

Boson-fermion correspondence in two-dimensional field theories

Myron Bander*

Department of Physics, University of California, Irvine, California 92664

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The correspondence between boson and fermion field theories in one space and one time dimension is examined in the context of a path-integral formulation of these theories. The advantage of this formulation is that the translation, both for the Lagrangians and the field operators, is fairly automatic. Normalization of products of fields, which in more conventional formulations required careful manipulation of singular quantities, in this approach is a straightforward consequence of Lorentz invariance.

I. INTRODUCTION

A remarkable correspondence has been noted between boson and fermion theories in one space and one time dimension. Coleman¹ noted that the Green's functions of the massive Thirring model and those governed by the sine-Gordon equation are related. Kogut and Susskind² provided a dictionary whereby one could translate a fermion theory into an equivalent boson theory. Finally, Mandelstam³ gave the correspondence for the Fermi field operator itself. For the massless Thirring model, much of this correspondence was already noted by Dell'Antonio, Frishman, and Zwanziger,⁴ who constructed a solution to this model in terms of currents. For one-dimensional electron gas problems this correspondence was known to many-body physicists.⁵

In this article we investigate this correspondence systematically using the path-integral formalism.⁶ The advantages of this approach are that the delicacies of normal-ordering prominent in the other approaches become somewhat automatic. In fact we are able to present a machinery, without excessive subtleties, for translating fermion theories into boson theories, including the correct correspondence for the field operators themselves. We always work in a cutoff field theory where all mathematical expressions and manipulations are meaningful, and the limit of infinite cutoffs, both momentum and spatial, is taken at the end.

Section II is devoted to a review of the Green's functions for massless free fermion and boson theories. For the results on fermion theories, free and interacting, we rely heavily on the work of Klaiber.⁷ In Sec. III we set up the correspondence between free massless fermion Green's functions and path integrals over free Bose fields. These results are extended in Sec. IV to a correspondence between composite bilinear fermion operators and functionals of the Bose fields. Armed with the results of these two sections we

treat in some detail in Sec. V three interacting theories. These are the massive and massless Thirring models, quantum electrodynamics, and interactions with a massive vector meson. For the massless fermion cases results on the Green's functions are obtainable in closed form; the massive fermion case cannot be solved either in the fermion or corresponding boson language.

One advantage of the present treatment that will be seen in the section on interacting fields is that whereas the definitions of currents still require care, other details of these modifications are much more automatic than in other approaches. The magnitude of these modifications is determined by Lorentz invariance.

Several points we wish to emphasize before concluding this introduction. It is imperative to use a Hamiltonian rather than a Lagrangian formalism for the bosons. This is because we deal with functionals of derivatives of the boson fields, including time derivatives, and a naive application of Lagrangian formalism with path integrals would yield incorrect propagators lacking contact terms.⁶ A spatial and momentum cutoff must be provided in order to make any sense of this procedure. Dependence on the spatial cutoff disappears by itself when one looks at nonvanishing Green's functions and the momentum cutoff is absorbed in the normalization of various operators in a standard way. The last point concerns the ordering of Fermi fields. The sign of fermion Green's functions depends on their ordering. The correspondence we shall present will be valid for one definite ordering and the sign changes that may result by varying this ordering do not occur automatically in the boson language, but must be put in by hand. This difficulty becomes important in the construction of composite operators.

II. FREE-FIELD THEORIES

We shall summarize the notation and basic formulas needed in subsequent sections. All deriva-

tions are straightforward and will not be presented in detail. The properties of free massless Fermi and Bose fields will be discussed.

A. Massless Fermi fields

The massless Dirac equation is

$$i\not{\partial}\psi = 0. \quad (2.1)$$

With the convention

$$\gamma_0 = \sigma_x, \quad \gamma_1 = i\sigma_y, \quad -\gamma_5 = \gamma_0\gamma_1 = \sigma_z \quad (2.2)$$

the chiral components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (2.3)$$

satisfy

$$\begin{aligned} i(\partial_0 + \partial_1)\psi_1 &= i\partial_+ \psi_1 = 0, \\ i(\partial_0 - \partial_1)\psi_2 &= i\partial_- \psi_2 = 0. \end{aligned} \quad (2.4)$$

A compact form of the Green's function of this theory was obtained by Klaiber⁷:

$$\langle F[\pi, \varphi] \rangle = \frac{\int [d\varphi d\pi] \{ \exp[i \int d^2x (\pi \partial_0 \varphi - \frac{1}{2} [\pi^2 + (\partial_1 \varphi)^2])] F[\pi, \varphi] \}}{\int [d\varphi d\pi] \{ \exp[i \int d^2x (\pi \partial_0 \varphi - \frac{1}{2} [\pi^2 + (\partial_1 \varphi)^2])] \}}. \quad (2.6)$$

The basic propagators of the theory are

$$\begin{aligned} \langle [\pi(x) \pm \partial_1 \varphi(x)] [\pi(y) \pm \partial_1 \varphi(y)] \rangle &= i\Delta_{(\pm)}(x-y) = -\frac{i\Lambda^2}{(2\pi)^2} \int d^2k \frac{2k_1(k_1 \mp k_0) e^{-ik \cdot (x-y)}}{(k^2 + i\epsilon)(k^2 - \Lambda^2 + i\epsilon)}, \\ \langle [\pi(x) \pm \partial_1 \varphi(x)] [\pi(y) \mp \partial_1 \varphi(y)] \rangle &= 0. \end{aligned} \quad (2.7)$$

In order to make some of our subsequent discussions meaningful we softened the propagator by introducing a cutoff Λ . The limit of infinite cutoff will be taken at the end of all calculations.

A generating functional summarizing the above is

$$\begin{aligned} \left\langle \exp \left\{ i \int d^2x [\alpha(x)(\pi(x) + \partial_1 \varphi(x)) + \beta(x)(\pi(x) - \partial_1 \varphi(x))] \right\} \right\rangle \\ = \exp \left\{ -\frac{i}{2} \int d^2x d^2y [\alpha(x)\Delta_{(+)}(x-y)\alpha(y) + \beta(x)\Delta_{(-)}(x-y)\beta(y)] \right\}. \end{aligned} \quad (2.8)$$

III. FERMION-BOSON CORRESPONDENCE-FREE FIELDS

Following Mandelstam³ it would be tempting to make the identification

$$\psi_{1,2}(x) = C \exp \left\{ -i\sqrt{\pi} \int_{-\infty}^{x^1} d\xi [\pi(x^0, \xi) \pm \partial_1 \varphi(x^0, \xi)] \right\}. \quad (3.1)$$

However, it is straightforward to show that expectation values of such expressions are singular owing to the infinite range of the spatial integra-

$$\begin{aligned} \langle \psi_1(x_1) \cdots \psi_1(x_n) \psi_1^*(y_1) \cdots \psi_1^*(y_n) \rangle \\ = \left(\frac{1}{2\pi i} \right)^n \frac{\prod_{j < k} (x_j^{(+)} - x_k^{(+)})(y_j^{(+)} - y_k^{(+)})}{\prod_{j,k} (x_j^{(+)} - y_k^{(+)}),} \end{aligned} \quad (2.5)$$

$$\begin{aligned} \langle \psi_2(x_1) \cdots \psi_2(x_n) \psi_2^*(y_1) \cdots \psi_2^*(y_n) \rangle \\ = \left(\frac{1}{2\pi i} \right)^n \frac{\prod_{j < k} (x_j^{(-)} - x_k^{(-)})(y_j^{(-)} - y_k^{(-)})}{\prod_{j,k} (x_j^{(-)} - y_k^{(-)}),} \end{aligned}$$

with $x^{(\pm)} = x^0 \pm x^1$, and no mixed contractions between different chiral components.

B. Path integrals for massless bosons

Owing to infrared singularities the free massless boson field does not exist. However, differences and derivatives do. We shall evaluate path integrals for functionals of these derivatives. As time derivatives will occur care has to be exercised and instead of dealing with the usual path integrals over fields alone we shall go to the more fundamental functional integral over fields and their canonical momenta.⁶ In all subsequent discussions we shall use the notations

tion.

This difficulty is overcome by modifying the integral in (3.1) through the introduction of a spatial cutoff

$$\int_{-\infty}^{x^1} d\xi (\pi \pm \partial_1 \varphi) \rightarrow \int_{-\infty}^{x^1} d\xi e^{\xi/R} (\pi \pm \partial_1 \varphi). \quad (3.2)$$

At the very end of the calculations we will set $R = \infty$. It should be noted that this procedure would have been necessary even if the fields were massive.

All our expressions will be built up from

$$\Phi_{\pm}(x) = \int_{-\infty}^{x^1} d\xi e^{\xi/R} [\pi(x^0, \xi) \pm \partial_1 \varphi(x^0, \xi)]. \quad (3.3)$$

The Green's functions are generated by the basic two-point function [evaluated with the help of (2.7)]

$$\langle \Phi_{\pm}(x) \Phi_{\pm}(y) \rangle_{R, \Lambda \rightarrow \infty} \sim -\frac{1}{\pi} \ln \left[\frac{\epsilon(x_0)(x-y)^{(\pm)} - i\gamma/\Lambda}{-i\gamma R} \right], \quad (3.4)$$

$$\langle \Phi_+(x) \Phi_-(y) \rangle = 0, \quad (3.5)$$

with

$$\ln \gamma = \int_0^{\infty} e^{-y} \ln y \, dy = 0.577 \dots \quad (3.6)$$

We are thus led to consider

$$\left\langle \exp \left\{ -i\sqrt{\pi} \left[\sum_{i=1}^N \alpha_i \Phi_+(x_i) + \sum_{j=1}^M \beta_j \Phi_-(y_j) \right] \right\} \right\rangle = \left(\frac{\gamma}{i\Lambda} \right)^{(M+N)/2} (-i\gamma R)^{-\langle \sum \alpha_i \rangle^2} (-i\gamma R)^{-\langle \sum \beta_j \rangle^2} \prod_{i < j} |x_i^{(+)} - x_j^{(+)}|^{\alpha_i \alpha_j} \prod_{i < j} |y_i^{(-)} - y_j^{(-)}|^{\beta_i \beta_j}. \quad (3.7)$$

In the above $|\alpha_i| = |\beta_j| = 1$.

Now the limit $R \rightarrow \infty$ may be taken, and a non-vanishing contribution is obtained only in the case $\sum \alpha_i = \sum \beta_j = 0$.

In order to recover (2.5) we may identify

$$\begin{aligned} \psi_1(x) &= e^{i\delta_1} \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp[-i\sqrt{\pi} \Phi_+(x)], \\ \psi_2(x) &= e^{i\delta_2} \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp[-i\sqrt{\pi} \Phi_-(x)]. \end{aligned} \quad (3.8)$$

The phases are arbitrary and may be adjusted by a combination of gauge and chiral transformations. We shall choose $\delta_1 = \delta_2 = 0$ and thus obtain

$$\begin{aligned} \psi_1(x) &= \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp[-i\sqrt{\pi} \Phi_+(x)], \\ \psi_2(x) &= \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp[-i\sqrt{\pi} \Phi_-(x)]. \end{aligned} \quad (3.9)$$

We note that (3.7) agrees with (2.5) in the case the operators in (2.5) are in the correct ordering. The sign changes, which would be obtained by shifting the anticommuting Fermi fields, cannot be reproduced by expressions such as (3.7). The substitution implied by (3.9) is valid only when the string of ψ_1 's and ψ_2 's is separately time-ordered. This ambiguity will be important in Sec. IV when we construct composite operators.

IV. COMPOSITE OPERATORS

In order to build up interacting theories we need the correspondence between composite fields made up of fermion operators and functionals of the c -number boson fields. This discussion will be limited to bilinear operators, namely $\bar{\psi}\Gamma\psi$. These operators are sums of products of the form $\psi_{\alpha}^* \psi_{\beta}$. At this state the anticommutativity of the fermions begins to plague us. The correspondence between

$:\psi_{\alpha}^*(x)\psi_{\beta}(x):$ and the boson fields will be established by substituting (3.9) for $\psi_{\alpha}^*(x)$ and $\psi_{\beta}(x)$ and the over-all sign will be determined by ensuring the correct expression for

$$\langle : \psi_{\alpha}^*(x) \psi_{\beta}(x) : [\psi_1(y) \psi_1^*(z) + \psi_1(y) \psi_2^*(z) + \psi_1^*(y) \psi_2(z) + \psi_2(y) \psi_2^*(z)] \rangle,$$

with $y^0 > z^0$. In the case of singular products a similar expression will be used to fix the normalization of these operators. If the sign and magnitude of the Green's function for a composite operator and two fermion operators, as above, are correct then the Green's functions involving more fermion fields will likewise be correct. The one involving two Fermi operators is the only one that is connected; the more involved are made up of the simpler ones.

A. Mass operators

In terms of the chiral components we find

$$\begin{aligned} :\bar{\psi}\psi: &= : \psi_1^* \psi_2 + : \psi_2^* \psi_1 : , \\ :\bar{\psi}\gamma_5\psi: &= : \psi_1^* \psi_2 - : \psi_2^* \psi_1 : . \end{aligned} \quad (4.1)$$

As there are no mixed contractions, normal ordering introduces no infinities and we may, up to a sign, apply (3.9) without special care. Determining the sign by the argument discussed above we obtain

$$\begin{aligned} :\psi_1^*(x) \psi_2(x): &= \frac{\Lambda}{2\pi\gamma} \exp \left[2i\sqrt{\pi} \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi(x^0, \xi) \right], \\ :\psi_2^*(x) \psi_1(x): &= \frac{\Lambda}{2\pi\gamma} \exp \left[-2i\sqrt{\pi} \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi(x^0, \xi) \right], \end{aligned} \quad (4.2)$$

or equivalently

$$:\bar{\psi}(x)\psi(x): = \frac{\Lambda}{\pi\gamma} \cos \left[2\sqrt{\pi} \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi(x^0, \xi) \right], \tag{4.3}$$

$$:\bar{\psi}(x)\gamma_5\psi(x): = \frac{i\Lambda}{\pi\gamma} \sin \left[2\sqrt{\pi} \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi(x^0, \xi) \right].$$

If we naively let $R \rightarrow \infty$ at this stage we would obtain the results of Refs. 1-3, namely,

$$\begin{aligned} :\bar{\psi}\psi: &= \frac{\Lambda}{\pi\gamma} \cos 2\sqrt{\pi} \varphi, \\ :\bar{\psi}\gamma_5\psi: &= \frac{i\Lambda}{\pi\gamma} \sin 2\sqrt{\pi} \varphi. \end{aligned} \tag{4.4}$$

However, it must be emphasized that this limit may not always be taken before the end of all calculations.

B. Current operators

The situation is more complicated for operators of the form $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\gamma_\mu\gamma_5\psi$. These are sums of $\psi_1^*\psi_1$ and $\psi_2^*\psi_2$, singular products needing care in their definition. We shall choose the prescription

$$:\psi_\alpha^*(x)\psi_\alpha(x): = d_\alpha \left[\psi_\alpha^* \left(x^0, x^1 + \frac{a}{\Lambda} \right) \psi_\alpha(x) - c \right], \tag{4.5}$$

with a, c, d constants and Λ the cutoff introduced earlier. c is chosen to ensure that $\langle : \psi_\alpha^* \psi_\alpha : \rangle = 0$; a and d will be connected to the normalization of the composite operator. We assume Λ is large and expand in $1/\Lambda$ retaining the first nonvanishing term. Substituting (3.9) into (4.5) we obtain

$$\begin{aligned} :\psi_1^*\psi_1: &= \frac{d_1\Lambda}{2\pi\gamma} \left\{ \exp \left[i\sqrt{\pi} \int_{x^1}^{x^1+a/\Lambda} d\xi (\pi + \partial_1 \varphi) \right] - 1 \right\}, \\ :\psi_2^*\psi_2: &= \frac{d_2\Lambda}{2\pi\gamma} \left\{ \exp \left[i\sqrt{\pi} \int_{x^1}^{x^1+a/\Lambda} d\xi (\pi - \partial_1 \varphi) \right] - 1 \right\}. \end{aligned} \tag{4.6}$$

For large Λ this may be reduced to

$$\begin{aligned} :\psi_1^*\psi_1: &= \frac{id_1a}{2\gamma\sqrt{\pi}} (\pi + \partial_1 \varphi), \\ :\psi_2^*\psi_2: &= \frac{id_2a}{2\gamma\sqrt{\pi}} (\pi - \partial_1 \varphi). \end{aligned} \tag{4.7}$$

$d_\alpha a$ may be determined by ensuring proper nor-

malization by comparing

$$\langle : \psi_\alpha^*(x)\psi_\alpha(x) : [\psi_1(y)\psi_1^*(z) + \psi_2(y)\psi_2^*(z)] \rangle,$$

both by using (2.5) by substituting (4.7) and (3.9). The results are

$$\begin{aligned} :\psi_1^*\psi_1: &= \frac{1}{2\sqrt{\pi}} (\pi + \partial_1 \varphi), \\ :\psi_2^*\psi_2: &= -\frac{1}{2\sqrt{\pi}} (\pi - \partial_1 \varphi). \end{aligned} \tag{4.8}$$

The current operators have the form

$$\begin{aligned} :\bar{\psi}\gamma_0\psi: &= \frac{1}{\sqrt{\pi}} \partial_1 \varphi, \\ :\bar{\psi}\gamma_1\psi: &= \frac{1}{\sqrt{\pi}} \pi, \\ :\bar{\psi}\gamma_0\gamma_5\psi: &= \frac{1}{\sqrt{\pi}} \pi, \\ :\bar{\psi}\gamma_1\gamma_5\psi: &= \frac{1}{\sqrt{\pi}} \partial_1 \varphi. \end{aligned} \tag{4.9}$$

If we substitute $\partial_0 \varphi$ for π we find the results quoted in Refs. 1-3,

$$\begin{aligned} :\bar{\psi}\gamma_\mu\psi: &= \frac{\epsilon_{\mu\nu}}{\sqrt{\pi}} \partial^\nu \varphi, \\ :\bar{\psi}\gamma_\mu\gamma_5\psi: &= \frac{1}{\sqrt{\pi}} \partial_\mu \varphi. \end{aligned} \tag{4.10}$$

V. FERMION-BOSON CORRESPONDENCE-INTERACTING THEORIES

A. Massive Thirring model

The Lagrangian for the massive Thirring model⁸ is

$$L = \bar{\psi}(i\not{\partial} - m)\psi - \frac{g}{2} j_\mu j^\mu. \tag{5.1}$$

It is tempting to let $j_\mu = \bar{\psi}\gamma_\mu\psi$; owing to the asymmetry between space and time introduced in the definition of the currents, the interactions modify the normalization of j_1 relative to j_0 . Thus in (5.1) we shall let $j_0 = \bar{\psi}\gamma_0\psi$; however, $j_1 = \lambda\bar{\psi}\gamma_1\psi$ and let Lorentz invariance choose λ for us. Implementing the correspondence discussed in the last two sections the Green's functions of this theory are obtained, up to an over-all normalization, from

$$\begin{aligned} \langle F[\psi_1, \psi_2] \rangle &= N \int [d\varphi d\pi] \left\{ \exp \left\{ i \int d^2x \left[\pi \partial_0 \varphi - \frac{\pi^2}{2} - \frac{(\partial_1 \varphi)^2}{2} - \frac{m\Lambda}{\pi\gamma} \cos \left(2\sqrt{\pi} \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi \right) \right] + \frac{g}{2\pi} [\lambda^2 \pi^2 - (\partial_1 \varphi)^2] \right\} \right. \\ &\quad \left. \times F \left[\left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp(-i\sqrt{\pi}\Phi_+), \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp(-i\sqrt{\pi}\Phi_-) \right] \right\}. \end{aligned} \tag{5.2}$$

We shall now rescale π by $(1 - \lambda g/\pi)^{-1/2}$ and φ by $(1 + g/\pi)^{-1/2}$. The terms quadratic in π and $\partial_1\varphi$ have the usual coefficients; however, $\pi\partial_0\varphi$ is multiplied by $(1 + g/\pi)^{-1/2}(1 - g\lambda/\pi)^{-1/2}$. Lorentz invariance is recovered only if this product is unity. Imposing this condition we find

$$\lambda = 1/(1 + g/\pi). \tag{5.3}$$

Introducing

$$\beta = 2\sqrt{\pi}(1 + g/\pi)^{-1/2} \tag{5.4}$$

we find that the current is

$$j_\mu = [\delta_\mu^0 + (\beta^2/4\pi)\delta_\mu^1]\bar{\psi}\gamma_\mu\psi, \tag{5.5}$$

which agrees with known results.

Returning to (5.2) we obtain

$$\begin{aligned} \langle F[\psi_1, \psi_2] \rangle = N' \int [d\varphi d\pi] & \left(\exp \left\{ i \int_{-\infty}^{x^1} d^2x \left[\pi\partial_0\varphi - \frac{\pi^2}{2} - \frac{(\partial_1\varphi)^2}{2} - \frac{m\Lambda}{\pi\gamma} \cos \left(\beta \int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1\varphi \right) \right] \right\} \right. \\ & \times F \left[\left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp \left\{ -i \int_{-\infty}^{x^1} d\xi e^{\xi/R} \left(\frac{2\pi}{\beta} \pi + \frac{\beta}{2} \partial_1\varphi \right) \right\}, \right. \\ & \left. \left. \left(\frac{\Lambda}{2\pi\gamma} \right)^{1/2} \exp \left\{ -i \int_{-\infty}^{x^1} d\xi e^{\xi/R} \left(\frac{2\pi}{\beta} \pi - \frac{\beta}{2} \partial_1\varphi \right) \right\} \right] \right). \end{aligned} \tag{5.6}$$

From this we note that the Green's functions of the massive Thirring model are obtainable as those of the boson theory with the Lagrangian

$$L = \frac{1}{2}(\partial\varphi)^2 + \frac{m\Lambda}{\pi\gamma} \cos \left(\beta \int_{-\infty}^{x^1} e^{\xi/R} \partial_1\varphi(x^0, \xi) \right) \tag{5.7}$$

with Mandelstam's identification

$$\psi_1 = c_1 \exp \left[-i \int_{-\infty}^{x^1} d\xi e^{\xi/R} \left(\frac{2\pi}{\beta} \pi + \frac{\beta}{2} \partial_1\varphi \right) \right], \tag{5.8}$$

$$\psi_2 = c_2 \exp \left[-i \int_{-\infty}^{x^1} d\xi e^{\xi/R} \left(\frac{2\pi}{\beta} \pi - \frac{\beta}{2} \partial_1\varphi \right) \right],$$

where c_1 and c_2 are constants to be determined by some normalization condition.

In the case $m = 0$ the Green's functions may be explicitly evaluated; requiring a finite answer in the limit $\Lambda \rightarrow \infty$ yields

$$c_\alpha \sim \Lambda^{(\pi^2/\beta^2 + \beta^2/4\pi)}, \tag{5.9}$$

and the Green's functions agree with those presented in Ref. 7 for currents defined in the manner of Schwinger.⁹

B. Quantum electrodynamics

We shall first consider a massless fermion interacting with the electromagnetic field.^{10,11} In two dimensions there are no photons and thus we shall look only at the fermion Green's functions. The Lagrangian of this theory is

$$L = i\bar{\psi}\not{\partial}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{5.10}$$

It is convenient to work in the gauge $A_1 = 0$. We

may solve for

$$A_0 = -\frac{e}{\partial_1^2} \bar{\psi}\gamma_0\psi \tag{5.11}$$

and find the effective Lagrangian for the fermions

$$L_{\text{eff}} = i\bar{\psi}\not{\partial}\psi - \frac{e^2}{2} \bar{\psi}\gamma_0\psi \frac{1}{\partial_1^2} \bar{\psi}\gamma_0\psi. \tag{5.12}$$

The correspondence with the boson fields yields an action

$$A = \pi_0\partial_0\varphi - \frac{\pi^2}{2} - \frac{(\partial_1\varphi)^2}{2} - \frac{e^2}{2\pi} (\partial_1\varphi) \frac{1}{\partial_1^2} (\partial_1\varphi), \tag{5.13}$$

which may be simplified to that of a massive boson field with mass

$$\mu^2 = e^2/\pi, \tag{5.14}$$

$$A = \pi\partial_0\varphi - \frac{\pi^2}{2} - \frac{(\partial_1\varphi)^2}{2} - \frac{\mu^2}{2} \varphi^2. \tag{5.15}$$

The substitution for the fermions remains as in (3.9). The Green's functions may be evaluated as in the free-field case except that the propagators are modified:

$$\begin{aligned} \langle \Phi_\pm(x)\Phi_\pm(y) \rangle_{\text{QED}} &= \langle \Phi_\pm(x)\Phi_\pm(y) \rangle_{\text{free}} \\ &+ F_\pm(x-y) + c \ln \mu R + d, \end{aligned} \tag{5.16}$$

$$\langle \Phi_+(x)\Phi_-(y) \rangle = F'(x-y) + c \ln \mu R,$$

where c, d are some constants and

$$F_{\pm}(x) = \frac{i\mu^2}{(2\pi)^2} \int d^2k \frac{(k_0 \mp k_1)^2 (e^{-ik \cdot x} - 1)(k_1^2 - \epsilon^2)}{(k^2 - i\epsilon)(k^2 - \mu^2 + i\epsilon)(k_1^2 + \epsilon^2)^2}, \quad (5.17)$$

$$F'(x) = \frac{i\mu^2}{(2\pi)^2} \int d^2k \frac{(e^{-ik \cdot x} - 1)(k_1^2 - \epsilon^2)}{(k^2 - \mu^2 + i\epsilon)(k_1^2 + \epsilon^2)^2}.$$

$$\begin{aligned} & \langle \psi_1(x_1) \psi_1^*(y_1) \psi_2(x_2) \psi_2^*(y_2) \rangle_{\text{QED}} \\ &= \exp\left(\frac{i\pi}{2} [F_+(x_1 - y_1) + F_-(x_2 - y_2) - F'(x_1 - x_2) + F'(x_1 - y_2) + F'(y_1 - x_2) - F'(y_1 - y_2)]\right) \\ & \quad \times \langle \psi_1(x_1) \psi_1^*(y_1) \psi_2(x_2) \psi_2^*(y_2) \rangle_{\text{free}}. \end{aligned} \quad (5.18)$$

This expression agrees with that given by Brown.¹¹

Giving the fermions a mass changes the Lagrangian of the boson field to

$$L = \frac{(\partial\varphi)^2}{2} - \frac{\mu^2}{2} \varphi^2 - \frac{m\Lambda}{\pi\gamma} \cos 2\sqrt{\pi}\varphi. \quad (5.19)$$

As we are now dealing with a massive boson theory it is permissible to replace $\int d\xi e^{z/R} \partial_1 \varphi$ by $\varphi - \varphi(\infty)$.¹²

C. Interaction with massive vector mesons

The last interacting theory we shall treat is that of a fermion interacting with a massive vector meson.^{13,14} The massless fermion case will be treated first, and the inclusion of the mass term left for the end. The Lagrangian of the theory is

$$L = L_A + L_f, \quad (5.20)$$

with

$$L_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mu_0^2 A^2, \quad (5.21)$$

$$L_f = i\bar{\psi}\not{\partial}\psi - ef_\mu A^\mu.$$

Again, the current f_μ is not given simply by $\bar{\psi}\gamma_\mu\psi$. It is defined as a gauge-invariant limit of $\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x)\exp(ie\epsilon A)$. For spacelike ϵ we find that

$$f_0 = \bar{\psi}\gamma_0\psi, \quad (5.22)$$

$$f_1 = \bar{\psi}\gamma_1\psi + \delta A_1.$$

Again, we shall let Lorentz invariance fix δ for us. In the case of QED, treated in Sec. VB, no such modification was necessary in the gauge $A_1=0$. Let us discuss the correspondence for L_f first:

Thus the Green's functions for the interacting theory are equal to those for free fields multiplied by an exponential of a sum of the F 's. Terms in $\ln\mu R$ cancel as long as the number of conjugated and unconjugated fields is the same. Instead of writing down the most general expression we shall give as an example a four-point function:

$$\begin{aligned} A_f &= \pi\partial_0\varphi - \frac{\pi^2 + (\partial_1\varphi)^2}{2} \\ & \quad - \frac{e}{\sqrt{\pi}} (A_0\partial_1\varphi - A_1\pi) + e\delta A_1^2. \end{aligned} \quad (5.23)$$

Changing variables in the functional integration

$$\begin{aligned} \pi &\rightarrow \pi + \frac{e}{\sqrt{\pi}} \frac{\partial_1}{\partial^2} (\partial \cdot A), \\ \varphi &\rightarrow \varphi - \frac{e}{\sqrt{\pi}} \frac{1}{\partial^2} (\partial_0 A_1 - \partial_1 A_0) \end{aligned} \quad (5.24)$$

yields

$$\begin{aligned} A_f &\rightarrow \pi\partial_0\varphi - \frac{\pi^2 + \partial_1\varphi}{2} - \frac{e^2}{4\pi} F_{\mu\nu} \frac{1}{\partial^2} F^{\mu\nu} + \frac{e^2}{2\pi} A_1^2 \\ & \quad + e\delta A_1^2. \end{aligned} \quad (5.25)$$

Choosing $\delta = -e/2\pi$ eliminates the noncovariant part of A_f , and combining with L_A we find the action corresponding to (5.19)

$$\begin{aligned} A &= \pi\partial_0\varphi - \frac{\pi^2 + (\partial_1\varphi)^2}{2} - \frac{1}{4} F_{\mu\nu} \left(1 + \frac{e^2}{\pi\partial^2}\right) F^{\mu\nu} \\ & \quad + \frac{1}{2} \mu_0^2 A^2. \end{aligned} \quad (5.26)$$

The change of variables proposed in (5.23) induces a modification in the correspondences for the fermion fields

$$\psi_1 = \left(\frac{\Lambda}{2\pi\gamma}\right)^{1/2} \exp\left\{-i\sqrt{\pi} \left[\Phi_1(x) + \frac{e}{\sqrt{\pi}} \frac{\partial_+(A_0 - A_1)}{\partial^2}\right]\right\}, \quad (5.27)$$

$$\psi_2 = \left(\frac{\Lambda}{2\pi\gamma}\right)^{1/2} \exp\left\{-i\sqrt{\pi} \left[\Phi_2(x) + \frac{e}{\sqrt{\pi}} \frac{\partial_-(A_0 + A_1)}{\partial^2}\right]\right\}.$$

It is straightforward to evaluate all Green's functions once the propagator for the vector fields is known. Inverting the part of (5.25) involving A_μ and being careful about contact terms one finds

$$\langle A_\mu(x)A_\nu(0) \rangle = \frac{-i}{(2\pi)^2} \int d^2k e^{-ik \cdot x} \left[\frac{g_{\mu\nu} - (k_\mu k_\nu / \mu^2)(1 - e^2/\pi k^2)}{k^2 - M^2 + i\epsilon} + \frac{\delta_{\mu 0} \delta_{\nu 0}}{\mu^2} \right] \quad (5.28)$$

with

$$M^2 = \mu^2 + e^2/\pi. \quad (5.29)$$

These results agree with those of Ref. 13. If the fermion has a mass m we have to add

$$L_m = \frac{m\Lambda}{\pi\gamma} \cos \left\{ 2\sqrt{\pi} \left[\int_{-\infty}^{x^1} d\xi e^{\xi/R} \partial_1 \varphi + \frac{e}{\sqrt{\pi}} \frac{1}{\partial^2} (\partial_0 A_1 - \partial_1 A_0) \right] \right\} \quad (5.30)$$

to (5.25).

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