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New approach to the singularities of Feynman amplitudes in the zero-mass limit*

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We present an improved method for studying the behavior of Feynman amplitudes in the limit in which some or all of the masses of the internal lines vanish. Feynman amplitudes develop singularities in such a limit for certain configurations of external momenta. Our method enables us to determine these configurations, be they Euclidean or Minkowskian, on or off the mass shell. They are worked out explicitly in a few relatively simple cases which include most of the physically interesting singularities. Power-counting rules for these singularities are briefly discussed.

I. INTRODUCTION

Analyticity of Feynman amplitudes as functions of external momenta has been studied extensively in the past.[1] In most of these investigations, masses of internal lines are treated as parameters fixed to some finite values. However, some aspects of Feynman amplitudes can be understood more clearly if masses are also treated as variables. The most intriguing among them is the singularity of Feynman amplitudes at the origin of some (or all) of the mass variables, the so-called *mass singularities*.[2,3]

The infrared divergence in quantum electrodynamics is a well-known example of a mass singularity. They appear in the exchange or emission of low-energy photons between or from electrons. The structure of such singularities in quantum electrodynamics has been analyzed in great detail.[4] In particular, infrared divergences arising from real and virtual soft photons cancel each other in the transition probability, if we sum over all final states containing an arbitrary number of emitted photons with their total energy below some fixed value.[2-4]

On the other hand, much less is known about the infrared properties of non-Abelian gauge field theories. It is not known, for example, whether the cancellation mechanism noted above works also for such theories. In fact, it has been conjectured[5] that the infrared divergence is so violent in these theories that the true asymptotic states may be quite different from those of free fields. If this in fact occurs as it seems in the two-dimensional Schwinger model,[6] it will have an interesting implication on hadron physics.

Of course the conventional perturbation expansion may not be reliable in treating problems of this sort. For this reason some authors have explored alternative approaches.[7] In view of the poor status of our knowledge at present, however, it may still be necessary and valuable to see what one can learn from perturbation theory on such a problem.[8]

Another important problem to which mass singularities are relevant is the behavior of Feynman amplitudes in various large-momentum limits. A basic result in this respect is the Weinberg theorem[9] which enables us to determine the leading power behavior of convergent Feynman integrals when the external momenta become very large. This theorem was later extended to nonleading terms by Slavnov,[10] and the rule for the logarithmic factor of the leading term was elaborated by Fink[11] and Westwater.[12] More recently, Bergère and Lam[13] studied the asymptotic expansion of arbitrary (even divergent) Feynman amplitudes with respect to the scale parameter λ introduced in all external momenta, and determined all logarithms associated with the highest power of λ. This problem was also investigated by Pohlmeyer[14] who, extending Speer's analytic renormalization scheme,[15] succeeded to obtain the complete asymptotic expansion in λ.

Most of these results concern the limit where external momenta are nonexceptional[16] and become large uniformly in the Euclidean direction.[17] Such a limit can be converted by scaling to one in which all internal masses become uniformly small while external momenta are kept fixed. We may thus be able to obtain the results listed above by determining the singularities of Feynman amplitudes at the origin of mass variables.

The mass singularity also becomes important in the large-momentum limit of another kind in which some or all of the external momenta lie on their mass shells (e.g., the Regge limit of two-body scattering amplitudes,[18] the behavior of form factors at large spacelike momentum transfer,[19] the

Bjorken limit of electroproduction structure functions,[20] etc.[21]). Because of the on-mass-shell restrictions, it often occurs for such limits that the leading behavior of the asymptotic expansion is determined by mass singularities rather than ultraviolet singularities. In a theory of charged fermions and massive neutral vector mesons, for example, the leading logarithms of fermion form-factor amplitudes come from the infrared-singular region of loop integrations, while the ultraviolet-singular region gives only nonleading logarithms.[19]

Thus far we have considered only individual Feynman amplitudes. To study the large-momentum behavior of *Green's functions,* it is at present most efficient to make use of the renormalization-group equations,[22] their variations,[23] or the Callan-Symanzik equations.[24] For nonexceptional momenta, these equations enable us to sum up the leading terms of perturbation expansion in a fairly straightforward way.[25] They are applicable even to the cases of exceptional momenta[26] if they are combined with the Wilson short-distance expansion.[27] Except in some simple cases,[28] however, it seems difficult to apply these techniques to the large-momentum limits of on-mass-shell amplitudes. In this respect, the knowledge of mass singularities will be useful to understand the limitation of their applicability and explore possible extensions, if any.

All these examples point to the necessity of understanding the nature of mass singularities that arise for the external momenta in the Minkowskian, exceptional, and/or on-the-mass-shell configurations. In view of various shortcomings of existing approaches to these problems, there is a strong demand for a systematic and effective method for handling such problems. It is the purpose of our present work to satisfy such a need.

This article, which is the first of a sequence, is concerned with the general treatment of mass singularities whether they are associated with exceptional or nonexceptional external momenta, Euclidean or Minkowskian, on or off the mass shell. Because of an advantage in handling Minkowskian external momenta, we start as in Ref. 2 from a parametric representation of Feynman amplitudes rather than the coordinate or momentum representation. On the other hand, instead of the cut-set representation[2] which turned out to be rather clumsy for dealing with exceptional momenta, we start here from a more flexible parametric representation developed recently by one of us.[29]

In Sec. II we examine carefully how mass- and momentum-dependent singularities appear in Feynman amplitudes. This leads to the definition of two fairly simple types of mass singularities (type I and II) which, however, include most of the physically interesting cases. We shall briefly touch upon more complicated mass singularities at the end of Sec. II. The explicit conditions that the internal masses and external momenta must satisfy at a type-I mass singularity are derived in Secs. III and IV, while those for a type-II mass singularity are discussed in Sec. V. Derivation of power-counting rules is outlined in Secs. III and IV for a restricted class of integrals.

In the present article, we ignore complications associated with the ultraviolet divergences. In order to find mass singularities of renormalized amplitudes, we still have to learn how ultraviolet divergences and their renormalization affect the strength of mass singularities. This applies in particular to the powers of logarithmic factors of mass singularities. We shall investigate this problem in subsequent articles where the technique developed here is applied to some representative models of renormalizable field theories.

II. DEFINITION OF MASS SINGULARITY

Let G be an arbitrary irreducible (i.e., one-particle-irreducible and one-vertex-irreducible[30]) Feynman diagram with N_e external lines of momenta P_a ($a=1,2,\ldots,N_e$) (all directed inwards), N internal lines of momenta p_i and masses[31] m_i ($i=1,2,\ldots,N$), and L loop integration momenta r_s ($s=1,2,\ldots,L$). The general form of the corresponding Feynman integral μ_G is given by[32]

$$\mu_G=\alpha_G\int\frac{F(P,p)}{\prod_{i=1}^{N}(p_i^2-m_i^2+i\epsilon)}\prod_{s=1}^{L}\frac{d^4r_s}{(2\pi)^4}\,, \qquad (2.1)$$

where α_G consists of coupling constants and numerical factors, and $F(P,p)$ is a polynomial of external and internal momenta generated by derivative couplings and numerator factors of propagators.[33] The internal momentum p_i may be decomposed as

$$p_i=q_i+k_i\,,\quad k_i=\sum_{s=1}^{L}\eta_{is}r_s\,, \qquad (2.2)$$

where η_{is} is the circuit matrix, namely, the projection $(\pm1,0)$ of p_i along r_s. The constant momentum q_i can be chosen freely subject only to the momentum conservation law at each vertex:

$$\sum_{j=1}^{N}\epsilon_{vj}q_j+P_v=0\,, \qquad (2.3)$$

where ϵ_{vj} is the incidence matrix [$\epsilon_{vj}=+1$ (-1) if the line j enters (leaves) the vertex v, and $=0$ otherwise] and P_v is the sum of the external momenta entering G at the vertex v.

Introducing the Feynman parameters $z_1,z_2,\ldots,z_N$ and carrying out the loop integrations, we obtain from (2.1) the parametric integral[34]

$$\mu_G=\alpha_G(-1)^N\left(\frac{i}{(4\pi)^2}\right)^L\sum_{k=0}^{K}\Gamma(N-2L-k)\int\frac{F_k(z,m,q)\delta\left(1-\sum_{i=1}^{N}z_i\right)\prod_{i=1}^{N}dz_i}{U(z)^{2+k}[V(z,m,q)-i\epsilon]^{N-2L-k}}\,, \tag{2.4}$$

where $U(z)$ is the determinant

$$U(z)=\det(U_{st}) \tag{2.5}$$

for the matrix (U_{st}),

$$U_{st}=\sum_{i=1}^{N}\eta_{is}\eta_{it}z_i\,, \tag{2.6}$$

and

$$V(z,m,q)=\sum_{i=1}^{N}z_i\,(m_i^2-q_i^2)+\frac{1}{U(z)}\sum_{i,j=1}^{N}z_iz_jB_{ij}(z)q_i\cdot q_j\,, \tag{2.7}$$

$B_{ij}(z)$ being defined by

$$B_{ij}(z)=\sum_{c}\eta_{ic}\eta_{jc}U_c(z)\,. \tag{2.8}$$

The summation in (2.8) goes over all (not necessarily independent) self-nonintersecting loops c of G and $U_c(z)$ is the U function[35] for the diagram obtained from G by shrinking the loop c to a point. The function F_k is calculated as follows: Pick out k pairs of internal momenta $(p_i^\mu, p_j^\nu),\ldots$ from $F(P,p)$ and replace them by $-\frac{1}{2}g^{\mu\nu}B_{ij},\ldots$. Replace the rest of p_i in $F(P,p)$ by Q_i' defined by

$$Q_i'=q_i-\frac{1}{U}\sum_{j=1}^{N}z_jB_{ij}q_j\,. \tag{2.9}$$

Sum up the result of this operation over all possible k pairs. If $F(P,p)$ is of order n in the internal momenta, K is equal to $[n/2]$.

Throughout this work we are interested in the singularities of Feynman integrals which are specifically associated with vanishing internal masses. We therefore treat μ_G as a function of both internal masses $\underline{m}=\{m_1,\ldots,m_N\}$ and external momenta $\underline{P}=\{P_v\,|\,v\in G\}$.

Formulas (2.4)–(2.8) show that a mass- and momentum-dependent singularity may appear in $\mu_G(\underline{m},\underline{P})$ when the V function vanishes.[36] To be more precise, let us define two hypersurfaces $S(\underline{m},\underline{P})$ and I in the complex Feynman parameter space $\mathbb{C}^N=\{(z_1,\ldots,z_N)\,|\,z_1,\ldots,z_N\in\mathbb{C}\}$ according to

$$S(\underline{m},\underline{P})\equiv\{(z_1,\ldots,z_N)\,|\,V(z,m,q)=0\}\,,$$

$$I\equiv\left\{(z_1,\ldots,z_N)\,\middle|\,\sum_{i=1}^{N}z_i=1,\; z_1,\ldots,z_N \text{ real and} \geq 0\right\}. \tag{2.10}$$

The vanishing of V generates a singularity of $\mu_G(\underline{m},\underline{P})$ if (i) the entire surface of some boundary plane of I is contained in the intersection $S_t(\underline{m},\underline{P})\equiv S(\underline{m},\underline{P})\cap I$, and/or (ii) $S(\underline{m},\underline{P})$ pinches a subhypersurface of I. As is well known, this leads to the Landau equation

$$z_i=0\quad\text{for } i\in A\,, \tag{2.11a}$$

$$\partial V/\partial z_i=0\quad\text{for } i\in G-A\,, \tag{2.11b}$$

A being a subdiagram of G.

If (2.11) has a unique[37] solution, we obtain the usual threshold singularity. This is of no particular interest to us since it is not necessary for its appearance that any of the internal masses vanish. Under some circumstances, however, the pinch condition (2.11b) becomes degenerate and ceases to determine z_i, $i\in G-A$, uniquely. It turns out that, in many cases, such a degeneracy occurs if and only if some of the masses of internal lines vanish. This indicates that mass singularities correspond to certain degenerate solutions of the Landau equation (2.11). Unfortunately, general analysis of such degeneracies is rather complicated and would require powerful mathematical tools. Instead of following such a path, we shall concentrate here on a few simplest cases which, however, cover most of the physically interesting mass singularities.

The simplest type of degeneracy occurs when the intersection $S_t(\underline{m},\underline{P})$ does not depend at all on $z_k\in G-A$, $A\neq\varnothing$. In this case, the pinch condition (2.11b) is equivalent to $V|_{z_i=0,\,i\in A}\equiv 0$ (identity in $z_k\in G-A$) because of the relation

$$V=\sum_{i=1}^{N}z_i\frac{\partial V}{\partial z_i}\,.$$

As we shall show in Sec. III, this leads to vanishing masses for all the internal lines of $G-A$. To distinguish the associated mass singularity from others, let us call it mass singularity of type I. To be more precise, let us define it as follows:

Definition I. Let G be an arbitrary irreducible diagram and $\overline{M}$ be a *proper* subdiagram of G. The Feynman amplitude $\mu_G(\underline{m},\underline{P})$ has a type-I mass singularity associated with the reduced diagram[38] $M\equiv G/\overline{M}$ if

$$V(z,m,q)|_{z_i=0,\,i\in\overline{M}}=0$$

holds for *any* values of $z_k\in G-\overline{M}$.

Two situations (without and with enhancement) arise for mass singularities of this type. They will be studied in Secs. III and IV, respectively.

The type-I mass singularity is a pure end-surface-type singularity. The next simplest situation will be encountered if we allow the presence of a pinch, but in such a restricted way that its position depends only on *some* of $z_k \in G-A$. This will be called a mass singularity of type II:

Definition II. Let $\bar{M}$ be a proper subdiagram of G and $\bar{M}'$ be a proper (possibly empty) subdiagram of $\bar{M}$. The Feynman amplitude $\mu_G(\underline{m}, \underline{P})$ has a type-II mass singularity associated with the reduced diagram $M \equiv G/\bar{M}$ if the Landau equation given by

$$z_i = 0 \quad \text{for } i \in \bar{M}',$$
$$\partial V/\partial z_j = 0 \quad \text{for } j \in G-\bar{M}'$$

has a degenerate solution such that $z_k \in G-\bar{M}$ are completely arbitrary, while $z_i \in \bar{M}$ have uniquely[37] determined values that do not depend on $z_k \in G-\bar{M}$ except through the normalization

$$\sum_{i=1}^{N} z_i = 1 .$$

Geometrically, a type-II mass singularity appears when $S(\underline{m}, \underline{P})$ pinches I along a hypersurface that extends uniformly in the direction of $z_k \in G-\bar{M}$. We shall examine its structure in Sec. V.

Of course one can find mass singularities much more complicated than those considered above. One example is the Landshoff diagram which appears in the composite hadron models for wide-angle elastic scattering.[39] In this case, $S(\underline{m}, \underline{P})$ pinches I along a multidimensional hypersurface which depends on *all* $z_k \in G-A$. A systematic analysis of such possibilities is beyond the scope of this article.

III. TYPE-I MASS SINGULARITY

According to the definition I, the Feynman integral μ_G develops a mass singularity of type I if the V function vanishes identically on the boundary plane $D(\bar{M})$ defined by

$$D(\bar{M}) \equiv \left\{ (z_1, \ldots, z_N) \middle| z_i = 0 \text{ for } i \in \bar{M}, \sum_{j \in G-\bar{M}} z_j = 1 \right\}.$$

Of course this happens only for certain values of internal masses and external momenta. Furthermore, the nature of such a singularity depends strongly on how fast V vanishes at $D(\bar{M})$. To analyze these problems it is useful to introduce a vanishingly small parameter δ which regulates the approach of Feynman parameters $z_1, \ldots, z_N$ to $D(\bar{M})$ in the following manner:

$$z_i = \begin{cases} O(\delta^2) & \text{for } i \in \bar{X} \ (\subset \bar{M}), \\ O(\delta) & \text{for } i \in \bar{M}-\bar{X}, \\ O(1) & \text{for } i \in G-\bar{M}, \end{cases} \tag{3.1}$$

and expand V in powers of δ. Here, $\bar{X}$ denotes a (possibly empty) *proper* subdiagram of $\bar{M}$ introduced to allow for the possibility that some $z_i \in \bar{M}$ may vanish faster than others at $D(\bar{M})$.[40]

Since V is a homogeneous function of Feynman parameters, it is generally not possible to expand it into ordinary Taylor series. We note, however, that any homogeneous function $f(z)$ of Feynman parameters of G may be expanded into a series of functions such that each term in this expansion is homogeneous with respect to the Feynman parameters of a subdiagram S of G. We denote such an expansion as

$$f(z) = [f(z)]_S^0 + [f(z)]_S^1 + \cdots + [f(z)]_S^n + \cdots , \tag{3.2}$$

where $[f(z)]_S^n$, called the nth S limit, is a homogeneous function of degree $n_0 + n$ with respect to $z_i \in S$, n_0 being a constant characteristic of the function $f(z)$. For the zeroth S limit $[f(z)]_S^0$ we often drop the superscript 0 for simplicity.

Corresponding to the hierarchial approach (3.1) of Feynman parameters to $D(\bar{M})$, let us expand V in two steps. First, we expand it with respect to $z_i \in \bar{X}$ assuming that $z_i = O(\epsilon)$ for $i \in \bar{X}$ and $z_j = O(1)$ for $j \in G-\bar{X}$ where $\epsilon = \delta^2$. In order to calculate the zeroth $\bar{X}$ limit of V, it is necessary to find the zeroth $\bar{X}$ limits of U and B_{ij}. For this purpose, note that $\bar{X}$ in general consists of several irreducible components $\bar{X}_\Gamma (\Gamma = 1, 2, \ldots)$ and a set $\bar{S}$ of lines connecting them. Let the number of loops in $\bar{X}_\Gamma$ be $L(\bar{X}_\Gamma)$ and let $L(\bar{X}) \equiv \sum_\Gamma L(\bar{X}_\Gamma)$. Then we find from (2.5) that[41]

$$[U]_{\bar{X}} = U^H \tilde{U}^{\bar{X}} \quad [= O(\epsilon^{L(\bar{X})})] , \tag{3.3}$$

where

$$\tilde{U}^{\bar{X}} = \prod_\Gamma U^{\bar{X}_\Gamma} , \tag{3.4}$$

$H \equiv G/\bar{X}$,[38] and U^H, etc. denote the U functions for the diagram H, etc. From (2.8) and (3.4), we obtain similarly[41]

$$[B_{ij}]_{\bar{X}} = \tilde{U}^{\bar{X}} B_{ij}^H \quad [= O(\epsilon^{L(\bar{X})})] \quad \text{for } i, j \in H . \tag{3.5}$$

Taking account of (3.3) and (3.5), and noting that $z_i = O(\epsilon)$ for $i \in \bar{X}$, we find that the zeroth $\bar{X}$ limit of V agrees with the V function for the reduced diagram H:[41]

$$\begin{aligned} [V]_{\bar{X}} &= \sum_{i \in H} z_i (m_i^2 - q_i^2) \\ &\quad + \frac{1}{[U]_{\bar{X}}} \sum_{i,j \in H} z_i z_j [B_{ij}]_{\bar{X}} q_i \cdot q_j \\ &= \sum_{i \in H} z_i (m_i^2 - q_i^2) + \frac{1}{U^H} \sum_{i,j \in H} z_i z_j B_{ij}^H q_i \cdot q_j \\ &= V_H . \end{aligned} \tag{3.6}$$

Since V_H is clearly of order ϵ^0, we find (recall $\epsilon = \delta^2$)

$$V = V_H + O(\delta^2). \tag{3.7}$$

The second step is to expand V_H with respect to $z_i \in X \equiv \overline{M}/\overline{X}$. By an argument similar to that leading to (3.6), we find that the zeroth X limit of V_H agrees with V_M, the V function for the reduced diagram $M \equiv H/X = G/\overline{M}$:

$$\begin{aligned}[V_H]_X &= V_M \\ &= \sum_{i \in M} z_i (m_i^2 - q_i^2) \\ &\quad + \frac{1}{U^M} \sum_{i,j \in M} z_i z_j B_{ij}^M q_i \cdot q_j .\end{aligned} \tag{3.8}$$

Since $z_i = O(\delta)$ for $i \in X$ by definition (3.1), $[V_H]_X^1 = O(\delta)$ and the expansion of V up to terms of order δ is given by

$$V = V_M + [V_H]_X^1 + O(\delta^2). \tag{3.9}$$

In deriving this formula, we have not taken account of the constraint

$$\sum_{i=1}^{N} z_i = 1 .$$

To incorporate it most easily, we rescale Feynman parameters according to

$$z_i = \begin{cases} \sigma z_i' & \text{for } i \in \overline{X}, \ \sum_{i \in \overline{X}} z_i' = 1 , \\ \rho z_i' & \text{for } i \in \overline{M} - \overline{X}, \ \sum_{i \in \overline{M}-\overline{X}} z_i' = 1 , \\ (1-\sigma-\rho) z_i' & \text{for } i \in G - \overline{M}, \ \sum_{i \in G-\overline{M}} z_i' = 1 . \end{cases} \tag{3.10}$$

The correct expansion is obtained by substituting (3.10) into (3.9) and re-expanding the result in powers of ρ and σ [regarding $\rho = O(\delta)$, $\sigma = O(\delta^2)$]. Because of the homogeneity of V, V_M, $[V_H]_X^1$, etc., we find

$$V = (1-\rho) V_M' + \rho [V_H']_X^1 + O(\rho^2, \sigma) , \tag{3.11}$$

where V_M' and $[V_H']_X^1$ are obtained from V_M and $[V_H]_X^1$, respectively, by the replacement $z_i \to z_i'$ ($i \in G$). In the following, we shall usually drop primes for simplicity, with the understanding that Feynman parameters then satisfy

$$\sum_{i \in X} z_i = 1$$

and

$$\sum_{i \in M} z_i = 1 .$$

The formula (3.11) shows that V becomes equal to V_M on the boundary plane $D(\overline{M})$ (i.e., $\delta = 0$). Therefore, it is at least necessary for the appearance of a mass singularity of type I that V_M vanishes identically.

In order to find the restrictions the vanishing of V_M imposes on internal masses and external momenta, we note that M, being the reduced diagram $G/\overline{M}$, in general consists of several irreducible components M_α ($\alpha = 1, 2, \ldots$). By definition no pair of lines of different irreducible components can belong to the same loop of M. Thus, we find from (2.5) and (2.8) that

$$U^M = \prod_\alpha U^{M_\alpha} , \tag{3.12}$$

$$B_{ij}^M = \begin{cases} 0 \quad \text{if } i \in M_\alpha, \ j \in M_\beta, \text{ and } M_\alpha \neq M_\beta , & (3.13) \\ B_{ij}^{M_\alpha} \prod_{\beta \neq \alpha} U^{M_\beta} \quad \text{if } i,j \in M_\alpha . & (3.14) \end{cases}$$

Substituting these results in (3.8) we obtain

$$V_M = \sum_\alpha V_{M_\alpha} , \tag{3.15}$$

where

$$\begin{aligned} V_{M_\alpha} &= \sum_{i \in M_\alpha} z_i (m_i^2 - q_i^2) \\ &\quad + \frac{1}{U^{M_\alpha}} \sum_{i,j \in M_\alpha} z_i z_j B_{ij}^{M_\alpha} q_i \cdot q_j \end{aligned} \tag{3.16}$$

is the V function for M_α and $B_{ij}^{M_\alpha}$ are all nonvanishing.

Note that, if V_M were to vanish identically under the constraint

$$\sum_{i \in M} z_i = 1 ,$$

it must also vanish without it since V_M is homogeneous in z. Also, since Feynman parameters of different V_{M_α}'s are all independent, V_M vanishes identically if and only if each V_{M_α} vanishes identically.

In order to find the necessary and sufficient conditions for the identical vanishing of V_{M_α}, recall that the constant momenta q_i can be routed arbitrarily as far as they satisfy the conservation law (2.3). Each choice of routing defines a subdiagram T of M, which we shall call routing diagram, according to

$$\begin{aligned} q_i &\neq 0 \quad \text{for } i \in T , \\ q_i &= 0 \quad \text{for } i \in M - T . \end{aligned} \tag{3.17}$$

Dropping terms which vanish trivially, we can rewrite (3.16) as

$$V_{M_\alpha}=\sum_{i\in M_\alpha-T_\alpha} z_i m_i^2+\sum_{i\in T_\alpha} z_i\,(m_i^2-q_i^2)+\frac{1}{U^{M_\alpha}}\sum_{i,j\in T_\alpha} z_i z_j B_{ij}^{M_\alpha} q_i\cdot q_j\,, \qquad (3.18)$$

where $T_\alpha\equiv T\cap M_\alpha$.

It is important to note here that, among all possible routings of constant momenta, we can always find one such that the corresponding T (and hence each T_α) contains *no loop*. For such a routing diagram T, we can prove the following:

(i) $z_i z_j B_{ij}^{M_\alpha}$ $(i,j\in T_\alpha)$ are all independent (see Appendix for proof).[42]

(ii) $z_i U^{M_\alpha}$ $(i\in T_\alpha)$ is independent of $z_i z_j B_{ij}^{M_\alpha}$ $(i,j\in T_\alpha)$. (This is a consequence of the second Kirchhoff law [Ref. 29, formula (50)].)

(iii) $z_k U^{M_\alpha}$ $(k\in M_\alpha-T_\alpha)$ is independent of other terms in $U^{M_\alpha}V_{M_\alpha}$ since it is quadratic in z_k for $k\in M_\alpha-T_\alpha$ while no other term is.

It follows that all terms of (3.18) are mutually independent. Hence V_{M_α} vanishes identically if and only if

$$m_i=0 \quad\text{for } i\in M_\alpha-T_\alpha\,, \qquad (3.19a)$$

$$m_i^2-q_i^2=0 \quad\text{for } i\in T_\alpha\,, \qquad (3.19b)$$

$$q_i\cdot q_j=0 \quad\text{for } i,j\in T_\alpha\,. \qquad (3.19c)$$

Setting $i=j$ in (3.19c), we find $q_i^2=0$ for $i\in T_\alpha$. Thus, (3.19) can be simplified to[42]

$$m_i=0 \quad\text{for } i\in M_\alpha\,, \qquad (3.20a)$$

$$q_i\cdot q_j=0 \quad\text{for } i,j\in T_\alpha\,. \qquad (3.20b)$$

The result (3.20a) shows that our definition in fact gives a singularity at zero mass. In order to see the significance of (3.20b) more clearly, it is useful to rewrite it in terms of external momenta. For this purpose suppose all lines of G except those belonging to the irreducible component M_α have been shrunk to points. Then each vertex w of the resulting diagram M_α^* is composed of several vertices of G, and the sum of the external momenta

$$P_w^\alpha\equiv\sum_{v\in w}P_v \qquad (3.21)$$

can be regarded as "relative" external momentum entering M_α^* at the vertex w. Thus, (2.3) reduces to the momentum conservation law for M_α^*

$$\sum_{i\in M_\alpha^*}\epsilon_{wi}^\alpha q_i+P_w^\alpha=0,\quad w\in M_\alpha^*\,, \qquad (3.22)$$

where ϵ_{wi}^α is the incidence matrix defined on M_α^*. If we ignore the "external" lines, M_α^* clearly agrees with M_α. Therefore, in (3.22), the summation actually extends only over $i\in T_\alpha$, and we obtain

$$\sum_{i\in T_\alpha}\epsilon_{wi}^\alpha q_i+P_w^\alpha=0,\quad w\in M_\alpha^*\,. \qquad (3.23)$$

On the other hand, since T_α contains no loop, any line i of T_α divides it into three disjoint[43] parts T_α', $\{i\}$, T_α''.[44] The constant momentum q_i, $i\in T_\alpha$, can therefore be expressed as

$$q_i=\pm\sum_{w\in T_\alpha'}P_w^\alpha=\mp\sum_{w\in T_\alpha''}P_w^\alpha\,. \qquad (3.24)$$

The relations (3.23) and (3.24) show clearly that (3.20b) holds if and only if

$$P_w^\alpha\cdot P_{w'}^\alpha=0\,,\quad w,w'\in M_\alpha^*\,. \qquad (3.25)$$

We note that this result does not depend on the choice of T.

The results obtained thus far may be summarized in a slightly modified form as follows:

Theorem I. In the limit

$$m_i=0 \text{ for all } i\in M=G/\overline{M}\,,$$

where $\overline{M}$ is a proper subdiagram of G, the Feynman integral μ_G develops a type-I mass singularity associated with the reduced diagram M if and only if the external momenta satisfy the condition (3.25) for each reduced diagram $M_\alpha^*=M/(M-M_\alpha)$ $(\alpha=1,2,\ldots)$, where P_w^α are the external momenta relative to M_α^* defined by (3.21). A mass singularity then arises from the integration over the boundary plane

$$D(\overline{M})=\left\{(z_1,\ldots,z_N)\,\middle|\, z_i=0 \text{ for } i\in\overline{M},\ \sum_{j\in G-\overline{M}}z_j=1\right\}.$$

Let us now see how this theorem enables us to determine the strength of the leading mass singularity of the Feynman integral μ_G. Suppose $\overline{M}_0$ is the subdiagram of G consisting of all internal lines with nonzero masses. (In general, $\overline{M}_0$ contains some internal lines. However, in the case where the external momenta P_a go to ∞ as $\lambda P_a'$ with P_a' fixed and $\lambda\to\infty$, $\overline{M}_0$ is empty since all mass terms vanish as m_i/λ after scaling.) Let $\mathcal{S}_0^G$ be the set of all subdiagrams of G that contain $\overline{M}_0$. Then each element $\overline{M}$ of $\mathcal{S}_0^G$ determines a (mass-singular) configuration $C_{\overline{M}}$ of external momenta of the form (3.25). Let $\mathfrak{C}^G$ be the set of all possible configurations $C_{\overline{M}}$, $\overline{M}\in\mathcal{S}_0^G$. Since several $\overline{M}$ may correspond to the same configuration in general, let us decompose $\mathcal{S}_0^G$ as

$$\mathcal{S}_0^G=\bigoplus_{C\in\mathfrak{C}^G}\mathcal{S}_C^G\,, \qquad (3.26)$$

where $\mathcal{S}_C^G$ consists of all $\overline{M}\in\mathcal{S}_0^G$ such that $C_{\overline{M}}=C$.

Clearly, each $\mathcal{S}_C^G$ contains a minimal subdiagram $\overline{M}_m$.

From now on we shall concentrate on a particular configuration C and the leading mass singularity arising from the boundary plane $D(\overline{M})$, where $\overline{M}=\overline{M}_m$ is the minimal subdiagram in $\mathcal{S}_C^G$. (It will be seen in the following that mass singularities arising from nonminimal subdiagrams are not as strong as that of $\overline{M}_m$.) In the rest of this section we shall assume further that $[V_H]_X^1$ does not vanish for any choice of $\overline{X}$. (Otherwise the strength of the mass singularity will be enhanced. Such a case is treated in Sec. IV.) It is then reasonable to suppose that $\overline{X}$ is empty since nonempty $\overline{X}$ only cuts down the size of the available phase space, thus leading to weaker singularities. Whether this mass singularity does actually lead to a divergent integral or not can be determined by carrying out the integration over z_i, $i\in\overline{M}$, around $z_i=0$. In so doing, however, we run into two problems: (i) The integration requires the knowledge of the behavior of F_k on $D(\overline{M})$. (ii) The integral (2.4) may be divergent reflecting the presence of ultraviolet-divergent subdiagrams.

On the boundary plane $D(\overline{M})$, the zeroth $\overline{M}$ limit of F_k will behave as δ^{f_k} [δ defined in (3.1) with $\overline{X}$ empty], where f_k depends on the specific model or theory. Since we have not chosen any particular theory, the best we can do here is to leave f_k as a free (non-negative) parameter. In order to deal with the problem (ii), we must replace the formula (2.4) by one in which ultraviolet divergences are removed by renormalization. This will be one of the main subjects of subsequent articles. Until then we shall be concerned only with those integrals that are free from ultraviolet divergences. For simplicity we shall also assume that $\overline{M}$ has no loop[45] and M is irreducible.[46]

In order to perform integration over $D(\overline{M})$ we shall make use of the parametrization (3.10) noting, however, that, by assumption, $\overline{X}$ is empty and hence σ is absent. The numerator function F_k can be expanded around $\rho=0$ as

$$F_k=\rho^{f_k}F_k'+\cdots, \tag{3.27}$$

where F_k' depends on z_i', external momenta and internal masses. Making use of (3.3) with $\overline{X}$ and H replaced, respectively, by $\overline{M}$ and $M=G/\overline{M}$, recalling the assumption that $\overline{M}$ has no loop (i.e., $U^{\overline{M}}=1$), and also using (3.11), we obtain

$$\frac{F_k}{U^{2+k}(V-i\epsilon)^{N-2L-k}}=\frac{\rho^{f_k}F_k'}{(U'^M)^{2+k}\{(1-\rho)V_M'+\rho[V']_{\overline{M}}^1-i\epsilon\}^{N-2L-k}}+\cdots, \tag{3.28}$$

where the neglected terms are of higher order in ρ, and V_M', etc., are obtained from V_M, etc., by the replacement $z_i\to z_i'$ $(i\in G)$.

In order to examine the behavior of the integral (2.4) at the mass singularity, let us suppose that all m_i^2, $i\in M$, are of order δ while all scalar products $P_w^\alpha\cdot P_{w'}^\alpha$, $w,w'\in M_\alpha^*$, satisfy (3.25) exactly,[47] and consider the limit $\delta\to 0$. Then, V_M' is also of order δ for arbitrary values of z_i', $i\in M$, and serves as a cutoff for the singularity of the integrand (3.28) at $\delta=0$. Obviously the effect of this cutoff is very similar to that of restricting z_i, $i\in\overline{M}$, to the domain $z_i>\delta>0$. For this reason we have used here the same notation δ as in (3.1).

Substituting (3.28) into (2.4), and performing the ρ integration around $\rho=0$, we obtain finally[46]

$$\int dz_M'\int dz_{\overline{M}}'\frac{F_k'}{(U'^M)^{2+k}([V']_{\overline{M}}^1)^{f_k+N_{\overline{M}}}}\times\frac{1}{(V_M'-i\epsilon)^{N-2L-k-f_k-N_{\overline{M}}}}+\cdots, \tag{3.29}$$

where $N_{\overline{M}}$ denotes the number of internal lines of $\overline{M}$, and

$$dz_M'\equiv\delta\left(1-\sum_{i\in M}z_i'\right)\prod_{i\in M}dz_i',$$

etc.

Now suppose $F_k'=O(\delta^{g_k})$ for $\delta\to 0$. Then, since $[V']_{\overline{M}}^1$ does not vanish identically by assumption, the most singular part of (3.29) behaves in the limit $\delta\to 0$ as[46]

$$\delta^{-d_k}\times(\text{powers of }\ln\delta), \tag{3.30}$$

where d_k determines the strength of the mass singularity of the kth term of the Feynman integral μ_G:

$$d_k=N-2L-k-N_{\overline{M}}-f_k-g_k. \tag{3.31}$$

The degree of divergence of μ_G as a whole is given by

$$d=\max_k(d_k). \tag{3.32}$$

Usually the $k=0$ term is the most singular and d is equal to d_0. The mass singularity is divergent if $d\geq 0$.

IV. ENHANCEMENT OF TYPE-I MASS SINGULARITY

In the last section we obtained the conditions (3.20a) and (3.25) that are necessary and sufficient for the appearance of a type-I mass singularity. In estimating its strength we assumed that $[V_H]_X^1$ does not vanish for any $\overline{X}$. Under some circumstances, however, $[V_H]_X^1$ may vanish identically in addition to V_M. If this happens, the V function behaves as $O(\delta^2)$ in the limit (3.1), leading to a stronger mass singularity. In this section we shall study the mechanism of such an enhancement. We shall find the constraints, besides (3.20a) and (3.25), that the external momenta and internal masses must satisfy in order that $[V_H]_X^1$ vanishes identically for an *appropriate* choice of $\overline{X}$. We shall then examine how this affects the behavior of Feynman integrals at the mass singularity.

Let us first note that the reduced diagram $H = G/\overline{X}$, since it is *not irreducible* in general, can be decomposed into irreducible components H_A $(A=1,2,\ldots)$. Correspondingly we obtain

$$V_H = \sum_A V_{H_A} \tag{4.1}$$

in parallel with (3.15), where

$$V_{H_A} = \sum_{i\in H_A} z_i(m_i^2 - q_i^2) + \frac{1}{U^{H_A}} \sum_{i,j\in H_A} z_i z_j B_{ij}^{H_A} q_i \cdot q_j \,. \tag{4.2}$$

Taking the first X limit of (4.1) we find

$$[V_H]_X^1 = \sum_A [V_{H_A}]_{X_A}^1 \,, \tag{4.3}$$

where X_A are components of the reduced diagram $X = \overline{M}/\overline{X}\,(\subset H)$ defined by

$$X_A = X \cap H_A\,, \quad X = \bigoplus_A X_A \,.$$

To proceed further, let $X_{A,b}$ $(b=1,2,\ldots)$ be the irreducible components of X_A, and S_A the set of lines of X_A that connect them. Recalling that $z_i = O(\delta)$ for $i\in X_A$, we find from (2.5) and (2.8) that

$$
\begin{aligned}
[U^{H_A}]_{X_A} &= U^{H_A/X_A} \prod_b U^{X_{A,b}} \quad [=O(\delta^{L(X_A)})]\,,\\
[B_{ij}^{H_A}]_{X_A} &= U^{H_A/X_A} \prod_{b'\neq b} U^{X_{A,b'}} B_{ij}^{X_{A,b}} \quad [=O(\delta^{L(X_A)-1})] \quad \text{if } i,j\in X_{A,b}\,,\\
&= B_{ij}^{H_A/X_A} \prod_{b'} U^{X_{A,b'}} \quad [=O(\delta^{L(X_A)})] \quad \text{if } i,j\in H_A - X_A\,,\\
&= O(\delta^{L(X_A)}) \quad \text{otherwise,}
\end{aligned} \tag{4.4}
$$

where $L(X_A)$ denotes the number of independent loops in X_A. With the help of these formulas we obtain from (4.2) the following equation:

$$
\begin{aligned}
[V_{H_A}]_{X_A}^1 &= \sum_{i\in X_A} z_i(m_i^2 - q_i^2) + \sum_b \frac{1}{U^{X_{A,b}}} \sum_{i,j\in X_{A,b}} z_i z_j B_{ij}^{X_{A,b}} q_i\cdot q_j + 2\frac{1}{[U^{H_A}]_{X_A}} \sum_{i\in X_A}\sum_{j\in H_A - X_A} z_i z_j [B_{ij}^{H_A}]_{X_A}^0 q_i\cdot q_j\\
&\quad + \frac{1}{[U^{H_A}]_{X_A}} \sum_{i,j\in H_A - X_A} z_i z_j [B_{ij}^{H_A}]_{X_A}^1 q_i\cdot q_j - \frac{[U^{H_A}]_{X_A}^1}{[U^{H_A}]_{X_A}} \frac{1}{U^{H_A/X_A}} \sum_{i,j\in H_A/X_A} z_i z_j B_{ij}^{H_A/X_A} q_i\cdot q_j \,.
\end{aligned} \tag{4.5}
$$

Now H_A/X_A, which is a subdiagram of $M = H/X = G/\overline{M}$, consists of several irreducible components of M: $H_A/X_A = \bigoplus_{\alpha\in A} M_\alpha$. Thus, the last term of (4.5) can be written as

$$-\frac{[U^{H_A}]_{X_A}^1}{[U^{H_A}]_{X_A}} \sum_{\alpha\in A} \frac{1}{U^{M_\alpha}} \sum_{i,j\in M_\alpha} z_i z_j B_{ij}^{M_\alpha} q_i\cdot q_j \,. \tag{4.6}$$

For subsequent discussions, it is convenient to fix the routing of constant momenta within H and introduce the corresponding routing diagram T as in Sec. III. Instead of (3.17), however, we shall choose a routing such that both $T(\subset H)$ and $T/(T\cap X)$ $(\subset M)$ contain *no loop*, which is always possible. For such a T, let us define $T_A \equiv T\cap H_A$ and decompose it into two disjoint parts $T_A \equiv T_A^X \oplus T_A^M$, where $T_A^X \equiv T_A \cap X_A$ and $T_A^M \equiv T_A \cap (H_A - X_A)$. Finally, corresponding to

$$H_A/X_A = \bigoplus_{\alpha\in A} M_\alpha \,,$$

we write

$$T_A^M \equiv \bigoplus_{\alpha\in A} T_\alpha^M \,.$$

In the reduced diagram M, T_α^M characterizes the routing of constant momenta within the irreducible component M_α.

Since the reduced routing diagram $T/(T\cap X)$ contains no loop by definition, we can repeat the argument preceding formula (3.19) and find the condition for identical vanishing of V_M:

$$m_i=0 \quad \text{for } i\in M_\alpha\,, \qquad (\alpha\in A) \tag{4.7a}$$

$$q_i\cdot q_j=0 \quad \text{for } i,j\in T_\alpha^M\,. \tag{4.7b}$$

Comparing (4.6) and (4.7b), we see that the last term of (4.5) vanishes automatically whenever V_M does. For our choice of constant momenta, (4.5) thus becomes

$$[V_{H_A}]^1_{X_A}=\sum_{i\in X_A-T_A^X} z_i m_i^2+\sum_{i\in T_A^X} z_i(m_i^2-q_i^2)+\sum_b \frac{1}{U^{X_{A,b}}}\sum_{i,j\in X_{A,b}\cap T_A^X} z_i z_j B_{ij}^{X_{A,b}} q_i\cdot q_j$$
$$+\frac{2}{[U^{H_A}]_{X_A}}\sum_{i\in T_A^X}\sum_{j\in T_A^M} z_i z_j [B_{ij}^{H_A}]_{X_A} q_i\cdot q_j+\frac{1}{[U^{H_A}]_{X_A}}\sum_{i,j\in T_A^M} z_i z_j [B_{ij}^{H_A}]^1_{X_A} q_i\cdot q_j\,. \tag{4.8}$$

As is shown in the Appendix, all terms in (4.8) are mutually independent. Therefore, $[V_{H_A}]^1_{X_A}$ vanishes identically if and only if[48]

$$m_i=0 \quad \text{for } i\in X_A-S_A\cap T_A^X\,, \tag{4.9a}$$

$$m_i^2=q_i^2 \quad \text{for } i\in S_A\cap T_A^X\,, \tag{4.9b}$$

$$q_i\cdot q_j=0 \quad \text{for } i,j\in X_{A,b}\cap T_A^X \ (b=1,2,\ldots)\,, \tag{4.9c}$$

$$q_j\cdot q_k=0 \quad \text{for } j\in T_A^X,\ k\in T_A^M\,, \tag{4.9d}$$

$$q_k\cdot q_l=0 \quad \text{for } k,\,l\in T_A^M\,. \tag{4.9e}$$

We note that (4.9a) and (4.9b) are consequences of some simplification in which (4.9c) plays a role similar to that of (3.19c) in reducing (3.19) to (3.20).

It may seem at first sight that (4.9a)–(4.9e) are highly dependent on the choice of T. Of course, this should not be the case since $[V_{H_A}]^1_{X_A}$ itself is independent of T. This circumstance may be seen most clearly if we express (4.9b)–(4.9e) in terms of external momenta. Instead of treating the general case, however, we shall carry it out only for a particular class of mass singularities with enhancement, called an *infrared-type mass singularity*, which is characterized by $m_i\neq 0$ for *all* $i\in X_A$ $(A=1,2,\ldots)$.

For such a singularity X_A must satisfy the following properties:

(a) Each connected component $X_{A,c}$ $(c=1,2,\ldots)$ of X_A is *minimal* with respect to its external vertices. By this we mean (i) $X_{A,c}$ contains no loop, (ii) any pair of external vertices of $X_{A,c}$ are connected by a set of lines in $X_{A,c}$, and (iii) no proper subdiagram of $X_{A,c}$ satisfies the property (ii). [Note that $X_{A,c}$ is different from the irreducible component $X_{A,b}$ introduced in (4.4).]

(b) Each $X_{A,c}$ possesses at least *two* external vertices or else consists of only one vertex (i.e., no internal line).

Proof of (a). Since $m_i\neq 0$ for all $i\in X_A$ by assumption, the set $X_A-S_A\cap T_A^X$ in (4.9a) is empty. This means $X_A=S_A=T_A^X$ which is equivalent to (i). The property (ii) follows from the definition of $X_{A,c}$. Next, suppose that $X_{A,c}-\{i\}$ is also minimal for some $i\in X_A$. Since $X_{A,c}$ has no loop, $X_{A,c}-\{i\}$ consists of two disconnected parts $X'_{A,c}$ and $X''_{A,c}$. By the assumed minimality of $X_{A,c}-\{i\}$, all external vertices of $X_{A,c}$ are connected by lines of $X_{A,c}-\{i\}$. Hence, they all belong to either $X'_{A,c}$ or $X''_{A,c}$. Therefore, q_i is either equal to 0, or a linear combination of q_j, $j\in T_A^M$. In either case, we obtain $m_i^2=q_i^2=0$ by (4.9b), (4.9e) and the property $X_A=T_A^X$ shown above, which contradicts the definition of the infrared-type mass singularity.

Proof of (b). Suppose $X_{A,c}$ consists of a number of internal lines, but contains at most one external vertex. Then for any line $i\in X_{A,c}$, q_i is either equal to 0 or a linear combination of q_j, $j\in T_A^M$, which leads to a contradiction as we have shown above.

We are now ready to rewrite (4.9) in terms of external momenta. Since $X_A=S_A=T_A^X$ according to the proof of (a), we can simplify the constraint on the external momenta as

$$q_i^2=m_i^2 \quad \text{for } i\in T_A^X\,, \tag{4.10a}$$

$$q_i\cdot q_k=0 \quad \text{for } i\in T_A^X,\ k\in T_A^M\,, \tag{4.10b}$$

$$q_k\cdot q_l=0 \quad \text{for } k,\,l\in T_A^M\,. \tag{4.10c}$$

For q_k $(k\in T_A^M)$, we obtain from (2.3) the conservation law

$$\sum_{k\in T_A^M}\epsilon_{wk}^A q_k+P_w^A=0\,, \tag{4.11}$$

where P_w^A is the "relative" external momentum at the vertex w of the reduced diagram M_A^* obtained from H by shrinking X_A and the irreducible components other than H_A to points. On the other hand, since T_A^M contains no loop, q_i $(i\in T_A^M)$ can be expressed as

$$q_i = \pm \sum_{w\in T_A^{M'}} P_w^A$$
$$= \mp \sum_{w\in T_A^{M''}} P_w^A \quad \text{for } i\in T_A^M, \tag{4.12}$$

as in (3.24), where $T_A^M \equiv T_A^{M'} \oplus \{i\} \oplus T_A^{M''}$.[49]

For q_i $(i\in X_{A,c})$, noting that $X_A = T_A^X$, we can write the conservation law as

$$\sum_{i\in X_{A,c}} \epsilon_{vi} q_i + \sum_{k\in T_A^M} \epsilon_{vk} q_k + P_v^{A,c} = 0 \quad \text{for } v\in X_{A,c}, \tag{4.13}$$

where $P_v^{A,c}$ denotes the external momentum at the vertex $v\in X_{A,c}$. Using the decomposition $X_{A,c} = X'_{A,c} \oplus \{i\} \oplus X''_{A,c}$, we can also write q_i $(i\in X_{A,c})$ explicitly as

$$q_i = \pm \sum_{v\in X'_{A,c}} P_v^{A,c} \pm \sum_{v\in X'_{A,c}} \sum_{k\in T_A^M} \epsilon_{vk} q_k$$
$$= \mp \sum_{v\in X''_{A,c}} P_v^{A,c} \mp \sum_{v\in X''_{A,c}} \sum_{k\in T_A^M} \epsilon_{vk} q_k . \tag{4.14}$$

It is now easy to verify that (4.10a)–(4.10c) are equivalent to the following three conditions[50]:

$$m_i^2 = \Big(\sum_{v\in X'_{A,c}} P_v^{A,c} \Big)^2$$
$$= \Big(\sum_{v\in X''_{A,c}} P_v^{A,c} \Big)^2 \quad \text{for } i\in X_{A,c}\ (c=1,2,\dots), \tag{4.15a}$$

$$P_v^{A,c} \cdot P_w^A = 0 \quad \text{for } v\in X_{A,c},\ w\in M_A^*, \tag{4.15b}$$

$$P_w^A \cdot P_{w'}^A = 0 \quad \text{for } w, w'\in M_A^*. \tag{4.15c}$$

As expected, they depend only on the topological structure of the diagram. Also, note the *on-mass-shell* conditions (4.15a) which are characteristic of infrared-type mass singularities.

Let us summarize the results obtained above as

Theorem II. A type-I mass singularity of the Feynman integral μ_G which arises under the conditions stated in theorem I receives an enhancement [in the sense that, for an appropriate choice of the subdiagram $\overline{X}$ of $\overline{M}$, the V function vanishes as $O(\delta^2)$ in the limit (3.1)] if and only if external momenta and internal masses satisfy the additional condition (4.9).[48] For infrared-type mass singularities [i.e., $m_i \neq 0$ for *all* $i\in X_A$ $(A = 1,2,\dots)$], this condition reduces to (4.15)[50] which is written explicitly in terms of external momenta.

Let us now see how the condition (4.15) works by applying it to some simple diagrams. It will be found that the infrared-type mass singularity is indeed a natural generalization of the usual infrared singularities.

For simplicity, suppose the reduced diagram $H = G/\overline{X}$ *itself* is irreducible and external vertices of H are *all* contained in $X = \overline{M}/\overline{X}$. Furthermore, we shall consider only the cases where X consists of at most two connected components.

(1) Connected X. The reduced diagram $M = H/X = G/\overline{M}$ has only *one* "external" vertex in this case. Since the sum of the external momenta at this vertex vanishes by over-all momentum conservation, a mass singularity of type I arises if and only if

$$m_i = 0 \quad \text{for } i\in M.$$

Let κ be the number of external vertices in X. According to the property b mentioned above, it is necessary for the presence of enhancement that $\kappa \geq 2$.

(i) $\kappa = 2$. The minimality requires that X is a continuous set of lines connecting the two external vertices [Fig. 1(a)]. The condition (4.15a) gives

$$P^2 = m_i^2 \quad \text{for } i\in X.$$

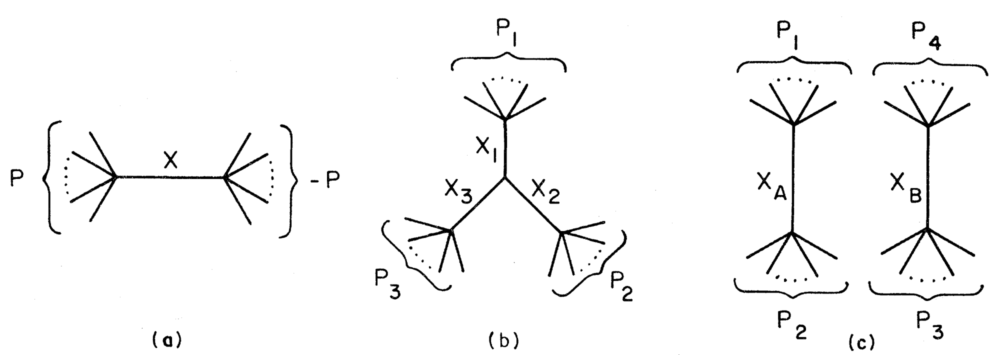

FIG. 1. General structure of connected subdiagram X with (a) two or (b) three external vertices satisfying the minimality requirement. (c) is an example where X consists of two separate connected components X_A and X_B.

If only *one* external line is attached to each external vertex, this represents the infrared singularity of the self-energy diagram.

(ii) $\kappa=3$. The general structrue of X that satisfies the minimality requirement is depicted in Fig. 1(b). From (4.15a) we find

$$P_\alpha^2=m_i^2 \quad \text{for } i\in X_\alpha\ (\alpha=1,2,3)$$

as the condition for enhancement. If X_1 is empty (i.e., no constraint on P_1) and only one external line is attached to the endpoint of X_2 and of X_3 this corresponds to the usual infrared singularity of the form factor.

(2) X consisting of two connected components X_A and X_B. The reduced diagram M now has two external vertices with momentum P and $-P$. We find from (3.20a) and (3.25) that a mass singularity appears if

$$m_i=0 \quad \text{for } i\in M \text{ and } P^2=0.$$

Let us assume that both X_A and X_B have two external vertices. By minimality, X_A and X_B have the structure shown in Fig. 1(c), where $P=P_1+P_2=-(P_3+P_4)$. We then find from (4.15) that

$$P_1^2=P_2^2=m_i^2 \quad \text{for } i\in X_A,$$

$$P_3^2=P_4^2=m_i^2 \quad \text{for } i\in X_B.$$

If only *one* external line is attached to each external vertex, this is nothing but the well-known infrared singularity of the forward Coulomb scattering amplitude of fourth or higher order.

The behavior of the Feynman integral at the mass singularity can be found by carrying out necessary integrations. We make the scale transformation (3.2) and suppose that F_k is expanded around $\rho=\sigma=0$ as

$$F_k=\rho^{f_k'}\sigma^{f_k''}F_k''+\cdots. \tag{4.16}$$

If we make, as in Sec. III, the simplifying assumption that $\overline{X}$ and $X=\overline{M}/\overline{X}$ contain no loop and that $H=G/\overline{X}$ and $M=H/X$ are irreducible,[51] the most singular part of the kth term of (2.4) is given by

$$\int dz_M' \int dz_X' \int dz_{\overline{X}}' \int \rho^{N_X-1}d\rho \int \sigma^{N_{\overline{X}}-1}d\sigma \, \frac{\rho^{f_k'}\sigma^{f_k''}F_k''}{(U'^M)^{2+k}\{(1-\rho)V_M'+\rho[V_H']_X^1+\sigma V_1'+\rho^2V_2'-i\epsilon\}^{N-2L-k}}+\cdots, \tag{4.17}$$

where N_X and $N_{\overline{X}}$ denote the number of internal lines of the diagrams $X=\overline{M}/\overline{X}$ and $\overline{X}$, V_1' and V_2' are terms of order unity and do not vanish for general values of z', and

$$dz_M'\equiv\delta\left(1-\sum_{i\in M}z_i'\right)\prod_{i\in M}dz_i', \text{ etc.}$$

To find out the strength of the mass singularity, we assume as in Sec. III that $V_M'=O(\delta)$ for arbitrary values of z_i', $i\in M$. In addition, we assume that $[V_H']_X^1$ is of order $\delta^{1/2}$ or smaller for arbitrary z_i', $i\in H$, so that $\rho[V_H']_X^1$ can be disregarded in comparison with V_M' for $\rho<\delta^{1/2}$. [Note that this δ is not equivalent to the δ in (3.1). The latter must be replaced by $\delta^{1/2}$ for proper correspondence.] Then the mass singularity of (4.17) at $\rho=\sigma=0$ is effectively regularized by V_M'. Carrying out the σ and ρ integrations around $\sigma=0$ and $\rho=0$ successively, we find[51]

$$\int dz_M' \int dz_X' \int dz_{\overline{X}}' \, \frac{F_k''}{(U'^M)^{2+k}} \, \frac{1}{(V_1')^{f_k''+N_{\overline{X}}}(V_2')^{(f_k'+N_X)/2}(V_M'-i\epsilon)^\alpha}+\cdots, \tag{4.18}$$

where

$$\alpha=N-2L-k-N_{\overline{X}}-\tfrac{1}{2}N_X-f_k''-\tfrac{1}{2}f_k'. \tag{4.19}$$

If we assume further that $F_k''=O(\delta^{g_k'})$ for $\delta\to 0$, the integral (4.18) behaves as

$$\delta^{-d_k'}\times(\text{powers of } \ln\delta), \tag{4.20}$$

where

$$d_k'=N-2L-k-N_{\overline{X}}-f_k''-\tfrac{1}{2}(N_X+f_k')-g_k'. \tag{4.21}$$

The Feynman integral (2.4) as a whole develops a divergent singularity if[51]

$$d'=\max_k(d_k')\geq 0. \tag{4.22}$$

Let us compare (3.32) and (4.22) for a theory for which $F(P,p)\equiv 1$ (i.e., f_k, g_k, etc. $=0$). Using the relation $N_{\overline{M}}=N_{\overline{X}}+N_X$, we find

$$d=N-2L-N_{\overline{M}}, \tag{4.23}$$

$$d'=N-2L-N_{\overline{M}}+\tfrac{1}{2}N_X. \tag{4.24}$$

These formulas show clearly that a convergent mass singularity ($d<0$) may become divergent ($d'\geq 0$) if sufficiently enhanced.

V. TYPE-II MASS SINGULARITY

Thus far we have examined the mass singularity of type I which arises from some boundary plane of the hypercontour of integration. Let us now turn our attention to mass singularities of type II which are the simplest mass singularities containing pinches. Although a complete analysis of these singularities is still beyond our reach, we have at least found an interesting interplay between zero mass and threshold singularities.

According to the definition II (see Sec. II), it is necessary for the appearance of a type-II mass singularity that, for arbitrary values $z_k \in G - \overline{M}$, the equations[52]

$$\partial V/\partial z_j |_{z_i=0\ (i\in \overline{M}')} = m_j^2 - (Q_j'^K)^2 = 0\,, \qquad j\in K \equiv G/\overline{M}'\,, \tag{5.1}$$

have a unique solution[37] $z_l \in \overline{M} - \overline{M}'$ which does not depend on $z_k \in G - \overline{M}$ except through the normalization

$$\sum_{i\in G} z_i = 1\,.$$

[The first equality in (5.1) follows from (2.7), (2.9), and formulas analogous to (3.3)–(3.5).] In general, K consists of several irreducible components K_A $(A=1,2,\ldots)$. Hence, by (3.12)–(3.14), (5.1) is equivalent to

$$\partial V_{K_A}/\partial z_j = m_j^2 - (Q_j'^{K_A})^2 = 0 \qquad \text{for } j\in K_A\ (A=1,2,\ldots)\,, \tag{5.2}$$

V_{K_A} being the V function for the irreducible component K_A. This leads to

$$V_{K_A} = 0 \quad (A=1,2,\ldots)\,, \tag{5.3}$$

because of the identity

$$V_{K_A} = \sum_{j\in K_A} z_j \partial V_{K_A}/\partial z_j\,.$$

Let us first see how the values of $z_j \in \overline{M} - \overline{M}'$ at the singularity are determined. For this purpose recall that (5.2) must hold for arbitrary values of $z_k \in K_A - Y_A$ by definition where $Y_A \equiv Y \cap K_A$ and $Y \equiv \overline{M}/\overline{M}'$. Taking the limit $z_k \to 0$ (for all $k\in K_A - Y_A$) in (5.2), we therefore obtain

$$\partial V_{\tilde{M}_A}/\partial z_j = m_j^2 - (Q_j'^{\tilde{M}_A})^2 = 0 \quad \text{for } j\in \tilde{M}_A\,, \tag{5.4}$$

where $\tilde{M}_A \equiv K_A/(K_A - Y_A)$ is the reduced diagram obtained from K_A by shrinking the lines of $K_A - Y_A$ to points, and $V_{\tilde{M}_A}$ denotes the V function for $\tilde{M}_A$. This is nothing but the ordinary threshold equation for $\tilde{M}_A$. If $\tilde{M}_A$ consists of several irreducible components $\tilde{M}_{A,\gamma}$ $(\gamma=1,2,\ldots)$, (5.4) decomposes into the threshold equations for each irreducible component:

$$\partial V_{\tilde{M}_{A,\gamma}}/\partial z_j = m_j^2 - (Q_j'^{\tilde{M}_{A,\gamma}})^2 = 0 \quad \text{for } j\in \tilde{M}_{A,\gamma}\,. \tag{5.5}$$

Let us assume in the following that all irreducible components $\tilde{M}_{A,\gamma}$ $(A=1,2,\ldots;\ \gamma=1,2,\ldots)$ have *nontrivial leading*[1] threshold singularities.[53]

Then (5.4) completely determines the ratio of Feynman parameters within each $\tilde{M}_{A,\gamma}$. The over-all normalization factor

$$\lambda_{A,\gamma} \equiv \sum_{i\in \tilde{M}_{A,\gamma}} z_i$$

is left undetermined, however, since (5.5) are homogeneous equations of degree zero. Since the set of internal lines of the diagram $\oplus_A \oplus_\gamma \tilde{M}_{A,\gamma}$ is identical with that of $\overline{M} - \overline{M}'$, (5.5) determines the values of all $z_i \in \overline{M} - \overline{M}'$ at the singularity except for normalization factors $\lambda_{A,\gamma}$ $(A=1,2,\ldots;\ \gamma=1,2,\ldots)$.

The next problem is to find the conditions that must be satisfied by the internal masses and external momenta. Unfortunately, because of its complexity, we have been able to solve it only partially; we have deduced necessary conditions but not sufficient ones.

The assumption that each $\tilde{M}_{A,\gamma}$ has the leading threshold singularity requires that the "relative" external momenta of $\tilde{M}_{A,\gamma}$ satisfy corresponding threshold relations. They are determined by solving (5.5) explicitly. If we substitute these threshold relations and the values of Feynman parameters of $\tilde{M}_{A,\gamma}$ $(\gamma=1,2,\ldots)$ determined by (5.5) into (5.3), it must vanish identically with respect to $z_k \in K_A - Y_A$. Since the over-all normalization factors $\lambda_{A,\gamma}$ $(\gamma=1,2,\ldots)$ are left undetermined as free parameters, each term in the expansion of V_{K_A} for small $\lambda_{A,\gamma}$ $(\gamma=1,2,\ldots)$ must vanish. For simplicity let us write this expansion as if it were an expansion with respect to the sum $\lambda_A \equiv \sum_\gamma \lambda_{A,\gamma}$. Then, in parallel with (3.11), we find

$$V_{K_A} = (1-\lambda_A)V_{K_A/Y_A} + \lambda_A[V_{K_A}]^1_{Y_A} + O(\lambda_A^2)\,, \tag{5.6}$$

where, on the right-hand side, the scale transformation

$$z_i \to \lambda_A z_i \quad \text{for } i\in Y_A\,, \quad \sum_{i\in Y_A} z_i = 1\,,$$

$$z_j \to (1-\lambda_A) z_j \quad \text{for } j\in K_A - Y_A\,, \quad \sum_{j\in K_A - Y_A} z_j = 1\,,$$

is understood.

Clearly the condition for the identical vanishing of V_{K_A/Y_A} is the same as that for the appearance of a type-I mass singularity at K_A/Y_A. Referring to (3.20a) and (3.25), we therefore find

$$m_i = 0 \quad \text{for } i\in M_\alpha \tag{5.7a}$$

$$P_w^\alpha \cdot P_{w'}^\alpha = 0 \quad \text{for } w, w'\in M_\alpha \quad (\alpha=1,2,\ldots)\,, \tag{5.7b}$$

where M_α $(\alpha=1,2,\ldots)$ are the irreducible components of K_A/Y_A and P_w^α denotes the "relative" external momenta of M_α defined as in Sec. III.

If we choose a routing of constant momenta within K such that the corresponding routing subdiagram T of K (i.e., $q_i \neq 0$ for $i \in T$ but $=0$ for $i \in K-T$) and the reduced diagram $T/(T \cap Y)$ both contain no loop, the explicit expression for $[V_{K_A}]_{Y_A}^1$ can be directly obtained from (4.8) and (5.7),

$$[V_{K_A}]_{Y_A}^1 = \sum_{i \in S_A - S_A \cap T_A^Y} z_i m_i^2 + \sum_{i \in S_A \cap T_A^Y} z_i (m_i^2 - q_i^2) + \sum_b V_{Y_{A,b}} + \frac{2}{[U^{K_A}]_{Y_A}} \sum_{i \in T_A^Y} \sum_{j \in T_A^M} z_i z_j [B_{ij}^{K_A}]_{Y_A} q_i \cdot q_j + \frac{1}{[U^{K_A}]_{Y_A}} \sum_{i,j \in T_A^M} z_i z_j [B_{ij}^{K_A}]_{Y_A}^1 q_i \cdot q_j , \tag{5.8}$$

where $Y_{A,b}$ $(b=1,2,\ldots)$ are the irreducible components of Y_A, $V_{Y_{A,b}}$ are the corresponding V functions, S_A denotes the set of lines of Y_A connecting $Y_{A,b}$ $(b=1,2,\ldots)$, $T_A^Y \equiv T \cap Y_A$, and $T_A^M \equiv T \cap (K_A - Y_A)$. Although (5.8) has the same form as (4.8), there is one important difference. While (4.8) must vanish for arbitrary variation of $z_k \in K_A - Y_A$ *and* $z_i \in Y_A$, the latter set of Feynman parameters are fixed in (5.8) by the threshold equations (5.5). Thus (5.8) vanishes for arbitrary $z_k \in K_A - Y_A$ if and only if[54]

$$\sum_{i \in S_A - S_A \cap T_A^Y} z_i m_i^2 + \sum_{i \in S_A \cap T_A^Y} z_i (m_i^2 - q_i^2) + \sum_b V_{Y_{A,b}} = 0 , \tag{5.9a}$$

$$q_i \cdot q_j = 0 \quad \text{for } i \in T_A^Y,\ j \in T_A^M , \tag{5.9b}$$

$$q_j \cdot q_k = 0 \quad \text{for } j,k \in T_A^M . \tag{5.9c}$$

The first condition (5.9a) can be sharpened with the help of (5.2). In fact, if we substitute (5.9b) and (5.9c) into (5.2) for $j \in Y_A$ and expand the result with respect to λ_A, the vanishing of the leading term in this expansion gives

$$m_i^2 = 0 \quad \text{for } i \in S_A - S_A \cap T_A^Y , \tag{5.10a}$$

$$m_i^2 - q_i^2 = 0 \quad \text{for } i \in S_A \cap T_A^Y , \tag{5.10b}$$

$$\partial V_{Y_{A,b}} / \partial z_i = 0 \quad \text{for } i \in Y_{A,b} \quad (b=1,2,\ldots) . \tag{5.10c}$$

If (5.9b), (5.9c) and (5.10a), (5.10b) are substituted back into V_{K_A}, it becomes

$$V_{K_A} = \sum_b \sum_{i \in Y_{A,b}} z_i (m_i^2 - q_i^2) + \frac{1}{U^{K_A}} \sum_{i,j \in T_A^Y} z_i z_j B_{ij}^{K_A} q_i \cdot q_j . \tag{5.11}$$

It is quite difficult to determine whether or not (5.5) and (5.10c) are sufficient for making (5.11) and its derivative (5.2) vanish identically. Since we have not been successful in this attempt, we have to be satisfied with the conditions obtained thus far. Although they represent only necessary conditions, one can learn several interesting aspects of type-II mass singularities from them:

(1) The nature of type-II mass singularities as a mixture (or a "product") of an end-surface singularity and a pinch singularity is manifest in (5.7) and (5.5). The former indicates that they are a generalization of type-I mass singularities studied in Sec. III, which are purely end-surface-type singularities. The latter says that they can also be viewed as a generalization of threshold singularities which are essentially pinch-type. The simultaneous occurrence of these two different types of singularities is made possible by the "complementarity" of the diagram $M \equiv K/Y$ (type-I mass singularity) and the diagram $\tilde{M} \equiv K/(K-Y)$ (threshold singularity).

(2) It is interesting to compare (5.9b), (5.9c) and (5.10) with (4.9). We find that, apart from notations, they agree with each other if and only if X_A and Y_A $(A=1,2,\ldots)$ contain no loop. This implies that a type-I mass singularity with enhancement corresponding to no-loop X_A $(A=1, 2,\ldots)$ turns into a type-II mass singularity if the external momenta satisfy threshold relations for $\tilde{M} \equiv K/(K-Y)$. In the Feynman parameter space, we may interpret this as follows: For a type-I mass singularity with enhancement, the hypersurface $S_t(\underline{m}, \underline{P})$ is located on some boundary

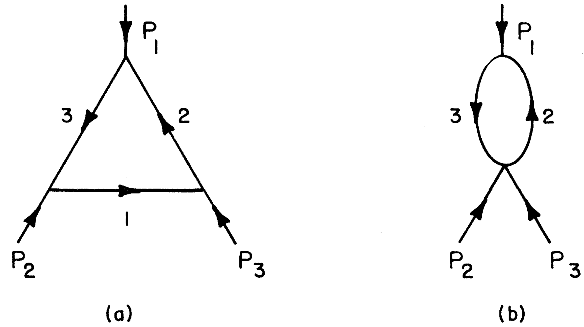

FIG. 2. (a) Feynman diagram with three internal lines. (b) Reduced diagram obtained from (a) by contracting the internal line 1.

plane of the hypercontour of integration I [see (2.10) for definitions of S_t and I]. If X_A ($A=1, 2, \ldots$) contain no loop and the external momenta satisfy threshold relations for $\tilde{M}$, then $S_t(\underline{m}, \underline{P})$ "extends" itself into the interior of I in the manner specified by (5.5).

Finally, let us illustrate the somewhat abstract discussions given above by simple examples.

Consider first the triangular diagram shown in Fig. 2(a). Suppose $\bar{M}=\{\text{lines 2 and 3}\}$ and $\bar{M}'$ = empty. If we choose a routing of constant momenta such that $q_1=0$, $q_2=P_3$, and $q_3=-P_2$, we find from (5.7), (5.9), and (5.10) that

$$m_1=0,\ P_2{}^2=m_3{}^2,\ P_3{}^2=m_2{}^2\ . \tag{5.11}$$

On the other hand, the reduce diagram $\tilde{M}$ looks like Fig. 2(b) and hence (5.5) is solved as[55]

$$\begin{aligned}&(P_2+P_3)^2=P_1{}^2=(m_2+m_3)^2\ ,\\&m_2z_2-m_3z_3=0\ .\end{aligned} \tag{5.12}$$

Clearly this mass singularity may be interpreted as a product of an infrared singularity (5.11) and a threshold singularity (5.12).

As the second example, let us consider a nonplanar diagram [Fig. 3(a)]. If we tentatively choose $\bar{M}=\{\text{lines 1, 2, 3, and 4}\}$ and $\bar{M}'$= empty, (5.7), (5.9), and (5.10) give

$$m_5=m_6=0\ , \tag{5.13a}$$

$$P_2{}^2=m_1{}^2=m_2{}^2,\ P_3{}^2=m_3{}^2=m_4{}^2\ . \tag{5.13b}$$

The threshold of the reduced diagram $\tilde{M}$ shown in Fig. 3(b) is determined by its dual diagram [Fig. 3(c)], which is compatible with (5.13b) only if $z_1=z_4=0$. Thus, a type-II mass singularity cannot exist for the present choice of $\bar{M}'$. Instead, we have to choose $\bar{M}'=\{\text{lines 1 and 4}\}$. Then a type-II mass singularity arises if $P_1{}^2=(m_2+m_3)^2$. This example shows that (5.5), (5.7), (5.9), and (5.10) may in some cases be overdetermined, and the condition for the presence of a leading threshold singularity for $\tilde{M}_{A,\gamma}$ may not be compatible with other conditions.

VI. CONCLUDING REMARKS

In this article we have investigated the condition for the presence of mass singularities of types I and II in arbitrary irreducible Feynman amplitudes starting from the definitions given in Sec. II. For the simplest of type-I mass singularities, which lack any enhancement mechanism, we have explicitly worked out the necessary and sufficient conditions that the external momenta and internal masses must satisfy. This result is summarized as theorem I. Mass singularities of type I whose strengths are enhanced by further restrictions on the external momenta have also been studied, although we have carried out the analysis completely only for the singularities of infrared type (theorem II). The main features of type-II mass singularities are examined in Sec. V. It is in principle possible, if it ever becomes necessary, to analyze more complicated cases by our method, although it will certainly be extremely tedious.

In general, a configuration of external momenta is called exceptional if the Feynman amplitude develops a divergent singularity for that configuration.[56] Such a definition is not very practical, however, unless it is implemented by an explicit rule that enables us to determine the nature of the divergence. This is a particularly troublesome problem for Minkowskian external momenta. It is one of the utilities of our results that they serve to clarify this point at least for type-I mass singularities. In fact, theorems I and II indicate that the external momenta are exceptional if they satisfy (3.25). Note, however, that satisfying (3.25) by itself is not a sufficient criterion for exceptional momenta. For instance, if (3.25) is satisfied trivially (i.e., if none of M_α^* has more than one relative external vertices), μ_G

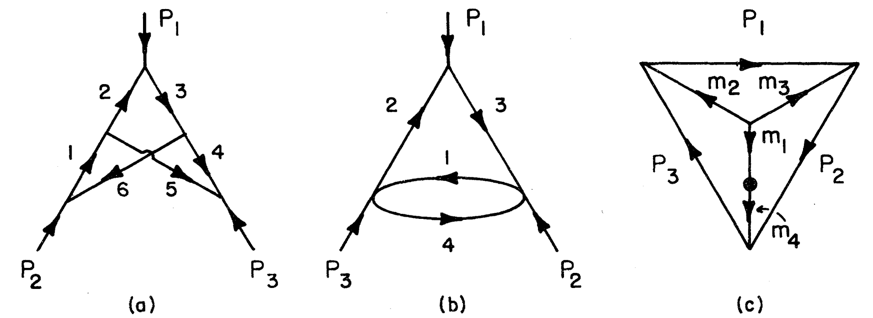

FIG. 3. (a) Feynman diagram with six internal lines. (b) Reduced diagram obtained from (a) by contracting the internal lines 5 and 6. (c) Diagram dual to the reduced diagram (b).

usually does not diverge and hence such momenta are nonexceptional.[57] They become exceptional only if the corresponding singularity is enhanced sufficiently by the additional constraint (4.9). On the other hand, μ_G diverges in general if the external momenta satisfy (3.25) nontrivially (i.e., at least one of M_α^* contains two or more relative external vertices). Such configurations are hence exceptional in general. Of course, there exist yet other types of exceptional configurations corresponding to type-II or more complicated mass singularities.[58] At present, we are unable to determine them completely.

In order to work out these considerations explicitly, it is necessary to obtain a general rule for determining the degree of zero-mass divergences of Feynman integrals. In view of the fact that the strength of the mass singularity (in particular, the power of logarithmic factors) is affected in general by the renormalization of ultraviolet divergences, we postpone the full treatment of power-counting rules to subsequent articles where technical problems associated with renormalization are analyzed in detail. The power-counting rules derived in Secs. III and IV are applicable to a restricted class of diagrams only.

In spite of the limited applicability, it will be instructive to see how these rules actually work since their property will be basically not too dissimilar to that of the general ones. Let us therefore choose as an example the large-momentum behavior of Feynman integrals, one of the physical problems we are most interested in, and examine how it is determined by the mass-singularity analysis.

To avoid unnecessary complications, let us choose the ϕ^4 theory and consider a diagram G containing no superficially divergent subdiagrams (including G itself). The Feynman integral corresponding to this diagram is given by

$$\mu_G(\underline{m}, \underline{P}) = \alpha_G \int \frac{\delta\left(1 - \sum_{i \in G} z_i\right) \prod_{i \in G} dz_i}{U^2[V(z, m, q) - i\epsilon]^{N-2L}} , \quad (6.1)$$

where U and V are defined by (2.5) and (2.7), and α_G consists of numerical and coupling-constant factors.

We examine the behavior of $\mu_G(\underline{m}, \lambda\underline{P})$ in the limit $\lambda \to \infty$ where λ is a scale parameter multiplied into all external momenta. The large-momentum limit of this type can be transformed into a mass-singularity problem as is seen from

$$\mu_G(\underline{m}, \lambda\underline{P}) = \alpha_G \lambda^{-2(N-2L)} \int \frac{\delta\left(1 - \sum_{i \in G} z_i\right) \prod_{i \in G} dz_i}{U^2[V(z, m/\lambda, q) - i\epsilon]^{N-2L}} . \quad (6.2)$$

The behavior of $\mu_G(\underline{m}, \lambda\underline{P})$ for large λ is thus controlled by the singularity of the integral of (6.2) in the "zero-mass limit" $m_i/\lambda \to 0$ $(i \in G)$.

In general, the integral (6.2) can develop various types of mass singularities ranging from types I or II to more complicated ones such as Landshoff-type pinches. Instead of exhausting all possiblities, which surely require a quite a complicated analysis, let us examine one simple case. We suppose that the external momenta $\underline{P}$ are fixed to some configuration C of the form (3.25) and no other squares of linear combinations of $\underline{P}$ vanish. In this case, the only singularity that can emerge is the type-I mass singularity without enhancement corresponding to the configuration C. The singular behavior then arises from the integration over the region defined by $z_i = 0$, $i \in \overline{M}$, and

$$\sum_{i \in G - \overline{M}} z_i = 1 ,$$

where $\overline{M}$ denotes a subdiagram in the set $\mathcal{S}_C^G$ introduced in (3.26). According to (3.30), the leading contribution of this singularity to the integral behaves as

$$\lambda^{2d_{\overline{M}}} \times (\text{powers of } \ln\lambda) , \quad (6.3)$$

$d_{\overline{M}}$ being given by

$$d_{\overline{M}} = N - 2L - (N_{\overline{M}} - 2L_{\overline{M}}) . \quad (6.4)$$

[$L_{\overline{M}}$ denotes the numbers of loops in $\overline{M}$. The term $2L_{\overline{M}}$ in (6.4) comes from the U function in the denominator.] If

$$d_C \equiv \max_{\overline{M} \in \mathcal{S}_C^G} d_{\overline{M}} \geq 0 ,$$

the integral (6.2), excluding the factors in front, diverges as $\lambda^{2d_C} \times$ (powers of $\ln\lambda$), while it is finite if $d_C < 0$. Therefore, the Feynman integral $\mu_G(\underline{m}, \lambda\underline{P})$ as a whole behaves as

$$\lambda^{D_C} \times (\text{powers of } \ln\lambda \text{ if } d_C \geq 0) , \quad (6.5)$$

D_C being given by

$$D_C = 4 - N_e + 2\max(0, d_C) . \quad (6.6)$$

Here we have used the simple topological relation $N - 2L = \frac{1}{2}(N_e - 4)$.

In order to obtain more precise information on D_C, let us suppose that the set of all external momenta is divided into γ subsets such that

$$P_\sigma \cdot P_{\sigma'} = 0 \quad \sigma, \sigma' = 1, 2, \ldots, \gamma , \quad (6.7)$$

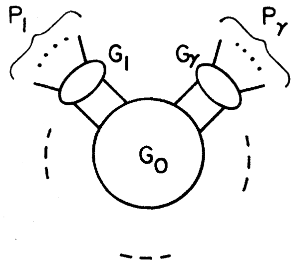

FIG. 4. Structure of a diagram having the maximum value of d_c allowed by (6.8). Each one of the subdiagrams $G_1, G_2, \ldots, G_\gamma$ is attached to G_0 by two lines only.

is satisfied, where P_σ denotes the sum of external momenta within the subset labeled σ. For this configuration, it can be shown that[59]

$$d_C \leq \gamma - 2\ , \tag{6.8}$$

where the maximum is achieved when the diagram has the structure shown in Fig. 4. Thus we obtain

$$D_C = 4 - N_e \quad \text{for } \gamma = 1\ , \tag{6.9a}$$

$$D_C \leq 4 - N_e + 2\gamma - 4 \quad \text{for } \gamma \geq 2\ . \tag{6.9b}$$

The case $\gamma = 1$ represents a nonexceptional configuration since the mass singularity of the integral (6.2) is convergent ($d_C \leq -1$). In this case (6.9a) agrees with that obtained by the Weinberg theorem. If $\gamma \geq 2$, on the other hand, we are generally dealing with the exceptional configuration and the power-counting rule has to be modified as given in (6.9b). Note that the estimates (6.9) apply to Minkowskian as well as Euclidean momenta since it is not necessary in our method to restrict the external momenta to the Euclidean region.

ACKNOWLEDGMENT

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APPENDIX: LINEAR INDEPENDENCE OF $z_i z_j B_{ij}$ IN (3.18) AND (4.8)

To examine the linear dependence of $z_i z_j B_{ij}^{M_\alpha}$ in (3.18), let us assume that

$$\sum_{i,j \in T_\alpha} a_{ij} z_i z_j B_{ij}^{M_\alpha} = 0 \tag{A1}$$

holds identically for some constants a_{ij}. Making use of (2.8) we can rewrite this as

$$\sum_c \Big(\sum_{i,j \in T_\alpha} a_{ij} z_i z_j \eta_{ic} \eta_{jc} \Big) U_c^{M_\alpha} = 0\ , \tag{A2}$$

where the summation $\sum_c$ is over all self-nonintersecting loops of M_α such that $c \cap T_\alpha \neq \varnothing$.

Since (A2) is an identity in z_i, the derivatives of its left-hand side with respect to any z_i must also be equal to zero. In this connection it is useful to recall that, if U^G is the U function for the diagram G, $\partial U^G/\partial z_i$ is the U function for the diagram $G(i)$ obtained from G by removing the chain i (i.e., the set of all lines carrying the same integration momentum as the line i). Similarly, when i and j belong to different chains of G, $\partial^2 U^G/\partial z_i \partial z_j$ is the U function for the diagram $G(i,j)$ obtained by removing the chain j from $G(i)$, or equivalently the chain i from $G(j)$. [Note that the chain j of $G(i)$ may consist of several chains of G].

Now, let L_α be the number of loops of M_α. Then we can choose $L_\alpha - 1$ lines $a(1), a(2), \ldots, a(L_\alpha - 1)$ from the set $M_\alpha - T_\alpha$ in such a way that successive removal of chains containing these lines leaves us with only one closed loop c. (Recall that T_α has no closed loop.)

This is equivalent to differentiating (A2) with respect to the variables $z_{a(1)}, z_{a(2)}, \ldots, z_{a(L_\alpha - 1)}$, and leads to

$$\sum_{i,j \in T_\alpha} a_{ij} z_i z_j \eta_{ic} \eta_{jc} = 0\ . \tag{A3}$$

Obviously (A3) holds for arbitrary $i \in c \cap T_\alpha$ if and only if $a_{ij} \equiv 0$ for $i,j \in c \cap T_\alpha$. Repeating this for all possible choices of lines $a(1), a(2), \ldots, a(L_\alpha - 1)$, we conclude that $a_{ij} \equiv 0$ for all $i,j \in T_\alpha$.

Let us next consider the linear dependence of $z_i z_j [B_{ij}^{H_A}]_{X_A}^{0,1}$ in (4.8). Suppose

$$2\sum_c \sum_{k \in T_A^X} \sum_{i \in T_A^M} a_{ki} z_k z_i \eta_{kc} \eta_{ic} [U_c^{H_A}]_{X_A}^0 + \sum_c \sum_{i,j \in T_A^M} a_{ij} z_i z_j \eta_{ic} \eta_{jc} [U_c^{H_A}]_{X_A}^1 = 0 \tag{A4}$$

holds for some constants a_{ki} and a_{ij}. Let L_H be the number of loops of H_A. Then we can successively remove $L_H - 2$ chains containing the lines $b(1), b(2), \ldots, b(L_H - 2)$ from the set $H_A - T_A$ in such a way that we are left with only two loops c_1 (containing two lines $k \in T_A^X$, $i \in T_A^M$) and c_2 (containing two lines $i,j \in T_A^M$). Noting that this is equivalent to differentiating (A4) with respect to the variables $z_{b(1)}, z_{b(2)}, \ldots, z_{b(L_H - 2)}$, we obtain

$$2\sum_{k\in T_A^X}\sum_{i\in T_A^M} a_{ki}z_k z_i \eta_{kc_1}\eta_{ic_1}\Big(\sum_{j\in T_A^M}\eta_{jc_2'}{}^2 z_j\Big)$$

$$+2\sum_{k\in T_A^X}\sum_{i\in T_A^M} a_{ki}z_k z_i \eta_{kc_3}\eta_{ic_3}\Big(\sum_{j\in T_A^M}\eta_{jc_1'}{}^2 z_j\Big)+\sum_{i,j\in T_A^M} a_{ij}z_i z_j \eta_{ic_2}\eta_{jc_2}\Big(\sum_{k\in T_A^X}\eta_{kc_1'}{}^2 z_k\Big)=0\,, \quad \text{(A5)}$$

where $c_2'=c_2-c_2\cap c_1$, $c_1'=c_1-c_1\cap c_2$, $c_3=c_1'+c_2'$. This trilinear form vanishes identically if and only if $a_{ki}=0$, $a_{ij}=0$ for all lines $k\in c_1'\cap T_A^X$, $i,j\in c_2\cap T_A^M$. Repeating this for all possible choices of $b(1)$, $b(2),\ldots,b(L_H-2)$, we find that $a_{ki}=0$, $a_{ij}=0$ for all $k\in T_A^X$, $i,j\in T_A^M$.

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[30]A connected diagram is said to be one-vertex (particle) -irreducible if it is connected after removing any one of its vertices (internal lines).

[31]They may be either unrenormalized or renormalized masses.

[32]If the integral (2.1) is ultraviolet divergent, some kind of regularization (e.g., Pauli-Villars, n-dimensional, or analytic regularization) is required. We shall not write down the necessary modifications explicitly in this and subsequent formulas since the main results of the present paper are not very sensitive to the structure of ultraviolet singularities of the Feynmann integral.

[33]In general, the numerator of the vector field propagator is not a polynomial of momentum. In such a case (2.1) must be modified accordingly. For example, write $g_{\mu\nu}-\alpha p_\mu p_\nu/p^2=(p^2 g_{\mu\nu}-\alpha p_\mu p_\nu)/p^2$ and include $1/p^2$ in the denominator of (2.1).

[34]See Ref. 29 for the details of derivation.

[35]For simplicity we shall frequently omit the arguments of various parametric functions, e.g., we write U for $U(z)$ and V for $V(z,m,q)$.

[36]Vanishing $U(z)$ also generates singularities of μ_G. For real non-negative $z_1,\ldots,z_N$, they are familiar ultraviolet divergences that can be removed by renormalization. A treatment of these singularities has been given in Ref. 29. For other methods, see the references cited there.

[37]By "unique" we mean uniqueness up to the over-all normalization factor since (2.11) are homogeneous equations.

[38]Let S be an arbitrary subdiagram of a diagram G. By reduced diagram G/S we mean the diagram which is obtained from G by shrinking the internal lines of S to points.

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[42]If M_α contains a self-energy insertion between the lines i and i', then we have $q_i=q_{i'}$. These q's yield identical information on the mass singularity. Also, T_α is empty if the number of external vertices relative to M_α is ≤ 1. Then (3.20b) gives no constraint on the external momenta.

[43]A set of subdiagrams is called disjoint if any pair of these subdiagrams have no more than one vertex in common and if the diagram obtained by putting them together contains no closed paths besides those already present in each subdiagram. If a diagram G consists of several disjoint subdiagrams $G_1,\ldots,G_n$, we write

$$G=G_1\oplus\cdots\oplus G_n=\bigoplus_{k=1}^{n} G_k\,.$$

[44]Strictly speaking, this is true if and only if T_α is connected. If T_α consists of several connected components $T_{\alpha,c}$ $(1\le c\le\gamma)$, the relative external momenta satisfy

$$\sum_{w\in T_{\alpha,c}} P_w^\alpha=0,\quad c=1,\ldots,\gamma,$$

and (3.24) is rewritten using the decomposition

$$T_{\alpha,c}=T'_{\alpha,c}\oplus\{i\}\oplus T''_{\alpha,c}\quad(i\in T_{\alpha,c},\ c=1,\ldots,\gamma)$$

as

$$q_i=\pm\sum_{w\in T'_{\alpha,c}} P_u^\alpha=\mp\sum_{w\in T''_{\alpha,c}} P_w^\alpha,\quad i\in T_{\alpha,c}\,.$$

This does not affect the final result (3.25).

[45]This restriction can be easily lifted. The example (6.4) indicates how the formula (3.31) should be modified when $\overline{M}$ contains loops.

[46]If M consists of several irreducible components $M=\oplus_\alpha M_\alpha$, the integrand of (3.29) can be decomposed by (3.15) and Feynman's formula into a sum of products of the form $\prod_\alpha (V_{M_\alpha})^{-\beta_\alpha}$, where β_α are appropriate integers. In such a case the mass singularity of $\mu_G(m,P)$ can be regarded as arising from the vanishing of individual V_{M_α} independently of other $V_{M_{\alpha'}}$'s. The power-counting rule (3.32) must be modified accordingly.

[47]Alternatively, one may choose $P_w^\alpha\cdot P_{w'}^\alpha=O(\delta)$. This will not affect our consideration if the positivity $V'_M=O(\delta)>0$ is maintained for all z_i, $i\in M$. In some cases, however, it may be necessary to allow both signs for V'_M. This will complicate the analytic property of the Feynmann integral, but will not alter the strength of mass singularity insofar as the dangerous region can be avoided by distorting the hypercontour of integration.

[48]To be precise, because of (4.7b), $q_k\cdot q_l$ terms for $k,l\in T_\alpha^M$, $\alpha\in A$, should be excluded from the last term of (4.8). Thus, a part of (4.9e) is redundant. These terms are included here for convenience in handling (4.15).

[49]A remark similar to Ref. 44 applies here, too.

[50]The condition (3.25) is contained in (4.15c). This redundancy originates from that of (4.9e). See Ref. 48.

[51]A remark similar to Ref. 46 applies here, too.

[52]Hereafter we use the following notation for any diagram S:

$$Q_j'^{\,S}\equiv q_j-\frac{1}{U^S}\sum_{k\in S} z_k B_{kj}^S q_k\quad(j\in S).$$

[53]If this assumption is not satisfied for some choice of subdiagrams $\overline{M}$ and $\overline{M}'$, it implies that the mass singularity of type II does not exist for such $\overline{M}$ and $\overline{M}'$. It is then necessary to make appropriate redefinitions:
(1) Suppose an irreducible component $\tilde{M}_{A,\gamma}$ has no nonvanishing relative external momenta. Such an $\tilde{M}_{A,\gamma}$ has no nontrivial thresholds. In fact, since $q_j=0$ for any $j\in\tilde{M}_{A,\gamma}$ and hence $Q_j'^{\tilde{M}_{A,\gamma}}=0$ in this case, the solution of (5.5) is given by $m_j=0$, $z_j=$ arbitrary for any $j\in\tilde{M}_{A,\gamma}$. In order to find a type-II mass singularity in the proper sense, we have to remove the lines of such $\tilde{M}_{A,\gamma}$ from $\overline{M}$ and put them in $G-\overline{M}$.
(2) If $\tilde{M}_{A,\gamma}$ has some nonvanishing relative external momenta, it will have nontrivial thresholds. It may

happen, however, that $\tilde{M}_{A,\gamma}$ has only *nonleading* threshold singularity. For such a singularity, some z_j of $\tilde{M}_{A,\gamma}$ vanish. Then we must remove the corresponding lines from $\bar{M}$ putting them in $\bar{M}'$.

[54]A remark similar to Ref. 48 applies to (5.9c).

[55]We ignore the pseudo threshold singularity at $P_1^2 = (m_1-m_2)^2, m_2z_2+m_3z_3=0$, and the second-type singularity at $P_1^2=0$, since they are not real singularities.

[56]We are slightly extending this concept in the sense that we apply it to individual Feynman amplitudes rather than Green's functions. See Ref. 16.

[57]For example, it can be shown using (3.31) that the total cross section for production of particles by a timelike virtual photon is free of mass singularities in most renormalizable field theories, including non-Abelian gauge theories.

[58]See. K. Symanzik, Refs. 16 and 25.

[59]See T. Kinoshita and A. Ukawa (unpublished).