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Classification of unquantized Yang-Mills fields*

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A classification of unquantized Yang-Mills fields is given according to their algebraic and differential properties. A method is used which is similar to the one employed by Petrov and Schell in their classification of the vacuum gravitational field. It is proved that there is no inherently nonlinear solution to a sourceless Yang-Mills field in a vacuum.

The classification of electromagnetic fields according to their algebraic properties is well known.[1] We have essentially two types of Maxwell fields: nonradiative and radiative. In the nonradiative case we can find a Lorentz frame where the electric and magnetic fields are parallel to each other and the energy-momentum tensor has a diagonal form. On the other hand, in the radiative case the electric and magnetic fields are of the same strength and are perpendicular to each other. There is a flow of energy with the velocity of light along the direction $\vec{E}\times\vec{H}$.

In the case of a vacuum gravitational field a similar classification has been obtained by Petrov[2] according to algebraic properties of the Riemann curvature tensor. In this case there are three types of fields, I, II, and III, the latter two of which have been shown to include gravitational radiations.[3] Petrov's classification has subsequently been improved by Schell[4,5] by taking into account the differential properties of the curvature tensor. The theory of the infinitesimal holonomy group (ihg) has been introduced and Petrov types have been further subdivided into classes having different ihg's.

In this note we apply similar methods to the classification of unquantized Yang-Mills fields. We first look at the algebraic properties of the Yang-Mills field and show that there are these fundamental types of fields: nonradiative, radiative, and mixtures of the two. Then we further subdivide these types by examining their ihg's. In the case of SU(2) we find ten classes of fields. The class IV_a corresponds to a sourceless vacuum Yang-Mills field and the class I_a to a field produced by a classical point source either electrically or magnetically charged. In both cases the ihg is simply U(1) and hence there are no inherently nonlinear solutions to these fields.

In the following we start with the descriptions of the electromagnetic field and then reformulate the theory in a manner useful in the non-Abelian case. We next present a preliminary classification of Yang-Mills fields into nonradiative, radiative, and mixed types. This is followed by an elementary discussion of the ihg and the full classification of Yang-Mills fields. Finally we make some comments and discussions.

The classification of the Maxwell field is usually formulated in terms of eigenproperties of the electromagnetic field strength $F_{\mu\nu}$ or the energy-momentum tensor $T_{\mu\nu}$.[1] If we take, for instance, the energy-momentum tensor, and consider its characteristic equation,

$$T_{\mu\nu}\,\xi^{\lambda} = x\xi_{\mu}\,, \tag{1}$$

its eigenvalues are given by[6]

$$x=\pm\delta\,,\quad \delta=\tfrac{1}{4}[(F_{\rho\sigma}F^{\rho\sigma})^2+(F_{\rho\sigma}F^{*\rho\sigma})^2]^{1/2}\,. \tag{2}$$

Therefore we have two distinct types of Maxwell fields, corresponding to $\delta\neq 0$ and $\delta=0$.

It is easy to show that in the case $\delta\neq 0$ we can find a particular Lorentz frame, called a canonical frame, where the electric and magnetic fields become parallel to each other. If, for instance, the z axis is chosen along the direction of fields, E_z and H_z are given by

$$\begin{aligned} E_z &= \pm(\delta-\tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma})^{1/2}\,,\\ H_z &= \pm(\delta+\tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma})^{1/2}\,. \end{aligned} \tag{3}$$

Hence these parameters are Lorentz invariants. If we introduce four independent vectors ξ_μ, η_μ, p_μ, and q_μ, where

$$\begin{aligned} \xi_\mu &= (1,0,0,-1)\,,\quad \eta_\mu=(1,0,0,1)\,,\\ p_\mu &= (0,-1,0,0)\,,\quad q_\mu=(0,0,1,0)\,, \end{aligned} \tag{4}$$

we obtain the following expressions for $F_{\mu\nu}$ and $T_{\mu\nu}$ in the canonical frame:

$$F_{\mu\nu}=\tfrac{1}{2}E_z[\xi_\mu,\eta_\nu]+H_z[p_\mu,q_\nu]\,, \tag{5}$$

$$T_{\mu\nu}=\tfrac{1}{2}(E_z^{\,2}+H_z^{\,2})(\{\xi_\mu,\eta_\nu\}-g_{\mu\nu})\,. \tag{6}$$

We have used the notations

$$[\xi_\mu,\eta_\nu]\equiv\xi_\mu\eta_\nu-\xi_\nu\eta_\mu\,,\quad \{\xi_\mu,\eta_\nu\}\equiv\xi_\mu\eta_\nu+\xi_\nu\eta_\mu\,. \tag{7}$$

Then it is not difficult to see that the above four vectors span the manifold made of eigenvectors of $T_{\mu\nu}$. The energy-momentum tensor has in this frame a diagonalized form, and hence there are no shearing stresses nor energy flows in any direction. The Maxwell field looks like a fluid at rest except for the sign of T_{zz}.

Expressions for $F_{\mu\nu}$ or $T_{\mu\nu}$ in other frames are easily obtained from Eq. (5) or (6) by Lorentz transformations. Denoting the Lorentz transformation from the canonical frame to the laboratory frame as $a_{\mu\nu}$, we have the following expression for $F_{\mu\nu}$ in the laboratory frame:

$$F_{\mu\nu} = \tfrac{1}{2}E_z[a_{\mu\lambda}\xi^\lambda, a_{\nu\kappa}\eta^\kappa] + H_z[a_{\mu\lambda}p^\lambda, a_{\nu\kappa}q^\kappa]. \tag{8}$$

We have used the fact that the parameters E_z and H_z are Lorentz invariants. If we also make use of the inverse Lorentz transformation $a_{\nu\mu}$, these parameters can be written in terms of $F_{\mu\nu}$:

$$E_z = a_{\rho 0}a_{\sigma 3}F^{\rho\sigma}, \quad H_z = -a_{\rho 1}a_{\sigma 2}F^{\rho\sigma}. \tag{9}$$

Then by combining Eqs. (8) and (9) and making use of Eq. (4) we find

$$F_{\mu\nu} = G_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}, \tag{10}$$

$$G_{\mu\nu\rho\sigma} = \tfrac{1}{2}[-(a_{\mu 0}a_{\nu 3} - a_{\nu 0}a_{\mu 3})(a_{\rho 0}a_{\sigma 3} - a_{\sigma 0}a_{\rho 3}) + (a_{\mu 1}a_{\nu 2} - a_{\nu 1}a_{\mu 2})(a_{\rho 1}a_{\sigma 2} - a_{\sigma 1}a_{\rho 2})]. \tag{11}$$

The physical meaning of Eqs. (10) and (11) is quite apparent. We have made a round trip by means of successive Lorentz transformations—starting from the laboratory, going to the canonical frame, and then coming back to the laboratory. The net result of the trip is $G_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$, which is equal to $F_{\mu\nu}$. Thus we have arrived at an interesting characterization of a Maxwell field as an eigenbivector of $G_{\mu\nu\rho\sigma}$ with eigenvalue unity.

On the other hand, in the case of the $\delta = 0$ field the electric and magnetic fields are of the same strength and are perpendicular to each other. Then after a suitable spatial rotation we find a canonical frame where, for instance, $\vec{E}\parallel x$ axis and $\vec{H}\parallel y$ axis and $E_x = H_y$. If we introduce two independent vectors ξ_μ and p_μ, where

$$\xi_\mu = (1, 0, 0, -1) \text{ and } p_\mu = (0, -1, 0, 0), \tag{12}$$

$F_{\mu\nu}$ and $T_{\mu\nu}$ can be expressed as

$$F_{\mu\nu} = -E_x[\xi_\mu, p_\nu], \tag{13}$$

$$T_{\mu\nu} = E_x{}^2\{\xi_\mu, \xi_\nu\}. \tag{14}$$

In this case the parameter $E_x = H_y$ varies under Lorentz boosts; however, it is invariant under spatial rotations. In contrast with the case of $\delta \neq 0$, the manifold of eigenvectors of $T_{\mu\nu}$ is spanned only by two vectors ξ_μ and p_μ. The energy-momentum tensor takes the form of a flow of an ensemble of massless noninteracting particles directed along the z axis.

Expressions for $F_{\mu\nu}$ in noncanonical frames are obtained from Eq. (13) by suitable spatial rotations. By denoting the spatial rotation from the canonical frame to the laboratory frame as $b_{\mu\nu}$, we have the following expression for $F_{\mu\nu}$ in the laboratory frame:

$$F_{\mu\nu} = H_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}, \tag{15}$$

$$H_{\mu\nu\rho\sigma} = -\tfrac{1}{4}[(b_{\mu 0} + b_{\mu 3})b_{\nu 1} - (b_{\nu 0} + b_{\nu 3})b_{\mu 1}] \times [(b_{\rho 0}b_{\sigma 1} - b_{\sigma 0}b_{\rho 1}) - (b_{\rho 3}b_{\sigma 1} - b_{\sigma 3}b_{\rho 1})]. \tag{16}$$

Thus the $\delta = 0$ electromagnetic field has also been expressed as an eigenbivector of $H_{\mu\nu\rho\sigma}$. It follows from Eqs. (10) and (15) that

$$G_{\mu\nu}{}^{\lambda\kappa}G_{\lambda\kappa\rho\sigma} = G_{\mu\nu\rho\sigma}, \quad H_{\mu\nu}{}^{\lambda\kappa}H_{\lambda\kappa\rho\sigma} = H_{\mu\nu\rho\sigma}. \tag{17}$$

These equations are in fact satisfied with Eqs. (11) and (16).

Let us next consider the problem of finding the canonical forms of $G_{\mu\nu\rho\sigma}$ and $H_{\mu\nu\rho\sigma}$. From the definition of $G_{\mu\nu\rho\sigma}$, Eq. (11), we easily derive its algebraic properties as follow:

$$\text{(i) } G_{\mu\nu\rho\sigma} = -G_{\nu\mu\rho\sigma}, \tag{18}$$

$$\text{(ii) } G_{\mu\nu\rho\sigma} = -G_{\mu\nu\sigma\rho}, \tag{19}$$

$$\text{(iii) } G_{\mu\nu\rho\sigma} = G_{\rho\sigma\mu\nu}, \tag{20}$$

$$\text{(iv) } G_{\mu\nu\rho\sigma} + G_{\mu\sigma\nu\rho} + G_{\mu\rho\sigma\nu} = 0, \tag{21}$$

$$\text{(v) } G^\rho{}_{\mu\rho\nu} = \tfrac{1}{2}g_{\mu\nu}. \tag{22}$$

These are just the properties of the curvature tensor in the vacuum Einstein space (the corresponding cosmological constant happens to be $\frac{1}{2}$). Thus the classification of $G_{\mu\nu\rho\sigma}$ becomes the same as Petrov's classification of vacuum gravitational fields. Because of the additional constraint (17) in our case, however, Petrov type II and III fields are excluded and only one special case of type I remains possible. In the 6-dimensional notation $G_{AB} = G_{\mu\nu\rho\sigma}$, where A, B run over (0,1), (0,2), (0,3), (2,3), (3,1), and (1,2), $G_{\mu\nu\rho\sigma}$ is uniquely given by[7]

$$G_A{}^B = \tfrac{1}{2}\begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix}. \qquad (23)$$

Then the general solution to Eq. (10) is given by

$$F_A = (0, 0, \alpha, 0, 0, \beta), \quad \alpha, \beta \text{ arbitrary}. \qquad (24)$$

Hence the fields have the form $\vec{E}\|\vec{H}\|z$ axis as we expected.

In the case of the $\delta = 0$ field it is also eady to find a canonical form of $H_{\mu\nu\rho\sigma}$. Again in the 6-dimensional notation, $H_{\mu\nu\rho\sigma}$ is given by

$$H_A{}^B = \tfrac{1}{4}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} (1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0)$$

$$= \tfrac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & & & & 0 & \\ 0 & & & & 0 & \\ 0 & & & & 0 & \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & & & & 0 & \end{bmatrix}, \qquad (25)$$

where we have chosen $b_{i1}\|x$ axis and $b_{i2}\|y$ axis ($i = 1, 2, 3$).[7] Eigenvectors are given by

$$F_A = \alpha(1, 0, 0, 0, -1, 0), \quad \alpha \text{ arbitrary}. \qquad (26)$$

Hence $\vec{E}\|x$ axis and $\vec{H}\|y$ axis and $E_x = H_y$ as we expected. Thus we have checked that Eqs. (10) and (15) reproduce known results. So far as electromagnetism is concerned the above method is merely a reformulation of the theory. If we turn to the Yang-Mills field, however, we find it particularly useful to determine the mathematical structure of non-Abelian fields. For the definiteness of arguments we discuss in the following only the SU(2) Yang-Mills field. Our method is, however, not restricted to SU(2), and its generalizations to other groups are straightforward.

Our idea is essentially to classify the Yang-Mills field according to the radiative or nonradiative character of its component fields $F^i_{\mu\nu}$ ($i = 1, 2, 3$). We note, however, that quantities like $F^i_{\rho\sigma}F^{i\rho\sigma}$ or $F^i_{\rho\sigma}F^{i*\rho\sigma}$ (not summed over i) are not gauge invariant by themselves, and F^i's may in general change their types under SU(2) rotations. Thus, if we require F^i's to remain of the same type under SU(2) rotations, we obtain a strong constraint on the possible form of $\vec{F}_{\mu\nu}$.

Let us first consider the case where all F^i's remain of the nonradiative type in all isospin gauges. In this case we have

$$F^i_{\mu\nu} = G^i_{\mu\nu\rho\sigma}F^{i\rho\sigma} \quad (i = 1, 2, 3), \qquad (27)$$

where each G^i is determined by the round trip to the canonical frame of $F^i_{\mu\nu}$. At first we may expect that the canonical frames for different F^i's differ from each other. It is easy to see, however, that the correct transformation property of Eq. (27) under SU(2) is guaranteed only when G^i's are isoscalars. Hence G^i's are in fact independent of i. By a single and common Lorentz transformation F^i's are simultaneously brought to their canonical forms. $\vec{F}_{\mu\nu}$ is expressed as

$$\text{I:} \quad \vec{F}_{\mu\nu} = \vec{\rho}A_{\mu\nu} + \vec{\zeta}A^*_{\mu\nu}, \qquad (28)$$

where $A_{\mu\nu} = \frac{1}{2}[\xi_\mu, \eta_\nu]$, $A^*_{\mu\nu} = \frac{1}{2}[\xi_\mu, \eta_\nu]^* = [p_\mu, q_\nu]$, $\vec{\rho} = \vec{E}_z$, and $\vec{\zeta} = \vec{H}_z$ in the canonical frame. $A_{\mu\nu}$ in arbitrary frames is characterized by $A_{\rho\sigma}A^{\rho\sigma} = -2$, $A_{\rho\sigma}A^{*\rho\sigma} = 0$.

We next consider the case where only one of $F^i_{\mu\nu}$ ($i = 1, 2, 3$) can be made radiative by a particular SU(2)-gauge choice. In this case by using arguments similar to those above it is possible to show that the two remaining nonradiative fields have a common canonical frame. The form of $\vec{F}_{\mu\nu}$ is given by

$$\text{II:} \quad \vec{F}_{\mu\nu} = \vec{\rho}A_{\mu\nu} + \vec{\zeta}A^*_{\mu\nu} + \vec{\sigma}B_{\mu\nu}, \qquad (29)$$

where $\vec{\rho}\cdot\vec{\sigma} = \vec{\zeta}\cdot\vec{\sigma} = 0$ and $B_{\mu\nu}$ satisfies $B_{\rho\sigma}B^{\rho\sigma} = B_{\rho\sigma}B^{*\rho\sigma} = 0$.

The remaining possible types of fields are enumerated as follows:

$$\text{III:} \quad \vec{F}_{\mu\nu} = \vec{\rho}A_{\mu\nu} + \vec{\zeta}A^*_{\mu\nu} + \vec{\sigma}_1B^1_{\mu\nu} + \vec{\sigma}_2B^2_{\mu\nu}, \quad \vec{\rho}\cdot\vec{\sigma}_1 = \vec{\rho}\cdot\vec{\sigma}_2 = \vec{\zeta}\cdot\vec{\sigma}_1 = \vec{\zeta}\cdot\vec{\sigma}_2 = \vec{\sigma}_1\cdot\vec{\sigma}_2 = 0. \qquad (30)$$

$$\text{IV:} \quad \vec{F}_{\mu\nu} = \vec{\sigma}_1B^1_{\mu\nu} + \vec{\sigma}_2B^2_{\mu\nu} + \vec{\sigma}_3B^3_{\mu\nu}, \quad \vec{\sigma}_1\cdot\vec{\sigma}_2 = \vec{\sigma}_2\cdot\vec{\sigma}_3 = \vec{\sigma}_3\cdot\vec{\sigma}_1 = 0. \qquad (31)$$

In type III (IV) two (three) of the $F^i_{\mu\nu}$ become radiative under a particular gauge choice. The B^i's satisfy $B^i_{\rho\sigma}B^{i\rho\sigma} = B^i_{\rho\sigma}B^{i*\rho\sigma} = 0$ and may or may not be equal to each other. In the extreme case of type IV where $B^1 = B^2 = B^3$, all components $F^i_{\mu\nu}$ ($i = 1, 2, 3$) become radiative in all SU(2) gauges. This special class is obtained directly if we assume that

$$F^i_{\mu\nu} = H^i_{\mu\nu\rho\sigma}F^{i\rho\sigma} \quad (i = 1, 2, 3). \qquad (32)$$

In this case $\vec{F}_{\mu\nu}$ has a particularly simple form,

$$\vec{F}_{\mu\nu} = \vec{\sigma} B_{\mu\nu} . \tag{33}$$

By looking at the energy-momentum tensor of the above fields we recognize that type I is purely a nonradiative field, type IV is a pure radiation field, and types III and IV are composed of both components. These four are the fundamental types of SU(2) Yang-Mills fields.

Our next step is to introduce the concept of an ihg[8] in order to examine the differential properties of $\vec{F}_{\mu\nu}$. The ihg of a field $\vec{F}_{\mu\nu}(x)$ at a point x is defined as a subgroup of SU(2) generated by a set consisting of $\vec{F}_{\mu\nu}$ and its covariant derivatives,

$$\vec{F}_{\mu\nu}(x),\ \nabla_\lambda \vec{F}_{\mu\nu}(x),\ \nabla_\kappa \nabla_\lambda \vec{F}_{\mu\nu}(x), \ldots , \tag{34}$$

where Lorentz indices are to be contracted with arbitrary tensor quantities. If the entire ihg is generated solely from $\vec{F}_{\mu\nu}$, then the ihg is said to be perfect; otherwise it is imperfect. If the field is analytic in a certain region R, the same ihg obtains at each point $x \in R$. If there are singularities in the field, the ihg may in general be different on the manifold of singular points from that in analytic regions. For simplicity we do not consider such possibilities in this paper.

Let us consider the extreme case of a type-IV field where $B^1 = B^2 = B^3$ and $\vec{F}_{\mu\nu} = \vec{\sigma} B_{\mu\nu}$ in order to illustrate the method. In this case we discriminate two kinds of possibilities, $\nabla_\lambda \vec{\sigma} \times \vec{\sigma} = 0$ and $\nabla_\lambda \vec{\sigma} \times \vec{\sigma} \neq 0$. In the first case we have

$$\nabla_\lambda \vec{\sigma} = a_\lambda \vec{\sigma} \tag{35}$$

with a certain vector field a_λ. It is obvious that in this case the ihg is one-parameter perfect. By taking a commutator of derivatives we have

$$[\nabla_\nu, \nabla_\lambda]\vec{\sigma} = (\partial_\nu a_\lambda - \partial_\lambda a_\nu)\vec{\sigma} = g \vec{F}_{\nu\lambda} \times \vec{\sigma} = 0 . \tag{36}$$

It follows that a_λ is a gradient $\partial_\lambda \phi$ and $\vec{\sigma}' \equiv e^{-\phi}\vec{\sigma}$ is a covariant constant,

$$\nabla_\lambda \vec{\sigma}' = 0 . \tag{37}$$

Thus the one-parameter perfect ihg is characterized by the presence of a covariant constant isovector field.

On the other hand, if $\nabla_\lambda \vec{\sigma} \times \vec{\sigma} \neq 0$, $\vec{F}_{\mu\nu}$ and $\nabla_\lambda \vec{F}_{\mu\nu}$ generate a three-parameter imperfect ihg. In this way we arrive at Table I, which gives the classification of SU(2) Yang-Mills fields.

In order to illustrate the physical meaning of this classification let us consider a solution to a sourceless Yang-Mills field. Its equation of motion is given by

$$\begin{aligned} &\nabla^\mu \vec{F}_{\mu\nu}(x) = 0 , \\ &\nabla^\mu \vec{F}^*_{\mu\nu}(x) = 0 \end{aligned} \tag{38}$$

TABLE I. Classification of SU(2) Yang-Mills fields. The dimension of $\vec{F}_{\mu\nu}$ means the number of independent isovectors given by $\vec{F}_{\mu\nu}$ when it is contracted with arbitrary Lorentz tensors.

type	dimension of $\vec{F}_{\mu\nu}$	dimension of ihg	class
I	1	1	I_a
	1	3	I_b
	2	3	I_c
II	2	3	II_a
	3	3	II_b
III	2	3	III_a
	3	3	III_b
IV	1	1	IV_a
	1	3	IV_b
	2	3	IV_c
	3	3	IV_d

at all x. It is tedious but possible to show that $\vec{F}_{\mu\nu}$ belongs to the extreme case of type IV Eq. (33). Then the above equation can be rewritten as

$$\begin{aligned} &\vec{\sigma}\partial^\mu B_{\mu\nu} + (\nabla^\mu \vec{\sigma}) B_{\mu\nu} = 0 , \\ &\vec{\sigma}\partial^\mu B^*_{\mu\nu} + (\nabla^\mu \vec{\sigma}) B^*_{\mu\nu} = 0 . \end{aligned} \tag{39}$$

Hence $\vec{\sigma} \times \nabla^\mu \vec{\sigma} = 0$ and we arrive at the class IV_a. This proves that the vacuum Yang-Mills solution has only a trivial kind of symmetry, U(1), and there are no inherently nonlinear solutions to it.

Similarly, it can also be proved that a Yang-Mills field produced by classical point charges, either electric or magnetic, necessarily belongs to the class I_a. It then follows that $\vec{F}_{\mu\nu}$ is proportional to a covariant constant. For instance, if we consider the magnetic case, we have

$$\vec{F}_{\mu\nu} = \vec{\zeta}' \exp(\phi) A^*_{\mu\nu} \equiv \vec{\zeta}' F^M_{\mu\nu} , \tag{40}$$

where $F^M_{\mu\nu}$ is identical to the Dirac monopole field. The existence of a covariant constant in the non-Abelian extension of Dirac's monopoles has been pointed out previously.[9]

If there is a covariant constant field $\vec{\zeta}'$, it is always possible to find a gauge where $\vec{\zeta}' \times \vec{A}_\lambda = 0$. In fact the desired gauge transformation is given by the nonlocal transformation

$$\vec{\tau} \cdot \vec{\zeta}'^M(x,P) = U(x,P)\vec{\tau} \cdot \vec{\zeta}'(x) U^{-1}(x,P) , \tag{41}$$

$$U(x,P) = T_\xi \exp\left(ig \int_{-\infty}^{x} \vec{A}_\mu(\xi) \cdot \vec{\tau}\, d\xi^\mu \right), \tag{42}$$

where the integral is along a certain path P and T_ξ denotes an ordering of τ matrices. If $\nabla_\lambda \vec{\zeta}' = 0$ holds, then ζ'^M becomes independent of path and also independent of x. Hence $\vec{\zeta}'^M \times \vec{A}^M_\lambda = 0$. This means that if a non-Abelian field allows a covariant constant, then the symmetry is necessarily reduced to Abelian, i.e., U(1).

In the above we have given a general analysis of algebraic and differential properties of unquantized Yang-Mills fields. It has been found that they have relatively simple mathematical structures. An arbitrary Yang-Mills field is expressed as a sum of a few terms each of which is a product of two factors relevant to the isospin and Lorentz structure, respectively. In the simplest case $\vec{F}_{\mu\nu}$ becomes proportional to an isovector which can be made constant by a suitable gauge choice.

As compared with the full-fledged quantized theory of Yang-Mills fields, the knowledge of their classical aspects has been relatively poor. Our general examination of unquantized Yang-Mills fields may, we hope, be useful in further understanding the physics of non-Abelian fields.

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[1]See for example J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1956).

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[6]Our metric is $g_{00}=-g_{ii}=1$. $F_{\mu\nu}$ is defined as $F_{0i}=E_i$, $F_{ij}=-\epsilon_{ijk}H_k$. $F^*_{\mu\nu}$ is a dual of $F_{\mu\nu}$.

[7]Other forms obtained from this by permuting x, y, and z are, of course, possible.

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