Resonance-sum model for Reggeization in the scattering of particles with arbitrary spin

Marcia J. King
Applied Physics Division, Argonne National Laboratory, Argonne, Illinois 60439

Loyal Durand*
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706

Kameshwar C. Wali†
Department of Physics, Syracuse University, Syracuse, New York 13210
(Received 21 July 1975)

Using a field-theoretic description of nonzero-spin particles, we have obtained center-of-mass helicity amplitudes corresponding to pole terms in four-particle reactions with arbitrary-spin external particles. We have discussed how to construct a van Hove-Durand–type model starting from these helicity amplitudes (which have a well specified kinematic structure in the field-theoretic description). Special attention has been paid to boson-fermion scattering. Straightforward Reggeization of helicity amplitudes assuming linear trajectories is known to produce parity doubling. Because of the generalized MacDowell symmetry of the helicity amplitudes, cuts appear in the J plane which prevent the occurrence in our model of parity-degenerate states in the physical J plane. One cannot have a pure fermion Regge pole unaccompanied by cuts. This conclusion has important consequences on both fitting data using Regge formulas in, say, backward scattering in boson-fermion scattering and theoretical considerations such as dual bootstrap models.

I. INTRODUCTION

Regge formulas for four-particle scattering and production amplitudes involving one or more high-spin external particles are becoming increasingly necessary from both the experimental and theoretical point of view. Experimental information concerning reactions such as $\pi N \to \pi$ + isobars or $pp \to$ two isobars is accumulating. Theoretically, the spin dependence of the Regge residues must be taken into account when imposing constraints such as the combination of SU(3) symmetry, crossing, and the absence of exotic states. Several people[1] in the last few years have derived important consequences on the hadronic mass spectrum and hadronic couplings using such constraints.

For arbitrary-spin external particles, the helicity amplitudes form a natural starting point for Reggeization. This procedure is well known,[2] but suffers from a lack of clear-cut separation of kinematic singularities and constraints. It is necessary in particular to define kinematic-singularity–free or "regularized" helicity amplitudes and impose kinematic constraints at appropriate points.[3] Some of these difficulties are avoided if one uses the invariant amplitudes. However, for other than the simplest low-spin reactions, the decomposition of the scattering amplitudes into invariant amplitudes becomes a formidable task. Further, such a formalism does not lend itself to the simple partial-wave expansion inherent in the helicity formalism.

We consider in this paper a Reggeization procedure for arbitrary-spin helicity amplitudes which is the generalization of the method used by Durand and van Hove[4] for zero-spin amplitudes. In such a method the Sommerfeld-Watson transformation is performed on the sum of particle-exchange contributions from all particles lying on a Regge trajectory. If we use a proper field-theoretic description of the spins of the external and exchanged particle, we obtain helicity amplitudes with explicit kinematic structure which automatically satisfy the various analyticity conditions including the kinematic constraints. The conditions necessary to implement the analytic continuation in the complex J plane become transparent. Also in such a model, since one starts with a Feynman diagram, the relation between the particle couplings (and hence the resonance widths) and the Regge residues is clearly exhibited.

The present work also has an important bearing on another longstanding problem concerning parity doubling in boson-fermion scattering. Straightforward Reggeization within the helicity formalism[2] and with linear trajectories leads to parity doubling[5] owing to the MacDowell symmetry. There is no experimental evidence for such a phenomenon, at least for low-lying baryonic states. There are mechanisms by means of which any finite number of unwanted parity partners can be eliminated. However, such mechanisms are somewhat *ad hoc* and artificial. Carlitz and Kislinger[6] and Durand and Lipinski[7] have shown that parity doubling is avoided in a very natural way in πN scattering if one uses the Durand–van Hove type model under discussion. There is a price one has to pay, however, and that is the appearance of Regge cuts associated with the Regge poles. It will be clear from our analysis that such cuts have a very simple origin and are natural in any boson-fermion scattering.

The plan of the paper is as follows: In Sec. II, we review the construction and properties of local fields with arbitrary spin. The reader familiar with this construction may wish to skip directly to Sec. III.

In Sec. III A, we construct a three-particle interaction Lagrangian which describes the coupling of particles with arbitrary spins S_1, S_2, and J and general values of the Lorentz quantum number j_0 of the spin-J particle. (The couplings are extended to arbitrary values of j_0 for all three particles in Appendix B). The results are new. The corresponding vertex functions are calculated in Sec. III B, and the two-particle scattering amplitude resulting from the process $S_1+S_2\to J\to S_3+S_4$ is calculated in Sec. III C. The contributions of intermediate particles and antiparticles are obtained separately. This separation leads to a remarkably simple interpretation of the generalized MacDowell symmetry considered in Sec. III D.

The two-particle scattering amplitude for general spin is Reggeized in Sec. IV A using the methods of Durand and van Hove. The reader interested mainly in the applications of Regge-type models can skip the derivations and begin with this section if desired. We find that for boson-fermion scattering with a single intermediate fermion trajectory, the usual parity-conserving helicity amplitudes obtain contributions only from intermediate particles ($\overline{G}^{+\sigma}$) and intermediate antiparticles ($\overline{G}^{-\sigma}$). We complete the discussion of boson-fermion scattering in Sec. IV B by considering a generalization of the Carlitz-Kislinger model proposed by Durand and Lipinski. This model contains a Regge pole with a linear trajectory, and a moving Regge cut. There is no parity-doubling of the resonances on the Fermion trajectory.

In Appendix A, we give a particularly simple form for the three-particle interaction vertex for arbitrary spins. This form displays clearly how factors associated with orbital angular momentum affect the vertex function, and is appropriate for consideration of particle decays. It is not appropriate for Reggeization in scattering. In Appendix B, the coupling considered in Sec. III A is generalized to the case of arbitrary Lorentz representations for the external particle.

II. COVARIANT FIELD OPERATORS—A REVIEW

A. Unitary representations of the Poincaré group—helicity representation

Any relativistically invariant theory of elementary-particle interactions contains as an underlying element unitary irreducible representations of the Poincaré group as discussed by Wigner.[8] In the helicity representation[9] of the Poincaré group, the states of a single particle of mass $m>0$, spin S, momentum $\vec{p}$ and positive energy $E=(\vec{p}^{\,2}+m^2)^{1/2}$ are described by

$$|\vec{p},\sigma\rangle=\left(\frac{m}{E}\right)^{1/2}U(H(\vec{p}))|\sigma\rangle, \tag{2.1}$$

where $|\sigma\rangle$ is the state of the particle at rest with $S_z=\sigma$, and $U(H(\vec{p}))$ is a unitary operator that corresponds to the Lorentz transformation $H(\vec{p})$ that takes the particle from rest to a state in which it has momentum $\vec{p}$ and helicity σ. $H(\vec{p})$ is taken to be

$$H(\vec{p})=R(\hat{p})L(|\vec{p}|\hat{z}), \tag{2.2}$$

where $L(|\vec{p}|\hat{z})$ is the boost which takes the particle at rest to a state in which it has momentum $|\vec{p}|\hat{z}$, and $R(\hat{p})$ is the rotation which takes it to the final state $|\vec{p},\sigma\rangle$. This construction and the normalization condition,

$$\langle\vec{p}\,',\sigma'|\vec{p},\sigma\rangle=\delta^3(\vec{p}\,'-\vec{p})\delta_{\sigma'\sigma}, \tag{2.3}$$

define the representation completely. Under an arbitrary Poincaré transformation (a,Λ), the transformation property of the state (2.1) is given by

$$U(a,\Lambda)|\vec{p},\sigma\rangle=e^{ia\cdot p'}\left(\frac{E'}{E}\right)^{1/2}\sum_{\sigma'}|\vec{p}\,',\sigma'\rangle D^S_{\sigma'\sigma}(R_W), \tag{2.4}$$

where a,Λ represent, respectively, an arbitrary space-time translation and a homogeneous Lorentz transformation, $U(a,\Lambda)$ is a unitary operator, $p'=\Lambda p$, and R_W is the well-known Wigner rotation $R_W=H^{-1}(\vec{p}\,')\Lambda H(\vec{p})$. The coefficients $D^S_{\sigma'\sigma}(R_W)$ are the matrix elements of the familiar $(2S+1)$-dimensional unitary matrix representation of the rotation group.[10]

To build multiparticle states, one introduces creation and annihilation operators $a^\dagger(\vec{p},\sigma)$ and $a(\vec{p},\sigma)$ satisfying commutation (anticommutation) rules for Bose (Fermi) particles:

$$[a(\vec{p},\sigma),a^\dagger(\vec{p}\,',\sigma')]_\pm=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,'), \tag{2.5}$$

$$|\vec{p},\sigma\rangle = a^\dagger(\vec{p},\sigma)|0\rangle, \quad a(\vec{p},\sigma)|0\rangle = 0. \tag{2.6}$$

It follows from (2.4) that the transformation properties of these operators are given by

$$\begin{aligned}\sqrt{E}\, U(a,\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(a,\Lambda) &= e^{ia\cdot p'}\sqrt{E'}\sum_{\sigma'} a^\dagger(\vec{p}',\sigma')D^S_{\sigma'\sigma}(R_W),\\ \sqrt{E}\, U(a,\Lambda)a(\vec{p},\sigma)U^{-1}(a,\Lambda) &= e^{-ia\cdot p'}\sqrt{E'}\sum_{\sigma'} D^S_{\sigma\sigma'}(R_W{}^{-1})a(\vec{p}',\sigma').\end{aligned} \tag{2.7}$$

B. Construction of local fields

Since R_W depends on the momentum $\vec{p}$, it is clear that the transformation coefficients in (2.7) depend on spin and momentum mixed in an intricate manner. Hence, although it is possible to construct the S matrix[11] using the above canonical representation, it has proved more convenient in practice to use local fields such as Dirac, Klein-Gordon, Proca, Pauli-Fierz, and their generalizations to higher spins to write relativistically invariant interactions. Such fields have simpler transformation properties in that the transformation coefficients are independent of momenta. They also give rise to manifestly covariant S-matrix elements with suitable analyticity properties and well-specified "kinematic" factors. There is a vast amount of literature[12,13] on the subject. Our purpose here is to collect together some main results necessary for our discussion.

The central idea underlying the construction of a local field describing an arbitrary-spin particle is to introduce a finite-dimensional nonunitary representation of the proper Lorentz group. One standard way is to use $\vec{J}$ and $\vec{K}$, the generators of rotations and pure Lorentz transformation, respectively, to define new operators

$$\vec{A} = \tfrac{1}{2}(\vec{J} + i\vec{K}), \quad \vec{B} = \tfrac{1}{2}(\vec{J} - i\vec{K}). \tag{2.8}$$

$\vec{A}$ and $\vec{B}$ have the commutation properties of two commuting angular momenta,

$$\begin{aligned}[A_i, A_j] &= i\epsilon_{ijk}A_k,\\ [B_i, B_j] &= i\epsilon_{ijk}B_k,\\ [A_i, B_j] &= 0.\end{aligned} \tag{2.9}$$

Finite-dimensional irreducible representations of the proper Lorentz group are characterized by two numbers (A, B), where $2A, 2B$ are integers and $A^2 = A(A+1)$, $B^2 = B(B+1)$ in the representation (A, B). The representation (A, B) is the direct product of the representations $(A, 0)\otimes(0, B)$. States in the (A, B) representation carry additional indices a, b, corresponding to the eigenvalues of A_z, B_z. Following Weinberg[12] we denote the matrices which represent a finite Lorentz transformation Λ by $D^A(\Lambda)$ and $\bar{D}^B(\Lambda)$ in the $(A, 0)$ and $(0, B)$ representations, respectively. The general representation matrix $D^{A,B}_{ab,a'b'}(\Lambda)$ is then given by

$$D^{A,B}_{ab,a'b'}(\Lambda) = D^A_{aa'}(\Lambda)\bar{D}^B_{bb'}(\Lambda). \tag{2.10}$$

The two representations $D^A(\Lambda)$ and $\bar{D}^A(\Lambda)$ are related by

$$D^A(\Lambda) = \bar{D}^A(\Lambda^{-1})^\dagger. \tag{2.11}$$

We note for future use that a four-vector x^μ can be associated with a 2×2 Hermitian matrix in either of the forms[14]

$$X = x_\mu\sigma^\mu = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2)\\ -(x^1 + ix^2) & x^0 + x^3\end{pmatrix}, \tag{2.12}$$

or

$$\tilde{X} = x^\mu\sigma^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2\\ x^1 + ix^2 & x^0 - x^3\end{pmatrix}, \tag{2.13}$$

where σ^0 is the unit matrix and the σ^i are the Pauli matrices. The effect of a Lorentz transformation $x'^\mu = \Lambda^\mu_\nu x^\nu$ is described in this representation by matrix multiplication,

$$X' = D^{1/2}(\Lambda)XD^{1/2\dagger}(\Lambda), \tag{2.14}$$

where $D(\Lambda)\in SL(2, C)$ and $\pm D(\Lambda)$ corresponds to Λ. Similarly,

$$\tilde{X}' = \bar{D}^{1/2}(\Lambda)\tilde{X}\bar{D}^{1/2\dagger}(\Lambda). \tag{2.15}$$

The gradient operators $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu = g^{\mu\nu}\partial_\mu$ can be represented in an analogous fashion by matrices ∂ and $\bar{\partial}$

$$\begin{aligned}\partial_{ab} &= \partial_\mu\sigma^\mu{}_{ab},\\ \bar{\partial}_{ab} &= \partial^\mu\sigma^\mu{}_{ab},\end{aligned} \tag{2.16}$$

with the transformation properties

$$\begin{aligned}\partial' &= D^{1/2}(\Lambda)\partial D^{1/2\dagger}(\Lambda),\\ \bar{\partial}' &= \bar{D}^{1/2}(\Lambda)\bar{\partial}\bar{D}^{1/2\dagger}(\Lambda).\end{aligned} \tag{2.17}$$

Local field operators are basically linear combinations of annihilation and creation operators such that the resultant fields transform according to finite-dimensional nonunitary representations of the Lorentz group, and thus have simple transformation properties. A simple way to understand how one constructs such fields is to start from the elementary or fundamental spinor representations of SL(2, C), the covering group of the proper Lorentz group. The elementary spinor representations have two components and can be used to describe particles of spin $\frac{1}{2}$. Higher-spin fields can be constructed from these by taking suitable linear combinations of direct products of such elementary spin-$\frac{1}{2}$ fields.

Let $\xi_\sigma(x)$ and $\eta^\sigma(x)$ denote elementary spinor fields which transform according to the fundamental representations of SL(2, C). Their transformation laws under an arbitrary Lorentz transformation Λ are given by

$$\begin{aligned} U(\Lambda)\xi_\sigma(x)U^{-1}(\Lambda) &= \sum_\alpha D^{1/2}_{\sigma\alpha}(\Lambda^{-1})\xi_\alpha(\Lambda x)\,, \\ U(\Lambda)\eta^\sigma(x)U^{-1}(\Lambda) &= \sum_\alpha \bar{D}^{1/2}_{\sigma\alpha}(\Lambda^{-1})\eta^\alpha(\Lambda x)\,. \end{aligned} \tag{2.18}$$

They will be assumed also to satisfy the free Klein-Gordon equation for particles of mass m, $(\Box + m^2)\xi = 0$, $(\Box + m^2)\eta = 0$, thus assuring that the mass-shell condition $p^2 = m^2$ is satisfied for plane wave states.

The transformation properties of the fields ξ and η are interchanged by the parity operation, $\mathcal{P}\xi\mathcal{P}^{-1} \propto \eta$, $\mathcal{P}\eta\mathcal{P}^{-1} \propto \xi$. Fields of both types are therefore required for the construction of interactions invariant under space inversion. These fields are not independent. The relations connecting them (the "subsidiary conditions") can be obtained by factoring the Klein-Gordon equations for ξ and η using the identity $\partial\bar{\partial} = \Box$ and the fact that $\bar{\partial}\xi$ and $\partial\eta$ transform respectively like η and ξ under Lorentz transformations,

$$\begin{aligned} i\partial_{ab}\eta^b(x) &= m\xi_a(x)\,, \\ i\bar{\partial}_{ab}\xi_b(x) &= m\eta^a(x)\,. \end{aligned} \tag{2.19}$$

Equations (2.19) can be combined into the single equation

$$i\gamma^\mu\partial_\mu\psi(x) = m\psi(x)\,, \tag{2.20}$$

where

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}$$

and

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}. \tag{2.21}$$

Equation (2.20) is the familiar Dirac equation in the Weyl representation.

Spinor fields of higher rank can be defined as objects which satisfy the Klein-Gordon equation and transform under Lorentz transformations the same way as direct products of the fundamental spinors ξ and η. We note in particular that the fields $\xi_{\{\sigma_1\sigma_2\cdots\sigma_{2S}\}}(x)$ and $\eta^{\{\sigma_1\sigma_2\cdots\sigma_{2S}\}}(x)$ which are totally symmetric in the $2S$ indices $\sigma_1\sigma_2\cdots\sigma_{2S}$ separately describe a particle of spin S without the imposition of any subsidiary conditions.

C. Local fields for general spin

For the purpose of constructing general interactions and propagators, we need local fields $\psi^{A,B}_{a,b}(x)$ discussed by Weinberg.[12] To see the connection between these fields and the spinor fields, first consider the $(2S+1)$-component fields $\phi^S_\sigma(x) \equiv \psi^{S,0}_{\sigma,0}(x)$ and $\chi^S_\sigma(x) \equiv \psi^{0,S}_{0,\sigma}(x)$. We can easily show that ϕ^S_σ and χ^S_σ are related to the spinor fields $\xi_{\{\sigma_1\sigma_2\cdots\sigma_{2S}\}}$ and $\eta^{\{\sigma_1\sigma_2\cdots\sigma_{2S}\}}$ by a simple change of basis. Suppose we start by defining $\phi^S_S(x)$ and $\chi^S_S(x)$ as

$$\phi^S_S(x) \equiv \xi_{1/2,\,1/2,\ldots,1/2}(x)\,, \tag{2.22}$$

$$\chi^S_S(x) \equiv \eta^{1/2,\,1/2,\ldots,1/2}(x)\,. \tag{2.23}$$

These are clearly fields with spin S and $S_z = S$. By applying the angular momentum lowering operators, we can obtain the remaining $\phi^S_\sigma(x)$ and $\chi^S_\sigma(x)$ fields as linear combinations of the $\xi(x)$ and $\eta(x)$ fields, respectively. The fields take the form

$$\phi^S_\sigma(x) = \sum_{\sigma_1\cdots\sigma_{2S}} \langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle \xi_{\{\sigma_1\cdots\sigma_{2S}\}}(x)\,, \tag{2.24}$$

and

$$\chi^S_\sigma(x) = \sum_{\sigma_1\cdots\sigma_{2S}} \langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle \eta^{\{\sigma_1\cdots\sigma_{2S}\}}(x)\,, \tag{2.25}$$

where the "parallel coupling" coefficient[15] $\langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle$ is defined by

$$\begin{aligned} \langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle = {} & C(S-\tfrac{1}{2},\tfrac{1}{2},S;\sigma-\sigma_{2S},\sigma_{2S})C(S-1,\tfrac{1}{2},S-\tfrac{1}{2};\sigma-\sigma_{2S}-\sigma_{2S-1},\sigma_{2S-1})\cdots \\ & \times C(\tfrac{1}{2},\tfrac{1}{2},1;\sigma-\sigma_{2S}-\cdots-\sigma_2,\sigma_2)C(0,\tfrac{1}{2},\tfrac{1}{2};\sigma-\sigma_{2S}-\cdots-\sigma_1,\sigma_1)\,. \end{aligned} \tag{2.26}$$

Note that the last Clebsch-Gordan coefficient in (2.26) is given by $C(0,\frac{1}{2},\frac{1}{2};\sigma-\sum_{i=1}^{2S}\sigma_i,\sigma_1) = \delta_{\sigma,\Sigma\sigma_i}$. Also, since the coefficient is real, $\langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle=\langle\sigma_1\cdots\sigma_{2S}|S\sigma\rangle$. Evaluation of (2.26) gives

$$\langle S\sigma|\sigma_1\cdots\sigma_{2S}\rangle=\delta_{\sigma,\Sigma\sigma_i}\left[\frac{(S+\sigma)!\,(S-\sigma)!}{(2S)!}\right]^{1/2}. \tag{2.27}$$

The coefficients also trivially satisfy the orthogonality relation

$$\sum_{\sigma_1\cdots\sigma_{2S}}\langle\sigma_1\cdots\sigma_{2S}|S\sigma\rangle\langle\sigma_1\cdots\sigma_{2S}|S\sigma'\rangle=\delta_{\sigma\sigma'}. \tag{2.28}$$

Using these coefficients, we can show that

$$D^S_{\sigma\sigma'}(\Lambda)=\sum_{\sigma_1\cdots\sigma_{2S}}\sum_{\sigma_1'\cdots\sigma_{2S}'}\langle\sigma_1\cdots\sigma_{2S}|S\sigma\rangle\langle\sigma_1'\cdots\sigma_{2S}'|S\sigma'\rangle D_{\sigma_1\sigma_1'}{}^{1/2}(\Lambda)\cdots D_{\sigma_{2S}\sigma_{2S}'}{}^{1/2}(\Lambda). \tag{2.29}$$

Then it follows from (2.18) and (2.29) that under an arbitrary Lorentz transformation Λ

$$U(\Lambda)\phi^S_\sigma(x)U^{-1}(\Lambda)=\sum_\alpha D^S_{\sigma\alpha}(\Lambda^{-1})\phi^S_\alpha(\Lambda x), \tag{2.30}$$

$$U(\Lambda)\chi^S_\sigma(x)U^{-1}(\Lambda)=\sum_\alpha \bar{D}^S_{\sigma\alpha}(\Lambda^{-1})\chi^S_\alpha(\Lambda x). \tag{2.31}$$

The general local fields[12] $\psi^{A,B}_{a,b}(x)$ transform like the product of $\phi^A_a(x)$ and $\chi^B_b(x)$. Thus

$$U(\Lambda)\psi^{A,B}_{a,b}(x)U^{-1}(\Lambda)=\sum_{a'b'}D^{A,B}_{ab,a'b'}(\Lambda^{-1})\psi^{A,B}_{a',b'}(\Lambda x), \tag{2.32}$$

where $D^{A,B}_{ab,a'b'}(\Lambda^{-1})$ is defined in (2.10). The fields $\psi^{A,B}_{a,b}(x)$ can be used to describe a particle of spin S when $|A-B|\leq S\leq A+B$.

Using the parallel-coupling coefficients, we can also generalize the differential operators $\partial,\bar\partial$ in (2.16). Define

$$\Pi^{AA}_{aa'}(i\partial_\mu\sigma^\mu)=\sum_{a_1\cdots a_{2A}}\sum_{a_1'\cdots a_{2A}'}\langle a_1\cdots a_{2A}|Aa\rangle\langle a_1'\cdots a_{2A}'|Aa'\rangle(i\partial_\mu\sigma^\mu)_{a_1a_1'}\cdots(i\partial_\mu\sigma^\mu)_{a_{2A}a_{2A}'}. \tag{2.33}$$

$$\bar\Pi^{AA}_{aa'}(i\partial^\mu\sigma^\mu)=\sum_{a_1\cdots a_{2A}}\sum_{a_1'\cdots a_{2A}'}\langle a_1\cdots a_{2A}|Aa\rangle\langle a_1'\cdots a_{2A}'|Aa'\rangle(i\partial^\mu\sigma^\mu)_{a_1a_1'}\cdots(i\partial^\mu\sigma^\mu)_{a_{2A}a_{2A}'}. \tag{2.34}$$

With the aid of (2.33) and (2.34), Eqs. (2.19) can be generalized for the free fields to

$$\sum_{\sigma'}\Pi^{SS}_{\sigma\sigma'}(i\partial_\mu\sigma^\mu)\chi^S_{\sigma'}(x)=(-\Box)^S\phi^S_\sigma(x)=m^{2S}\phi^S_\sigma(x), \tag{2.35}$$

$$\sum_{\sigma'}\bar\Pi^{SS}_{\sigma\sigma'}(i\partial^\mu\sigma^\mu)\phi^S_{\sigma'}(x)=(-\Box)^S\chi^S_\sigma(x)=m^{2S}\chi^S_\sigma(x). \tag{2.36}$$

The operators Π and $\bar\Pi$ obey the orthogonality relation

$$\sum_a\Pi^{AA}_{a'a}(i\partial_\mu\sigma^\mu)\bar\Pi^{AA}_{aa''}(i\partial^\mu\sigma^\mu)=\delta_{a'a''}(-\Box)^{2A} \tag{2.37}$$

and transform under Lorentz transformations as (mixed) tensors,

$$U(\Lambda)\Pi^{AA}_{aa'}(i\partial_\mu\sigma^\mu)U^{-1}(\Lambda)=\sum_{\alpha\alpha'}D^A_{a\alpha}(\Lambda^{-1})D^{A*}_{a'\alpha'}(\Lambda^{-1})\times\Pi^A_{\alpha\alpha'}(i\partial_\mu'\sigma^\mu),$$

$$U(\Lambda)\bar\Pi^{AA}_{aa'}(i\partial^\mu\sigma^\mu)U^{-1}(\Lambda)=\sum_{\alpha\alpha'}\bar D^A_{a\alpha}(\Lambda^{-1})\bar D^{A*}_{a'\alpha'}(\Lambda^{-1})\times\bar\Pi^A_{\alpha\alpha'}(i\partial'^\mu\sigma^\mu), \tag{2.38}$$

where $\partial_\mu'=\partial/\partial x'^\mu$.

D. Momentum-space expansions of $\psi^{A,B}_{a,b}(x)$

Our next task is to express the general fields in terms of annihilation and creation operators. Following Weinberg,[12] define

$$\alpha(\vec{p};a,b)=\sum_{a'b'\lambda}C(ABJ;a'b'\lambda)\times D^{A,B}_{ab,a'b'}(H(\vec{p}))\,a(\vec{p},\lambda), \tag{2.39}$$

$$\beta(\vec{p}; a, b) = \sum_{a'b'\lambda} C(ABJ; a'b'\lambda) \times \{D^{A,B}(H(\vec{p}))C^{-1}\}_{ab,a'b'}\, b^\dagger(\vec{p}, \lambda), \tag{2.40}$$

where $a(\vec{p}, \lambda)$ and $b^\dagger(\vec{p}, \lambda)$ are annihilation and creation operators for particles and antiparticles of spin J, respectively. C is a $(2J+1)$-dimensional matrix which satisfies the relations

$$C^\dagger C = 1, \quad C^* C = (-1)^{2J}. \tag{2.41}$$

It can be chosen to be real,

$$\begin{aligned} &C = e^{-i\pi J_y^{(J)}},\\ &C_{ab} = (-1)^{J+a}\,\delta_{a,-b},\\ &C^{-1}{}_{ab} = (-1)^{J+b}\,\delta_{a,-b}. \end{aligned} \tag{2.42}$$

In (2.40),

$$\begin{aligned} (D^{A,B}C^{-1})_{ab;a'b'} &= (D^A C^{-1})_{aa'}(\bar{D}^B C^{-1})_{bb'}\\ &= (-1)^{A+a'}(-1)^{B+b} D^A_{a,-a'}\bar{D}^B_{b,-b'}. \end{aligned} \tag{2.43}$$

Now let

$$\psi^{A,B,+}_{a,b}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\alpha(\vec{p}; a, b)e^{-ip\cdot x}, \tag{2.44}$$

$$\psi^{A,B,-}_{a,b}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\beta(\vec{p}; a, b)e^{ip\cdot x}, \tag{2.45}$$

and

$$\psi^{A,B}_{a,b}(x) = \psi^{AB,+}_{ab}(x) + (-1)^{2B}\psi^{AB,-}_{ab}(x). \tag{2.46}$$

It is a straightforward matter to check the transformation properties of $\alpha(\vec{p}; a, b)$ and $\beta(\vec{p}; a, b)$ and verify that the linear combination (2.46) has the correct Lorentz transformation property (2.31). Also as noted by Weinberg,[12] (2.46) leads to a causal propagator.

For later convenience, we note here that we can write the momentum space expansions (2.44) and (2.45) in a form analogous to that familiar for the Dirac field by introducing the "wave functions"

$$u^{A,B}_{a,b}(\vec{p}, \lambda) = \sum_{a'b'} C(ABJ; a'b'\lambda) \times D^A_{aa'}(H(\vec{p}))\bar{D}^B_{bb'}(H(\vec{p})), \tag{2.47}$$

$$v^{A,B}_{a,b}(\vec{p}, \lambda) = (-1)^{J-\lambda}u^{A,B}_{a,b}(\vec{p}, \lambda). \tag{2.48}$$

Then,

$$\psi^{A,B}_{a,b}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda\left[u^{A,B}_{a,b}(\vec{p}, \lambda)a(\vec{p}, \lambda)e^{-ip\cdot x} + (-1)^{2B}v^{A,B}_{a,b}(\vec{p}, \lambda)b^\dagger(\vec{p}, -\lambda)e^{ip\cdot x}\right]. \tag{2.49}$$

The momentum-space expansions of the $(2S+1)$-component fields $\phi^S_\sigma(x)$ and $\chi^S_\sigma(x)$ can easily be obtained from those of $\psi^{A,B}_{a,b}(x)$ by setting $B=0$ and $A=0$, respectively. Thus

$$\phi^S_\sigma(x) = \phi^{S,+}_\sigma(x) + \phi^{S,-}_\sigma(x), \tag{2.50}$$

$$\chi^S_\sigma(x) = \chi^{S,+}_\sigma(x) + (-1)^{2S}\chi^{S,-}_\sigma(x), \tag{2.51}$$

where

$$\phi^{S,+}_\sigma(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda D^S_{\sigma\lambda}(H(\vec{p}))\,a(\vec{p}, \lambda)e^{-ip\cdot x}, \tag{2.52}$$

$$\phi^{S,-}_\sigma(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda [D^S(H(\vec{p}))C^{-1}]_{\sigma\lambda}b^\dagger(\vec{p}, -\lambda)e^{ip\cdot x} \tag{2.53}$$

and

$$\chi^{S,+}_\sigma(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda \bar{D}^S(H(\vec{p}))a(\vec{p}, \lambda)e^{-ip\cdot x}, \tag{2.54}$$

$$\chi^{S,-}_\sigma(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda [\bar{D}^S(H(\vec{p}))C^{-1}]_{\sigma\lambda}b^\dagger(\vec{p}, -\lambda)e^{ip\cdot x}. \tag{2.55}$$

For later use, we note here that the momentum-space expansions of products such as $\Pi\chi^S$ and $\bar{\Pi}\phi^S$ are easily obtained by using the relations

$$\Pi^{AA}_{aa'}(i\partial_\mu\sigma^\mu)e^{-ip\cdot x}=(p^2)^A D^{AA}_{aa'}(R(\hat p)L(|\vec p|\hat z)L(|\vec p|\hat z)R^{-1}(\hat p))e^{-ip\cdot x}$$

and (2.56)

$$\overline{\Pi}^{AA}_{aa'}(i\partial^\mu\sigma^\mu)e^{-ip\cdot x}=(p^2)^A \overline{D}^{AA}_{aa'}(R(\hat p)L(|\vec p|z)L(|\vec p|\hat z)R^{-1}(\hat p))e^{-ip\cdot x}\,.$$

These follow from the definitions (2.33) and (2.34) and from the relations

$$\begin{aligned}p_\mu\sigma^\mu&=(p^2)^{1/2}D^{1/2}(L(\vec p)L(\vec p))\\&=(p^2)^{1/2}D^{1/2}(H(\vec p)R^{-1}(\hat p)H(\vec p)R^{-1}(\hat p))\,,\\p^\mu\sigma^\mu&=(p^2)^{1/2}\overline{D}^{1/2}(L(\vec p)L(\vec p))\\&=(p^2)^{1/2}\overline{D}^{1/2}(H(\vec p)R^{-1}(\hat p)H(\vec p)R^{-1}(\hat p))\,,\end{aligned}\tag{2.57}$$

where $L(\vec p)$ is the Lorentz boost to the frame in which the particle of mass $(p^2)^{1/2}$ has momentum $\vec p$.

Using (2.56) and the orthogonality relation for Π and $\overline{\Pi}$ given in (2.37), we find in momentum space

$$\begin{aligned}\sum_a D^A_{a'a}(R(\hat p)L^2(|\vec p|\hat z)R^{-1}(\hat p))\overline{D}^A_{aa''}(R(\hat p)L^2(|\vec p|\hat z)R^{-1}(\hat p))&=\sum_a D^A_{a'a}(L^2(\vec p))\overline{D}^A_{aa''}(L^2(\vec p))\\&=\delta_{a'a''}\,.\end{aligned}\tag{2.58}$$

This relation may also be derived by noting that for a pure boost L, $D(L)^\dagger=D(L)$, hence, that $\overline{D}(L^2(\vec p))=D(L^{-2}(\vec p))^\dagger=D(L^{-2}(\vec p))$.

E. Properties of $\psi^{A,B}_{a,b}(x)$ under space inversion

Under space inversion $\mathcal{P}$, the annihilation operators for particles and antiparticles transform according to

$$\mathcal{P}a(\vec p,\lambda)\mathcal{P}^{-1}=(-1)^{J+\lambda}\exp[i\lambda\phi(\hat p)]\eta_P a(-\vec p,-\lambda)\,,\tag{2.59}$$

$$\mathcal{P}b(\vec p,\lambda)\mathcal{P}^{-1}=(-1)^{J+\lambda}\exp[i\lambda\phi(\hat p)]\overline{\eta}_P b(-\vec p,-\lambda)\,,$$

where η is the intrinsic parity. Then it follows that

$$\mathcal{P}\psi^{A,B}_{a,b}(t,\vec x)\mathcal{P}^{-1}=\eta_P(-1)^{A+B-J}\psi^{B,A}_{b,a}(t,-\vec x)\,,\tag{2.60}$$

where we have used[12]

$$\overline{\eta}_P=(-1)^{2J}\eta^*_P\,.\tag{2.61}$$

To construct parity-conserving interactions, it is convenient to define composite fields $\Psi^{A,B}_{a,b}(x)$, where

$$\Psi^{A,B}_{a,b}(x)=\begin{pmatrix}\psi^{A,B}_{a,b}(x)\\ \psi^{B,A}_{b,a}(x)\end{pmatrix}\,.\tag{2.62}$$

Under an arbitrary Lorentz transformation,

$$U(\Lambda)\Psi^{A,B}_{a,b}(x)U^{-1}(\Lambda)=\sum_{a'b'}\mathfrak{D}^{A,B}_{ab,a'b'}(\Lambda^{-1})\Psi^{A,B}_{a',b'}(\Lambda x)\,,\tag{2.63}$$

where the matrix $\mathfrak{D}^{A,B}_{ab,a'b'}$ is defined by

$$\mathfrak{D}^{A,B}_{ab,a'b'}(\Lambda^{-1})=\begin{pmatrix}D^{A,B}_{ab,a'b'}(\Lambda^{-1})&0\\0&D^{B,A}_{ba,b'a'}(\Lambda^{-1})\end{pmatrix}\,.\tag{2.64}$$

Under space inversion $\mathcal{P}$,

$$\mathcal{P}\Psi^{A,B}_{a,b}(t,\vec x)\mathcal{P}^{-1}=\eta_P(-1)^{A+B-J}\Psi^{B,A}_{b,a}(t,-\vec x)\,.\tag{2.65}$$

For momentum-space expansions of Ψ fields, we shall introduce

$$U^{A,B}_{a,b}(\vec p,\lambda)=\begin{pmatrix}u^{A,B}_{a,b}(\vec p,\lambda)\\u^{B,A}_{b,a}(\vec p,\lambda)\end{pmatrix}\,,\tag{2.66}$$

$$V^{A,B}_{a,b}(\vec p,\lambda)=\begin{pmatrix}(-1)^{2B}v^{A,B}_{a,b}(\vec p,\lambda)\\(-1)^{2A}v^{B,A}_{b,a}(\vec p,\lambda)\end{pmatrix}\,.\tag{2.67}$$

Then

$$\Psi^{A,B}_{a,b}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_\lambda[U^{A,B}_{a,b}(\vec p,\lambda)\,a(\vec p,\lambda)e^{-ip\cdot x}+V^{A,B}_{a,b}(\vec p,\lambda)\,b^\dagger(\vec p,-\lambda)e^{ip\cdot x}]\,.\tag{2.68}$$

F. Related fields $\tilde\psi$ and $\tilde\Psi$

The Lorentz-transformation properties of the Hermitian conjugate field $\psi^\dagger(x)$ differ from those of $\psi(x)$. Therefore, in constructing Lorentz-invariant interactions involving the destruction and creation of particles, it proves convenient to use fields related to the Hermitian conjugate fields but which have transformation properties identical to the fields themselves.[12] Such fields, which we denote by $\tilde{\psi}$ and $\tilde{\Psi}$ may be defined by writing

$$\psi_{a,b}^{A,B\dagger}(x)=(-1)^{2B-J+a+b}\tilde{\psi}_{-b,-a}^{B,A}(x)\,, \tag{2.69}$$

or

$$\tilde{\psi}_{a,b}^{A,B}(x)=(-1)^{2A-J-a-b}\psi_{-b,-a}^{B,A\dagger}(x)\,, \tag{2.70}$$

and

$$\tilde{\Psi}_{a,b}^{A,B}=\begin{pmatrix}(-1)^{2J}\tilde{\psi}_{a,b}^{A,B}\\ \tilde{\psi}_{b,a}^{B,A}\end{pmatrix}. \tag{2.71}$$

Under space inversion,

$$\mathcal{P}\tilde{\Psi}_{a,b}^{A,B}(t,\vec{x})\mathcal{P}^{-1}=\eta_P^*(-1)^{A+B-J}\tilde{\Psi}_{b,a}^{B,A}(t,-\vec{x})\,. \tag{2.72}$$

The $\tilde{\psi}$ fields are linear combinations of antiparticle-annihilation and particle-creation operators,

$$\tilde{\psi}_{a,b}^{A,B}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_{\lambda}\left[u_{a,b}^{A,B}(\vec{p},\lambda)b(p,\lambda)e^{-ip\cdot x}+(-1)^{2B}v_{a,b}^{A,B}(\vec{p},\lambda)a^\dagger(\vec{p},-\lambda)e^{ip\cdot x}\right] \tag{2.73}$$

$$\tilde{\Psi}_{a,b}^{A,B}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p\left(\frac{M}{2E}\right)^{1/2}\sum_{\lambda}\left[\tilde{U}_{a,b}^{A,B}(\vec{p},\lambda)b(\vec{p},\lambda)e^{-ip\cdot x}+\tilde{V}_{a,b}^{A,B}(\vec{p},\lambda)a^\dagger(\vec{p},-\lambda)e^{ip\cdot x}\right], \tag{2.74}$$

where

$$\tilde{U}_{a,b}^{A,B}(\vec{p},\lambda)=\begin{pmatrix}(-1)^{2J}u_{a,b}^{A,B}(\vec{p},\lambda)\\ u_{b,a}^{B,A}(\vec{p},\lambda)\end{pmatrix}, \tag{2.75}$$

$$\tilde{V}_{a,b}^{A,B}(\vec{p},\lambda)=\begin{pmatrix}(-1)^{2(J+B)}v_{a,b}^{A,B}(\vec{p},\lambda)\\ (-1)^{2A}v_{b,a}^{B,A}(\vec{p},\lambda)\end{pmatrix}. \tag{2.76}$$

Note in particular that the expansions of ψ and $\tilde{\psi}$ (2.49) and (2.73) differ only by the interchange of particle and antiparticle operators, $a(\vec{p},\lambda) \to b(\vec{p},\lambda)$, $b^\dagger(\vec{p},-\lambda)\to a^\dagger(\vec{p},-\lambda)$.

III. INTERACTION LAGRANGIAN DENSITIES AND SCATTERING MATRIX ELEMENTS

A. Three-particle interaction Lagrangians

From Sec. II, we see that we can use any of a number of fields to describe a particle with a given spin. There are consequently an arbitrary number of ways to construct interaction Lagrangian densities. It is not our intention to develop a full-fledged field theory of high-spin particles. Our aim is limited to obtaining the contributions associated with single-particle intermediate states lying on Regge trajectories to the scattering amplitudes for external particles with fixed spin. It will be sufficient for this purpose to consider one choice of fields, and discuss the results in some detail.

The external particles will be represented for simplicity using the $2(2S+1)$-component fields discussed in Sec. II. This restriction is not essential. The results given here are easily generalized, but with some loss in simplicity. The general coupling is given in Appendix B.

The possible choices for the fields used to describe the intermediate particle are more restricted. We are primarily interested in the leading trajectory in a parent-daughter sequence. This trajectory can be characterized by the value of the (complex) angular momentum J and the Lorentz quantum number j_0 which specifies the minimum angular momentum contained in the representation chosen, $j_0=|A-B|$. The highest spin (leading trajectory) for a given A,B has $J=A+B$. We will therefore use the A,B representation for the intermediate particle with

$$A=\tfrac{1}{2}(J+j_0)\,,\qquad B=\tfrac{1}{2}(J-j_0)\,. \tag{3.1}$$

The scattering amplitude will be Reggeized by continuing to complex J,A,B with j_0 fixed. As is well known,[16] an amplitude constructed in this way, with a single fixed value of j_0, will factor properly, and will have the correct analyticity properties at $W=0$. The value of j_0 determines the dominant helicity amplitude at that point. The cases $j_0=0,\frac{1}{2}$ have been considered in detail by Morrow,[17] who used the explicit tensor constructions of Bose and Fermi fields in the $(\frac{1}{2}J,\frac{1}{2}J)$ and $(\frac{1}{2}(J+1),\frac{1}{2}(J-1))$ representations to construct symmetric three-particle couplings and Reggeized scattering amplitudes for particles of arbitrary spin and parity. The present construction extends these results by a different method to arbitrary values of j_0.

The coupling scheme we have adopted is as follows. We first couple the external fields in the $(S_1, 0)$ and $(S_2, 0)$ representations into a combined spin-S field in the $(S, 0)$ representation. This is combined with the spin-J field in the (A, B) representation and the derivative tensors $\Pi(\partial_3), \Pi(\partial_{12})$ to obtain a Lorentz scalar. The number of derivatives in the coupling scheme is taken as the minimum number necessary to obtain the most general behavior of the vertex function at thresholds and pseudothresholds,[3] that is, as the number necessary to obtain the most general JLS coupling schemes at thresholds and pseudothresholds independently in the corresponding nonrelativistic limits. This coupling scheme is new.

The total-spin field $\Psi^{S,0}_{m,0}(x)$ is defined as follows:

$$\Psi^{S,0}_{m,0}(x) = \begin{pmatrix} \psi^{S,0}_{m,0}(x) \\ \psi^{0,S}_{0,m}(x) \end{pmatrix}, \tag{3.2}$$

with

$$\psi^{S,0}_{m,0}(x) = \sum_\mu C(S_1 S_2 S; m-\mu, \mu)\psi^{S_1,0}_{m-\mu,0}(x)\psi^{S_2,0}_{\mu,0}(x) \tag{3.3}$$

and

$$\psi^{0,S}_{0,m}(x) = \sum_\mu C(S_1 S_2 S; m-\mu, \mu)\psi^{0,S_1}_{0,m-\mu}(x)\psi^{0,S_2}_{0,\mu}(x)\,. \tag{3.4}$$

The composite field $\Psi^{S,0}_{m,0}$ transforms under Lorentz transformations according to the representation $\mathfrak{D}^{S,0}$, (2.64). The general interaction Lagrangian is now given in matrix notation by

$$\mathcal{L}_I^{J j_0 S_1 S_2}(x) = \sum_{S=|S_1-S_2|}^{S_1+S_2} \sum_{j_0'=-S}^{S} \mathcal{L}_I^{J j_0 j_0' S}(x)\,, \tag{3.5}$$

with

$$\mathcal{L}_I^{J j_0 j_0' S}(x) = g^{J j_0 j_0' S} \sum_{a b a' k_1 k_2 k_1' k_2' m} C(BB0; b, -b, 0)C(ASA'; a, m, a')C(k'kA'; k_2', k_2, a')C(k'kB; k_1', k_1, -b)$$
$$\times (\mu^2)^{-k-k'}[\underline{\Pi}^{k',k'}_{k_1',k_2'}(\partial_3)\tilde{\Psi}^{A,B}_{a,b}(x)]^T \Gamma[\underline{\Pi}^{k,k}_{k_1,k_2}(\partial_{12})\Psi^{S,0}_{m,0}(x)] + \text{H.c.}\,, \tag{3.6}$$

where $\tilde{\psi}^{A,B}_{a,b}(x)$ is defined in (2.71), μ is an arbitrary scale mass, and

$$A' = [(J+j_0)/2] - j_0',$$
$$k \equiv (J - j_0')/2, \quad k' \equiv |j_0 - j_0'|/2\,. \tag{3.7}$$

Γ is a numerical matrix

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & \eta_P \end{pmatrix}, \quad \eta_P = \eta_J \eta_1 \eta_2\,. \tag{3.8}$$

The derivative matrix $\underline{\Pi}$ is defined as

$$\underline{\Pi}^{k\,k}_{k_1 k_2}(\partial_{12}) = \begin{pmatrix} \overline{\Pi}^{k\,k}_{k_1 k_2}(i\partial^\mu_{12}\sigma^\mu) & 0 \\ 0 & \Pi^{k\,k}_{k_1 k_2}(i\partial_{\mu,12}\sigma^\mu) \end{pmatrix}, \tag{3.9}$$

with the corresponding definition of $\underline{\Pi}^{k'k'}_{k_1'k_2'}(\partial_3)$. In (3.9), the notation $\partial^\mu_{12}\sigma^\mu$ denotes $(\partial^\mu_1 - \partial^\mu_2)\sigma^\mu$. Note that the independent coupling constants are introduced through the sums on j_0' and S. The remaining structure and sums produce a Lorentz scalar [(0, 0) representation] by coupling the A and B indices to zero separately using the coupling schemes $[(kk')A', A, S] = 0$, $[(kk')B, B, 0] = 0$. The fields $\psi^{A,B}$ and $\psi^{B,A}$ and the derivative operators Π and $\overline{\Pi}$ have been combined so that $\mathcal{L}_I^{J,j_0}$ transforms properly under the discrete operations of $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$,

$$\mathcal{P}\mathcal{L}_I^{J,j_0}(t, \vec{x})\mathcal{P}^{-1} = \mathcal{L}_I^{J,j_0}(t, -\vec{x})\,,$$
$$\mathcal{C}\mathcal{L}_I^{J,j_0}(t, \vec{x})\mathcal{C}^{-1} = \mathcal{L}_I^{J,j_0}(t, \vec{x})\,, \tag{3.10}$$
$$\mathcal{T}\mathcal{L}_I^{J,j_0}(t, \vec{x})\mathcal{T}^{-1} = \mathcal{L}_I^{J,j_0}(-t, \vec{x})\,.$$

As remarked earlier, we have chosen a form for $\mathcal{L}_I^{J,j_0}(x)$ appropriate for continuation to complex J. If one is not interested in the analytic continuation of the matrix elements, but instead, say, in the form of the decay matrix elements of high-spin particles, a simpler choice of the interaction Lagrangian may be made. In particular, by setting $B = 0$ in (3.6), we obtain a form of the interaction in which the orbital angular momentum of the final-state particles is explicit. This is discussed in detail in Appendix A. Unfortunately, this simple form of the interaction Lagrangian requires the choice $j_0 = J$, and is not suitable for continuation. (If both J and j_0 are continued, with $j_0 = J$, the resulting amplitude violates the known analyticity requirements on the scattering amplitude at zero total energy, $W = 0$.)

B. Calculation of three-particle vertex functions

Let us now turn our attention to the evaluation of the matrix element for the vertex function for $S_1+S_2 \to J$ (see Fig. 1) defined by

$$V(kJ\Lambda;p_1S_1\lambda_1,p_2S_2\lambda_2)\equiv\int d^4x\langle kJ\Lambda|\mathcal{L}_I^{J\,j_0S_1\,S_2}(x)|p_1S_1\lambda_1,p_2S_2\lambda_2\rangle\,. \tag{3.11}$$

By substituting the momentum-space expansions of the various fields, we find

$$\begin{aligned}V(kJ\Lambda;p_1S_1\lambda_1,p_2S_2\lambda_2)=&(2\pi)^4\delta^4(k-p_1-p_2)\frac{\mathfrak{N}_J\mathfrak{N}_1\mathfrak{N}_2}{(2\pi)^{9/2}}(-1)^{S_2-\lambda_2}\exp[i\lambda_2\phi(\hat{p}_2)]\\&\times\sum_{j_0'S}g^{J,j_0j_0'S}\sum_{aba'k_1k_2k_1'k_2'm}C(BB0;b,-b,0)C(ASA';a,m,a')C(k'kB;k_1',k_1,k-b)\\&\times C(k'kA;k_2'k_2a')\sum_{\bar{a}\bar{b}\mu}\delta_{\mu\Lambda}C(ABJ;\bar{a}\bar{b}\mu)\sum_{\beta}C(S_1S_2S;m-\beta,\beta,m)\\&\times(\mu^2)^{-k-k'}[(p_1+p_2)^2]^{k'}(p_{12}{}^2)^k(-1)^{2k'}\\&\times\{(-1)^{2(J+B)}\bar{D}^{k'}_{k_1'k_2'}(L^2(\vec{k}))[D^A(H(\vec{k}))C^{-1}]_{a\bar{a}}[\bar{D}^B(H(\vec{k}))C^{-1}]_{b\bar{b}}\\&\quad\times\bar{D}^k_{k_1k_2}(L^2(\vec{p}_{12}))D^{S_1}_{m-\beta,\lambda_1}(H(\vec{p}_1))D^{S_2}_{\beta\lambda_2}(H(\vec{p}_2))\\&\quad+(-1)^{2A+k+k'-B}D^{k'}_{k_1'k_2'}(L^2(\vec{k}))[D^B(H(\vec{k}))C^{-1}]_{b\bar{b}}[\bar{D}^A(H(\vec{k}))C^{-1}]_{a\bar{a}}\\&\quad\times D^k_{k_1k_2}(L^2(\vec{p}_{12}))\bar{D}^{S_1}_{m-\beta,\lambda_1}(H(\vec{p}_1))\bar{D}^{S_2}_{\beta\lambda_2}(H(\vec{p}_2))\}\,,\end{aligned} \tag{3.12}$$

where $p_{12}\equiv p_1-p_2$ and $\mathfrak{N}_i=[(M_i/2E_i)]^{1/2}$. In the above, we have adopted the "particle 2" phase convention used by Jacob and Wick[9]; that is, we add a phase to the usual definition of the single-particle state (2.1), and define

$$|p_2S_2\lambda_2\rangle\equiv(-1)^{S_2-\lambda_2}\exp[i\lambda_2\phi(\hat{p}_2)]\left(\frac{m_2}{E_2}\right)^{1/2}U(R(\hat{p}_2)L(|\vec{p}_2|\hat{z}))|\lambda_2\rangle\,. \tag{3.13}$$

This is purely a matter of convenience in order to avoid the presence of a similar phase in the final center-of-mass helicity amplitude.

Equation (3.12) takes on a simpler form in the center-of-mass system of particles 1 and 2. By separating the boosts and rotations in the various D functions, and then by using the Clebsch-Gordan series for the rotation functions,[10] we finally obtain the vertex function,

$$V(M_JJ\Lambda;p_1S_1\lambda_1,p_2S_2\lambda_2)=(2\pi)^4\delta^4(k-p_1-p_2)\frac{\mathfrak{N}_1\mathfrak{N}_2\mathfrak{N}_J}{(2\pi)^{9/2}}D^J_{\Lambda,\lambda}(R(\hat{p}))f^{J,j_0}_{\lambda_1\lambda_2}(W), \tag{3.14}$$

with

$$\begin{aligned}f^{J,j_0}_{\lambda_1\lambda_2}(W)=&\sum_{S=|S_1-S_2|}^{S_1+S_2}C(S_1S_2S;\lambda_1,-\lambda_2)\\&\times\sum_{j_0'=-S}^{S}\sum_{\kappa=j_0'}^{S}(2J-2\kappa+1)^{1/2}C(J-\kappa,S,J;0\lambda)\\&\qquad\times[e^{-(\lambda_1\theta_1+\lambda_2\theta_2)}+\eta_J\eta_1\eta_2(-1)^{J-\kappa}e^{(\lambda_1\theta_1+\lambda_2\theta_2)}](\mu^2)^{-k-k'}W^{2k'}\\&\qquad\times(p_{12}{}^2)^k(-1)^{B-k'-k}(2A'+1)(2k+1)^{1/2}\begin{Bmatrix}k&A'&k'\\B&k&J-\kappa\end{Bmatrix}\begin{Bmatrix}A&B&J\\J-\kappa&S&A'\end{Bmatrix}\\&\qquad\times d^{kk}_{J-\kappa,0,0}(\theta_{12})g^{Jj_0j_0'S}.\end{aligned} \tag{3.15}$$

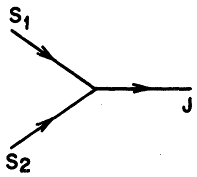

FIG. 1. Vertex involving particles of spins S_1, S_2, and a particle of spin J.

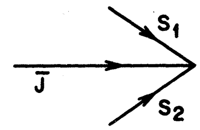

FIG. 2. Vertex involving particles of spins S_1, S_2, a and an antiparticle of spin J.

The bracketed quantities $\{\cdots\}$ are $6-j$ symbols,[10] and $\lambda \equiv \lambda_1 - \lambda_2$. The limits of summation given in (3.15) are those appropriate for large J ($J \geq 2S$). For $J \leq 2S$, some terms vanish. The function $d^{kk}(\theta_{12})$ is the boost d function.[16,18] W and the hyperbolic angles θ_i are defined in terms of the center-of-mass four-momenta by

$$\begin{aligned} p_1 &\equiv (E_1, \vec{p}) \equiv m_1(\cosh\theta_1, \hat{p}\sinh\theta_1), \\ p_2 &\equiv (E_2, -\vec{p}) \equiv m_2(\cosh\theta_2, -\hat{p}\sinh\theta_2), \\ p_{12} &\equiv (E_1 - E_2, 2\vec{p}) \equiv (p_{12}{}^2)^{1/2}(\cosh\theta_{12}, \hat{p}\sinh\theta_{12}), \\ p_1 + p_2 &\equiv (E_1 + E_2, \vec{0}) \equiv (W, \vec{0}). \end{aligned} \tag{3.16}$$

The decay matrix element for $J \to S_1 + S_2$ is, of course, obtained from the complex conjugate of (3.14).

The function $f^{J,j_0}_{\lambda_1,\lambda_2}(W)$ transforms simply under parity or, equivalently, under the reflection Y in the xz plane considered by Jacob and Wick[9] which changes λ_i to $-\lambda_i$. It follows from (3.15) and the symmetries of the Clebsch-Gordan coefficients that

$$f^{J,j_0}_{-\lambda_1,-\lambda_2}(W) = (-1)^{J-S_1-S_2}\eta_J\eta_1\eta_2 f^{J,j_0}_{\lambda_1,\lambda_2}(W). \tag{3.17}$$

The kinematic structure of the vertex functions is displayed explicitly in Eqs. (3.14) and (3.15). We will not enter a discussion of the well-known kinematic behavior of vertex functions in the helicity representation.[3] However, the factor $W^{2k'}$ in Eq. (3.15) requires some comment. The matrix elements of the derivative tensor $\Pi^{k'k'}$ in Eq. (3.6) introduce a factor $(k^2)^{k'}$ in the vertex function [comp. Eq. (2.53)]. For the decay of a real spin-J particle, $k^2 = M_J{}^2 = (p_1 + p_2)^2$. However, when we use Eq. (3.15) to describe a vertex in a scattering amplitude, it is necessary to use the second form of the equality, $k^2 = (p_1 + p_2)^2 = W^2$, to obtain an amplitude with the correct analytic structure for $W \to 0$. (Alternatively, if we wish to avoid the appearance of noncovariant terms in the vertex function or S matrix, we must arrange by partial integration that the derivatives ∂_3 act on the external fields, and not on the spin-J field or its propagator. See, for example, the discussion of similar problems in Weinberg.[12])

Finally, we note that in calculating the single-intermediate-particle contribution to two-particle scattering amplitude, we also need the vertex shown in Fig. 2. This is easily computed along the same lines as the vertex in Fig. 1. We have

$$\begin{aligned} \overline{V}(kJ\Lambda; p_1S_1\lambda_1, p_2S_2\lambda_2) &\equiv \int d^4x \langle 0|\mathfrak{L}_I^{Jj_0S_1S_2}(x)|kJ\Lambda, p_1S_1\lambda_1, p_2S_2\lambda_2\rangle \\ &= (2\pi)^4\delta^4(k + p_1 + p_2)\frac{\mathfrak{N}_1\mathfrak{N}_2\mathfrak{N}_J}{(2\pi)^{9/2}} D^J_{\Lambda,\lambda}[R(\hat{p})]\bar{f}^{J,j_0}_{\lambda_1\lambda_2}(W), \end{aligned} \tag{3.18}$$

with

$$\begin{aligned} \bar{f}^{J,j_0}_{\lambda_1\lambda_2}(W) = \sum_{S=|S_1-S_2|}^{S_1S_2} & C(S_1S_2S; \lambda_1, -\lambda_2) \\ & \times \sum_{j_0'=-S}^{S} \sum_{\kappa=j_0'}^{S} (2J - 2\kappa + 1)^{1/2} C(J-\kappa, S, J; 0\lambda) \\ & \times [e^{-(\lambda_1\theta_1 + \lambda_2\theta_2)} + \eta_J\eta_1\eta_2(-1)^{J+\kappa}e^{(\lambda_1\theta_1+\lambda_2\theta_2)}](\mu^2)^{-k-k'} \\ & \times W^{2k'}(p_{12}{}^2)^k(-1)^{B+k'+k}(2A'+1)(2k+1)^{1/2} \begin{Bmatrix} k & A' & k' \\ B & k & J-\kappa \end{Bmatrix} \begin{Bmatrix} A & B & J \\ J-\kappa & S & A' \end{Bmatrix} \\ & \times d^{kk}_{J-\kappa,0,0}(\theta_{12})\, g^{Jj_0j_0'S}. \end{aligned} \tag{3.19}$$

The functions $\bar{f}^{J,j_0}_{\lambda_1,\lambda_2}(W)$ and $\bar{f}^{J,j_0}_{-\lambda_1,-\lambda_2}(W)$ are related by

$$\bar{f}^{J,j_0}_{-\lambda_1,-\lambda_2}(W)=(-1)^{J+S_1+S_2}\eta_J\eta_1\eta_2\bar{f}^{J,j_0}_{\lambda_1,\lambda_2}(W). \tag{3.20}$$

C. Two-particle scattering amplitudes

We shall now proceed to evaluate the center-of-mass helicity amplitudes corresponding to the graph of Fig. 3 in which the four-momenta, masses, spins, and helicities of the external particles are denoted by p_i, m_i, S_i, and λ_i $(i=1, 2, 3, 4)$, respectively. The intermediate particle has mass M_J and spin J. The amplitude to be evaluated, then, is given by

$$G_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(u,s)=(-i)\int d^4x\int d^4x'\langle p_3\lambda_3;p_4\lambda_4|\,T(\mathcal{L}_I^{Jj_0S_3S_4}(x')\mathcal{L}_I^{Jj_0S_1S_2}(x))\,|p_1\lambda_1;p_2\lambda_2\rangle, \tag{3.21}$$

where the interaction Lagrangian $\mathcal{L}_I$ is defined in (3.5) and (3.6). T denotes the time-ordered product. Finally,

$$u=(p_1+p_2)^2=(p_3+p_4)^2,\quad s=(p_1-p_4)^2=(p_3-p_2)^2. \tag{3.22}$$

Evaluation of the amplitude (3.21) involves the calculation of the propagator of the intermediate spin-J particle,

$$[S_F^{A,B;J}(x'-x)]_{a'b',ab}=-i\langle 0|\,T(\tilde{\Psi}^{A,B\dagger}_{a',b'}(x')\tilde{\Psi}^{A,B}_{a,b}(x))\,|0\rangle, \tag{3.23}$$

Instead of writing a manifestly covariant expression for S_F, we shall find it convenient to keep the particle and antiparticle contributions separate on the right-hand side of (3.23). Introduction of the momentum-space expansions (2.74) for $\tilde{\Psi}^{A,B}$ into (3.23) allows us to express the propagator (3.23) as

$$\begin{aligned}[S_F^{A,B;J}(x'-x)_{a'b',ab}=-i\Big\{&\theta(t'-t)\int\frac{d^3k}{(2\pi)^3}\mathfrak{N}_J{}^2\sum_\Lambda\tilde{U}^{A,B\dagger}_{a',b'}(\vec{k},E_k,\Lambda)\tilde{U}^{A,B}_{a,b}(\vec{k},E_k,\Lambda)e^{+i\vec{k}\cdot(\vec{x}'-\vec{x})-iE_k(t'-t)}\\&+(-1)^{2J}\theta(t-t')\int\frac{d^3k}{(2\pi)^3}\mathfrak{N}_J{}^2\sum_\Lambda\tilde{V}^{A,B\dagger}_{a',b'}(\vec{k},E_k,\Lambda)\tilde{V}^{A,B}_{a,b}(\vec{k},E_k\Lambda)e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})+iE_k(t'-t)}\Big\},\end{aligned} \tag{3.24}$$

with $E_k=(\vec{k}^2+M_J{}^2)^{1/2}$. If we use the integral representation

$$\theta(t'-t)=-\frac{1}{2\pi i}\int dk^0\frac{e^{-i(k^0-E_k)(t'-t)}}{k^0-E_k+i\epsilon}, \tag{3.25}$$

the propagator can be put into the form

$$\begin{aligned}[S_F^{A,B;J}(x'-x)]_{a'b',ab}=&\frac{1}{(2\pi)^4}\int d^4k\,\mathfrak{N}_J{}^2e^{+ik\cdot(x'-x)}\\&\times\sum_\Lambda\left[\frac{\tilde{U}^{A,B\dagger}_{a',b'}(\vec{k},E_k,\Lambda)\tilde{U}^{A,B}_{a,b}(\vec{k},E_k,\Lambda)}{k^0-E_k+i\epsilon}-(-1)^{2J}\frac{\tilde{V}^{A,B\dagger}_{a',b'}(-\vec{k},E_k,\Lambda)\tilde{V}^{A,B}_{a,b}(-\vec{k},E_k,\Lambda)}{k^0+E_k-i\epsilon}\right],\end{aligned} \tag{3.26}$$

with $k=(k^0,\vec{k})$. In (3.26), $\tilde{U}$ and $\tilde{V}$ are evaluated on the mass shell, and the variable k^0 which appears in the exponential and denominator is unrestricted.

It may be helpful to see that the expression (3.26) corresponds to the familiar spin-$\frac{1}{2}$ propagator in the Dirac representation, namely,

$$\begin{aligned}[S_F(x'-x)]_{\sigma'\sigma}&=-i\langle 0|\,T(\psi_{\sigma'}(x')\bar{\psi}_\sigma(x))\,|0\rangle\\&=\frac{1}{(2\pi)^4}\int d^4k\,e^{-ik\cdot(x'-x)}\frac{m}{E_k}\sum_\lambda\left[\frac{u_{\sigma'}(\vec{k},E_k,\lambda)\bar{u}_\sigma(\vec{k},E_k,\lambda)}{k^0-E_k+i\epsilon}+\frac{v_{\sigma'}(-\vec{k},E_k,\lambda)\bar{v}_\sigma(-\vec{k},E_k,\lambda)}{k^0+E_k-i\epsilon}\right].\end{aligned} \tag{3.27}$$

Equation (3.27) can be cast in the more familiar form if we substitute

$$\sum_{\lambda} u(\vec{k}, E_k, \lambda)\bar{u}(\vec{k}, E_k, \lambda) = \frac{\gamma^0 E_k - \vec{\gamma}\cdot\vec{k} + m}{2m}, \tag{3.28}$$

$$\sum_{\lambda} v(-\vec{k}, E_k, \lambda)\bar{v}(-\vec{k}, E_k, \lambda) = \frac{\gamma^0 E_k + \vec{\gamma}\cdot\vec{k} - m}{2m}. \tag{3.29}$$

Then we obtain

$$S_F(x'-x) = \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik\cdot(x'-x)}\frac{\gamma\cdot k + m}{k^2 - m^2 + i\epsilon}. \tag{3.30}$$

Now we return to the calculation of the helicity amplitude $G^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ in (3.21), which we shall evaluate in the center-of-mass system where $\vec{k}=0$, $E_k = M_J$, and we put $k^0 = W = \sqrt{u}$. As noted after (3.17) all derivatives which appear in the coupling must be taken as acting on the external fields. Using the form of the propagator expressed in (3.24), one can carry out the calculation of the center-of-mass helicity amplitude in much the same way as the calculation of the vertex functions described earlier. The contribution to $G_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ of an intermediate particle of spin J is given by

$$G^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,u) = \frac{\mathfrak{N}_1\mathfrak{N}_2\mathfrak{N}_3\mathfrak{N}_4}{(2\pi)^9}(2\pi)^4 \times\delta^4(p_1+p_2-p_3-p_4)(J+\tfrac{1}{2}) \times g^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)\, D^{J*}_{\lambda\lambda'}(\phi_u, \theta_u, -\phi_u), \tag{3.31}$$

where $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_3 - \lambda_4$, ϕ_u, θ_u are center-of-mass scattering angles, and the partial-wave amplitude is

$$(J+\tfrac{1}{2})\, g^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W) = \frac{f^{J,j_0}_{\lambda_1\lambda_2}(W) f^{J,j_0}_{\lambda_3\lambda_4}(W)}{W - M + i\epsilon} - (-1)^{2J}\frac{\bar{f}^{J,j_0}_{\lambda_1\lambda_2}(W)\bar{f}^{J,j_0}_{\lambda_3\lambda_4}(W)}{W + M - i\epsilon}, \tag{3.32}$$

with f^{J,j_0} and $\bar{f}^{J,j_0}$ given by Eqs. (3.15) and (3.19), respectively.

Note that when the azimuthal angle $\phi_u = 0$, we have the relation

$$G^J_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(s,u) = (-1)^{s_1+s_2-s_3-s_4}(-1)^{\lambda-\lambda'} \times \frac{\eta_1\eta_2}{\eta_3\eta_4} G^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,u), \tag{3.33}$$

which is the usual expression of the conservation parity.

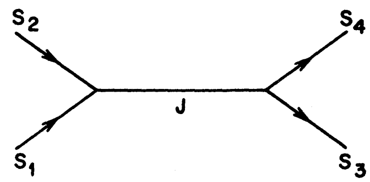

FIG. 3. Two-particle scattering with intermediate particle of spin J.

D. MacDowell symmetry in boson-fermion scattering

From now on, for simplicity, we shall concentrate on the case of boson-fermion scattering. The results are easily generalized to boson-boson and fermion-fermion scattering. It is customary to use parity-conserving partial-wave amplitudes defined by

$$g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \equiv g^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \sigma\eta_3\eta_4(-1)^{s_3+s_4-1/2} g^J_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}, \tag{3.34}$$

where for FB scattering for a definite spin J

$$\sigma \equiv \tau P, \quad \tau \equiv (-1)^{J-1/2}, \tag{3.35}$$

and $P = \eta_J$ is the parity of the given J state. A complete set of independent amplitudes also includes

$$g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \equiv g^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \sigma\eta_3\eta_4(-1)^{s_3+s_4-1/2} g^J_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}. \tag{3.36}$$

From (3.17), (3.20), and (3.32), we find that this somewhat obscure construction simple separates the particle and antiparticle contributions to $g^J_{\lambda_3\lambda_4,\lambda_1\lambda_2}$,

$$(J+\tfrac{1}{2})\, g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W) = \frac{f^{J,j_0}_{\lambda_1\lambda_2}(W) f^{J,j_0}_{\lambda_3\lambda_4}(W)}{W - M + i\epsilon}, \tag{3.37}$$

$$\sigma = \eta_J(-1)^{J-1/2}$$

and

$$(J+\tfrac{1}{2})\, g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W) = \frac{\bar{f}^{J,j_0}_{\lambda_1\lambda_2}(W)\bar{f}^{J,j_0}_{\lambda_3\lambda_4}(W)}{W + M - i\epsilon} \tag{3.38}$$

$$-\sigma = \eta_J(-1)^{J+1/2},$$

The $g^{J,\pm\sigma}$ satisfy the simple symmetry relations

$$g^{J,\sigma}_{-\lambda_3,-\lambda_4;\lambda_1\lambda_2}=(-1)^{J-s_3-s_4}\eta_J\eta_3\eta_4 g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}, \tag{3.39}$$

$$g^{J,-\sigma}_{-\lambda_3,-\lambda_4;\lambda_1\lambda_2}=(-1)^{J+s_3+s_4}\eta_J\eta_3\eta_4 g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}, \tag{3.40}$$

Note also that the $g^{J,\pm\sigma}$ consist of a product of two factors which depend separately on the initial and final helicities λ_1, λ_2 and λ_3, λ_4. The complete partial-wave amplitude g^J does not have this property.

The MacDowell symmetry relating $g^{J,\sigma}(W)$ to $g^{J,-\sigma}(-W)$ is derived with the aid of the relations given below. From

$$e^{\pm\lambda\theta}=\left(\frac{E+p}{m}\right)^{\pm\lambda}, \tag{3.41}$$

$$E(W)=-E(-W), \tag{3.42}$$

and

$$p(W)=-p(-W),$$

we have

$$e^{\pm\lambda\theta(W)}=(-1)^{\pm\lambda}e^{\pm\lambda\theta(-W)}. \tag{3.43}$$

In turn, the Lorentz boost function $d^{kk}(\theta_{12})$, which can be expressed as[16]

$$d^{kk}_{J-\kappa,0,0}(\theta_{12})=\sum_\mu C(kkJ-\kappa;\mu,-\mu)\times C(kk0;\mu,-\mu)e^{-2\mu\theta_{12}} \tag{3.44}$$

obeys the relation

$$d^{kk}_{J-\kappa,0,0}(\theta_{12}(W))=(-1)^{2k}d^{kk}_{J-\kappa,0,0}(\theta_{12}(-W)). \tag{3.45}$$

Finally,

$$(W)^{2k'}[p_{12}{}^2(W)]^k=(-1)^{2(k'+k)}(-W)^{2k'}[p_{12}{}^2(-W)]^k =(-1)^{J+j_0}(-W)^{2k'}[p_{12}{}^2(-W)]^k. \tag{3.46}$$

It follows that

$$g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)=-(-1)^{\lambda-\lambda'}g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-W). \tag{3.47}$$

This is simply the MacDowell symmetry relation generalized to arbitrary $FB\to FB$ scattering.[19] We shall explore the consequences of this symmetry on the Reggeization of the contributions from a sequence of resonances lying on a Regge trajectory.

IV. REGGEIZATION OF A SEQUENCE OF FERMION-RESONANCE AMPLITUDES: REGGE POLES AND REGGE CUTS

A. General considerations

The partial-wave helicity amplitudes derived in Sec. III will be the starting point for a Regge representation to be obtained by summing resonance contributions lying on an infinitely rising linear Regge trajectory with definite signature τ.[4] We will drop the kinematic factor in (3.31), and work with the usual center-of-mass scattering amplitude. The full scattering amplitude corresponding to this sequence of resonances is then given by

$$G_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,\cos\theta_u)=\sum_J(J+\tfrac{1}{2})g^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)d^J_{\lambda\lambda'}(\theta_u) =\tfrac{1}{2}\sum_J(J+\tfrac{1}{2})[g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)+g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)]d^J_{\lambda\lambda'}(\theta_u), \tag{4.1}$$

$$(-1)^{J-1/2}=\tau$$

where, we recall, $\lambda=\lambda_1-\lambda_2$, $\lambda'=\lambda_3-\lambda_4$, and $\phi_u=0$. The sum is restricted to physical values of J such that $(-1)^{J-1/2}=\tau$. A resonance of spin J, parity P, and mass $M(J)$ leads to a pole in the amplitude $g^{J,\sigma}$ with $\sigma=\tau P$ [cf. (3.37)]. It is important to note, however, that the resonance also gives a finite contribution to the amplitude $g^{J,-\sigma}$. It is necessary to retain this contribution if we wish to obtain a complete Reggeized amplitude which satisfies the MacDowell symmetry relation for $W\sim 0$ (backward FB scattering).

Although there is no need with the Van Hove–Durand construction to follow the usual procedures to obtain a properly Reggeized amplitude, it is illuminating to do so. The Reggeization is normally carried out using the "parity conserving" helicity amplitudes defined by

$$\overline{G}^{\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}=\overline{G}_{\lambda_3\lambda_4;\lambda_1\lambda_2}+\sigma\eta_3\eta_4(-1)^{s_3 s_4-1/2}\overline{G}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}, \tag{4.2}$$

$$\overline{G}^{-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}=\overline{G}_{\lambda_3\lambda_4;\lambda_1\lambda_2}-\sigma\eta_3\eta_4(-1)^{s_3+s_4-1/2}\overline{G}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}. \tag{4.3}$$

Here $\overline{G}$ denotes the helicity amplitude with the half-angle factor removed,

$$\bar{G}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,\cos\theta_u)=\left(\cos\frac{\theta_u}{2}\right)^{-(\lambda+\lambda')}\left(\sin\frac{\theta_u}{2}\right)^{-(\lambda-\lambda')}G_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,\cos\theta_u)$$
$$=\tfrac{1}{2}\sum_J(J+\tfrac{1}{2})[g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)+g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)]\hat{d}^J_{\lambda\lambda'}(\theta_u), \tag{4.4}$$

with

$$\hat{d}^J_{\lambda\lambda'}(\theta_u)=\left(\cos\frac{\theta_u}{2}\right)^{-(\lambda+\lambda')}\left(\sin\frac{\theta_u}{2}\right)^{-(\lambda-\lambda')}d^J_{\lambda\lambda'}(\theta_u). \tag{4.5}$$

(We assume for convenience that the helicities are such that $\lambda\geq|\lambda'|$. Other cases can be treated using the symmetries of the d's and the parity relations given in Sec. III.)

The restriction of the sum in (4.4) to values of $J\geq\lambda$ such that $(-1)^{J-1/2}=\tau$ can be removed by replacing $\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)$ by the signatured d-function $\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)$ defined by

$$\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)=\tfrac{1}{2}[\hat{d}^J_{\lambda\lambda'}(\theta_u)+\tau(-1)^{\lambda+1/2}\hat{d}^J_{\lambda,-\lambda'}(\pi-\theta_u)] . \tag{4.6}$$

Because of the identity

$$\hat{d}^J_{\lambda,-\lambda'}(\pi-\theta_u)=(-1)^{J+\lambda}\hat{d}^J_{\lambda\lambda'}(\theta_u) \tag{4.7}$$

satisfied by the $\hat{d}^J$ for physical values of J, $\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)$ is equal to $\hat{d}^J_{\lambda\lambda'}(\theta_u)$ for physical J with $(-1)^{J-1/2}=\tau$, and is zero for $(-1)^{J-1/2}=-\tau$. The partial-wave expansion for $\bar{G}^\sigma$ can be written using the foregoing definitions and the relations (3.39) and (3.40) as

$$\bar{G}^{\pm\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,\cos\theta_u)=\tfrac{1}{2}\sum_J(J+\tfrac{1}{2})\{[\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)]_\pm g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)+[\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)]_\mp g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W)\},\quad \sigma=\tau P, \tag{4.8}$$

where

$$[d^{J,\tau}_{\lambda\lambda'}(\theta_u)]_\pm=\hat{d}^{J,\tau}_{\lambda\lambda'}(\theta_u)\pm\hat{d}^{J,\tau}_{\lambda,-\lambda'}(\theta_u) . \tag{4.9}$$

For $\lambda\geq|\lambda'|$, $|\cos\theta_u|\to\infty$, and general values of J, $\hat{d}^J_{\lambda\lambda'}$ and $\hat{d}^J_{\lambda,-\lambda'}$ approach a common limit,

$$\hat{d}^J_{\lambda\lambda'}(\theta_u)\sim\hat{d}^J_{\lambda,-\lambda'}(\theta_u)\sim\frac{2^{-J+\lambda}\Gamma(2J+1)}{[\Gamma(J+\lambda+1)\Gamma(J-\lambda+1)\Gamma(J+\lambda'+1)\Gamma(J-\lambda'+1)]^{1/2}}\times(\cos\theta_u)^{J-\lambda} . \tag{4.10}$$

The corrections are of order $(\cos\theta_u)^{-1}$. As a result, $[\hat{d}^{J,\tau}_{\lambda\lambda'}]_+$ is of order $(\cos\theta_u)^{J-\lambda}$ for $|\cos\theta_u|\to\infty$, while $[\hat{d}^{J,\tau}_{\lambda\lambda'}]_-$ is of order $(\cos\theta_u)^{J-\lambda-1}$ and can be neglected in (4.8) if we retain only the leading contributions in $\cos\theta_u$ or s. If we use the relation

$$4pp'\cos\theta_u\sim -s,\quad s\to\infty \tag{4.11}$$

$$\cos(\pi-\theta_u)=e^{-i\pi}\cos\theta_u\sim s,\quad s\to\infty, \tag{4.12}$$

the sum of the leading contributions to (4.5) in powers of s gives the asymptotic relation

$$\bar{G}^{\pm\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}\sim\tfrac{1}{2}\sum_J(J+\tfrac{1}{2})(1+\tau e^{-i\pi(J-1/2)})\frac{2^{-J+\lambda}\Gamma(2J+1)}{[\Gamma(J+\lambda+1)\Gamma(J-\lambda+1)\Gamma(J+\lambda'+1)\Gamma(J-\lambda'+1)]^{1/2}}\times(-s)^{J-\lambda}(4pp')^{-J+\lambda}g^{J,\pm\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(W) . \tag{4.13}$$

We thus find that the leading contributions to $\bar{G}^{\pm\sigma}$ are provided by the particle and the antiparticle contributions from the intermediate state, a result which is not obvious in the usual approach. This result could, of course, be obtained directly from (4.1) without the formal construction.

The pole structure of (4.13) which results from the sum over resonances can be displayed by substituting (3.37) and (3.38) in (4.13). We will introduce the definitions

$$\beta^\sigma_{\lambda_1\lambda_2}(J,W)=\left[\frac{2^{-J+\lambda-1}\Gamma(2J+2)}{\Gamma(J+\lambda+1)\Gamma(J-\lambda+1)}\right]^{1/2}\times(4p)^{-J+\lambda}f^{J,\pm\sigma}_{\lambda_1\lambda_2}(W), \tag{4.15}$$

$$\beta^{-\sigma}_{\lambda_1\lambda_2}(J,W)=\left[\frac{2^{-J+\lambda-1}\Gamma(2J+2)}{\Gamma(J+\lambda+1)\Gamma(J-\lambda+1)}\right]^{1/2}\times(4p)^{-J+\lambda}f^{J,j_0}_{\lambda_1\lambda_2}(W)\,, \qquad (4.16)$$

and similar definitions for $\beta^{\pm\sigma}_{\lambda_3\lambda_4}(J,W)$. Then

$$\overline{G}^{\pm\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,\cos\theta_u)\sim\sum_J\tfrac{1}{2}(1+\tau e^{-i\pi(J-1/2)})(-s)^{J-\lambda}\times\beta^{\pm\sigma}_{\lambda_1\lambda_2}(J,W)\beta^{\pm\sigma}_{\lambda_3\lambda_4}(J,W)\times\frac{1}{W\mp[M(J)-i\epsilon]}\,. \qquad (4.17)$$

We will assume that the $\beta^{\pm\sigma}(J,W)$ and $M(J)$ are functions of J which satisfy the requirements of Carlson's theorem, and can be continued uniquely to complex J. The continuation is performed with the summation indices j_0' and κ in (3.15) and (3.19) fixed at physical values, $-S\leq j_0'\leq S$, $j_0'\leq\kappa\leq S$. The Clebsch-Gordan coefficients and 6-j symbols in (3.15) and (3.19) consist of finite sums of ratios of Γ functions, and can be continued using the explicit expressions given, for example, by Edmonds.[20] Application of the Sommerfeld-Watson transformation to (4.17) now gives the desired result for the leading contribution to G for $s\to\infty$ with u fixed (backward scattering in the s channel),

$$\overline{G}^{\pm\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,s)\sim\frac{i}{4}\int_C dJ\,\xi(J)s^{J-\lambda}\beta^{\pm\sigma}_{\lambda_1\lambda_2}(J,W)\beta^{\pm\sigma}_{\lambda_3\lambda_4}(J,W)\times\frac{1}{W\mp[M(J)-i\epsilon]}\,, \qquad (4.18)$$

where $\xi(J)$ is the usual signature factor,

$$\xi(J)=(1+\tau e^{-i\pi(J-1/2)})/\sin\pi(J-\lambda)\,. \qquad (4.19)$$

B. Models with moving poles and cuts

The discussion so far has been quite general. We turn next to specific models which illustrate the phenomenological content of our analysis. We will require that our models contain a fermion trajectory $\alpha^+(W)$ which is a linear function of W^2, as suggested by experiment. This will be assured if $M^2(J)$ is a linear function of J [see (4.18)]. Note, however, that the denominator function in (4.18) depends on $M(J)$ rather than $M^2(J)$, and will therefore have a square-root branch point in the J plane. The customary models with linear parity-degenerate trajectories eliminate this branch point by introducing a second trajectory with the same signature, but opposite parity, $\alpha^-(W)=\alpha^+(W)$, and with $\beta'^{\pm}(12)\beta'^{\pm}(34)=\beta^{\mp}(12)\beta^{\mp}(34)$. This construction satisfies the requirements of MacDowell symmetry, and eliminates the linear dependence of the complete amplitude on $M(J)$. This model, though widely used, has the obvious disadvantage of requiring parity doubling of the fermion resonances, a phenomenon which is not observed.

Carlitz and Kislinger[6] showed that the appearance of physical resonances of the "wrong" parity could be avoided if the J-plane cut associated with $M(J)$ was retained. As we shall see, there are still two Regge trajectories $\alpha^+(W)$ and $\alpha^-(W)$ as required by MacDowell symmetry, but for $W^2>0$, the wrong parity pole is on the second (unphysical) sheet of the cut J plane reached through the cut, and does not lead to resonances at physical values of J. The Carlitz-Kislinger model uses a mass function $M(J)=M_0(J-\alpha_0)^{1/2}$, α_0 fixed, and has a fixed cut at $J=\alpha_0$. Durand and Lipinski[7] noted that the model can be generalized to obtain a moving (Regge) cut by taking M^2 as a linear function of J and W^2.[21] We will use the latter model, with

$$M(J,W^2)=M_0[J-\alpha_c(W^2)]^{1/2}\,, \qquad (4.20)$$

where M_0 is a constant and the function $\alpha_c(W^2)$ is given by

$$\alpha_c(W^2)=\alpha_0+\alpha_c'W^2,\quad \alpha_0<\tfrac{1}{2}. \qquad (4.21)$$

The last factor in the integrand for $\overline{G}^\sigma$ then has the form

$$\frac{1}{W-(M_J-i\epsilon)}=-\frac{1}{M_0^2}\,\frac{W+M_0[J-\alpha_c(W^2)]^{1/2}}{J-\alpha^+(W^2)}\,, \qquad (4.22)$$

with

$$\alpha^+(W^2)=\alpha_0+(\alpha_c'+M_0^{-2})W^2+i\eta. \qquad (4.23)$$

The integrand thus has a moving pole at $J=\alpha^+(W^2)$, and moving branch point at $J=\alpha_c(W^2)$. The fixed-cut model is recovered for $\alpha_c'=0$.

The J-plane structure of the integrand in $\overline{G}^{-\sigma}$ in (4.18) is somewhat more subtle. There is the expected branch point at $J=\alpha_c(W^2)$. In addition, there is a pole corresponding to a value of J on an unphysical sheet of the J plane, namely, for J satisfying

$$M_0[J-\alpha_c(W^2)]^{1/2}=-W+i\epsilon\,. \qquad (4.24)$$

This corresponds to a linear Regge trajectory

$$J=\alpha^-(W^2)=\alpha_0+(\alpha_c'+M_0^{-2})W^2-i\eta\,. \qquad (4.25)$$

If the contour integral is deformed in the physical J plane before continuing to $W^2<0$, it appears to pick up only the branch-cut contribution due to $\alpha_c(W^2)$. However, the unphysical pole (4.25) contribution can be considered as "buried" in the branch-cut integral. In order to see this more clearly, consider the position of the pole after continuation to $W^2<0$ as indicated in Fig. 4. Here the pole has shifted to the lower part of the path of the branch-cut integral and must be taken into account.

It should be emphasized that for physical values of W, $W>0$, the fact that the pole at $J=\alpha^-(W^2)$ is on an unphysical sheet in J means that this pole does not give rise to physical resonances. There is consequently no parity doubling of resonances despite the fact that $\alpha^-(W^2)=\alpha^+(W^2)$ in our model. This can be seen somewhat differently if we write the last factor in the integrand for $\bar{G}^{-\sigma}$ in the form

$$\frac{1}{W+[M(J)-i\epsilon]}=\frac{1}{M_0^2}\,\frac{M_0[J-\alpha_c(W^2)]^{1/2}-W}{J-\alpha^-(W^2)}\quad . \tag{4.26}$$

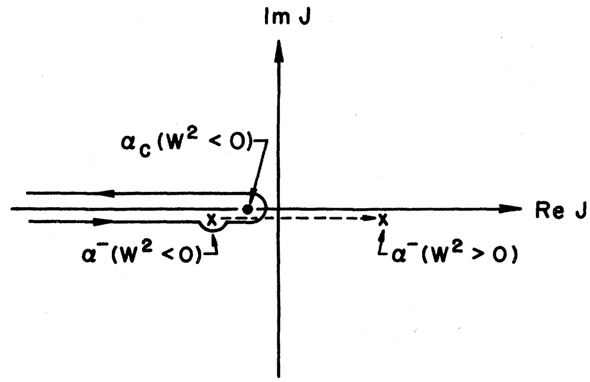

FIG. 4. Contour of integral for $\bar{G}^{-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ when $W^2<0$. The dashed line shows the position of the pole α^- on the unphysical sheet when $W^2>0$.

It is then clear that the residue of the pole at $J=\alpha^-(W^2)$ vanishes for $W>0$.

It is now straightforward to extract the Regge-pole contributions from (4.18) for $W^2<0$, and express $\bar{G}^{\pm\sigma}$ as

$$\bar{G}^{\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,s)\sim\pi\frac{W}{M_0^2}\,\xi(\alpha^+(W^2))\,s^{\alpha^+(W^2)-\lambda}\,\beta^{\sigma}_{\lambda_1\lambda_2}(\alpha^+,W)\beta^{\sigma}_{\lambda_3\lambda_4}(\alpha^+,W)$$
$$-\frac{1}{2M_0}\int_{-\infty}^{\alpha_c(W^2)}dJ\;\xi(J)s^{J-\lambda}\beta^{\sigma}_{\lambda_1\lambda_2}(J,W)\beta^{\sigma}_{\lambda_3\lambda_4}(J,W)\frac{[\alpha_c(W^2)-J]^{1/2}}{J-\alpha^+(W^2)} \tag{4.27}$$

and

$$\bar{G}^{-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(u,s)\sim\pi\frac{W}{M_0^2}\,\xi(\alpha^-(W^2))s^{\alpha^-(W^2)-\lambda}\beta^{-\sigma}_{\lambda_1\lambda_2}(\alpha^-,W)\beta^{-\sigma}_{\lambda_3\lambda_4}(\alpha^-,W)$$
$$+\frac{1}{2M_0}\int_{-\infty}^{\alpha_c(W^2)}dJ\;\xi(J)s^{J-\lambda}\beta^{-\sigma}_{\lambda_1\lambda_2}(J,W)\beta^{-\sigma}_{\lambda_3\lambda_4}(J,W)\frac{[\alpha_c(W^2)-J]^{1/2}}{J-\alpha^-(W^2)}\;, \tag{4.28}$$

$s\to\infty$, $u=W^2$ fixed.

Equations (4.27) and (4.28) give the final results of our paper. They are generalizations of the results of Carlitz and Kislinger[6] and Durand and Lipinski[7] to arbitrary spins. The factored residues $\beta^{\pm\sigma}$ defined by (4.15), (4.16), (3.15), and (3.19) satisfy all the kinematic constraints on the helicity amplitudes at thresholds, pseudothresholds, and $W=0$.[3] The coupling constants $g^{J_0 J_0' S}$ which appear in (3.15) and (3.19) can be taken as functions of W^2 for purposes of phenomenology without disrupting these kinematic conditions. However, we shall leave the investigation of specific models for future papers.

As a final note to this section, we observe that cuts of the type considered here are specific to fermion Reggeization. For boson resonances, $g^{J,\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ is a function of W^2 and $g^{J,-\sigma}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ is identically zero. Thus, simple boson Regge poles can exist without parity doubling or J-plane cuts.

V. SUMMARY AND DISCUSSION

We have shown how to construct a Regge representation for four-particle reactions with arbitrary spins using a Van Hove–Durand type model. A field-theoretic description of the single-particle contributions in the intermediate state permits us to specify the kinematic structure of the center-of-mass helicity amplitudes. A judicious choice of fields and the form of interaction lead us to a relatively simple and physically transparent result for single-particle exchange contributions. We have shown, for example, that in fermion-boson scattering, the parity-conserving partial-wave amplitudes $g^{J,\sigma}$ and $g^{J,-\sigma}$ are simply related to the contributions to the scattering amplitude of intermediate states containing particles and antiparticles.

Our main result concerns boson-fermion scattering. In a field-theoretic description of such a reaction, the natural variable is $W=\sqrt{u}$ and the partial-wave helicity amplitudes obey a generalized MacDowell symmetry which relates $g^{J,\sigma}(W)$ to $g^{J,-\sigma}(-W)$. The consequences of the MacDowell symmetry are well known. Whenever there are resonance poles associated with given total angular momentum J and parity, there are also nonvanishing contributions to partial-wave amplitudes belonging to the same J but opposite parity. However, as we have seen, the existence of poles of given J and parity on a trajectory linear in W^2 does not necessarily imply the existence of physical parity partners for observed resonances. We have constructed a simple model without parity doubling in which a moving cut in the J plane prevents the appearance of the parity partners on the physical sheet of the J plane. The MacDowell-symmetric poles are present, but far from the physical region.

To our knowledge there is no definitive experimental information which rules out the existence of such cuts. Because of the flexibility afforded by our general parametrization of the functions $\beta^{\pm\sigma}$, one probably can make a variety of models in specific reactions. While this needs further investigation, we would like to emphasize the general result, that within the framework of our model one can always get rid of the conspiring opposite-parity trajectory necessary from MacDowell symmetry (and analyticity constraints) by putting it on the second sheet in a cut J plane. This holds for *any* external spins in fermion-boson scattering, and generalizes the result of Carlitz and Kislinger.[6]

Finally, as remarked in the Introduction, another consequence of our investigation has to do with the duality constraints. When such constraints are imposed, one generally assumes pure Regge poles. If there are associated cuts, since the cuts have an energy dependence different from that of the pole, it is necessary that the constraints should be applied to the cut contributions as well. The constraints on the cut contributions involve the same residue functions as those in the pole contributions. Consequently it would be interesting to see whether these two sets of constraints are mutually compatible. Such a study may lead to modifications in the conclusions concerning hadronic spectrum and hadronic couplings reached by several authors in recent years.

ACKNOWLEDGMENTS

The authors would like to express their appreciation to the Aspen Center for Physics for the hospitality afforded them during part of this work. Two of the authors, M. J. King and K. C. Wali, would also like to acknowledge many helpful conversations with A. P. Balachandran, Michael Creutz, C. Gardiner, Frank Paige, and Larry Trueman.

APPENDIX A: COUPLINGS FOR 2(2J+1)-COMPONENT FIELDS

In Sec. III, the requirements of analyticity on the scattering amplitude at zero total energy led us to choose a general $\psi^{A,B}(x)$ field for the intermediate particle with spin J. If we are interested, however, only in the decay vertex of three arbitrary-spin particles or the scattering amplitude in the neighborhood of a narrow-resonance pole with spin J, we can choose the $2(2J+1)$-component fields $\psi^{J,0}(x)$ for the intermediate particle. This particular case merits special attention because, as we shall see, it leads to results which have a simple physical interpretation.

For simplicity in notation we shall henceforth define

$$\psi^{A,0}(x)=\phi^A(x),\quad \psi^{0,A}(x)=\chi^A(x)\ ,$$

and

$$\Psi_a^A(x)=\Psi_{a,0}^{A,0}(x)=\begin{pmatrix}\phi_a^A(x)\\ \chi_a^A(x)\end{pmatrix},$$
$$\tilde{\Psi}_a^A(x)=\tilde{\Psi}_{a,0}^{A,0}(x)=\begin{pmatrix}(-1)^{2A}\tilde{\phi}_a^A(x)\\ \tilde{\chi}_a^A(x)\end{pmatrix}. \tag{A1}$$

If we set $B=0$ and hence $A=J$, $j_0=J$, $k=k'$, $l=2k$, in (3.6) we can easily show that the interaction (3.6) reduces to

$$\mathfrak{L}_I^{JS_1S_2}=\sum_{s=|S_1-S_2|}^{S_1+S_2}\ \sum_{l=J-s}^{J+s}\mathfrak{L}_I^{Jls}(x)\ , \tag{A2}$$

where

$$\mathfrak{L}_I^{Jls}(x)=g^{Jls}\sum_{aa'k_1k_2k_1'k_2'm}C(Jsl;a,ma')C(\tfrac{1}{2}l,\tfrac{1}{2}l,l;k_2'k_2a')C(\tfrac{1}{2}l,\tfrac{1}{2}l,0;k_1'k_10)(\mu^2)^{-l}$$
$$\times\{[\Pi_{k_1'k_2'}^{l/2,l/2}(\partial_3)\tilde{\Psi}_a^J]^T[\Pi_{k_1k_2}^{l/2,l/2}(\partial_{12})\Psi_m^s(x)]+\text{H.c.}\}\ . \tag{A3}$$

The sum on l in (3.2) can be restricted to values such that

$$(-1)^l = \eta_J \eta_1 \eta_2 \tag{A4}$$

without losing the most general behavior at the physical thresholds and above. We will use this restriction.

The vertex function is given by a matrix element similar to (3.11),

$$V(kJ\Lambda; p_1 S_1 \lambda_1, p_2 S_2 \lambda_2) = \int d^4x \langle kJ\Lambda | \mathcal{L}_l^{JS_1S_2}(x) | p_1 S_1 \lambda_1, p_2 S_2 \lambda_2 \rangle . \tag{A5}$$

By substituting the momentum-space expansions of the various fields and going to the rest frame of particle with spin J, we obtain

$$V(kJ\Lambda; p_1 S_1 \lambda_1, p_2 S_2 \lambda_2) = (2\pi)^4 \delta^4(k - p_1 - p_2) \frac{\mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{N}_J}{(2\pi)^{9/2}} \times D^J_{\Lambda, \lambda}(R(\hat{p})) f^J_{\lambda_1 \lambda_2}(W) , \tag{A6}$$

where

$$f^J_{\lambda_1 \lambda_2}(W) = \sum_{l,s} f^{Jls}_{\lambda_1 \lambda_2}(W) , \tag{A7}$$

$$f^{Jls}_{\lambda_1 \lambda_2}(W) = G(Jls) C(S_1 S_2 s; \lambda_1, -\lambda_2, \lambda) C(lsJ; 0\lambda\lambda) \times (pW/\mu^2)^l \cosh(\lambda_1 \theta_1 + \lambda_2 \theta_2),$$

$$(-1)^l = \eta_J \eta_1 \eta_2 \tag{A8}$$

and

$$G(Jls) = (-1)^{J+s} \left[\frac{2l+1}{2l!\,(l+1)(2J+1)} \right]^{1/2} 2^{l+1} l!\, g(Jls). \tag{A9}$$

The kinematic structure of f^{Jls} is explicit in the factor $(pW)^l \cosh(\lambda_1\theta_1 + \lambda_2\theta_2)$. Since $\theta_1 \to 0$ and $\theta_2 \to 0$ for $p \to 0$, $\cosh(\lambda_1\theta_1 + \lambda_2\theta_2) \to 1$ near threshold, and f^{Jls} vanishes there as p^l,

$$f^{Jls}_{\lambda_1 \lambda_2}(W) \underset{p \to 0}{\sim} G(Jls) C(S_1 S_2 s; \lambda_1, -\lambda_2, \lambda) \times C(lsJ; 0\lambda\lambda)(pW/\mu^2)^l . \tag{A10}$$

In this limit, we clearly have an effective "JLS" coupling. The Clebsch-Gordan coefficients describe the coupling of S_1 and S_2 to "total spin" s, and the coupling of s and the "orbital angular momentum" l (the number of derivatives in the coupling) to total angular momentum $\boldsymbol{J}$.

The result for $f^J_{\lambda_1 \lambda_2}$ in (A7)–(A9) is quite simple, incorporates the proper threshold behavior, and can be used to obtain the most general parametrization of decay matrix elements. However, we should emphasize that l and s are not the orbital angular momentum L and total spin S in the proper Russell-Saunders JLS coupling scheme for $f^J_{\lambda_1 \lambda_2}$. The latter coupling can be constructed, but the results are complicated and not nearly as easy to use as those above.

APPENDIX B: COUPLINGS FOR FIELDS OF ARBITRARY LORENTZ TYPE

The three-particle interaction Lagrangian considered in Sec. III was constructed using fields $\psi^{A,B}(x)$, $\psi^{S_1,0}(x)$, and $\psi^{S_2,0}(x)$ to describe the internal particle with spin J, and the external particles with spins S_1 and S_2. The choice of the (A, B) representation of the Lorentz group for the internal particle, with $A+B=J$ and $|A-B| = j_0$, was necessitated by our desire to obtain scattering amplitudes which could be continued to complex J while retaining the correct analytic properties at $W=0$. The choice of the $(S_1, 0)$ and $(S_2, 0)$ representations for the external particles was motivated simply by convenience.

In this appendix, we will generalize the interaction Lagrangian to include the case of external fields which transform according to general representations (A_1, B_1) and (A_2, B_2) of the Lorentz group, with $A_1+B_1=S_1$, $|A_1-B_1|=j_{0,1}$, and $A_2+B_2=S_2$, $|A_2-B_2|=j_{0,2}$. These couplings are inevitably more complicated than those of Sec. III, but provide the natural framework for the comparison of scattering or decay processes which involve different members of a Regge sequence as external particles. [For example, the set of reactions $\pi + N \to \pi + N'$, with N' any of the physical states on the nucleon Regge trajectory, $S=\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$, should be described using fields in the (A, B) representation with $A+B=S$, $|A-B|=\frac{1}{2}$ to describe the particles N'.] The general coupling is also a natural starting point for the consideration of multi-Regge couplings.

The scheme which we will follow involves coupling the external fields in the (A_1, B_1) and (A_2, B_2) representations into a combined intermediate field in the (S_A, S_B) representation, and then combining this field, the spin-J field in the (A, B) representation, and the derivative tensors $\Pi(\partial_3)$ and $\Pi(\partial_{12})$ to obtain a Lorentz scalar. The number of derivatives in the coupling is the minimum number necessary to obtain the most general behavior of the vertex functions at thresholds and pseudothresholds.

The combined field $\Psi^{S_A,S_B}_{m_A,m_B}(x)$ is defined as follows:

$$\Psi^{S_A,S_B}_{m_A,m_B} = \begin{pmatrix} \psi^{S_A,S_B}_{m_A,m_B}(x) \\ \psi^{S_B,S_A}_{m_B,m_A}(x) \end{pmatrix}, \tag{B1}$$

with

$$\psi^{S_A,S_B}_{m_A,m_B}(x) = \sum_{a_1 b_1 a_2 b_2} C(A_1 A_2 S_A; a_1 a_2 m_A) \times C(B_1 B_2 S_B; b_1 b_2 m_B) \times \psi^{A_1,B_1}_{a_1,b_1}(x)\psi^{A_2,B_2}_{a_2,b_2}(x), \tag{B2}$$

and

$$\psi^{S_B,S_A}_{m_B,m_A}(x) = \sum_{a_1 b_1 a_2 b_2} C(A_1 A_2 S_A; a_1 a_2 m_A) \times C(B_1 B_2 S_B; b_1 b_2 m_B) \times \psi^{B_1,A_1}_{b_1,a_1}(x)\psi^{B_2,A_2}_{b_2,a_2}(x). \tag{B3}$$

We assume, as noted above, that $A_1+B_1=S_1$ and $A_2+B_2=S_2$. The field $\Psi^{S_A,S_B}(x)$ transforms under proper Lorentz transformations according to the representation $\mathfrak{D}^{S_A,S_B}$, (2.64), and transforms under space inversions as

$$\mathcal{P}\Psi^{S_A,S_B}_{m_A,m_B}(t,\vec{x})\mathcal{P}^{-1} = \eta_1\eta_2\Psi^{S_B,S_A}_{m_B,m_A}(t,-\vec{x}). \tag{B4}$$

The generalized interaction Lagrangian is given in terms of the composite field $\Psi^{S_A,S_B}(x)$, the spin-J field $\Psi^{A,B}(x)$, (2.62), and the derivative operators $\underline{\Pi}$, (3.9), by

$$\mathcal{L}_I^{JS_1S_2}(x) = \sum_{S_A S_B A' B'} \mathcal{L}_I^{S_A S_B A' B'}(x), \tag{B5}$$

with

$$\mathcal{L}_I^{S_A S_B A' B'}(x) = g^{S_A S_B A' B'} \sum_{\substack{a a' b b' m_A m_B \\ k_1 k_1' k_2 k_2' m}} (-1)^{B'-b'} C(AS_A A'; a, m_A, a') C(BS_B B'; b, m_B, b') \times C(k'kA'; k_2', k_2, a') C(k'kB'; k_1', k_1 - b')(\mu^2)^{-k-k'} \times [\underline{\Pi}^{k',k'}_{k_1',k_2'}(\partial_3)\bar{\Psi}^{A,B}_{a,b}(x)]^T \Gamma [\underline{\Pi}^{k,k}_{k_1,k_2}(\partial_{12})\Psi^{S_A,S_B}_{m_A,m_B}(x)] + \text{H.c.}, \tag{B6}$$

where μ is an arbitrary scale mass, and Γ is the numerical matrix defined in (3.8). The parameters $A, B, \ldots$ are defined in terms of the spins and Lorentz quantum numbers j_0 of the various particles by

$$A = \tfrac{1}{2}(J+j_0), \quad B = \tfrac{1}{2}(J-j_0),$$
$$A_1 = \tfrac{1}{2}(S_1+j_{0,1}), \quad B_1 = \tfrac{1}{2}(S_1-j_{0,1}), \tag{B7}$$
$$A_2 = \tfrac{1}{2}(S_2+j_{0,2}), \quad B_2 = \tfrac{1}{2}(S_2-j_{0,2}).$$

Note that $|j_{0,i}|$ is equal to the lowest physical spin in the corresponding sequence of resonances. The parameters k, k' are defined in terms of A', B' by

$$k = \tfrac{1}{2}(A'+B'), \; k' = \tfrac{1}{2}|A'-B'|. \tag{B8}$$

The Lagrangian $\mathcal{L}^{JS_1S_2}(x)$ in (B5) and (B6) is related in the special case $B_1 = B_2 = S_B = 0$ to the Lagrangian $\mathcal{L}^{Jj_0S_1S_2}(x)$ of Sec. III, (3.5) and (3.6), by the identifications

$$j_0' = A - A', \quad S = S_A,$$
$$g^{Jj_0j_0'S} = (2B+1)^{1/2} g^{S_A 0 A' B}. \tag{B9}$$

The lack of symmetry between the A couplings and the B couplings in (B6) is associated with the fact that the derivative tensors Π and $\bar{\Pi}$ transform under Lorentz transformations as D^k and $\bar{D}^k$ on their first index, and as D^{k*} and $\bar{D}^{k*}$ rather than $\bar{D}^k$ and D^k on their second index, (2.38). One can obtain a symmetrical coupling by using instead modified tensors Π' and $\bar{\Pi}'$ defined by

$$\Pi'^{AA}_{aa'} = (-1)^{A-a'}\Pi^{AA}_{a,-a'},$$
$$\bar{\Pi}'^{AA}_{aa'} = (-1)^{A-a'}\bar{\Pi}^{AA}_{a,-a'}, \tag{B10}$$

with the transformation property

$$U(\Lambda)\Pi'^{AA}_{aa'}(i\partial_\mu\sigma^\mu)U^{-1}(\Lambda) = \sum_{\alpha\alpha'} D^A_{a\alpha}(\Lambda^{-1})\bar{D}^A_{a'\alpha'}(\Lambda^{-1})\Pi'^{AA}_{\alpha\alpha'}(i\partial_\mu'\sigma^\mu),$$
$$U(\Lambda)\bar{\Pi}'^{AA}_{aa'}(i\partial^\mu\sigma^\mu)U^{-1}(\Lambda) = \sum_{\alpha\alpha'} \bar{D}^A_{a\alpha}(\Lambda^{-1})D^A_{a'\alpha'}(\Lambda^{-1})\Pi'^{AA}_{\alpha\alpha'}(i\partial'^\mu\sigma^\mu). \tag{B11}$$

The evaluation of the vertex functions corresponding to $\mathcal{L}_l^{JS_1S_2}$ is straightforward but tedious. The results are of the forms given in (3.14) and (3.18), but with f and $\bar{f}$ given by expressions considerably more complicated than (3.15) and (3.19). These functions can be expressed, if desired, in terms of Lorentz d-functions summed with Clebsch-Gordan, $6j$, and $9j$ coefficients. However, it is as simple or simpler to express f and $\bar{f}$ entirely in terms of Clebsch-Gordan coefficients and exponentials. These expressions are easily obtained by evaluating the vertex functions in the special case $\vec{p}_1 = -\vec{p}_2 = p\hat{z}$, $\Lambda = \lambda = \lambda_1 - \lambda_2$ using the results of Sec. II [cf. (3.14), (3.18)]. The details will not be given here.

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*Work supported by funds granted by the U. S. Energy Research and Development Administration under Contract No. AT(11-1)-881.

†Work supported by funds granted by the U. S. Energy Research and Development Administration under Contract No. AT(11-1)-3533.

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