Comments and Addenda

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Comments on quark-confinement potentials

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Explicit expansions are derived for the wave functions, eigenenergies, and Regge trajectories of the nonrelativistic wave equation for quark-confining potentials.

The recent discovery of heavy resonances in electron-positron annihilation has generated considerable interest in dynamical calculations of the masses of quark-antiquark bound states.[1-3] Since free quarks have not been observed, the quark-binding potential is believed to increase with their separation so that the quarks cannot be pulled apart by any finite amount of energy.[4] The deeper theoretical justification[5] for a simple phenomenology based on such potentials comes from a field theory of quarks coupled minimally to non-Abelian gauge fields. Such a theory has the desirable property of being asymptotically free in the short-distance limit but coupled strongly in the large-distance domain.[6] The equivalent effective potential behaves as $1/(r\ln r)$ in the short-distance domain of quark-quark separations, and as r^λ, $\lambda \geqslant 1$ in the large-distance domain.

Confinement potentials have been used recently in numerous investigations[7] dealing with hadron spectroscopy. Lacking explicit solutions of wave equations for such potentials, many authors resort to numerical methods.

The purpose of this paper is to show that it is easy to find explicit perturbation expansions for both the eigenvalues and eigenfunctions of Schrödinger-type wave equations containing confinement potentials. By application to ψ spectroscopy one can check that even the few terms evaluated explicitly here for the eigenvalues yield very good agreement with the results obtained by numerical methods.

We consider the Schrödinger equation for the unscreened power potential $V(\vec{r}) = g|\vec{r}|^\lambda$. Separating off the motion of the center of mass in the usual way, we obtain the radial wave equation for the relative motion of two quarks of masses m_1, m_2

$$\frac{d^2\psi}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - V(r)\right]\psi = 0\,, \tag{1}$$

where, as usual, $\Psi = (1/r)\psi(r)P_l^m(\cos\theta)e^{im\varphi}$, $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass of the two quarks and r is their separation. Throughout we ignore an arbitrary additive constant in V.

Inserting the power potential, we have the equation

$$\frac{d^2\psi}{dr^2} + \left(\alpha - \beta r^\lambda - \frac{\gamma}{r^2}\right)\psi = 0\,, \tag{2}$$

where we have set

$$\alpha = \frac{2\mu E}{\hbar^2}\,, \quad \beta = \frac{2\mu g}{\hbar^2}\,, \quad \gamma = l(l+1) \tag{3}$$

and $\lambda \geqslant 1$ for confinement of the quarks. We now solve Eq. (2) in a region where the dimensionless quantity $\alpha^{1+2/\lambda}/\beta^{2/\lambda}$ is large but finite. The usefulness of this solution will be discussed below.

We set

$$r = e^z\,, \quad \psi = e^{z/2}\phi\,. \tag{4}$$

Equation (2) then becomes

$$\frac{d^2\phi}{dz^2} + \left[-\gamma - \tfrac{1}{4} + v(z)\right]\phi = 0\,, \tag{5}$$

where $-\infty < z < \infty$ and

$$v(z) = \alpha e^{2z} - \beta e^{(2+\lambda)z}\,. \tag{6}$$

Next we find that value of z, say z_0, for which $v(z)$ becomes maximal. In the vicinity of this maximum $v(z)-\gamma-\frac{1}{4}$ can become positive and the solutions therefore oscillatory as required for the existence of eigenvalues. Simple differentiation yields the result

$$z_0=\frac{1}{\lambda}\ln\left[\frac{2\alpha}{(2+\lambda)\beta}\right] \tag{7}$$

for $\alpha>0$, $\beta>0$. It may be noted that the spectrum of bound states lies in the domain $E>0$ in much the same way as for the simple harmonic oscillator.

Expanding $v(z)$ in the neighborhood of the maximum at z_0 we obtain

$$v(z)=v(z_0)+\sum_{i=2}^{\infty}\frac{(z-z_0)^i}{i!}v^{(i)}(z_0), \tag{8}$$

where

$$v^{(i)}(z_0)=2\alpha\left[\frac{2\alpha}{\beta(2+\lambda)}\right]^{2/\lambda}[2^{i-1}-(2+\lambda)^{i-1}] \tag{9}$$

for $i=0,1,2,\ldots$. For $i=0$ this expression is positive, for $i=1$ it is zero, and for $i>1$ it is negative [as required for a maximum of $v(z)$ at $z=z_0$ for $\alpha>0$]. We now set

$$h=\left\{4\alpha\lambda\left[\frac{2\alpha}{\beta(2+\lambda)}\right]^{2/\lambda}\right\}^{1/4}$$

$$=\left\{\frac{8\mu E}{\hbar^2}\lambda\left[\frac{2E}{(2+\lambda)g}\right]^{2/\lambda}\right\}^{1/4} \tag{10}$$

and change the independent variable in (5) to

$$\omega=h(z-z_0). \tag{11}$$

The equation then becomes

$$\frac{d^2\phi}{d\omega^2}+\left[\frac{-\frac{1}{4}-\gamma+h^4/[4(2+\lambda)]}{h^2}-\frac{\omega^2}{4}\right]\phi$$
$$=\frac{1}{2\lambda}\sum_{i=3}^{\infty}\left[\frac{(2+\lambda)^{i-1}-2^{i-1}}{i!}\right]\frac{\omega^i}{h^{i-2}}\phi. \tag{12}$$

For large values of h the right-hand side of this equation may—to a first approximation—be neglected. The corresponding behavior of the "eigenvalues" $(1/h^2)[-\frac{1}{4}-\gamma+h^4/4(2+\lambda)]$ may then be determined by comparing the equation with the equation of parabolic cylinder functions. The solutions are square-integrable only if

$$\frac{1}{h^2}\left[-\frac{1}{4}-\gamma+\frac{h^4}{4(2+\lambda)}\right]=\tfrac{1}{2}q,$$

where q is an odd integer, i.e., $2n+1$, $n=0,1,2,\ldots$ (provided the wave function is required to vanish at infinity; otherwise it is only approximately an odd integer). For the complete solution we set

$$\frac{1}{h^2}\left[-\frac{1}{4}-\gamma+\frac{h^4}{4(2+\lambda)}\right]=\tfrac{1}{2}q+\frac{\Delta}{h}. \tag{13}$$

The quantity Δ in (13) remains to be determined. We proceed as follows. Substituting (13) in (12) we have an equation which may be written

$$\mathfrak{D}_q\phi=\frac{2\Delta}{h}\phi-\frac{1}{\lambda}\sum_{i=3}^{\infty}\left[\frac{(2+\lambda)^{i-1}-2^{i-1}}{i!}\right]\frac{\omega^i}{h^{i-2}}\phi, \tag{14}$$

where

$$\mathfrak{D}_q\equiv-2\frac{d^2}{d\omega^2}-q+\tfrac{1}{2}\omega^2. \tag{15}$$

Equation (14) is now in a form suitable for the application of our perturbation method. To a first approximation $\phi=\phi^{(0)}$ is simply a parabolic cylinder function $D_{(q-1)/2}(\omega)$, i.e.,

$$\phi^{(0)}=\phi_q=D_{(q-1)/2}(\omega),\quad \mathfrak{D}_q\phi_q=0. \tag{16}$$

We have

$$D_{(q-1)/2}(\omega)=2^{(q-3)/4}e^{-\omega^2/4}\omega\Psi\left(\frac{3-q}{4},\tfrac{3}{2};\frac{\omega^2}{2}\right),$$

where Ψ is a confluent hypergeometric function. The function ϕ_q is well known to obey the recurrence formula

$$\omega\phi_q=(q,q+2)\phi_{q+2}+(q,q-2)\phi_{q-2}, \tag{17}$$

where

$$(q,q+2)=1,\quad (q,q-2)=\tfrac{1}{2}(q-1). \tag{18}$$

For higher powers we have

$$\omega^i\phi_q=\sum_{j=2i}^{-2i}S_i(q,j)\phi_{q+j}, \tag{19}$$

and a recurrence relation may be written down for the coefficients S_i. The first approximation $\phi=\phi^{(0)}$ then leaves uncompensated terms amounting to

$$R_q^{(0)}=\left\{\frac{2\Delta}{h}-\frac{1}{\lambda}\sum_{i=3}^{\infty}\left[\frac{(2+\lambda)^{i-1}-2^{i-1}}{i!}\right]\frac{\omega^i}{h^{i-2}}\right\}\phi_q(\omega)$$

$$=\frac{2\Delta}{h}\phi_q-\sum_{i=3}^{\infty}\frac{1}{h^{i-2}}\sum_{j=2i}^{-2i}\tilde{S}_i(q,j)\phi_{q+j}(\omega), \tag{20}$$

where we have set

$$\tilde{S}_i(q,j)=\frac{1}{\lambda}\left[\frac{(2+\lambda)^{i-1}-2^{i-1}}{i!}\right]S_i(q,j). \tag{21}$$

We rewrite (20) in the form

$$R_q^{(0)}=\sum_{i=3}^{\infty}\frac{1}{h^{i-2}}\sum_{j=2i}^{-2i}[q,q+j]_i\phi_{q+j}(\omega), \tag{22}$$

where

$$[q,q]_3 = 2\Delta - \tilde{S}_3(q,0)$$

and for $j \neq 0$,

$$[q,q+j]_3 = -\tilde{S}_3(q,j) \tag{23}$$

and for $i > 3$,

$$[q,q+j]_i = -\tilde{S}_i(q,j)\,.$$

Since $\mathfrak{D}_{q+j} = \mathfrak{D}_q - j$, $\mathfrak{D}_q \phi_{q+j} = j\phi_{q+j}$, a term $\mu\phi_{q+j}$ in $R_q^{(0)}$ may be removed by adding to $\phi^{(0)}$ the contribution $\mu\phi_{q+j}/j$ except, of course, when $j=0$.

Thus the next-order contribution of ϕ becomes

$$\phi^{(1)} = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=2i \\ j\neq 0}}^{-2i} \frac{[q,q+j]_i}{j} \phi_{q+j}(\omega)\,. \tag{24}$$

In its turn this contribution leaves uncompensated

$$R_q^{(1)} = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=2i \\ j\neq 0}}^{-2i} \frac{[q,q+j]_i}{j} R_{q+j}^{(0)}\,, \tag{25}$$

and yields the next contribution of ϕ:

$$\phi^{(2)} = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=2i \\ j\neq 0}}^{-2i} \frac{[q,q+j]_i}{j} \sum_{i'=3}^{\infty} \frac{1}{h^{i'-2}} \sum_{\substack{j'=2i' \\ j+j'\neq 0}}^{-2i'} \frac{[q+j,q+j+j']_{i'}}{j+j'} \phi_{q+j+j'}\,. \tag{26}$$

Proceeding in this way we obtain the solution $\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \cdots$, which is an asymptotic expansion in descending powers of h valid for

$$|\ln(r/r_0)| < O\left(\frac{1}{h}\right), \tag{27}$$

where $r_0 = e^{z_0} = [2E/(2+\lambda)g]^{1/\lambda}$ and jointly an eigenvalue equation from which Δ in (13) may be determined. The latter is obtained by setting equal to zero the sum of the terms in ϕ_q in $R_q^{(0)}, R_q^{(1)}, \ldots$ which have been unaccounted for so far. Thus

$$0 = \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} [q,q]_i + \sum_{i=3}^{\infty} \frac{1}{h^{i-2}} \sum_{\substack{j=2i \\ j\neq 0}}^{-2i} \frac{[q,q+j]_i}{j} \sum_{i'=3}^{\infty} \frac{1}{h^{i'-2}} [q+j,q]_{i'} + \cdots$$

or

$$0 = \frac{1}{h}[q,q]_3 + \frac{1}{h^2}\left\{[q,q]_4 + \sum_{\substack{j=6 \\ j\neq 0}}^{-6} \frac{[q,q+j]_3}{j}[q+j,q]_3\right\} + O\left(\frac{1}{h^3}\right). \tag{28}$$

This is the equation from which Δ and hence the eigenvalues are determined. The expansion on the right-hand side is much simpler than may appear on first sight because [owing to the fact that there are no terms in ϕ_{q+1}, ϕ_q, ϕ_{q-1} in (17)] many terms are zero. E.g., $\tilde{S}_3(q,0)$, $\tilde{S}_3(q,\pm 1)$, $\tilde{S}_3(q,\pm 3)$, $\tilde{S}_3(q,\pm 5)$ vanish. Thus

$$\begin{aligned}
\Delta &= \frac{1}{2h}[\tilde{S}_4(q,0) - \tfrac{1}{6}\tilde{S}_3(q,6)\tilde{S}_3(q+6,-6) + \tfrac{1}{6}\tilde{S}_3(q,-6)\tilde{S}_3(q-6,6) - \tfrac{1}{2}\tilde{S}_3(q,2)\tilde{S}_3(q+2,-2) + \tfrac{1}{2}\tilde{S}_3(q,-2)\tilde{S}_3(q-2,2)] + O\left(\frac{1}{h^2}\right) \\
&= \frac{1}{2^5 3^2 h}[9(q^2+1)(\lambda^2+6\lambda+12) - (15q^2+7)(\lambda+4)^2] + O\left(\frac{1}{h^2}\right) \\
&= -\frac{1}{2^3 3^2 h}(51q^2+1) + O\left(\frac{1}{h^2}\right) \text{ when } \lambda = 1 \\
&= -\frac{q^2}{h} + O\left(\frac{1}{h^2}\right) \text{ when } \lambda = 2\,.
\end{aligned} \tag{29}$$

We now consider the case $\lambda = 1$. Inserting (29) into (13) we obtain

$$l(l+1) = \frac{8\mu}{27}\frac{E^3}{g^2\hbar^2} - \frac{2\sqrt{2}\,q}{3}\frac{\mu^{1/2}}{\hbar}\frac{E^{3/2}}{g} + \frac{17(3q^2-1)}{72} + O\left(\frac{g^{1/2}}{E^{3/4}}\right).$$

Solving for l we obtain ($q=2n+1$)

$$l=-\tfrac{1}{2}\pm\tfrac{2}{3}\left(\frac{2\mu}{3}\right)^{1/2}\frac{E^{3/2}}{g\hbar}\left[1-\frac{9q\hbar}{4(2\mu)^{1/2}}\frac{g}{E^{3/2}}-\frac{3(3q^2-1)}{128}\frac{\hbar^2}{\mu}\frac{g^2}{E^3}+\cdots\right] \tag{30}$$

and solving for E we get ($q=2n+1$)

$$E\equiv E_q\simeq\left(\frac{3g\hbar}{4(2\mu)^{1/2}}\right)^{2/3}\left\{3q\underset{(-)}{+}\frac{1}{6^{1/2}}[(3q^2+17)+72l(l+1)]^{1/2}\right\}^{2/3}. \tag{31}$$

In (31) [and hence in (30)] we choose the upper sign in order to obtain a real value of E for the ground state.

The masses of the bound states are given by

$$M^{(q)}=m_1+m_2+E_q \tag{32}$$

apart from the arbitrary additive constant in V. In the literature[1,2,3,5,7] the case of S waves ($l=0$) has been discussed. There the formula corresponding to (32) is given in terms of zeros of Airy functions. It is interesting to see the connection. Thus, setting $l=0$ and $\lambda=1$ in (2) we have

$$\frac{d^2\psi}{dr'^2}+(\alpha'-r')\psi=0, \tag{33}$$

where $r'=\beta^{1/3}r$ and $\alpha'=\alpha/\beta^{2/3}$. The solution of this equation which is finite at both $r=0$ and $r=\infty$ is the Airy function[8] which is given by

$$\mathrm{Ai}(r'-\alpha')=\frac{1}{\pi}\left(\frac{r'-\alpha'}{3}\right)^{1/2}K_{1/3}\left(\tfrac{2}{3}(r'-\alpha')^{3/2}\right) \tag{34a}$$

for α' real and $r'>\alpha'$ where $K_\nu(z)$ is a modified Bessel function. For $r'<\alpha'$ this function is given by

$$\mathrm{Ai}(r'-\alpha')=\tfrac{1}{3}(\alpha'-r')^{1/2}\left[J_{1/3}\left(\tfrac{2}{3}(\alpha'-r')^{3/2}\right)+J_{-1/3}\left(\tfrac{2}{3}(\alpha'-r')^{3/2}\right)\right]. \tag{34b}$$

The eigenvalues are now given[9] by the zeros of the function $\mathrm{Ai}(-\alpha')$. If α_q' ($q=2n+1$, $n=0,1,2,\ldots$) are those values of α' for which $\mathrm{Ai}(-\alpha')=0$, we have

$$E=\left(\frac{g^2\hbar^2}{2\mu}\right)^{1/3}\alpha_q', \tag{35}$$

and so

$$M^{(q)}=m_1+m_2+\left(\frac{g^2\hbar^2}{2\mu}\right)^{1/3}\alpha_q' \quad (l=0,\lambda=1). \tag{36}$$

Comparing (35) with (31) we obtain an approximate formula for the zeros of the Airy function:

$$\alpha_q'\simeq(\tfrac{3}{4})^{2/3}\left[3q+\left(\frac{3q^2+17}{6}\right)^{1/2}\right]^{2/3}, \tag{37}$$

which is most accurate for low q (0.8% error for $q=1$).

The eigenfunctions normalized according to $\int\psi^2(r')dr'=1$ are[8,9]

$$\psi_q=-\frac{\mathrm{Ai}(r'-\alpha_q')}{\left[\frac{d}{dr}\mathrm{Ai}(r)\right]_{r=-\alpha_q'}}.$$

By expanding $\mathrm{Ai}(r'-\alpha_q')$ around $r'=0$ and realizing that the full wave function is normalized according to $\int|\Psi(r,\theta,\phi)|^2d\Omega dr=1$ we obtain at the origin

$$|\Psi(r=0)|^2=\frac{g\mu}{2\pi\hbar^2}. \tag{38}$$

Formulas (36) and (38) coincide with those given by Harrington *et al.*[2]

Thus, Eq. (30) gives the Regge trajectories and Eqs. (31) or (35) the eigenenergies of the nonrelativistic system for the linear confinement potential. The solution ϕ derived above is valid in the region (27). It is possible to derive other branches of the solution in other regions of validity but the eigenvalue expansion is the same. The method can also be applied to relativistic equations, or equations incorporating relativistic kinematics, although the algebra becomes more complicated.

Of course, the simple power potential considered here represents the unscreened quark-quark interaction. As the quarks separate, this unscreened potential between them grows until the energy is such that a pair of light quarks is created out of the vacuum, which then combine with the original (heavy) quarks to form a pair of mesons. This creation of quark pairs has the effect of screening the confinement potential. These screened interactions are roughly analogous to Yukawa or Gauss potentials, for which the wave equation has been solved in a similar way.[10]

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