Study of the allowed domains for the parameters of $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ chiral symmetry breaking*

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(Received 19 August 1975)

The allowed domains of the parameters describing a chiral-symmetry-breaking Hamiltonian belonging to the $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ representations of $SU(3)\times SU(3)$ are investigated. In addition to constraints on these parameters arising from spectral-function positivity we employ conditions corresponding to bounds on the ratio f_K/f_π of kaon- to pion-decay constants and on the $\pi\pi$ S-wave $I=0$ and 2 scattering lengths. It is found that these constraints are extremely effective in reducing the size of the allowed domains. Particular attention is focused on the Gell-Mann–Oakes–Renner, Okubo, and Sirlin-Weinstein forms of symmetry breaking, which are special cases of the general $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ scheme.

I. INTRODUCTION

The $SU(3)\times SU(3)$ structure of the strong-interaction Hamiltonian remains uncertain despite many attempts at its elucidation. In general, the Hamiltonian density is written

$$\mathcal{H}(x)=\mathcal{H}_0(x)+\mathcal{H}_{SB}(x)\,, \qquad (1.1)$$

where $\mathcal{H}_0(x)$ is an $SU(3)\times SU(3)$ invariant and $\mathcal{H}_{SB}(x)$ breaks $SU(3)\times SU(3)$ symmetry in a well-defined manner. In one of the most popular models[1-3] of symmetry breaking $\mathcal{H}_{SB}(x)$ transforms as the scalar isoscalar component of the $(3,3^*)\oplus(3^*,3)$ representation of $SU(3)\times SU(3)$. This scheme has been recently brought into question with the determination that the πN σ term may be as large as 60 MeV (see Ref. 4) and that the $\pi\pi$ S-wave $I=0$ scattering length (a_0^0) may be as large as 0.5 m_π^{-1}.[5] In addition, there is some difficulty with the decay $\eta\to\pi^+\pi^-\pi^0$ in this model.[6,7]

In light of these developments other representations of $SU(3)\times SU(3)$ have been considered for $\mathcal{H}_{SB}$. These include the $(1,8)\oplus(8,1)$, the $(8,8)$, and the $(6,6^*)\oplus(6^*,6)$ representations. The $(1,8)\oplus(8,1)$ form has been considered by Gell-Mann, Oakes, and Renner[2] and its contribution (based on soft-pion arguments) is expected to be small. The $(8,8)$ representation was introduced by Barnes and Isham[8] and studied in more detail by Genz and Katz[9] among others. The $(6,6^*)\oplus(6^*,6)$ representation has been investigated by Auvil, and McDonald, Rosen and Kuo.[10]

The entire symmetry-breaking situation is made more complicated because the symmetry seems to be realized in a Nambu-Goldstone manner.[11] Thus, even when all explicit symmetry-breaking terms are turned off, the chiral symmetry is not realized in the usual manner, but the solutions belong to some subgroup of chiral $SU(3)\times SU(3)$. This subgroup is usually assumed to be $SU(3)$.

One of the more interesting hypotheses of chiral symmetry breaking is that $\mathcal{H}_{SB}$ belongs to more than one representation of $SU(3)\times SU(3)$. This idea is especially interesting when one of the forms chosen is that of the $(3,3^*)\oplus(3^*,3)$ representation. Indeed, we can then let this part be the dominant contribution and conserve chiral $SU(2)\times SU(2)$ symmetry; the other representations provide small corrections and generate the pion mass. These ideas have been introduced by Okubo[12] and by Sirlin and Weinstein[13] among others. Okubo has argued that $\mathcal{H}_{SB}$ might contain the $SU(2)\times SU(2)$-invariant part of the $(3,3^*)\oplus(3^*,3)$ representation plus any other $SU(3)$ singlet. Sirlin and Weinstein suggest that $\mathcal{H}_{SB}$ contain the even-parity $SU(3)$ singlet and $I=0$ octet components in the $(8,8)$ representation in addition to the $SU(2)\times SU(2)$-invariant part of the $(3,3^*)\oplus(3^*,3)$.

In this paper we assign $\mathcal{H}_{SB}$ to the $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ representations of $SU(3)\times SU(3)$ and investigate the Okubo and Sirlin-Weinstein forms of symmetry breaking and other special cases. A $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry-breaking structure has been considered previously by a number of authors.[4,7,13-17] The present study is devoted to an analysis of the allowed domains of the various symmetry-breaking parameters contained in $\mathcal{H}_{SB}$. To this end we use the spectral representation technique of Okubo and Mathur,[18] who analyzed the allowed domains for the parameters describing the $(3,3^*)\oplus(3^*,3)$ symmetry-breaking scheme.

In order to outline the technique of Okubo and Mathur we first consider the Lehmann-Källén spectral representation for the chiral current commutators. One has[18,19]

$$\langle 0|[V_{\mu i}(x), V_{\nu j}(x')]|0\rangle = i\int_0^\infty dm^2\left[\rho_{ij}(m^2)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)-\frac{\sigma_{ij}(m^2)}{m^2}\partial_\mu\partial_\nu\right]\Delta(x-x';m^2) \tag{1.2}$$

and

$$\langle 0|[A_{\mu i}(x), A_{\nu j}(x')]|0\rangle = i\int_0^\infty dm^2\left[\rho^5_{ij}(m^2)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)-\frac{\sigma^5_{ij}(m^2)}{m^2}\partial_\mu\partial_\nu\right]\Delta(x-x';m^2)\,, \tag{1.3}$$

where $V_{\mu i}(x)$ and $A_{\mu i}(x)$ $(i=1,\ldots,8)$ are the usual vector and axial-vector current octets, the space integrals of whose time components are the generators of $\mathrm{SU}(3)\times\mathrm{SU}(3)$, F_i and F^5_i, respectively; ρ_{ij}, ρ^5_{ij} and σ_{ij}, $\sigma^5{}_{ij}$ are the transverse and longitudinal weight functions. Taking the divergence of Eqs. (1.2) and (1.3), setting $\nu=0$, and integrating over space leads to (assuming no subtraction is required)

$$K_{ij}\equiv\int_0^\infty dm^2\sigma_{ij}(m^2)=i\langle 0|[\partial^\mu V_{\mu i},F_j]|0\rangle$$

and (1.4)

$$I_{ij}\equiv\int_0^\infty dm^2\sigma^5_{ij}(m^2)=i\langle 0|[\partial^\mu A_{\mu i},F^5_j]|0\rangle\,.$$

The positivity constraints on σ_{ij} and σ^5_{ij} next allow one to write[18,19]

$$K_{ii}\geq 0\quad(\text{no sum})$$

and (1.5)

$$I_{ii}\geq 0\quad(\text{no sum})\,.$$

These conditions (Eq. 1.5) will be used as the primary constraints to limit the allowed domains of the symmetry-breaking parameters.

Following Deshpande and Dicus[20] the domains can be reduced considerably by investigating the form of σ^5_{ij}. Explicitly

$$q^2\sigma^5_{ij}(q^2) = (2\pi)^3\sum_n \delta^4(p_n-q)\langle 0|\partial^\mu A_{\mu i}|n\rangle\langle n|\partial^\nu A_{\nu j}|0\rangle\,. \tag{1.6}$$

If we then assume that the one-meson states saturate the sum over states and define, in the usual manner,

$$[(2\pi)^3 2\omega_j]^{1/2}\langle 0|A_{\mu i}|\phi_j(p)\rangle = if_i\delta_{ij}p_\mu \tag{1.7}$$

we have

$$I_{33}=m_\pi^2 f_\pi^2 \tag{1.8}$$

and

$$I_{44}=m_K^2 f_K^2\,. \tag{1.9}$$

Consequently

$$\left(\frac{f_K}{f_\pi}\right)^2=\frac{I_{44}}{I_{33}}\left(\frac{m_\pi}{m_K}\right)^2\,. \tag{1.10}$$

This relation can be used to impose severe constraints on the symmetry-breaking parameters, if we use the experimental values for the masses and require that

$$\frac{f_K}{f_\pi}>0.8 \tag{1.11}$$

and

$$\frac{f_K}{f_\pi}<1.7\,, \tag{1.12}$$

as suggested by experiment.[21] We do not make these requirements too restrictive as Eq. (1.10) is not an identity.[22]

We can restrict the domains further, if we make use of a soft-pion relation[13,23] for the π-π scattering amplitude. To first order in the chiral-symmetry-breaking parameters the $I=0$ and 2 S-wave π-π scattering lengths can be expressed as

$$a^0_0=\frac{1}{96\pi m_\pi}\left[5A+16\frac{m_\pi^2}{f_\pi^2}\right] \tag{1.13}$$

and

$$a^2_0=\frac{1}{48\pi m_\pi}\left[A-4\frac{m_\pi^2}{f_\pi^2}\right]; \tag{1.14}$$

A, which depends on the parameters which characterize the symmetry breaking in the Hamiltonian and of the vacuum, will be given in Sec. II. π-π phase-shift analyses are in general agreement[24] that a^0_0 and a^2_0 are restricted to the respective regions

$$a^0_0>0 \tag{1.15}$$

and

$$a^2_0<0\,. \tag{1.16}$$

The restriction in (1.15) seems especially secure. These bounds imply that A is confined to

$$-\frac{16}{5}\frac{m_\pi^2}{f_\pi^2}<A<4\frac{m_\pi^2}{f_\pi^2}\,. \tag{1.17}$$

Actually, we will relax (1.17) somewhat to take into account the approximate nature of Eqs. (1.13) and (1.14).

This completes the set of general constraints used to determine the allowed domains for the symmetry-breaking parameters. We will also restrict these allowed domains in several other ways. For example, for aesthetic reasons[12,13] and on the basis of past experience[25] we will usually assume that the $(3,3^*)\oplus(3^*,3)$ part of $\mathcal{H}_{SB}$ is dominant over that belonging to $(8,8)$. In addition, although considerable latitude is permitted, we will not concentrate on regions of the (otherwise) allowed domains corresponding to excessive (relative to the GMOR model) SU(3) breaking in the $(3,3^*)\oplus(3^*,3)$ part of $\mathcal{H}_{SB}$. We will be especially interested in the Okubo and Sirlin-Weinstein versions in which the $(3,3^*)\oplus(3^*,3)$ contribution is $SU(2)\times SU(2)$-invariant. Finally, we will assume that the vacuum is much more nearly SU(3), than $SU(3)\times SU(3)$-invariant.

Other studies undertaken using similar approaches include the work of Okubo and Mathur[18] who consider both the case where $\mathcal{H}_0$ is $U(3)\times U(3)$-invariant and where it is $SU(3)\times SU(3)$-invariant. They relate the domain boundaries to the realization of various subgroups of $U(3)\times U(3)$ and $SU(3)\times SU(3)$, respectively. They also briefly discuss the case where $\mathcal{H}_{SB}$ is contained in the $(3,3^*)\oplus(3^*,3)\oplus(1,8)\oplus(8,1)$ representation of $SU(3)\times SU(3)$. Carruthers and Haymaker[19] use this approach to discuss the manner in which $U(3)\times U(3)$ symmetry is reduced to $SU(3)\times SU(3)$. Hu[26] also employs this technique to investigate chiral symmetry breaking of the $(3,3^*)\oplus(3^*,3)$ type with the addition of an isospin-breaking term. More recently, Deshpande and Dicus[20] have investigated briefly the $(6,6^*)\oplus(6^*,6)$ and $(8,8)$ forms of symmetry breaking using this method.

In Sec. II we derive explicit forms for the K_{ij} and I_{ij} tensors for the case in which $\mathcal{H}_{SB}$ belongs to the $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ representations. In Sec. III we analyze the allowed domains of the symmetry-breaking parameters for some special cases. Section IV contains a discussion of our results.

II. CALCULATION OF K_{ij} AND I_{ij}

We begin this section by outlining the properties of the $(3,3^*)\oplus(3^*,3)$ and $(8,8)$ representations of chiral $SU(3)\times SU(3)$ that are used to form $\mathcal{H}_{SB}$. Then, with the form of $\mathcal{H}_{SB}$ specified, we derive the expressions for K_{ij} and I_{ij}. Some of the assumptions to be used in Sec. III, where various specific forms of symmetry breaking will be considered, are also discussed.

The $(3,3^*)\oplus(3^*,3)$ representation includes nonets of scalar and pseudoscalar densities, $u_\alpha(x)$ and $v_\alpha(x)$, respectively, which obey the following equal-time commutation relations[1]: $(i=1,2,\ldots,8;\ \alpha,\beta=0,1,2,\ldots,8)$:

$$[F_i,u_\alpha(x)]=if_{i\alpha\beta}u_\beta(x),\quad [F_i,v_\alpha(x)]=if_{i\alpha\beta}v_\beta(x),\quad [F_i^5,u_\alpha(x)]=-id_{i\alpha\beta}v_\beta(x),\tag{2.1}$$

and

$$[F_i^5,v_\alpha(x)]=id_{i\alpha\beta}u_\beta(x),$$

where F_i and F_i^5 are the usual generators of chiral $SU(3)\times SU(3)$.

We next discuss the $(8,8)$ representation in some detail. Consider the fields $\psi_i^L\in(8,1)$ and $\psi_i^R\in(1,8)$ $(i=1,2,\ldots,8)$. Then

$$\psi_{ij}=\psi_i^L\times\psi_j^R\in(8,8).\tag{2.2}$$

These tensors transform under $F_i^+=\frac{1}{2}(F_i+F_i^5)$ and $F_i^-=\frac{1}{2}(F_i-F_i^5)$, the generators of $SU(3)\times SU(3)$, as

$$[F_i^+,\psi_{jk}]=if_{ijl}\psi_{lk}$$

and

$$[F_i^-,\psi_{jk}]=if_{ikl}\psi_{jl}.\tag{2.3}$$

Consequently

$$[F_i,\psi_{jk}]=if_{ijl}\psi_{lk}+if_{ikl}\psi_{jl}$$

and

$$[F_i^5,\psi_{jk}]=if_{ijl}\psi_{lk}-if_{ikl}\psi_{jl}.\tag{2.4}$$

Operators of definite symmetry can be defined as

$$S_{ij}=\tfrac{1}{2}(\psi_{ij}+\psi_{ji})$$

and

$$A_{ij}=\tfrac{1}{2}(\psi_{ij}-\psi_{ji}).\tag{2.5}$$

These operators have the following transformation properties:

$$[F_i,S_{jk}]=if_{ijl}S_{kl}+if_{ikl}S_{jl},\quad [F_i,A_{jk}]=-if_{ijl}A_{kl}+if_{ikl}A_{jl},\quad [F_i^5,S_{jk}]=-if_{ijl}A_{kl}-if_{ikl}A_{jl},\tag{2.6}$$

and

$$[F_i^5,A_{jk}]=if_{ijl}S_{kl}-if_{ikl}S_{jl}.$$

If we assume that the S_{ij} and A_{ij} have definite parity, it is clear from the above commutation relations that they must have opposite parities. The only SU(3) singlet which can be formed from the S_{ij} and A_{ij} is S_{ii}. If this is to be used in (a nonweak part of) $\mathcal{H}_{SB}$, then it, and the S_{ij}, must have even parity. Recalling the identities[27]

$$d_{ijk}d_{ljk}=\tfrac{5}{3}\delta_{il}$$

and

$$f_{ijk}f_{ljk}=3\delta_{il},\tag{2.7}$$

we define the singlet and the even- and odd-parity octets in (8, 8) by[28]

$$z_0=(\tfrac{1}{8})^{1/2}S_{ii}\,,$$
$$z_i=(\tfrac{3}{5})^{1/2}d_{ijk}S_{jk}\,, \qquad (2.8)$$

and

$$w_i=(\tfrac{1}{3})^{1/2}f_{ijk}A_{jk}\,,$$

respectively. With these definitions the S_{ij} and A_{ij} decompose into irreducible representations of SU(3) in the following manner:

$$S_{ij}=(\tfrac{1}{8})^{1/2}\delta_{ij}z_0+(\tfrac{3}{5})^{1/2}d_{ijk}z_k+\text{operators from }\underline{27} \qquad (2.9)$$

and

$$A_{ij}=(\tfrac{1}{3})^{1/2}f_{ijk}w_k+\text{operators from }\underline{10}\oplus\underline{\overline{10}}\,. \qquad (2.10)$$

These reductions will be useful in projecting the various representations of SU(3) in which we are interested.

The SU(3) singlet and octets satisfy the following equal-time commutation relations with F_i:

$$[F_i,z_0(x)]=0\,,$$
$$[F_i,z_j(x)]=if_{ijk}z_k(x)\,, \qquad (2.11)$$

and

$$[F_i,w_j(x)]=if_{ijk}w_k(x)\,.$$

The commutation relations with F_i^5 are more difficult to evaluate. The easiest is

$$[F_i^5,z_0]=-i\sqrt{\tfrac{3}{2}}\,w_i\,, \qquad (2.12)$$

which can be calculated in a straight-forward manner. For the even-parity octet one obtains

$$[F_i^5,z_j]=-i2\sqrt{\tfrac{3}{5}}\,d_{jkl}f_{ikm}A_{lm}\,. \qquad (2.13)$$

Using the decomposition (2.10) and the identity[27]

$$d_{ijk}f_{klm}f_{mni}=-\tfrac{3}{2}d_{jln} \qquad (2.14)$$

gives

$$[F_i^5,z_j]=-i\frac{3}{\sqrt5}d_{ijk}w_k+\left(i\frac{3}{\sqrt5}d_{ijk}w_k-i2\frac{\sqrt3}{\sqrt5}d_{jkl}f_{ikm}A_{lm}\right)\,, \qquad (2.15)$$

where the latter term is contained in $10\oplus\overline{10}$. For the odd-parity octet

$$[F_i^5,w_j]=i\frac{2}{\sqrt3}f_{jkl}f_{ikm}S_{lm}\,. \qquad (2.16)$$

Using Eq. (2.9) and the identities of Eqs. (2.7) and (2.14) leads to

$$[F_i^5,w_j]=i(\tfrac{3}{2})^{1/2}\delta_{ij}z_0+i\frac{3}{\sqrt5}d_{ijk}z_k+\left[i\frac{2}{\sqrt3}f_{jkl}f_{ikm}S_{lm}-i(\tfrac{3}{2})^{1/2}\delta_{ij}z_0-i\frac{3}{\sqrt5}d_{ijk}z_k\right]\,, \qquad (2.17)$$

where the term in brackets belongs to the $\underline{27}$ representation of SU(3).

The symmetry-breaking Hamiltonian density studied here has the structure

$$\mathcal{H}_{SB}(x)=c_0u_0(x)+c_8u_8(x)+d_0z_0(x)+d_8z_8(x)\,, \qquad (2.18)$$

where the subscript zero denotes the SU(3) singlet and eight denotes the $I=0$ component of the octet. With the form of $\mathcal{H}_{SB}(x)$ specified we can proceed to obtain explicit expressions for the tensors K_{ij} and I_{ij}.

Recalling the definitions of these tensors given in Eq. (1.4) and using the divergence relations[1]

$$\partial_\mu V_i^\mu(x)=-i[F_i,\mathcal{H}_{SB}(x)]$$

and (2.19)

$$\partial_\mu A_i^\mu(x)=-i[F_i^5,\mathcal{H}_{SB}(x)]\,,$$

we can rewrite the tensors as

$$K_{ij}=-\langle 0|[F_j[F_i,\mathcal{H}_{SB}]]|0\rangle$$

and (2.20)

$$I_{ij}=-\langle 0|[F_j^5[F_i^5,\mathcal{H}_{SB}]]|0\rangle\,.$$

In this form one can demonstrate easily that both K_{ij} and I_{ij} are symmetric in i and j by means of the Jacobi identity.

Using the structure of $\mathcal{H}_{SB}$ given in Eq. (2.18) and the commutation relations in Eqs. (2.1) and (2.11) one obtains

$$K_{ij}=-f_{i8k}f_{j8k}\{c_8\xi_8+d_8\xi_8'\}\,, \qquad (2.21)$$

where

$$\xi_8=\langle 0|u_8|0\rangle$$

and (2.22)

$$\xi_8'=\langle 0|z_8|0\rangle\,.$$

First, note that $K_{ij}=0$ for $i\neq j$. Secondly, observe that, due to isospin invariance, we need only consider K_{33}, K_{44}, and K_{88}. These tensors are

$$K_{33}=0\,,$$
$$K_{44}=-\tfrac{3}{4}\{c_8\xi_8+d_8\xi_8'\}\,, \qquad (2.23)$$

and

$$K_{88}=0\,.$$

The calculations of I_{ij} must be considered in more detail. We start by evaluating $[F_i^5,\mathcal{H}_{SB}]$. Using Eqs. (2.1), (2.12), and (2.13) one finds

$$[F_i^5, \mathcal{H}_{SB}] = -i[(\tfrac{2}{3})^{1/2} c_0 v_i + c_8 d_{i8\alpha} v_\alpha + (\tfrac{3}{2})^{1/2} d_0 w_i + 2(\tfrac{3}{5})^{1/2} d_8 d_{8kl} f_{ikm} A_{lm}] . \tag{2.24}$$

The last term includes the components of $\underline{10} \oplus \overline{10}$ that must be retained. From Eqs. (2.1) and (2.17)

$$[F_j^5[F_i^5, \mathcal{H}_{SB}]] = (\tfrac{2}{3})^{1/2} c_0 d_{ij\alpha} u_\alpha + c_8 d_{8i\alpha} d_{j\alpha\beta} u_\beta + \tfrac{3}{2} d_0 \delta_{ij} z_0 + \frac{3\sqrt{3}}{\sqrt{10}} d_0 d_{ijk} z_k - i2(\tfrac{3}{5})^{1/2} d_8 d_{8kl} f_{ikm} [F_j^5, A_{lm}] + \text{terms from } \underline{27} . \tag{2.25}$$

Evaluating the remaining commutator expression using Eq. (2.6) and extracting the singlet and octet parts contained therein using Eq. (2.9), we find a value for this term of

$$d_8 \left[\frac{3\sqrt{3}}{\sqrt{10}} d_{8ij} z_0 + \tfrac{6}{5} d_{8kl} f_{ikm} (f_{jln} d_{mnp} - f_{jmn} d_{lnp}) z_p + \text{terms from } \underline{27} \right] . \tag{2.26}$$

By using various identities relating the f and d tensors[27] and the fact that $I_{ij} = I_{ji}$ this expression can be rewritten in the form

$$d_8 \frac{3\sqrt{3}}{\sqrt{10}} d_{8ij} z_0 + d_8 [f_{8ik} f_{kjl} - \tfrac{12}{5} d_{ilk} d_{j8k} + 2\delta_{8l}\delta_{ij} - \tfrac{9}{5} d_{8lk} d_{kij}] z_l + \text{terms from } \underline{27} . \tag{2.27}$$

Nonvanishing terms in I_{ij} occur only for u and z indices of 0 and 8. As in Eq. (2.22) we define

$$\xi_0 = \langle 0 | u_0 | 0 \rangle$$

and (2.28)

$$\xi_0' = \langle 0 | z_0 | 0 \rangle .$$

Then I_{ij} has the structure

$$I_{ij} = -\tfrac{2}{3} c_0 \delta_{ij} \xi_0 - (\tfrac{2}{3})^{1/2} c_0 d_{8ij} \xi_8 - (\tfrac{2}{3})^{1/2} c_8 d_{8ij} \xi_0 - c_8 d_{8ik} d_{8jk} \xi_8 - \tfrac{3}{2} d_0 \delta_{ij} \xi_0' - \frac{3\sqrt{3}}{\sqrt{10}} d_0 d_{8ij} \xi_8'$$
$$- \frac{3\sqrt{3}}{\sqrt{10}} d_8 d_{8ij} \xi_0' + d_8 \left(f_{8ik} f_{8jk} + \tfrac{12}{5} d_{8ik} d_{8jk} - 2\delta_{ij} - \frac{3\sqrt{3}}{5} d_{8ij} \right) \xi_8' + \text{terms from } \underline{27} . \tag{2.29}$$

At this stage we will assume that operators belonging to the $\underline{27}$ representation have negligible vacuum expectation values. We then note that $I_{ij} = 0$ for $i \neq j$. Again, using isospin invariance, only I_{33}, I_{44}, and I_{88} need be considered.

In order to make the analysis of the structure of the allowed domains of the symmetry-breaking parameters easier, we will rewrite the K_{ij}'s and I_{ij}'s of interest in a simpler form. To this end we first define

$$a = \frac{1}{\sqrt{2}} \frac{c_8}{c_0} ,$$
$$b = \frac{1}{\sqrt{2}} \frac{\xi_8}{\xi_0} ,$$
$$\gamma = -\tfrac{2}{3} c_0 \xi_0 ,$$
$$\alpha = \tfrac{9}{4} \frac{d_0 \xi_0'}{c_0 \xi_0} , \tag{2.30}$$
$$y = \frac{1}{\sqrt{10}} \frac{d_8}{d_0} ,$$
$$z = \frac{1}{\sqrt{10}} \frac{\xi_8'}{\xi_0'} ,$$

and

$$\rho = \frac{9}{4\sqrt{10}} \frac{d_8 \xi_0'}{c_0 \xi_0} .$$

The last definition will only be used in cases where $d_0 = 0$. From Eqs. (2.23) and (2.29) we obtain

$$K_{44} = \frac{\gamma}{4} (9ab + 20\alpha y z) , \tag{2.31}$$

$$I_{33} = \gamma [1 + a + b + ab + \alpha(1 + 2y + 2z + 12yz)] , \tag{2.32}$$

$$I_{44} = \gamma \left[1 - \frac{a}{2} - \frac{b}{2} + \frac{ab}{4} + \alpha(1 - y - z + 5yz)\right] , \tag{2.33}$$

and

$$I_{88} = \gamma [1 - a - b + 3ab + \alpha(1 - 2y - 2z + 4yz)] . \tag{2.34}$$

It is interesting to note that these expressions are symmetric under the interchange of a and b and of y and z.

In order to make use of the π-π scattering length bounds discussed in the Introduction, we need the expression, in terms of the symmetry-breaking parameters, for the quantity A appearing in Eqs. (1.13) and (1.14). A is the constant term in a low-energy expansion of the π-π scattering amplitude and was found to be given by[13,23]

$$A(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})=-\frac{1}{f_\pi^4}\{\langle 0|[F_l^5,[F_k^5,[F_j^5,[F_i^5,\mathcal{H}_{SB}(0)]]]]|0\rangle+(ljik)+(kjil)+O(\mathcal{H}_{SB}{}^2)\}$$

$$(i,j,k,l=1,2,3)\,. \qquad (2.35)$$

The commutation relations developed above, together with Eq. (2.18), lead to

$$A=\frac{\gamma}{f_\pi^4}[1+a+3\alpha(1+\tfrac{10}{3}y)]\,. \qquad (2.36)$$

We will eliminate γ from Eq. (2.36) using Eq. (2.32) and assuming that Eq. (1.8) holds, since we are more interested in the other symmetry-breaking parameters.[29] f_π will be taken to be 95 MeV.

To summarize from the Introduction, the symmetry-breaking parameters (2.30) will be restricted to those regions which are implied by the requirements that

$$\begin{aligned} &K_{44}\geq 0\,,\\ &I_{33}\geq 0\,,\\ &I_{44}\geq 0\,,\\ &I_{88}\geq 0\,, \end{aligned} \qquad (2.37)$$

$$0.64\leq\left(\frac{m_\pi}{m_K}\right)^2\frac{I_{44}}{I_{33}}\leq 2.89\,, \qquad (2.38)$$

and

$$-3.2\left(0.6\pi+\frac{m_\pi{}^2}{f_\pi^2}\right)\leq A\leq 4\left(1.2\pi+\frac{m_\pi{}^2}{f_\pi^2}\right), \qquad (2.39)$$

where the upper and lower bounds on A correspond to the somewhat conservative restrictions $a_2^0<0.1$ and $a_0^0>-0.1$, respectively.[30]

The above constraints involve as many as five parameters in whose domains we are interested. Consequently, some method of simplification must be employed to make a useful analysis feasible. Thus, we will consider many approximations and special cases that we feel may be physically interesting.

As indicated in the Introduction we expect the $(3,3^*)\oplus(3^*,3)$ terms to dominate those from the $(8,8)$ representation. Hence, the parameters α and ρ should not be too large. In addition, we will not consider the otherwise allowed portions of the domains with $|a|\gtrsim 2$. Also, because the vacuum is probably more nearly SU(3)-invariant than SU(3) $\times$ SU(3)-invariant, we will (where possible) restrict the parameters b and z such that $|b|$, $|z|$ $\lesssim 0.5$. Finally, the SU(3) octet and singlet nature of the symmetry breaking implies that we can neglect the $\underline{27}$-plet contributions to the K and I tensors (which we have done and will continue to do).

Studies of $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry breaking within the context of the linear SU(3) σ model[17] tend to support these assumptions. For instance, in this model, $b=-0.16$ and $z=-0.016$. In addition to the GMOR scheme the σ model also seems to prefer those solutions in which the $(3,3^*)\oplus(3^*,3)$ contribution to $\mathcal{H}_{SB}$ is SU(2) $\times$ SU(2)-invariant (i.e., the Okubo[12] and Sirlin-Weinstein[13] versions). Thus, we feel that there is some justification, other than merely that of simplicity, for the special cases to be considered.

III. THE ALLOWED DOMAINS FOR THE SYMMETRY-BREAKING PARAMETERS IN SPECIAL CASES

In the following analysis six possible variations of the basic SU(3) $\times$ SU(3)-symmetry-breaking Hamiltonian given in Eq. (2.18) are considered, including the forms suggested by Okubo[12] and by Sirlin and Weinstein.[13] Many parameters are involved in the analysis. In order to demonstrate the structure of the parameter domains clearly we will employ a diagrammatic technique and consider planes in a higher dimensional space. In most cases the behavior of the domains as a function of the symmetry-breaking parameters can be easily isolated and the basic features can be illustrated with a few diagrams.

In each of Figs. 1–4, used to illustrate the allowed domains, a sequence of constraints is indicated, progressing from what we believe are the most firmly based to those conditions which depend on stronger assumptions. Thus, within a particular figure, the largest allowed domain [shown as diagonal lines ($\gamma\geq 0$) and vertical lines ($\gamma\leq 0$)] corresponds to regions permitted by the general positivity constraints (2.37). Superimposed on the latter, the next larger domains (shown as diagonally cross-hatched) correspond to the imposition of conditions (2.37) together with the f_K/f_π bounds (2.38). Finally, the smallest domains (shown as dotted) result from the complete set of constraints (2.37), (2.38), and the π-π scattering length conditions (2.39).

The domains presented in Figs. 1–4 are not intended to reproduce the exact form of the domain of the symmetry-breaking parameters. Rather, they are intended to demonstrate the basic structure of these domains. Some changes in the domains as the f_K/f_π and π-π scattering length constraints are imposed are too small to be shown. In addition, some of the smaller domains presented in the figures have been slightly enlarged

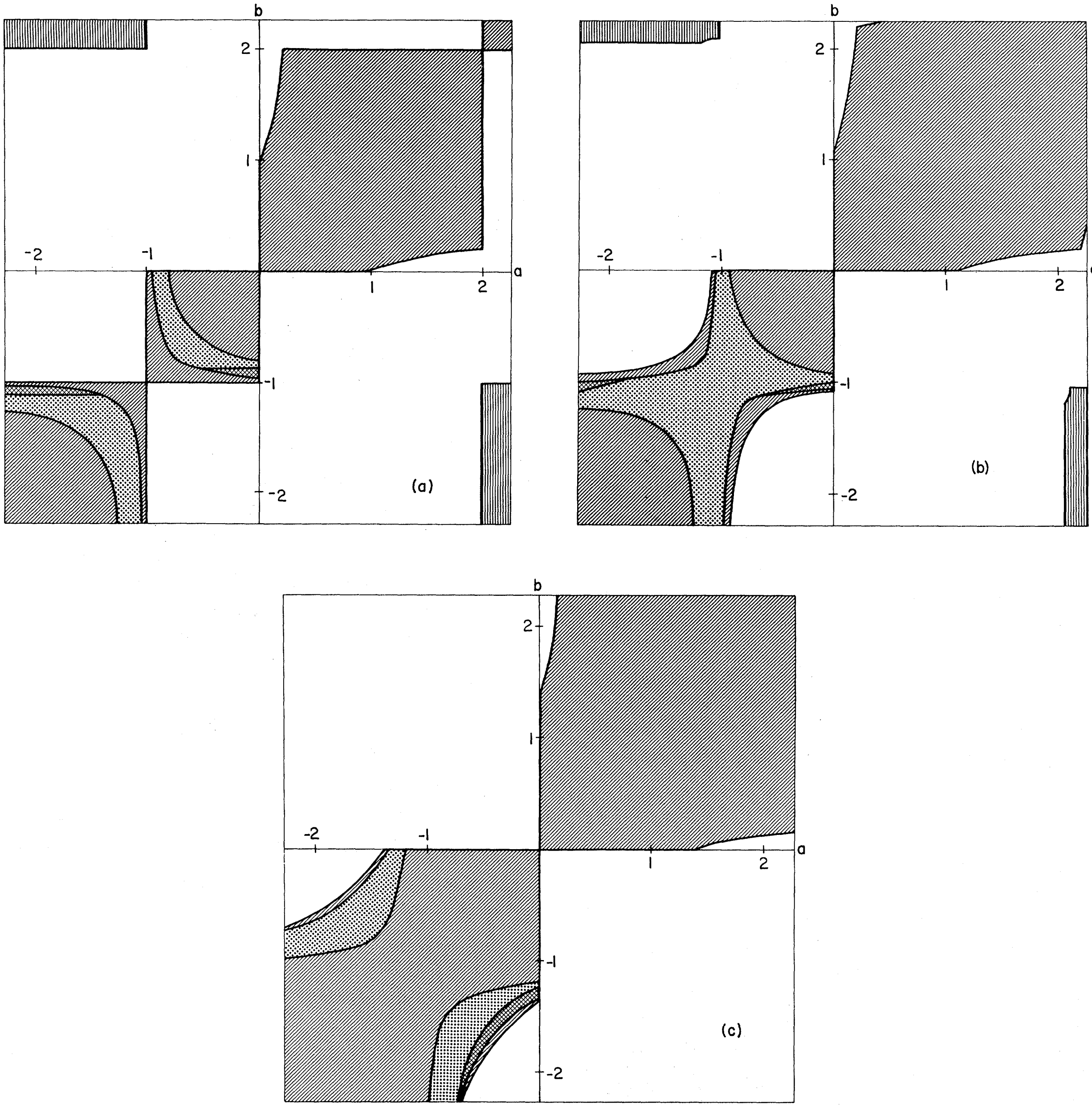

FIG. 1. (a) Allowed domains of a and b for $\mathcal{H}_{SB}=c_0u_0+c_8u_8+d_0z_0$ with $\alpha=0$ and $z=0$; (b) allowed domains of a and b for $\mathcal{H}_{SB}=c_0u_0+c_8u_8+d_0z_0$ with $\alpha=0.1$ and $z=0$; (c) allowed domains of a and b for $\mathcal{H}_{SB}=c_0u_0+c_8u_8+d_0z_0$ with $\alpha=0.4$ and $z=0$.

in order to be shown.

Our analyses of the various cases below will deal with the behavior of the fully constrained domains. Inasmuch as the constraint (2.38) seems to us to be reasonably loose, we will not discuss the sensitivity of the domains to changes in the (effective) bounds on f_K/f_π. Although the scattering-length conditions may be the least trustworthy of the constraints, they do not, in fact, appreciably decrease the size of the allowed domains in most cases. We will comment occasionally on the effect of strengthening the scattering length constraints.

Finally, for all the cases considered here, the allowed domains of interest (to us) correspond to positive values of the parameter γ [see (2.30)]. Its magnitude is not determined by the constraints.

Case 1

We begin our analysis by adding the (8, 8) SU(3) singlet term to the general $(3, 3^*)\oplus(3^*, 3)$ symmetry-breaking Hamiltonian so that

$$\mathcal{H}_{SB} = c_0 u_0 + c_8 u_8 + d_0 z_0 \,. \tag{3.1}$$

Hence $y=0$. We also take the (probably small) parameter $z=0$ for the sake of simplicity. The resulting tensors are (from Eqs. 2.31–2.34)

$$K_{44} = \tfrac{9}{4}\gamma ab \,,$$
$$I_{33} = \gamma(1+a+b+ab+\alpha)\,, \tag{3.2}$$
$$I_{44} = \gamma\left(1 - \frac{a}{2} - \frac{b}{2} + \frac{ab}{4} + \alpha\right),$$

and

$$I_{88} = \gamma(1-a-b+3ab+\alpha)\,.$$

For the diagrammatic analysis we will employ a three-dimensional space with coordinates (a, b, α) and investigate the (a, b) plane for various fixed values of α. The resulting domain structure is illustrated in Figs. 1(a) through 1(c).

We note first that for $\alpha = 0$ [Fig. 1(a)] the (8, 8) contribution vanishes. This case has been considered in detail by Okubo and Mathur.[18] It is evident that the restrictions of Eqs. (2.37), (2.38), and (2.39) severely limit the allowed domains of a and b. The GMOR solution[2] is within the allowed domains. The boundary of the domain closest to the origin is sensitive to the upper limit imposed on a_2^0, while the other domain is sensitive to the lower limit on a_0^0. For instance, the requirement $a_2^0 \leq 0.0$ will eliminate the Brandt-Preparata solution.[31] Imposing $a_0^0 \gtrsim 0.1$ eliminates the domain more remote from the origin. Imposing $a_0 \gtrsim 0.38$ eliminates the entire plane.

For $\alpha \neq 0$ the allowed values of a and b change dramatically as α varies. For α large and negative there are no allowed domains in the vicinity of the origin. As α becomes less negative this situation continues until $\alpha \approx -0.4$, when a small domain is allowed. This domain increases in size as α increases and goes through the structures of Figs. 1(a) and 1(b) to that of Fig. 1(c) with $\alpha = 0.0$, 0.1, and 0.4, respectively. Then as α continues to increase the domains remain in the latter form and move further away from the origin and axes.

For $b > -1$ as α increases a must rapidly become more negative. For example, at $\alpha \approx 0.75$ we must have $a < -1.5$. For small b and $-1.15 \leq a \leq -0.85$ we expect $0 \leq \alpha \leq 0.35$. In general the constraints favor positive values for α, since these lead to larger allowed areas of the domains.

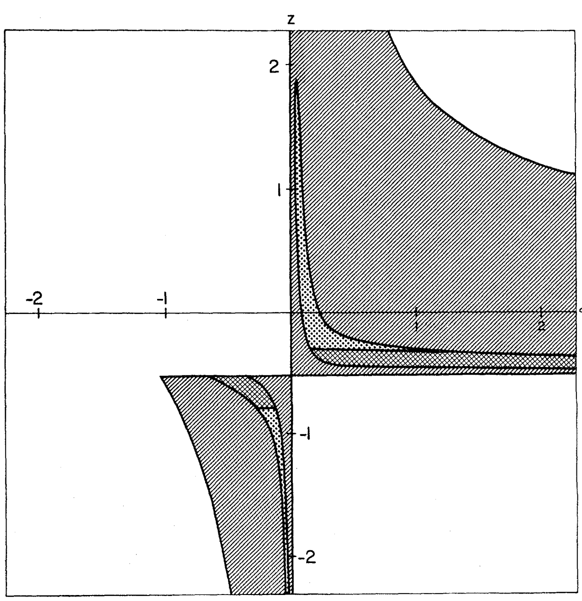

FIG. 2. Allowed domains of α and z for $\mathcal{H}_{SB} = c_0(u_0 - \sqrt{2}u_8) + d_0 z_0$ with $b = -0.2$.

Case 2

As the second special case the (8, 8) $I=0$ octet term is added to the general $(3, 3^*)\oplus(3^*, 3)$ symmetry-breaking Hamiltonian. The form of $\mathcal{H}_{SB}$ is then

$$\mathcal{H}_{SB} = c_0 u_0 + c_8 u_8 + d_8 z_8 \,. \tag{3.3}$$

For simplicity z is again set equal to zero. The four tensors are

$$K_{44} = \tfrac{9}{4}\gamma ab \,,$$
$$I_{33} = \gamma(1+a+b+ab+2\rho)\,, \tag{3.4}$$
$$I_{44} = \gamma\left(1 - \frac{a}{2} - \frac{b}{2} + \frac{ab}{4} - \rho\right),$$

and

$$I_{88} = \gamma(1-a-b+3ab-2\rho)\,.$$

The domains now lie in a three-dimensional space with coordinates (a, b, ρ). The diagrams in the (a, b) plane for fixed values of ρ have the same general structure as Figs. 1(a) through 1(c). In this case there are no domains in the vicinity of the origin until $\rho \gtrsim -0.3$. The domains then move through the three stages of Figs. 1(a) through 1(c) faster than they did in Case 1. For instance, at $\rho = 0.5$ $a \lesssim -2.0$ and $b \lesssim -0.2$. For ρ between 0.0 and 0.15 and $|b|$ small, a is allowed in the region $-1.15 \lesssim a \lesssim -0.85$. On the basis of the size of allowed areas, positive values of ρ are favored.

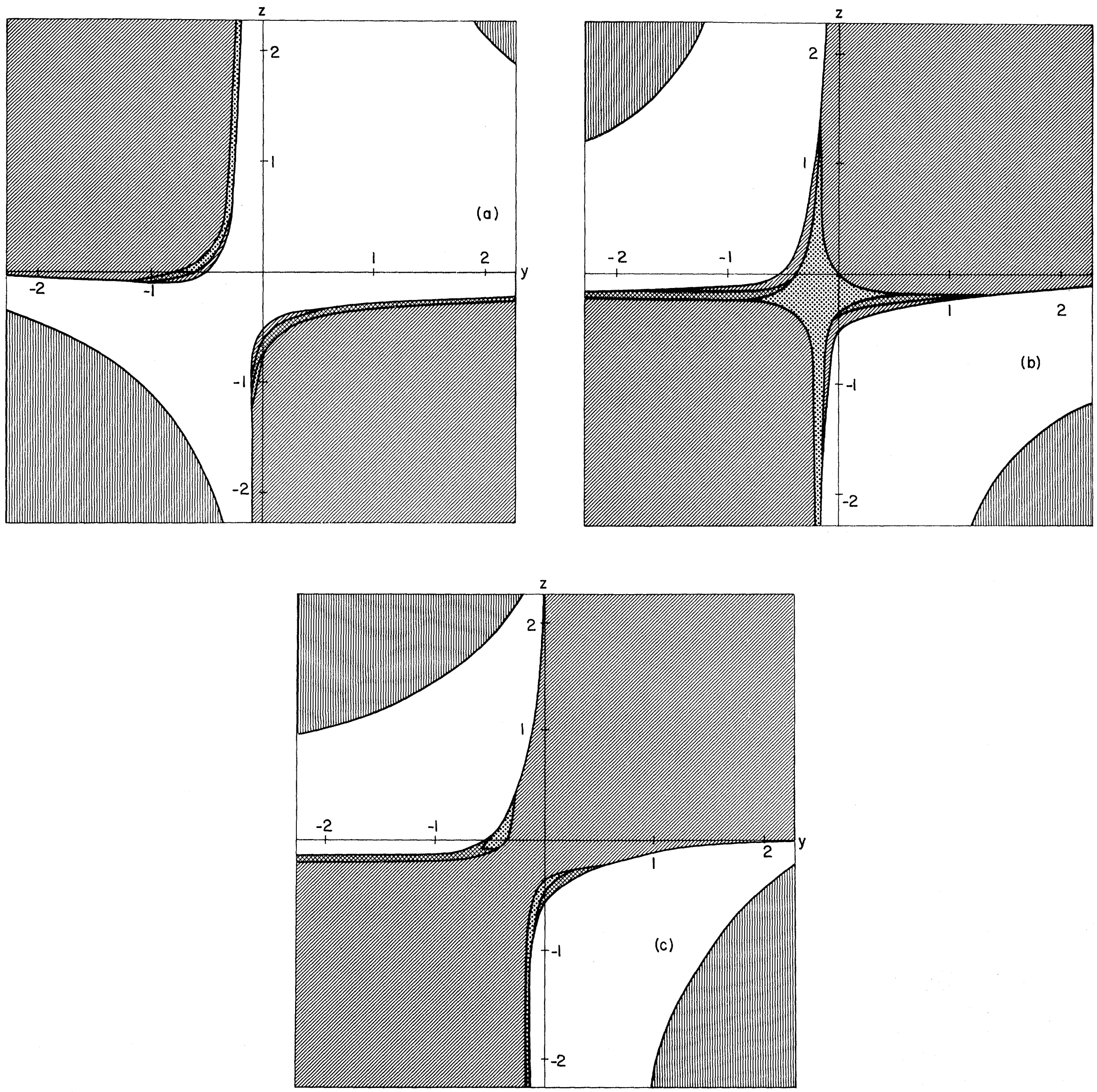

FIG. 3. (a) Allowed domains of y and z for $\mathcal{H}_{SB}=c_0(u_0-\sqrt{2}u_8)+d_0z_0+d_8z_8$ with $b=-0.1$ and $\alpha=-0.25$; (b) Allowed domains of y and z for $\mathcal{H}_{SB}=c_0(u_0-\sqrt{2}u_8)+d_0z_0+d_8z_8$ with $b=-0.1$ and $\alpha=0.2$; (c) Allowed domains of y and z for $\mathcal{H}_{SB}=c_0(u_0-\sqrt{2}u_8)+d_0z_0+d_8z_8$ with $b=-0.1$ and $\alpha=0.5$.

Case 3

In this case we force the $(3,3^*)\oplus(3^*,3)$ part of $\mathcal{H}_{SB}$ to be invariant under $SU(2)\times SU(2)$ and add the $(8,8)$ SU(3) singlet. This is the type of symmetry breaking suggested by Okubo.[12] Thus

$$\mathcal{H}_{SB}=c_0(u_0-\sqrt{2}u_8)+d_0z_0 \tag{3.5}$$

so that $a=-1$ and $y=0$. The tensors are

$$\begin{aligned} K_{44}&=-\tfrac{9}{4}\gamma b\,,\\ I_{33}&=\gamma\alpha(1+2z)\,,\\ I_{44}&=\gamma[\tfrac{3}{2}-\tfrac{3}{4}b+\alpha(1-z)]\,, \end{aligned} \tag{3.6}$$

and

$$I_{88}=\gamma[2-4b+\alpha(1-2z)]\,.$$

We consider a three-dimensional space with coordinates (α, z, b) and investigate the (α, z) plane for various fixed values of b. One immediately finds that the conditions $\gamma > 0$ and $b \leq 0$ are required. Figure 2 (with $b = -0.2$) illustrates the general domain structure in this case. These domains are relatively independent of b as b becomes more negative.

We expect that z will be small and, in particular, that $z > -0.5$. This, in turn, requires that α be positive. In addition, as the domains are relatively independent of b we expect (for $b \gtrsim -1.0$) that $\alpha \lesssim 1.5$. As z approaches zero α is further constrained to lie in the region $0.08 \leq \alpha \leq 0.2$.

If the scattering length constraint on a_0^0 is strengthened so that $a_0^0 > 0.30$, only small negative values of z $(-0.35 \lesssim z \lesssim -0.25)$ and positive values of α in the range $0.1 \lesssim \alpha \lesssim 1.0$ are allowed.

This type of symmetry breaking has been investigated using a linear SU(3) σ model.[17] In that model it was found that $b = -0.16$, $z = -0.016$, and $\alpha = 0.091$. These values are in the allowed domains.

Case 4

For this case the entire (8, 8) contribution is added to the SU(2) × SU(2)-invariant $(3, 3^*) \oplus (3^*, 3)$ part of $\mathcal{H}_{SB}$, giving

$$\mathcal{H}_{SB} = c_0(u_0 - \sqrt{2}u_8) + d_0 z_0 + d_8 z_8 . \quad (3.7)$$

This form for $\mathcal{H}_{SB}$ was suggested by Sirlin and Weinstein.[13] Again setting $z = 0$ for simplicity,

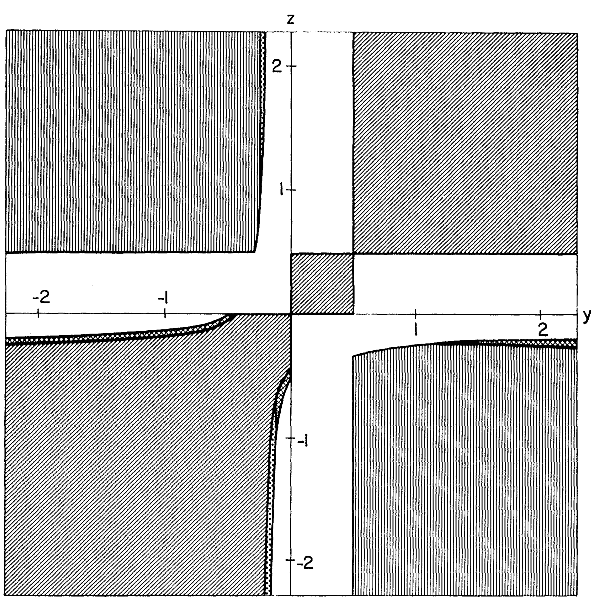

FIG. 4. Allowed domains of y and z for $\mathcal{H}_{SB} = d_0 z_0 + d_8 z_8$.

$$K_{44} = -\tfrac{9}{4}\gamma b ,$$
$$I_{33} = \gamma\alpha(1 + 2y) , \quad (3.8)$$
$$I_{44} = \gamma[\tfrac{3}{2} - \tfrac{3}{4}b + \alpha(1 - y)] ,$$

and

$$I_{88} = \gamma[2 - 4b + \alpha(1 - 2y)] .$$

Due to the y-z symmetry of the general tensors Eq. (3.8) has the same structure as Eq. (3.6). The resulting domain structure is nearly identical to that of Fig. 2 with the z axis becoming the y axis. The only difference is that the straight lines bounding the final domains are moved vertically in such a way that these domains become larger. This difference is due to the nonsymmetric condition imposed on y and z by the scattering length requirement.

It is again expected that $b \leq 0$, $\alpha \geq 0$, and $y \gtrsim -0.5$ following the analysis of the previous case.

Case 5

This case is identical to that of Case 4 except that z is no longer forced to be zero. Thus, $\mathcal{H}_{SB}$ is given by Eq. (3.7) and the tensors are now $(a = -1)$

$$K_{44} = \frac{\gamma}{4}(-9b + 20\alpha yz) ,$$
$$I_{33} = \gamma\alpha(1 + 2y + 2z + 12yz) ,$$
$$I_{44} = \gamma[\tfrac{3}{2} - \tfrac{3}{4}b + \alpha(1 - y - z + 5yz)] ,$$

and

$$I_{88} = \gamma[2 - 4b + \alpha(1 - 2y - 2z + 4yz)] .$$

The domain structure is analyzed by considering a four-dimensional space with coordinates (y, z, α, b). b is fixed and the domains in the (y, z) plane are investigated for various fixed values of α. First we note that, in order to get the allowed domains in the vicinity of the origin, it was necessary to have b negative or $|\alpha|$ large $(|\alpha| \gtrsim 2)$. Consequently we concentrated on the domains for negative b. As in Case 3 the domains are relatively independent of the value of b as b becomes more negative. Figs. 3(a) through 3(c) give the allowed domains for $b = -0.1$ and $\alpha = -0.25$, 0.2, and 0.5, respectively.

From the general form of the domain as a function of α it is found that the area of the allowed domains decreases rapidly as $|\alpha|$ increases. For $\alpha < 0$ we expect y and z to have opposite signs. For $z > -\frac{1}{6}$ $(\alpha < 0)$ note that y must be negative and, most likely, in the range $-1 \leq y \leq -\frac{1}{6}$.

For $\alpha > 0$ the domains go through the structure of Fig. 3(b), where there is a large area available near the origin, to that of Fig. 3(c). Again for

$z > -\frac{1}{6}$ we require $y < 0$ and expect $-1 \le y \le -\frac{1}{6}$.

This case has been investigated in the linear SU(3) σ model.[17] It was found that $b = -0.16, \alpha = 0.20$, $y = -0.36$, and $z = -0.016$. These values are in the allowed domains.

Case 6

We now consider the (8,8) contributions alone. The Hamiltonian density is then

$$\mathcal{H}_{SB} = d_0 z_0 + d_8 z_8 .$$

The tensors are

$$K_{44} = 5\gamma\alpha y z ,$$

$$I_{33} = \gamma\alpha(1 + 2y + 2z + 12yz) ,$$

$$I_{44} = \gamma\alpha(1 - y - z + 5yz) ,$$

and

$$I_{88} = \gamma\alpha(1 - 2y - 2z + 4yz) .$$

The allowed domains for this case are given in Fig. 4. We expect $\gamma\alpha$ to be positive and y and z both to be negative. The final allowed domain is exceedingly narrow, but requires $|z|$ to be rather large ($|z| \gtrsim 0.7$). The σ model[17] does not give an acceptable solution in this case as it cannot fit the pseudoscalar mass spectrum.

IV. DISCUSSION

It is obvious from the above analysis that the constraints (2.37)–(2.39) greatly restrict the ranges of the symmetry-breaking parameters. The positivity conditions (2.37) leave substantial domains open to the parameters. However, the condition (2.38) resulting from the bounds on f_K/f_π (especially the lower bound) proves very effective in reducing the sizes of these domains.

The π-π scattering length condition (2.39) usually produces rather small additional shrinkage, but its effects are important in Cases 3 (the Okubo[12] form), 5 (the Sirlin-Weinstein[13] form), and 6 [pure (8,8) symmetry breaking] as can be seen from Figs. 2, 3, and 4. In the Okubo case the constraint (2.39) sharply reduces the range of α [which is proportional to the ratio of the strength of the (8,8) contribution to that of the $(3,3^*)\oplus(3^*,3)$] from what is allowed by (2.37) and (2.38). In the Sirlin-Weinstein and pure (8,8) cases it is the ratio of SU(3) octet to singlet breaking within the (8,8) (the parameter y) which is impressively curtailed when (2.39) is applied after (2.37) and (2.38).

Finally, it is quite satisfying that the requirement of relatively small SU(3) breaking [compared with SU(3) × SU(3) breaking] of the vacuum (small $|b|, |z|$) can be used in each case in conjunction with the conditions (2.37)–(2.39) to place very tight bounds on another parameter (a, α, or y) which would otherwise not be nearly as well confined. For example, the analysis of (8,8) symmetry breaking by Genz and Katz[9] leads to perhaps unacceptably large values of $|z|$ in order to accomodate their value of $y = -0.446$ (see Fig. 4). This would not be true if constraint (2.39) (which is probably the least reliable of the three) were withdrawn. Also, of course, if one gives up the notion that, in the chiral-symmetric limit, SU(3) × SU(3) is spontaneously broken while SU(3) is not, then the relative sizes of SU(3) and SU(3) × SU(3) breaking of the vacuum (and, hence, the magnitude of z) are not *a priori* obvious.

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*Work supported in part by the National Research Council of Canada.

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