu_{\min}{}^2 = 4\beta + 3\pi^{1/3}\beta^{1/3} + O(\beta^{-1/3}) ,$$

$$\mu_{\min} = 2\sqrt{\beta} + \frac{3\pi^{1/3}}{4}\,\frac{1}{\beta^{1/6}} + O\left(\frac{1}{\beta^{5/6}}\right).$$

The first thing we conclude from this is that for large m_c, $\phi_{c\bar{c}}$ is centered at $x=\frac{1}{2}$ with width $\Delta x \sim m_c{}^{-2/3}$ and maximum amplitude proportional to $m_c{}^{1/3}$—these facts will be essential in our discussion of charmonium decay. More important, we are now in a position to ask whether phenomenologically interesting values of m_c are in fact "large" in the sense of this approximation scheme. The basic requirement is that the width of $\phi_{c\bar{c}}$ $[\Delta x \simeq (2/c)^{1/2} = \pi^{1/6}\beta^{-1/3}/2]$ is small compared to unity. According to 't Hooft, the Regge slope of this model is $\pi g^2 N$. One clearly must take this equal to 1 GeV^2 to fit standard meson phenomenology. Thus $\beta = \pi M_c^2/g^2N = \pi^2 M_c^2/$(1 GeV^2). The favored value for M_c is about 1.5 GeV, implying that $\Delta x \sim 0.25$. This is small enough that the sharply peaked approximation to $\phi_{c\bar{c}}$ makes sense, as will various conclusions we will draw from it. Furthermore, one should note that in the expansion of μ, the mass of the $\bar{c}c$ state, in powers of β, successive terms appear to decrease by an order of magnitude. Therefore, the zero-order mass formula (which amounts to just adding quark masses) should be good to 10% while the first-order mass formula (which amounts to including nonrelativistic potential theory corrections) should be good to 1%. It is on the face of it amazing that $m_c \sim 1.5$ GeV should be so "asymptotic," but the result follows inescapably from the curious relation: Regge slope $= \pi^2(g^2N/\pi)$, where g^2N/π is the natural coupling constant of the theory. The factor of π^2 makes the natural coupling unusually small and leads to the remarkable numerology just expounded. We find this encouraging, although we have no clear idea whether anything like this happens in three dimensions.

For the $\bar{c}\mathfrak{N}$ wave function, we must study the slightly more difficult limit $\beta_b \to \infty$, β_a fixed. The solution of the variational problem is concentrated at $x \sim (\beta_b)^{-1/2}$ with a width of the same order of magnitude. Consequently "edge effects" due to the boundary condition that ϕ vanish at $x=0$ are always important and make an accurate numerical evaluation of the variational principle more difficult than before. We have contented ourselves with very rough arguments which indicate that

$$\phi_{\bar{c}\mathfrak{N}}(x) \simeq x^r e^{-(c/2)x} [\Gamma(2r+1)C^{2r+1}]^{1/2} ,$$

$$C = \zeta(\beta_b)^{-1/2} ,$$

$$\mu \sim \sqrt{\beta_b} + \bar{\zeta} + O(\beta_b{}^{-1/2}) ,$$

where ζ, $\bar{\zeta}$ are constants of order unity, r is determined by the "finite" mass β_a and is not easy to evaluate, and the approximate form of ϕ makes sense only if $C \gg 1$. Again we note that the criteria for the asymptotic region are met if we use physically reasonable values for the various parameters, since it turns out that $\sqrt{\beta_b} \simeq 5$. The first correction to the zero-order mass formula ($m = m_c$) should be quite accurate, although it is not obviously susceptible to a simple potential theory interpretation since it arises from a finite-mass quark moving in a strong potential. Finally we note that $\phi_{\bar{c}\mathfrak{N}}$ can be described as being concentrated at $x_0 \sim m_c{}^{-1}$, having a width $\Delta x \sim m_c{}^{-1}$ and a peak value $\sim m_c{}^{1/2}$. Now we are ready to discuss the dependence of the charmonium decay amplitude on m_c, assuming m_c is large in the sense we have discussed.

Consider first the $c\bar{c} \to \bar{c}\mathfrak{N} + c\bar{\mathfrak{N}}$ vertex. According to Sec. III it is proportional to the overlap integral

$$A_{1\to 2+3} = \int_0^1 dx\, \phi_3(x)\phi_1\left(\frac{P_{3-}}{P_{1-}}x\right)\Phi_2\left(\frac{P_{1-}-xP_{3-}}{P_{2-}}\right) ,$$

when as usual

$$\Phi(x) = \int_0^1 dy \frac{\phi(y)}{(x-y)^2} .$$

First we note that since ϕ_3 is concentrated at $x=1$ and ϕ_1 is concentrated at $x=\frac{1}{2}$, the overlap integral will vanish unless $P_{3-}/P_{1-} \sim \frac{1}{2}$. Further, the range of values of $\zeta = P_{3-}/P_{1-}$ where $A_{1\to 2+3}$ does not vanish will be proportional to the smaller of the widths of ϕ_1 and ϕ_3 and so will go like m_c^{-1}. When $\zeta = P_{1-}/P_{3-}$ is within this allowed range we can see that $\Phi_2 \propto m_c^{3/2}$: Since $\zeta \sim \frac{1}{2}$, in the integral for Φ, $x - y \propto m_c^{-1}$; at the same time $\delta y \propto m_c^{-1}$ and $\phi_3 \propto m_c^{1/2}$. Thus $\Phi_3 \propto m_c^{3/2}$. Then in the integral for $A_{1\to 2+3}$, $\delta x \propto m_c^{-1}$, $\phi_1 \propto m_c^{1/3}$, $\phi_3 \propto m_c^{1/2}$, $\Phi_2 \propto m_c^{3/2}$ and we have the final result that

$$A_{1\to 2+3} \propto m_c^{4/3} .$$

By a similar, crude line of argument one finds that the amplitude, $A_{2+3\to 1'}$, for $(c\bar{\mathfrak{N}}) + (\bar{c}\mathfrak{N}) \to \mathfrak{N}\bar{\mathfrak{N}}$ behaves like m_c^{1}.

One is now in a position to evaluate the contribution of the $2+3$ intermediate state to the loop in Fig. 6. The integral d^2L over the product of propagators for particles 2 and 3 would just be $O(m_c^{-2})$ on dimensional grounds. However, the integral over L_- is restricted by the structure of the vertices such that $\delta(L_-/R_-) \propto m_c^{-1}$. Then the loop integration over the two propagators is proportional to m_c^{-3} while the two vertices are respectively proportional to m_c and $m_c^{4/3}$. *In toto,* this contribution to the amplitude behaves like $m_c^{-2/3}$. Further arguments of the type given here suggest that in the sum over states 2 and 3, only a finite number of low-lying states are important so that our estimate that the loop grows like $m_c^{-2/3}$ should be accurate. Actually, since the large dimensionless parameter governing the falloff with m_c of the various wave functions is $\sqrt{\beta_c} = (\pi m_c^2/g^2 N)^{1/2}$ we should probably ascribe to the loop a dimensionless suppression factor $(\sqrt{\beta_c})^{-2/3}$. There is a further kinematical suppression due to the propagator connecting the loop to the 3-normal-meson vertex. This propagator is evaluated at $q^2 = 4m_c^2$ and gives a suppression factor of $(2m_c/m_0)^{-2}$ if we compare with the decay of a normal meson of mass m_0 (say 1 GeV). The net result is

$$A_{\text{charmonium}} \sim N^{-1}(\beta_c)^{-1/3}\left(\frac{m_0}{2m_c}\right) \times A_{\text{normal}} .$$

Putting in standard parameters ($N=3$, $m_c = 1.5$ GeV, $m_0 = 1$ GeV, etc.) leads to a suppression in rate of a few thousand. This is probably a considerable overestimate because we have set all unknown factors of order unity equal to unity—nevertheless it is the right order of magnitude to be of interest.

The lesson of this discussion is fairly complicated. On the one hand there is a regime of large quark masses (large compared to the rather modest interaction strength) where charmonium masses and wave functions are given accurately by a nonrelativistic potential model. The spectroscopy of low-lying charmonium states should therefore be amenable to description in considerable detail. The decay of these low-lying charmonium states is another matter. Although the wave functions are effectively those of a weak-c coupling theory, the annihilation of the c quarks into $\mathfrak{N}$ quarks is a complicated process involving the interaction of bound states of various kinds and can in no obvious way be described as a weak-coupling process involving only a few gluons. In spite of this, for a complex of topological and dynamical reasons we can argue that charmonium decay is strongly suppressed relative to normal decays, thereby providing a dynamical basis for the OZI rule.

Finally it is worth noting that the same type of argument can be utilized to discuss other suppressed decays. For example, one might consider the decay of excited charmonium. Here there are two competing channels: the decay directly to hadrons as well as the decay into the charmonium ground state with hadron emission. Both processes are forbidden by the OZI rule, and both contain identical suppression factors of $1/N^2$. The difference in rates is therefore totally dynamical. In our model we find that the latter is enhanced, again by a power of β. Thus the mass suppression factors can distinguish between various processes forbidden by the OZI rule.

VIII. HIGHER ORDERS IN $1/N$

An important check of the consistency of the model is the evaluation of higher orders in $1/N$. One might be concerned that the radiative corrections to the gluon propagator and the the quark-gluon vertex might radically alter the properties of the model. For example, the lowest-order contribution to the gluon propagator, given in Fig. 17(a), is of order $1/N$. If one evaluates this graph one finds that the gluon propagator develops a pole at $q^2 \approx g^2/\pi$, at least for $m^2 \ll g^2$. Such an effect would indeed qualitatively change the nature of the model. For example, there would clearly exist finite energy colored states.

However, in the $1/N$ expansion one must sum *all* contributions of the same order in $1/N$. To leading order in $1/N$ these are given by Fig. 17(b). Using the fact that the gluons are in the adjoint representation of SU(N) it is easy to see that all other contributions of the gluon self-energy

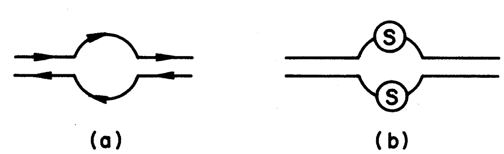

FIG. 17. Gluon propagator.

are of order $1/N^2$. The effect of this nonperturbative sum of graphs is to replace the free quark propagator by the dressed quark propagator, Eq. (1), calculated to leading order in $1/N$. Since the quark propagator vanishes like λ (as $\lambda \to 0$) and the p_+ loop momentum in Fig. 17(b) is of order $1/\lambda$, one concludes that the quark self-energy must behave like $(1/N)[\lambda + O(\lambda^2) + O(1/N)]$. Thus it would appear that through $1/N$ there is no correction to the gluon propagator. Actually one must be very careful in discussing the gluon propagator at zero momentum. Owing to the infrared singularities, terms of the order λ in the gluon self-energy cannot be neglected—however, the net effect of such terms, as shown below, is to multiplicatively change the infrared-divergent part of the gluon propagator. This will not cause qualitative changes in the model.

Thus we see that simplicity is induced in higher orders by the confinement produced to lowest order. (This of course only occurs with the singular cutoff. Calculations with the regular cutoff would be much more complicated.) In this section we consider in some detail the $1/N$ corrections to the gluon propagator, the quark-antiquark-gluon vertex, the quark propagator, and the Bethe-Salpeter kernel for quark-antiquark scattering. Our results are that to order $1/N$ only the infrared-divergent part of the gluon propagator is modified, that the vertex function is unmodified, and that to order $1/N$ only the infinite self-mass of the fermion is modified. We find three new contributions to the Bethe-Salpeter kernel to order $1/N$: a piece arising from the modification of the gluon propagator which will cancel with the corrections to the quark self-energy, a contribution from 2-gluon production, and a term arising from the production of hadronic bound states. In evaluating the $1/N$ corrections to hadronic masses only the last term survives. This means that the $1/N$ corrections to hadronic masses arise solely from the mixing of hadronic states induced by the hadronic couplings derived in Sec. III.

As we have seen, the gluon self-energy vanishes to order $1/N$ as $\lambda \to 0$. Since the inverse gluon propagator vanishes like q_-^2 one must keep terms in the self-energy of order $q_-\lambda$ or λ^2, which could contribute in the region $q_- \approx \lambda$. An evaluation of Fig. 17(b) yields

$$i\Pi(q) = \frac{i\lambda|q_-|}{2N} + O\left(\frac{\lambda^2}{N}\right).$$

[Actually one can show that this form, $\Pi(q) \sim \lambda|q_-| + O(\lambda^2)$, holds to all orders in $1/N$.] A careful evaluation of the terms of order $(1/N)\lambda^2$ shows that they can be neglected, even as $q_- \to 0$.

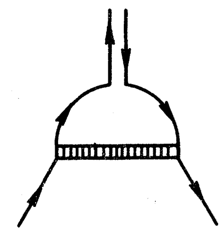

FIG. 18. Contribution of order $1/N$ to quark-antiquark-gluon vertex.

On the other hand, the term of order $\lambda|q_-|$ must be included. Its effect is to modify the infrared behavior of the gluon propagator, which to this order is

$$D(q) = \frac{-i}{q_-^2 + \lambda|q_-|/2N}. \tag{34}$$

With our infrared cutoff this means that integrals over $D(q)$ are now given by [if $F(q)$ is finite at $q_- = 0$]

$$\int dq_- D(q) F(q) = -i\frac{2}{\lambda}\left(1 - \frac{1}{4N}\right) F(q_+, q_- = 0) + \mathrm{P}\int dq \frac{-i}{q_-^2} F(q).$$

The contribution to the quark-antiquark-gluon vertex to order $1/N$ arises from the graphs in Fig. 18. At first sight this would appear to be finite as $\lambda \to 0$, since the bound-state wave functions yield a factor of $(1/\lambda)^2$ and the quark propagators give a factor of λ^2. Closer examination shows that if r_-, p_-^1, p_-^2, $r_- - p_-^1$, and $r_- - p_-^2$ are all simultaneously positive (negative), so as to give the maximal factor of $1/\lambda$, then the poles in r_+ lie all in the lower (upper) half plane. In addition, when $q_- = 0 = p_-^1 - p_-^2$, one loses both factors of $1/\lambda$ in the bound-state wave functions. Therefore, this contribution to the vertex behaves like

$$\Gamma \sim \frac{\lambda}{N}(q_- + O(\lambda))$$

and can be neglected. Again this form of the vertex can be shown to hold to all orders in $1/N$

The $1/N$ correction to the quark self-energy is now readily calculable. To this order it is given by Fig. 19, where the gluon propagator is given in Eq. (34). The $1/N$ correction only occurs in the infrared-divergent piece of the gluon propagator—

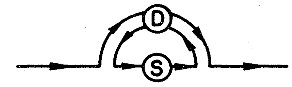

FIG. 19. $1/N$ correction to quark self-energy.

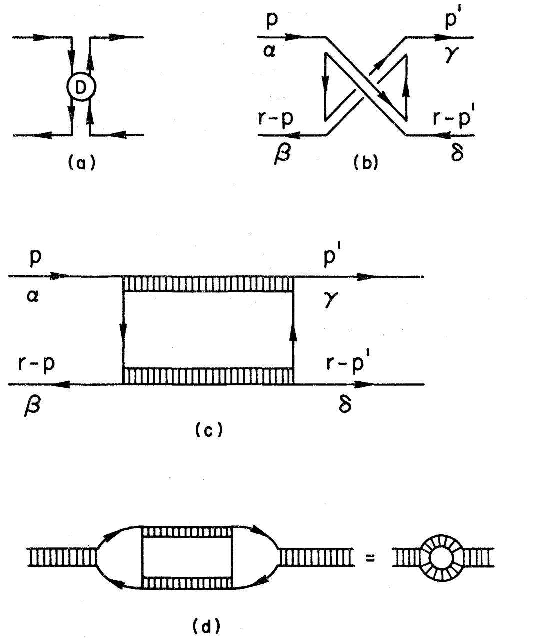

FIG. 20. Corrections to Bethe-Salpeter kernel from (a) gluon propagator correction, (b) 2-gluon exchange, and (c) production of two hadrons. [(d) Illustrates correction to mass of bound state.]

and their sole effect is to modify the infinite self-energy of the quark. Thus to this order the quark self-energy is

$$i\Sigma(p)=\gamma_-\frac{g^2N}{2\pi}\left[\frac{\operatorname{sgn}p_-}{\lambda}\left(1-\frac{1}{4N}\right)-\frac{1}{p_-}\right]. \tag{35}$$

We now consider the $1/N$ corrections to the Bethe-Salpeter kernel and the effect on the hadronic mass spectrum. The three nonzero contributions to the kernel of order $1/N$, which are pure SU(N) singlet in the quark-antiquark channel, are illustrated in Fig. 20. The modification of the gluon propagator, Fig. 20(a), only occurs in the infrared-divergent part of the potential and precisely cancels the $1/N$ contribution to the infinite self-energy of the quarks, Eq. (35). This cancellation will occur to all orders in $1/N$. Since the other $1/N$ corrections to the kernel are not infrared singular we conclude that, as in leading order, all infrared divergences cancel in the equation for hadronic bound states. There remain finite corrections to the hadron mass which might arise due to corrections to the kernel from Fig. 20.

The 2-gluon contribution, K^2, to the kernel, given in Fig. 20(b), is easily evaluated. One finds that

$$K^2_{\alpha\beta,\gamma\delta}(p,r-p;p',r-p') = -\frac{ig^2}{N}\frac{(\gamma_-)_{\alpha\beta}(\gamma_-)_{\gamma\delta}}{r_-^{\,2}}\left[\theta(p_-)\theta(p'_- - r_-)+\theta(-p_-)\theta(r_- - p'_-)+\theta(p_- - r_-)\theta(p'_-)+\theta(r_- - p_-)\theta(-p'_-)\right]+O\left(\frac{\lambda}{N}\right).$$

However, due to the θ functions this term cannot affect the hadron masses to order $1/N$. This is because to leading order the $1/N$ contribution to the mass of the Kth hadron, $\delta m_K^{\,2}$, will be given by first-order perturbation theory,

$$\delta m_K^{\,2}\sim\langle\phi_K|K^2|\phi_K\rangle,$$

where ϕ_K are the lowest-order bound-state wave functions. Since these wave functions vanish unless the quark-antiquark pair are moving in the same direction as the hadron, and K^2 vanishes unless at least one quark is moving in the opposite direction, this contribution to $\delta m_K^{\,2}$ is zero.

Thus the only $1/N$ contribution to the hadronic masses arise from the kernel given by Fig. 20(c). Lowest-order perturbation theory will yield a contribution represented by Fig. 20(d). Here we recognize, recalling the discussion of Sec. III where we derived the form of the 3-hadron vertex, a self-energy graph of the effective hadronic theory which arises to lowest order in $1/N$.

It is clear that in principle one would calculate the wave functions of the bound states, the hadronic scattering amplitudes, and current matrix elements to order $1/N$, and in fact to all orders in $1/N$.

Our final conclusion is that these higher-order corrections will change none of the qualitative features of the model.

IX. CONCLUSIONS

Our main conclusion is that two-dimensional Yang-Mills theory indicates the viability of infrared slavery as a confinement mechanism. The model possesses all of the properties required of a sensible thoery. The subspace of color-singlet hadronic states is unitary by itself. There are no long-range forces between hadrons, and colored states cannot be produced in hadronic collisions. The properties of gauge-invariant local operators are as expected. In particular the theory is asymptotically free at short distances, revealing the underlying quark model. The properties of gauge-variant Green's functions depend greatly on how one treats the infrared singularity. The choice of a singular cutoff is advantageous in making confinement manifest by producing an infinite self-energy for the quark, and causing many gauge-variant amplitudes to vanish.

The model also indicates that mass corrections to scaling might be governed by the quark masses and not the hadronic mass scale. It suggests a dynamical basis for the OZI rule.

The real world is of course four-dimensional and much more complicated. Confinement will not be automatic but will have to be produced dynamically. Our investigation suggests the importance of enhancing the infrared singularities to render the confinement manifest, and confirms the great utility of the $1/N$ expansion of 't Hooft. If the four-dimensional theory confines, it will produce an infinite number of discrete bound states to leading order in $1/N$. The resulting theory of hadrons should have many of the qualitative features discussed above.

ACKNOWLEDGMENTS

We would like to acknowledge stimulating conversations with many of our colleagues, in particular, A. Casher and E. Witten.

---

*Research supported in part by the National Science Foundation under Grant No. MPS73-04997.

[1]M. Y. Han and Y. Nambu, Phys. Rev. 139, B1006 (1965); H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[2]W. Bardeen, H. Fritzsch, and M. Gell-Mann, *Scale and Conformal Invariance in Hadron Physics* (Wiley, New York, 1973).

[3]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973).

[4]H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[5]S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[6]J. Schwinger, Phys. Rev. 125, 397 (1962); 128, 2425 (1962); J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971); A. Casher, J. Kogut, and L. Susskind, Phys. Rev. Lett. 31, 792 (1973); Phys. Rev. D 10, 732 (1974).

[7]G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[8]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[9]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975).

[10]G. Zweig, CERN Report No. 84191 Th 412, 1964 (unpublished); J. Iizuka, K. Okada, and O. Shito, Prog. Theor. Phys. 35, 1061 (1966); S. Okubo, Phys. Lett. 5, 165 (1963).

[11]C. R. Hagen, Nucl. Phys. B95, 477 (1975). Hagen argues that since the quark mass is finite in this gauge quarks will be unconfined. This is certainly true if one considers gauge-variant sectors of the theory. In Sec. VI we present arguments that in the gauge *invariant* sector quarks still do not appear despite their finite mass.