Possible ambiguities in πN phase-shift analysis*

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We discuss possible ambiguities in the reconstruction of πN scattering amplitudes from experimental data and the elastic unitarity condition below the threshold of inelastic processes. We restrict ourselves to partial waves with $l=0$ and 1 and show that certain twofold ambiguities can occur if only two experimental quantities are measured (the cross section plus polarization or one of the polarization parameters). We also show that these ambiguities can be removed by a further measurement, unless the polarization happens to be zero or the experiment is performed at an unfortunately chosen angle, for which both sets of amplitudes predict the same value for the corresponding experimental quantity.

I. INTRODUCTION

The general problem approached in this article is that of reconstructing scattering amplitudes from experimental data and general principles of quantum theory. More specifically, we consider the case of elastic pion-nucleon scattering below the threshold of inelastic processes (like pion production). We investigate whether it is possible to reconstruct completely the πN scattering matrix, including all phases, from the knowledge of the differential and total cross sections and some polarization parameters (the polarization of the recoil nucleon and/or the polarization rotation parameters for a polarized target), making use of elastic unitarity.

Generally speaking, from experiments alone it is in principle (assuming that experimental errors can be made arbitrarily small) possible to determine all elements of the scattering matrix, up to one over-all phase, for any fixed value of energy E and scattering angle θ. The optical theorem can be used to fix the value of this over-all phase for $\theta=0$, but otherwise it remains unknown (since all experimental quantities are bilinear real functions of the scattering amplitudes). The elastic unitarity equation (for energies below the first inelastic threshold) can be considered to be an integral equation (or set of equations) for the over-all phase. The kernel of the equation will, of course, involve data from experiments performed for all scattering angles. Since unitarity for the total scattering amplitudes is expressed in terms of nonlinear integral equations, it is, in general, not guaranteed that the unitarity equations have a solution at all or that a solution, if it exists, will be unique.

Let us just mention that the problem of a complete reconstruction of scattering amplitudes is of fundamental importance for several reasons. Indeed, in any dynamical theory (involving a Lagrangian, Hamiltonian, some sort of field theory, etc.) it is possible to calculate the scattering amplitudes completely. A complete check of the predictions of such a theory hence implies a knowledge of the amplitudes. On the other hand, in S-matrix theory, it is again the amplitudes that are fundamental quantities. Assumptions about analyticity, asymptotic behavior, Regge poles, various discrete or intrinsic symmetries, etc., always have stronger consequences for amplitudes than for the experimental quantities [thus an equation involving three amplitudes, following, e.g., from SU(3), will only lead to inequalities for cross sections]. If we want to relate one process to another one, e.g., perform three-body calculations in terms of two-body quantities, we need the two-body amplitudes with all phases. The interference between two interactions (like Coulomb and nuclear interactions) will also depend on the over-all phases.

The problem of possible ambiguities in scattering amplitudes, due to the nonlinear character of the unitarity equation, has been studied from two points of view.

The first approach is of a fundamentally mathematical nature. Topological and analytical methods in the theory of nonlinear integral equations are used to prove that under certain conditions on the experimental data the unitarity condition has a solution and under more stringent conditions the solution is unique. For the scattering of spinless particles we refer to Refs. 1–6; scattering of spin-zero on spin-$\frac{1}{2}$ particles is considered, e.g., in Refs. 7 and 8; nucleon-nucleon scattering is treated from this point of view in the Refs. 8 and

9. The results of this approach are very illuminating but the conditions for their applicability (for example, the "contraction mapping conditions") are much too restrictive to cover all cases of physical interest (e.g., they essentially exclude the existence of resonances[2] in the considered region).

A different approach is more simple and explicitly displays existing ambiguities. Instead of considering the total amplitudes a partial-wave expansion is performed. Elastic unitarity then simply implies that the phase shifts are real. The problem then is to find all different sets of real phase shifts that reproduce the same data. The first ambiguity of the above type was found for spin-zero particle scattering by Crichton and it involves only S, P, and D waves.[10] This problem was pursued for scalar particles in several other articles.[11-15] The disadvantage of this approach is that it is only manageable for a finite (and small) number of partial waves. Also, the question of whether ambiguities thus found are typical for experimental quantities that are strictly expressible as polynomials in $\cos\theta$, or whether they occur in general, remains open. See, however, some relevant comments in de Roo's thesis.[12]

Ambiguities in πN scattering (more generally in the scattering of spin-0 and spin-$\frac{1}{2}$ particles), specifically different sets of phase shifts giving the same cross section and polarization, were considered by Puzikov, Ryndin, and Smorodinskiĭ,[16] who also introduced the concept of a complete experiment. They point out the existence of a generalized Minami ambiguity, i.e., a transformation of the amplitudes, leaving the cross section, polarization, and unitarity condition invariant. Less obvious ambiguities, involving S and P waves only, were recently found by Berends and Ruijsenaars.[17] Related problems were studied by Dean and Lee,[18] Klepikov and Smorodinskiĭ,[19,20] *et al.*

In the present article we consider pion-nucleon scattering, i.e., an elastic reaction of the type $0+\frac{1}{2}\to 0+\frac{1}{2}$. We write the scattering matrix in the form

$$M=f(k,\theta)+g(k,\theta)\vec{\sigma}\cdot\vec{n}\ , \tag{1}$$

where f and g are two complex functions of the energy and scattering angle (say in the c.m. system), $\vec{\sigma}$ represents the three Pauli matrices and $\vec{n}$ is a unit vector along the normal to the scattering plane. Introducing three orthonormal vectors

$$\vec{l}=\frac{\vec{k}_i+\vec{k}_f}{|\vec{k}_i+\vec{k}_f|}\ ,\ \vec{m}=\frac{-\vec{k}_i+\vec{k}_f}{|-\vec{k}_i+\vec{k}_f|}\ ,\ \text{and}\ \ \vec{n}=\frac{\vec{k}_i\times\vec{k}_f}{|\vec{k}_i\times\vec{k}_f|}\ , \tag{2}$$

where $\vec{k}_i$ and $\vec{k}_f$ are the initial and final c.m. momenta, we can express four linearly independent experimental quantities as

$$\begin{aligned}
I&=\tfrac{1}{2}\,\mathrm{Tr}MM^\dagger=|f|^2+|g|^2\ ,\\
IP&=\tfrac{1}{2}\,\mathrm{Tr}\sigma_n MM^\dagger=2\,\mathrm{Re}fg^*\ ,\\
ID_{ll}&=ID_{mm}=\tfrac{1}{2}\,\mathrm{Tr}\sigma_l M\sigma_l M^\dagger=|f|^2-|g|^2\ ,\\
ID_{ml}&=-ID_{lm}=\tfrac{1}{2}\,\mathrm{Tr}\sigma_m M\sigma_l M^\dagger=2\,\mathrm{Im}fg^*\ .
\end{aligned} \tag{3}$$

Here I is the differential cross section, P is the recoil nucleon polarization, and D_{ik} is the polarization rotation tensor[21,22] (i.e., D_{ll} and D_{ml} are c.m. Wolfenstein parameters). These quantities satisfy the equation

$$P^2+D_{ll}{}^2+D_{lm}{}^2=1\ . \tag{4}$$

The problem thus is to reconstruct Ref, Imf, Reg, and Img from two or more of the quantities I, P, D_{ll}, and D_{lm} plus the elastic unitarity condition. Like the authors of Ref. 17, we restrict ourselves to $l=0$ and $l=1$ partial waves. We consider ambiguities in the phase shifts if I and P, I and D_{ll}, or I and D_{ml} are measured. In each case we explicitly construct the transformation from one set of phase shifts to the other one, leaving the pair of experimental quantities invariant. We also show how the remaining two quantities transform and show whether a measurement of one or both of them will always eliminate the ambiguity.

In Sec. II we consider some "global" ambiguities, existing for any number of partial waves, discuss the case of $P=0$ in some detail, and make some general comments on the method. In Sec. III we consider transformations of the amplitudes f and g leaving I and P fixed. In Sec. IV we do the same for I and D_{ll} fixed, in Sec. V for I and D_{lm}. The results and future outlook are summarized in Sec. VI.

Note that there is an overlap between our Sec. III and Ref. 17; however, the methods and some of the results differ.

II. GENERAL COMMENTS ON AMBIGUITIES

Throughout this article we make use of the standard partial-wave expansion[23,24] of the amplitudes f and g in (1):

$$\begin{aligned}
f(k,\theta)&=\frac{1}{k}\sum_{l=0}^{L}\left[(l+1)f_l^+ + lf_l^-\right]P_l(\cos\theta)\ ,\\
g(k,\theta)&=\frac{i}{k}\sum_{l=1}^{L}(f_l^+-f_l^-)\sin\theta\,\frac{dP_l(\cos\theta)}{d\cos\theta}\ .
\end{aligned} \tag{5}$$

Expansions (5) diagonalize the elastic unitarity equation which then simply implies that the partial-wave amplitudes satisfy

$$f_l^{\pm} = \frac{\zeta_l^{\pm} - 1}{2i}, \quad \zeta_l^{\pm} = e^{2i\delta_l^{\pm}} \tag{6}$$

with real phase shifts $\delta_l^{\pm}$.

Let us assume that a set of real phase shifts, describing the data, has been found. We wish to find all possible transformations

$$\delta_l \to \delta_l' \tag{7}$$

such that δ_l' are also real and reproduce the same data (exactly, i.e., assuming that the experimental errors vanish). Let us first consider "global" ambiguities, existing for an arbitrary number of partial waves. Three obvious transformations that come to mind, are the following.

(i) The first transformation is

$$\delta_l' = -\delta_l ,$$

i.e., (8)

$$f' = -f^*, \quad g' = g^*$$

(the asterisk implies complex conjugation). The experimental quantities (3) transform as follows:

$$I' = I, \quad D_{ll}' = D_{ll}, \quad D_{lm}' = D_{lm}, \text{ and } P' = -P. \tag{9}$$

Thus, measurements of I, D_{ll}, and D_{lm} will not distinguish between the two sets in (8) whereas a measurement of the polarization will, unless we have $P=0$. For $P=0$ the two sets are indistinguishable.

(ii) The original Minami ambiguity[25] is

$$\delta_l^{+\prime} = \delta_{l+1}^-, \quad \delta_l^{-\prime} = \delta_{l-1}^+ \tag{10a}$$

i.e.,

$$\begin{aligned} f' &= f\cos\theta + ig\sin\theta , \\ g' &= -if\sin\theta - g\cos\theta . \end{aligned} \tag{10b}$$

The experimental quantities (3) transform as

$$\begin{aligned} &I' = I, \quad P' = -P , \\ &D_{ll}' = D_{ll}\cos 2\theta + D_{lm}\sin 2\theta , \\ &D_{ml}' = -D_{ml}\cos 2\theta + D_{ll}\sin 2\theta . \end{aligned} \tag{11}$$

The cross section is always invariant; the polarization is invariant only if it is equal to zero. Each of the polarization rotation parameters will distinguish between the two sets in (10), unless $\theta=0$, when $D_{ll}'=D_{ll}$ (and $D_{ml}'=-D_{ml}$) or $\theta=\pi/2$, when $D_{ml}'=D_{ml}$ (and $D_{ll}'=-D_{ll}$).

(iii) The modified Minami ambiguity[16]—a combination of the above two ambiguities—is

$$\delta_l^{+\prime} = -\delta_{l+1}^-, \quad \delta_l^{-\prime} = -\delta_{l-1}^+, \tag{12a}$$

i.e.,

$$\begin{aligned} f' &= -f^*\cos\theta + ig^*\sin\theta , \\ g' &= if^*\sin\theta - g^*\cos\theta . \end{aligned} \tag{12b}$$

The experimental quantities satisfy

$$\begin{aligned} &I' = I, \quad P' = P , \\ &D_{ll}' = D_{ll}\cos 2\theta + D_{ml}\sin 2\theta , \\ &D_{ml}' = -D_{ml}\cos 2\theta + D_{ll}\sin 2\theta . \end{aligned} \tag{13}$$

Both I and P are thus invariant, D_{ll} is invariant only for $\theta=0$, D_{ml} is invariant only for $\theta=\pi/2$.

In passing, let us note that it has been shown[7] that under certain stringent conditions (for example, the "contraction mapping conditions"), no other than the above ambiguities exist, even if D_{ll} and D_{lm} are not measured.

In order to search systematically for less obvious ambiguities, we modify methods used earlier.[11,17] For convenience we replace the c.m. scattering angle by the variable

$$t = \tan\tfrac{1}{2}\theta, \text{ i.e., } \sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2} \tag{14}$$

and then look for transformations conserving two quantities: (I, P), (I, D_{ml}), or (I, D_{ll}).

In the case (I, P) we follow Ref. 17 by introducing the function

$$G(t) = f(t) + g(t) \tag{15}$$

(we suppress the energy variable k). Restricting ourselves to L terms in the expansion (5) we can write

$$G(t) = \frac{f(0)}{(1+t^2)^L} \prod_{i=1}^{2L} (f_i t - 1) , \tag{16}$$

where $1/f_i$ are the complex roots of the polynomial $G(t)$. Since $I = \frac{1}{2}[|G(t)|^2 + |G(-t)|^2]$ and $IP = \frac{1}{2}[|G(t)|^2 - |G(-t)|^2]$, any transformation leaving $|G(t)|^2$ invariant will leave I and P invariant. Restricting ourselves to transformations leaving the number of partial waves L fixed, and also leaving the total cross section

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\,\mathrm{Im} f(0) = \frac{4\pi}{k}\,\mathrm{Im} G(0) \tag{17}$$

invariant, we find that the only allowed transformations are the following.

(i) S: $\mathrm{Re} f(0) \to -\mathrm{Re} f(0)$ $[f(0) \to -f^*(0)]$, (18)

(ii) T_i: $f_i \to f_i^*$ (complex conjugation of a root), (19)

and combinations of such transformations.

In the case when the pair (I, D_{ml}) is assumed to be known we introduce the function

$$H(t)=f(t)+ig(t)\ . \tag{20}$$

Since $I=\frac{1}{2}[|H(t)|^2+|H(-t)|^2]$ and $ID_{ml}=\frac{1}{2}[|H(t)|^2-|H(-t)|^2]$ we again search for transformations leaving the modulus of H invariant. We put

$$H(t)=\frac{f(0)}{(1+t^2)^L}\prod_{i=1}^{2L}(h_i t-1) \tag{21}$$

and find that transformations of $H(t)$, leaving $|H(t)|$, $\mathrm{Im}f(0)$, and the number of waves L invariant, are analogous to those in (18) and (19), i.e., the transformation S: $f(0)\to -f^*(0)$ and the following:

$$\text{(iii)}\ \ T_i':\ \ h_i\to h_i^*\ . \tag{22}$$

Finally, if I and D_{ll} are known, both the moduli $|f|^2$ and $|g|^2$ are fixed. We can write $f(t)$ and $g(t)$ as

$$\begin{aligned} f(t)&=\frac{f(0)}{(1+t^2)^L}\prod_{i=1}^{L}\frac{(t-a_i)(t+a_i)}{-a_i{}^2}\ ,\\ g(t)&=c\,\frac{i}{k}\ \frac{t}{(1+t^2)^L}\ \prod_{i=1}^{L-1}(t-b_i)(t+b_i)\ , \end{aligned} \tag{23}$$

where a_i, b_i, and c are constants. Transformations leaving $|f(t)|^2$, $|g(t)|^2$, $\mathrm{Im}f(0)$, and L invariant are again S: $f(0)\to -f^*(0)$ and also

$$\begin{aligned} &\text{(iv)}\ \ T_i'':\ \ a_i\to a_i^*\ ,\\ &\text{(v)}\ \ T_i''':\ \ b_i\to b_i^*,\\ &\text{(iv)}\ \ S^\phi:\ \ ce^{i\phi},\quad 0\le\phi<2\pi\ . \end{aligned} \tag{24}$$

The problem thus is to consider all the above transformations and their appropriate combinations, and to see how the remaining experimental quantities transform in each case. Further, we must assure that both the initial amplitudes and the ones obtained after the transformations obey the unitarity conditions.

From here on we restrict ourselves to S and P waves only, i.e., put

$$\begin{aligned} f(t)&=\frac{1}{2k(1+t^2)}(a+bt^2),\quad a\ne 0\\ g(t)&=\frac{i}{k}\ c\,\frac{t}{1+t^2}\ , \end{aligned} \tag{25}$$

where a, b, and c are complex constants, related to the phase factors in (6) by

$$\begin{aligned} a&=-i(\zeta_0+2\zeta_++\zeta_--4)\ ,\\ b&=-i(\zeta_0-2\zeta_+-\zeta_-+2)\ ,\\ c&=-i(\zeta_+-\zeta_-)\ . \end{aligned} \tag{26}$$

We have introduced the notation

$$\zeta_0^+\equiv\zeta_0,\quad \zeta_1^+\equiv\zeta_+,\quad \zeta_1^-\equiv\zeta_-\ . \tag{27}$$

The unitarity condition is simply

$$|\zeta_i|^2=1,\quad i=0,+,-\ . \tag{28}$$

The Minami ambiguity (10a) reduces to

$$\zeta_0'=\zeta_-,\quad \zeta_-'=\zeta_0,\quad \zeta_+'=\zeta_+=1\ , \tag{29}$$

and the modified ambiguity (12a) is

$$\zeta_0'=\zeta_-^*,\quad \zeta_-'=\zeta_0^*,\quad \zeta_+'=\zeta_+=1\ . \tag{30}$$

The sign ambiguity (8) $\delta_l\to-\delta_l$ exists always and the solutions δ_l and $-\delta_l$ (ζ_l and ζ_l^*) are indistinguishable if $P=0$. For $l=0$ and 1 only, the equation

$$P=2\,\mathrm{Re}fg^*=0 \text{ with } \sigma_{\text{tot}}=\frac{2\pi}{k^2}\,\mathrm{Im}a\ne 0 \tag{31}$$

can be easily solved. Indeed, if $g=0$ it is sufficient to put

$$\zeta_0=e^{2i\delta_0},\quad \zeta_+=\zeta_-=e^{2i\delta} \tag{32}$$

and to substitute into (26). If $g\ne 0$ (note that $\mathrm{Im}a\ne 0$ implies $f\ne 0$) then (31) implies that

$$\begin{aligned} &\mathrm{Im}a\,\mathrm{Re}c-\mathrm{Re}a\,\mathrm{Im}c=0\ \ ,\\ &\mathrm{Im}b\,\mathrm{Re}c-\mathrm{Re}b\,\mathrm{Im}c=0\ \ . \end{aligned} \tag{33}$$

Since the unitarity conditions (28) can always be rewritten in terms of the amplitudes a, b, and c as

$$\begin{aligned} &\mathrm{Im}(a+b)=\tfrac{1}{4}|a+b|^2\ ,\\ &\mathrm{Im}(a-b+2c)=\tfrac{1}{12}|a-b+2c|^2\ ,\\ &\mathrm{Im}(a-b-4c)=\tfrac{1}{12}|a-b-4c|^2\ . \end{aligned} \tag{34}$$

Using (33) we can eliminate $\mathrm{Re}b$ and $\mathrm{Re}c$ from (34) and use (34) to express all the entries in terms of a single suitably chosen parameter. The first equation in (34) can be seen to imply

$$b=-a \text{ or } \mathrm{Im}b=\frac{4(\mathrm{Im}a)^2}{|a|^2}-\mathrm{Im}a\ .$$

Similar relations are obtained from the other equations. Combining them all together, we find five types of solutions of (31), summarized in Table I. Notice that it is only in case (i) that all the polarization effects are trivial and that case (ii) corresponds to a constant differential cross section but nontrivial polarization rotation parameters.

We now go over to a consideration of ambiguities specific for the case of S and P waves only and corresponding to nonzero polarization.

III. TRANSFORMATIONS OF AMPLITUDES LEAVING I AND P INVARIANT

We use the amplitudes f and g in the form (25) and write (16) in the form

TABLE I. Phase factors, amplitudes and experimental quantities when $P=0$.

	Solution I	Solution II	Solution III	Solution IV	Solution V
ζ_0	$e^{2i\delta_0}$	1	1	$e^{2i\delta}$	$e^{2i\delta}$
ζ_+	$e^{2i\delta}$	1	$e^{2i\delta}$	1	$e^{2i\delta}$
ζ_-	$e^{2i\delta}$	$e^{2i\delta}$	1	$e^{2i\delta}$	1
f	$\frac{i}{2k(1+t^2)}[(4-e^{2i\delta_0}-3e^{2i\delta})$ $+(-2-e^{2i\delta_0}+3e^{2i\delta})t^2]$	$\frac{i(1-e^{2i\delta})}{2k}\frac{1-t^2}{1+t^2}$	$\frac{i(1-e^{2i\delta})}{k}\frac{1-t^2}{1+t^2}$	$\frac{i}{k}\frac{(1-e^{2i\delta})}{1+t^2}$	$\frac{i(1-e^{2i\delta})}{2k}\frac{3-t^2}{1+t^2}$
g	0	$\frac{1-e^{2i\delta}}{k}\frac{t}{1+t^2}$	$-\frac{1-e^{2i\delta}}{k}\frac{t}{1+t^2}$	$\frac{(1-e^{2i\delta})t}{k(1+t^2)}$	$-\frac{1-e^{2i\delta}}{k}\frac{t}{1+t^2}$
I	$\lvert f\rvert^2$	$\frac{1-\cos 2\delta}{2k^2}$	$\frac{2(1-\cos 2\delta)}{k^2}\frac{1-t^2+t^4}{(1+t^2)^2}$	$\frac{2(1-\cos 2\delta)}{k^2(1+t^2)}$	$\frac{1-\cos 2\delta}{2k^2}\frac{9-2t^2+t^4}{(1+t^2)^2}$
P	0	0	0	0	0
D_{ll}	1	$\frac{1-\cos 2\delta}{2k^2}\frac{1-6t^2+t^4}{(1+t^2)^2}$	$\frac{2(1-\cos 2\delta)}{k^2}\frac{1-3t^2+t^4}{(1+t^2)^2}$	$\frac{2(1-\cos 2\delta)}{k^2}\frac{1-t^2}{(1+t^2)^2}$	$\frac{1-\cos 2\delta}{2k^2}\frac{9-10t^2+t^4}{(1+t^2)^2}$
D_{ml}	0	$\frac{2(1-\cos 2\delta)}{k^2}\frac{t(1-t^2)}{(1+t^2)^2}$	$-\frac{4(1-\cos 2\delta)}{k^2}\frac{t(1-t^2)}{(1+t^2)^2}$	$\frac{4(1-\cos 2\delta)}{k^2}\frac{t}{(1+t^2)^2}$	$-\frac{2(1-\cos 2\delta)}{k^2}\frac{t(3-t)}{(1+t^2)^2}$
$\frac{k^2}{2\pi}\sigma_{tot}$	$4-\cos 2\delta_0-3\cos 2\delta$	$1-\cos\delta$	$2(1-\cos\delta)$	$2(1-\cos\delta)$	$3(1-\cos\delta)$

$$G(t)=f(t)+g(t)$$
$$=\frac{a}{2k}(f_1t-1)(f_2t-1)\ . \qquad (35)$$

Considering a, f_1, and f_2 as independent quantities, we shall use

$$a,\quad b=af_1f_2,\quad c=\frac{ia}{2}(f_1+f_2) \qquad (36)$$

and

$$\zeta_0=1+\frac{ia}{2}(1+f_1f_2)\ ,$$
$$\zeta_+=1+\frac{ia}{6}[1-f_1f_2+i(f_1+f_2)]\ , \qquad (37)$$
$$\zeta_-=1+\frac{ia}{6}[1-f_1f_2-2i(f_1+f_2)]\ .$$

Let us consider the transformations (18), (19), and combinations of these.

(1) ST_1T_2. We have

$$a\to -a^*,\quad f_1\to f_1^*,\quad f_2\to f_2^* \qquad (38)$$

and hence

$$f\to -f^*,\quad g\to -g^*\ . \qquad (39)$$

The new phase factors ζ_i' are obtained from ζ_i by performing the substitution (38) in (37). It is then easy to see that the conditions $|\zeta_i'|=|\zeta_i|=1$ imply that

$$g=0\ . \qquad (40)$$

Thus we obtain a special case of the "global ambiguity" (8), treated above.

(2) S. We have

$$a\to -a^*,\quad f_1\to f_1,\quad f_2\to f_2\ . \qquad (41)$$

The original and transformed phase factors can be written as[17]

$$\zeta_j=1+iaC_j,\quad \zeta_j'=1-ia^*C_j\ . \qquad (42)$$

The equations $|\zeta_j|^2=|\zeta_j'|^2=1$ imply that

$$i(a+a^*)(C_j-C_j^*)=0\ .$$

If $a=-a^*$ then S is an identity transformation (of no interest), if $C_j=C_j^*$ then we see from (37) that f_1f_2 and $i(f_1+f_2)$ are real. Transformation S, in view of (36), then reduces to $f\to -f^*$, $g\to g^*$, i.e., we again obtain the global ambiguity (8).

(3) T_1T_2. We have

$$a\to a,\quad f_1\to f_1^*,\quad f_2\to f_2^*\ . \qquad (43)$$

The condition $|\zeta_0|^2=|\zeta_0'|^2$ implies $\mathrm{Re}\,a\,\mathrm{Im}(f_1f_2)=0$. However, if $\mathrm{Re}\,a=0$ then T_1T_2 is equivalent to ST_1T_2 already considered above. Hence we need only consider the case when f_1f_2 is real. The condition $|\zeta_+|^2=|\zeta_+'|^2$ then implies either that $\mathrm{Im}(f_1+f_2)=0$, (but then T_1T_2 is an identity transformation) or that

$$f_1f_2=1-\frac{6\,\mathrm{Im}a}{|a|^2}\ . \qquad (44)$$

Substituting (44) into expression (37) for ζ_0 and requiring $|\zeta_0|^2=1$ we obtain

$$|a|^2=\kappa\,\mathrm{Im}a,\ \text{with}\ \kappa=\begin{cases}3 & (45a)\\ 5 & (45b)\end{cases}\ .$$

We now have

$$\zeta_0=1+ia\left(1-\frac{3}{\kappa}\right),\quad \zeta_+=1+\frac{ia}{\kappa}-\frac{a}{6}(f_1+f_2)\ ,$$
$$\zeta_-=1+\frac{ia}{\kappa}+\frac{a}{3}(f_1+f_2)\ . \qquad (46)$$

Requiring that $|\zeta_+|^2=|\zeta_-|^2=1$ and putting $\phi=\arg(f_1+f_2)$ we obtain

$$|f_1+f_2|=\frac{3\sqrt{2}}{\kappa}\ ;$$
$$\tan\phi=\frac{\mathrm{Im}(f_1+f_2)}{\mathrm{Re}(f_1+f_2)}=\pm\left(\frac{8\kappa^2-9|a|^2}{|a|^2}\right)^{1/2}\ . \qquad (47)$$

We thus obtain a sign ambiguity $\phi\to -\phi$ in the phase factors and amplitudes, which have the form

$$\zeta_0=1+ia\left(1-\frac{3}{\kappa}\right),$$
$$\zeta_+=1+\frac{ia}{\kappa}\left(1+\frac{1}{\sqrt{2}}e^{i\phi}\right) \qquad (48)$$
$$\zeta_-=1+\frac{ia}{\kappa}(1-i\sqrt{2}e^{i\phi})$$

and

$$f=\frac{a}{2k}\,\frac{1+(1-6/\kappa)t^2}{1+t^2}\ ,\quad g=-\frac{3a\sqrt{2}}{2k\kappa}e^{i\phi}\,\frac{t}{1+t^2}\ . \qquad (49)$$

The values of the parameter a are constrained by the condition (45a) or (45b) and from (47) also by the condition

$$|a|^2\leqslant\frac{8\kappa^2}{9}\ . \qquad (50)$$

The experimental quantities in this case are

$$I=I'=\frac{|a|^2}{4k^2}\frac{1}{(1+t^2)^2}\left[1+2\left(1-\frac{3}{\kappa}\right)^2t^2+\left(1-\frac{6}{\kappa}\right)^2t^4\right],$$
$$IP=I'P'=-\frac{|a|^2 3\sqrt{2}}{2k^2\kappa}\frac{t}{(1+t^2)^2}\left[1+\left(1-\frac{6}{\kappa}\right)t^2\right]\cos\phi,$$
$$ID_{ll}=I'D'_{ll}=\frac{|a|^2}{4k^2}\frac{1}{(1+t^2)^2}\left[1+2\left(1-\frac{6}{\kappa}-\frac{9}{\kappa^2}\right)t^2+\left(1-\frac{6}{\kappa}\right)^2t^4\right],\tag{51}$$
$$ID_{ml}=\frac{|a|^2 3\sqrt{2}}{2k^2\kappa}\frac{t}{(1+t^2)^2}\left[1+\left(1-\frac{6}{\kappa}\right)t^2\right]\sin\phi.$$

The transformation T_1T_2 can thus be seen to be

$$f\to f,\quad g\to ge^{-2i\phi}\tag{52}$$

leaving I, P, and D_{ll} invariant, but changing the sign of D_{ml}. The two sets of amplitudes become completely indistinguishable if $D_{ml}=0$. The cases $|a|^2=0$ and $\sin\phi=0$ are of no interest, since in the first case there is no scattering, and in the second there is no ambiguity. However, we can have $D_{ml}=0$ for special values of t:

$$t=0 \text{ or } t^2=\frac{\kappa}{6-\kappa};\tag{53}$$

the cases $t=0$ or $t^2=1$ ($\kappa=3$, $\theta_{\text{c.m.}}=0$, or $\pi/2$) are not very interesting; the case $t^2=5$ ($\kappa=5$) represents a genuine ambiguity for that particular scattering angle, when no "complete experiment"[16] exists to determine f and g uniquely.

The transformations T_1 (or analogously T_2) and ST_1 are more complicated to deal with. Using formulas (25) and (36) we write general expressions for the experimental quantities (3):

$$I=\frac{|a|^2}{4k^2(1+t^2)^2}\left[1+(|f_1|^2+|f_2|^2+4\text{Re}f_1\text{Re}f_2)t^2+|f_1|^2|f_2|^2t^4\right],$$
$$IP=-\frac{|a|^2t}{2k^2(1+t^2)^2}\left[\text{Re}f_1+\text{Re}f_2+(|f_1|^2\text{Re}f_2+|f_2|^2\text{Re}f_1)t^2\right],$$
$$ID_{ll}=\frac{|a|^2}{4k^2(1+t^2)^2}\left[1-(|f_1|^2+|f_2|^2+4\text{Im}f_1\text{Im}f_2)t^2+|f_1|^2|f_2|^2t^4\right],\tag{54}$$
$$ID_{ml}=\frac{|a|^2t}{2k^2(1+t^2)^2}\left[\text{Im}f_1+\text{Im}f_2-(|f_1|^2\text{Im}f_2+|f_2|^2\text{Im}f_1)t^2\right].$$

We shall now consider the transformations T_1 and ST_1 and show how the requirement of unitarity for the original and transformed amplitudes imposes severe restrictions on the quantities a, f_1, and f_2 and hence on the forms of the experimental quantities.

(4) T_1. Requiring that $|\zeta_0|^2=|\zeta_0'|^2$ and $|\zeta_+|^2=|\zeta_+'|^2$ we obtain two constraints, namely

$$\text{Re}a\,\text{Re}f_2-\text{Im}a\,\text{Im}f_2=-\tfrac{1}{2}|a|^2\text{Im}f_2,\tag{55}$$

$$\text{Re}a\,\text{Re}f_2+\text{Im}a(1-\text{Im}f_2)=\tfrac{1}{6}|a|^2(1+|f_2|^2-2\text{Im}f_2)\tag{56}$$

(these also assure that $|\zeta_-|^2=|\zeta_-'|^2$). We can solve these for Rea and Ima, unless the corresponding determinant, equal to Ref_2, is zero.

Let us first consider the case

$$\text{Re}f_2=0.\tag{57}$$

Since $\text{Im}f_2\neq 0$ (otherwise T_1 would be equivalent to T_1T_2) we can solve (55) and (56) for a and f_2 and obtain

$$a=i+e^{i\alpha},\quad 0\leq\alpha<2\pi\tag{58}$$

$$f_2=i\kappa,\quad \kappa=1 \text{ or } \kappa=-2.\tag{59}$$

Taking $\kappa=1$ we obtain from (37) that $\zeta_+=1$; requiring that $|\zeta_-|^2=|\zeta_0|^2=1$ we find that $f_1=e^{i\phi}$ and that $\alpha=\pm\pi/2$ or $\phi=\pm\pi/2$. Both of these can be seen to correspond to special cases of the Minami ambiguities (29) or (30). Taking $\kappa=-2$ we find that the conditions $|\zeta_-|^2=|\zeta_0|^2=1$ cannot be satisfied.

The second case is more interesting.

$$\text{Re}f_2\neq 0,\quad \text{Im}f_2\neq 0.\tag{60}$$

From (55) and (56) we obtain

$$\mathrm{Re}a=\frac{|a|^2}{6}z\,\mathrm{Im}f_2,\quad z=\frac{|f_2|^2+\mathrm{Im}f_2-2}{\mathrm{Re}f_2},$$
$$\mathrm{Im}a=\frac{|a|^2}{6}z\,\mathrm{Re}f_2+\frac{|a|^2}{2},\tag{61}$$
$$a=\frac{6i}{3+zf_2}.$$

Using (61) and the three conditions $|\zeta_i|^2=1$ we obtain three equations, linear in $\mathrm{Re}f_1$ and $|f_1|^2$:

$$\mathrm{Re}f_1(|f_2|^2+\mathrm{Im}f_2-2)-|f_1|^2\left(\tfrac{3}{2}\mathrm{Re}f_2\right)=\frac{\mathrm{Re}f_2}{|f_2|^2}\left(\tfrac{1}{2}-\mathrm{Im}f_2-|f_2|^2\right),\tag{62a}$$
$$2\mathrm{Re}f_1(|f_2|^2+2\mathrm{Im}f_2)+|f_1|^2\mathrm{Re}f_2=\mathrm{Re}f_2,\tag{62b}$$
$$2\mathrm{Re}f_1(-|f_2|^2+\mathrm{Im}f_2)-|f_1|^2\mathrm{Re}f_2=\mathrm{Re}f_2\,\frac{2|f_2|^2-10\mathrm{Im}f_2-1}{|f_2|^2+4\mathrm{Im}f_2+4}.\tag{62c}$$

In equations (62b) and (62c) we have divided both sides by certain expressions that are nonzero for $\mathrm{Re}f_2\neq 0$. Equations (62b) and (62c) can be solved for $\mathrm{Re}f_1$ and $|f_1|^2$ (the determinant is zero only if $\mathrm{Im}f_2=0$ or $\mathrm{Re}f_2=0$ which is excluded). Substituting the obtained expressions into the first of equations (62) we obtain a constraint on f_2 that can be factorized into the form

$$(|f_2|^2+2\mathrm{Im}f_2)(2|f_2|^2-\mathrm{Im}f_2-1)(|f_2|^2-2\mathrm{Im}f_2+1)=0.\tag{63}$$

Three cases must thus be considered.

(a) First consider

$$2\mathrm{Im}f_2+|f_2|^2=0.\tag{64}$$

We immediately find

$$f_2=-i+e^{i\eta},\quad 0\leqslant\eta<2\pi.\tag{65}$$

From (61) we have

$$a=\frac{6i}{2+ie^{i\eta}}\tag{66}$$

and from (62)

$$f_1=e^{i\phi},\quad \cos\phi=\frac{4\sin\eta-5}{4\cos\eta},\tag{67}$$

where the angle η is such that

$$\frac{5-\sqrt{7}}{8}\leqslant\sin\eta\leqslant\frac{5+\sqrt{7}}{8}.\tag{68}$$

Finally we obtain the amplitudes

$$f=\frac{a}{2k}\,\frac{1+e^{i\phi}(-i+e^{i\eta})t^2}{1+t^2},$$
$$g=-\frac{a}{2k}\,\frac{t}{1+t^2}\,(e^{i\phi}+e^{i\eta}-i)\tag{69}$$

with a and ϕ related to η by (66) and (67) and η in the region (68). The ambiguity T_1 corresponds to $\phi\to-\phi$. From (54) we see that $I=I'$, $P=P'$ but both D_{ll} and D_{ml} will distinguish between the two solutions. The only exceptions are for $t=0$ (then $D'_{ll}=D_{ll}$, $D'_{ml}=D_{ml}=0$) or for $1-|f_2|^2t^2=0$ (then $D'_{ml}=D_{ml}$, but $D'_{ll}\neq D_{ll}$).

(b) Consider the case

$$2|f_2|^2-\mathrm{Im}f_2-1=0.\tag{70}$$

From (70) we find that f_2 can be expressed in terms of one real parameter ρ as

$$f_2=\tfrac{1}{4}(i+3e^{i\rho}),$$
$$0\leqslant\rho<2\pi,\quad \rho\neq(2n+1)\frac{\pi}{2},\quad (n=\text{integer}).\tag{71}$$

From (61) we find that a is given as

$$a=16i\cos\rho[\cos\rho(5+3\sin\rho)-i(1+3\sin\rho)(1-\sin\rho)]^{-1}\tag{72}$$

and using (62) we express f_1 as $f_1=|f_1|e^{i\phi}$ with

$$|f_1|^2=\frac{33\sin^2\rho+30\sin\rho+1}{2(1+3\sin\rho)(5+3\sin\rho)},\tag{73}$$
$$|f_1|\cos\phi=\frac{3\cos\rho(1-\sin\rho)}{2(1+3\sin\rho)(5+3\sin\rho)}\tag{74}$$

(the denominators never vanish for $\mathrm{Im}f_2\neq 0$). In addition to being a real angle, ρ must be such that $|f_1|^2\geqslant 0$ and $-1\leqslant\cos\phi\leqslant 1$. This restricts $\sin\rho$ to the following regions:

$$-0.8253\leqslant\sin\rho\leqslant-0.4520$$
$$\text{or}\tag{75}$$
$$-0.0029\leqslant\sin\rho\leqslant 1.$$

To display the ambiguity explicitly we return to the amplitudes f and g:

$$f=\frac{a}{2k(1+t^2)}\left[1+\tfrac{1}{4}|f_1|(i+3e^{i\rho})e^{i\phi}t^2\right],$$
$$g=-\frac{a}{2k}\,\frac{t}{(1+t^2)}\left[|f_1|e^{i\phi}+\tfrac{1}{4}(i+3e^{i\rho})\right].\tag{76}$$

The amplitudes f, g and f', g', obtained by replacing $\phi\to-\phi$, give the same values of I and P. It follows from (54) that D_{ll} and D_{lm} will distinguish between the two sets.

(c) The case $|f_2|^2-2\mathrm{Im}f_2+1=0$ is of no interest, since it implies $f_2=i$ and leads only to the Minami ambiguity.

Let us mention that the case (b) coincides with an ambiguity found in Ref. 17; ambiguity (a) is new.

(5) ST_1. We write ζ_i in the form (37), obtain ζ'_i by replacing $a\to-a^*$, $f_1\to f_1^*$, $f_2\to f_2$ and require $|\zeta_0|^2=|\zeta'_0|^2$ and $|\zeta_+|^2=|\zeta'_+|^2$ (then $|\zeta_-|^2$

$=|\zeta'_-|^2$ is redundant). We assume $\mathrm{Im} f_2 \neq 0$ (otherwise ST_1 is equivalent to ST_1T_2) and obtain, after simple operations

$$\mathrm{Re}\,a\,\mathrm{Re}f_1 - \mathrm{Im}\,a\,\mathrm{Im}f_1 = -\frac{|a|^2}{2}\mathrm{Im}f_1\,, \tag{77a}$$

$$\mathrm{Re}\,a\,\mathrm{Re}f_2 = \frac{|a|^2}{6}\mathrm{Im}f_1(-|f_2|^2 - \mathrm{Im}f_2 + 2)\,. \tag{77b}$$

Two cases must be considered.

(a) Assume $\mathrm{Re}f_2 = 0$. Equation (77b) implies

$$f_2 = i\kappa, \quad \kappa = \begin{cases} 1 \\ -2 \end{cases}. \tag{78}$$

Using (77a) and imposing $|\zeta_i|^2 = 1$, we find that $\kappa = 1$ leads to the Minami ambiguity and that $\kappa = -2$ is incompatible with unitarity.

(b) Assume $\mathrm{Re}f_2 \neq 0$.

We solve (77) for $\mathrm{Re}\,a$ and $\mathrm{Im}\,a$ and find

$$a = \frac{6i}{3 - f_1 z}\,, \quad z = \frac{|f_2|^2 + \mathrm{Im}f_2 - 2}{\mathrm{Re}f_2}\,. \tag{79}$$

Imposing unitarity $|\zeta_i|^2 = 1$ itself, we find again three equations, linear in $\mathrm{Re}f_1$ and $|f_1|^2$:

$$\mathrm{Re}f_1(2 - x - y) + |f_1|^2\mathrm{Re}f_2(2 - x - \tfrac{5}{2}y) = -\tfrac{3}{2}\mathrm{Re}f_2\,, \tag{80a}$$

$$2\mathrm{Re}f_1(2x - y - xy) - |f_1|^2\mathrm{Re}f_2(2x+1) = (y - 4x)\mathrm{Re}f_2\,, \tag{80b}$$

$$2\mathrm{Re}f_1(2 - x - 5y + 4x^2) + |f_1|^2\mathrm{Re}f_2(2x + y - 6) = (2y - 5)\mathrm{Re}f_2\,, \tag{80c}$$

where we have put

$$x = \mathrm{Im}f_2, \quad y = |f_2|^2\,. \tag{81}$$

We wish to solve (80b) and (80c) for $\mathrm{Re}f_1$ and $|f_1|^2$. Consider first the case when the corresponding determinant is equal to zero, i.e.,

$$(2x - y - 1)(4x^2 + xy + 5x + y - 2) = 0\,. \tag{82}$$

The first case is $2x - y - 1 = 0$; this implies $f_2 = i$ and we obtain the Minami ambiguity. The second case implies

$$y = \frac{-4x^2 - 5x + 2}{x + 1} \tag{83}$$

[$x = -1$ is not compatible with (82)]. Putting (83) into (80b) and (80c) we find $y = |f_2|^2 = (1 \mp \sqrt{3})/(1 \pm \sqrt{3}) < 0$, which is impossible.

We can thus assume that the determinant is not zero and obtain

$$\mathrm{Re}f_1 = -\mathrm{Re}f_2\,\frac{4x + y - 5}{2(4x^2 + xy + 5x + y - 2)}\,, \tag{84}$$

$$|f_1|^2 = \frac{8x^2 + 2xy + 2x - 3y}{4x^2 + xy + 5x + y - 2}. \tag{85}$$

Substituting these expressions back into (80a), we obtain a relation between x and y, i.e., a constraint on f_2:

$$5xy^2 + 22x^2y - 6xy - 8y^2 + 8y + 8x^3 - 22x^2 - 5x - 2 = 0\,. \tag{86}$$

Contrary to equation (63) in the case T_1, (86) does not factorize. Since it is only quadratic in y we can solve it, obtaining two solutions

$$y_\pm = \frac{-22x^2 + 6x - 8 \pm \sqrt{D}}{2(5x - 8)} \equiv \frac{-B \pm \sqrt{D}}{2A}\,, \qquad D = 108x(3x^3 + 4x^2 - 2x - 2)\,. \tag{87}$$

Thus, all relevant quantities, i.e., a, f_1, and f_2 (and thus the amplitudes f and g) are functions of $x = \mathrm{Im}f_2$ alone. For each value of x in the allowed regions discussed below [see (88) and (89)] there exist four sets of values of ($\mathrm{Re}f_1, \mathrm{Re}f_2, \mathrm{Im}\,a; \mathrm{Im}f_2 = x$) for which the transformation ST_1 leads to an ambiguity. The ambiguity in f and g corresponds to the two possible signs of $\mathrm{Im}f_1 = \pm[|f_1|^2 - \mathrm{Re}f_1)^2]^{1/2}$, leading also to the two values a or $-a^*$ [see (79)]. Notice that (54) implies that D_{ll} and D_{ml} will resolve the ambiguity.

Constraints on the possible values of x are obtained by requiring $D \geq 0$ in (87) (i.e., $|f_2|^2$ real), further that $y - x^2 \geq 0$ [i.e., $(\mathrm{Re}f_2)^2 \geq 0$], and finally that $(\mathrm{Im}f_1)^2 = |f_1|^2 - (\mathrm{Re}f_1)^2 \geq 0$. These conditions have been investigated numerically and we find that the allowed values of x lie in the regions

$$-0.6022 \leq x \leq -0.0486 \quad \text{or} \quad 0.7483 \leq x \leq 1.0170 \tag{88}$$

for

$$y_- = \frac{-B - \sqrt{D}}{2A}\,,$$

and

$$-0.6022 \leq x \leq -0.0243 \quad \text{or} \quad 0.7483 \leq x \leq 1.000 \tag{89}$$

for

$$y_+ = \frac{-B + \sqrt{D}}{2A}\,.$$

The results of this section agree with those of Ref. 17, but our method, making use of the linearity of (80) in $\mathrm{Re}f_1$ and $|f_1|^2$, has allowed us to express f_1, f_2, and a explicitly in terms of x.

To summarize this section we can state the following. In addition to the "global" ambiguities, discussed in Sec. II, further ambiguities may exist if only I and P are measured and if I has one of several specific forms. These are the T_1T_2 ambiguity for I given in (51) and the T_1 or ST_1 ambiguities for I given in (54), with the constraints on f_1, f_2, and a given in (65) to (68), (71) to (75), or (79) to (89).

IV. TRANSFORMATIONS OF AMPLITUDES LEAVING I AND D_{ml} INVARIANT

We write $H(t)$ in the form (21) with $L=1$ and use the parameters a, h_1, and h_2. Three relevant quantities are

$$a, \quad b=ah_1h_2, \quad c=\frac{a}{2}(h_1+h_2) \tag{90}$$

so that

$$\zeta_0=1+\frac{ia}{2}(1+h_1h_2), \quad \zeta_+=1+\frac{ia}{6}(1-h_1h_2+h_1+h_2)\,, \qquad \zeta_-=1+\frac{ia}{6}(1-h_1h_2-2h_1-2h_2)\,. \tag{91}$$

Let us now consider the transformation S of (18), T_1', T_2' of (22), and combinations of these.

(1) $ST_1'T_2'$. This transformation directly gives $f\to -f^*$, $g\to g^*$, which is the "global ambiguity" of Sec. II.

(2) S. We put $\zeta_j=1+iaC_j$ and have $\zeta_j'=1-ia^*C_j$. The conditions $|\zeta_j|^2=|\zeta_j'|^2$ imply

$$i(a+a^*)(C_j-C_j^*)=0\,. \tag{92}$$

Since $a\neq -a^*$ (otherwise S is the identity transformation) we find $C_j=C_j^*$. Requiring that $|\zeta_j|^2=1$ we find that h_1h_2 and h_1+h_2 are real. It follows that transformation S is equivalent to ST_1T_2 and only provides the global ambiguity.

(3) $T_1'T_2'$. We again put $\zeta_j=1+iaC_j$ and have $\zeta_j'=1+iaC_j^*$. The requirement $|\zeta_j|^2=|\zeta_j'|^2$ again implies equation (92). If $a=-a^*$ then $T_1'T_2'$ is equivalent to $ST_1'T_2'$. If $C_j=C_j^*$ we find that h_1h_2 and h_1+h_2 are real and hence $T_1'T_2'$ is the identity transformation.

(4) T_1'. We use the general expressions

$$f=\frac{a}{2k(1+t^2)}(1+h_1h_2t^2), \quad g=\frac{ia}{2k}(h_1+h_2)\frac{t}{1+t^2} \tag{93}$$

for the amplitudes and express the experimental quantities as

$$\begin{aligned}
I&=\frac{|a|^2}{4k^2(1+t^2)^2}[1+(|h_1|^2+|h_2|^2+4\mathrm{Re}h_1\mathrm{Re}h_2)t^2+|h_1|^2|h_2|^2t^4]\,,\\
IP&=\frac{|a|^2}{2k^2}\frac{t}{(1+t^2)^2}[-\mathrm{Im}h_1-\mathrm{Im}h_2+(|h_1|^2\mathrm{Im}h_2+|h_2|^2\mathrm{Im}h_1)t^2]\,,\\
ID_{ml}&=-\frac{|a|^2}{2k^2}\frac{t}{(1+t^2)^2}[\mathrm{Re}h_1+\mathrm{Re}h_2+(|h_1|^2\mathrm{Re}h_2+|h_2|^2\mathrm{Re}h_1)t^2]\,,\\
ID_{ll}&=\frac{|a|^2}{4k^2(1+t^2)^2}[1+(-|h_1|^2-|h_2|^2-4\mathrm{Im}h_1\mathrm{Im}h_2)t^2+|h_1|^2|h_2|^2t^4]\,.
\end{aligned} \tag{94}$$

Unitarity for the original and transformed amplitudes will be used to obtain relations between the parameters a, h_1, and h_2 in (93) and (94). Indeed, the conditions $|\zeta_i|^2=|\zeta_i'|^2$ [where ζ_i are given in (91) and ζ_i' are obtained by the transformation $a\to a$, $h_1\to h_1^*$, $h_2=h_2$] imply

$$\mathrm{Re}a\,\mathrm{Re}h_2-\mathrm{Im}a\,\mathrm{Im}h_2=-\frac{|a|^2}{2}\mathrm{Im}h_2\,, \qquad \mathrm{Re}a(1-\mathrm{Re}h_2)+\mathrm{Im}a\,\mathrm{Im}h_2=\frac{|a|^2}{3}\mathrm{Im}h_2 \tag{95}$$

(we assume $h_1\neq h_1^*$—otherwise T_1' is an identical transformation; and $h_2\neq h_2^*$—otherwise T_1' is equivalent to $T_1'T_2'$). We can solve (95) for Rea and Ima and obtain

$$h_2=3\left(1-\frac{2i}{a}\right). \tag{96}$$

The phase factors ζ_i can now be written as

$$\zeta_0=\frac{i}{2}(a-2i)(1+3h_1)\,, \qquad \zeta_+=\frac{i}{3}(a-3i)(2-h_1)\,, \qquad \zeta_-=-\frac{i}{6}(5a-6i)(1+h_1)\,. \tag{97}$$

We introduce a new variable

$$\rho=a-2i \tag{98}$$

and require that $|\zeta_i|^2=1$. We obtain three equations

$$1+9|h_1|^2+6\mathrm{Re}h_1=\frac{4}{|\rho|^2}, \tag{99a}$$

$$4+|h_1|^2-4\mathrm{Re}h_1=\frac{9}{|\rho-i|^2}, \tag{99b}$$

$$1+|h_1|^2+2\mathrm{Re}h_1=\frac{36}{|5\rho+4i|^2}. \tag{99c}$$

Since the left-hand sides are again linear in $\mathrm{Re}h_1$ and $|h_1|^2$, we can easily solve two of these equations, say (99a) and (99c) (note that $\rho=0$, $\rho=i$, and $\rho=-4i/5$ are incompatible with unitarity). We obtain

$$|h_1|^2=\frac{1}{3}+\frac{2}{3x}-\frac{18}{25x^2+40y+16},$$
$$\mathrm{Re}h_1=-\frac{2}{3}-\frac{1}{3x}+\frac{27}{25x^2+40y+16}, \tag{100}$$

where

$$x=|\rho|^2, \quad y=\mathrm{Im}\rho. \tag{101}$$

The condition that the three Eqs. (99) should be compatible provides a relation between y and x, namely

$$Ay^2+By+C=0,$$
$$A=-80(7x+2)$$
$$B=-2(35x^2+36x-8), \tag{102}$$
$$C=175x^3-14x^2-76x+32.$$

Since (102) is quadratic in y it can be solved and we find

$$y_\pm=\frac{-B\pm\sqrt{D}}{2A},$$
$$D=B^2-4AC \tag{103}$$
$$=324(1225x^4+280x^3-544x^2+64x+64).$$

Finally, we see that using (103) we obtain h_2 [Eq. (96)], h_1 [Eq. (100)], and a [Eq. (101) gives $|a|^2=x+4y+4$, $\mathrm{Im}a=y+2$] as functions of a single parameter x. For a chosen value of x in the allowed regions discussed below [see (104) and (105)] there exist four sets of values of $\{\mathrm{Re}a, \mathrm{Im}a, \mathrm{Re}h_1, |h_1|^2, \mathrm{Re}h_2, \mathrm{Im}h_2\}$ for which the transformation T_1' leads to an ambiguity. The ambiguity T_1' then corresponds to the sign ambiguity in $\mathrm{Im}h_1$ for each given set. Obviously T_1' leaves I and D_{ml} invariant. It follows from equations (94) that a measurement of P will distinguish between $\pm\mathrm{Im}h_1$, unless $t=0$ or $|h_2|^2t^2=1$. A measurement of D_{ll} will always distinguish between the two sets of amplitudes, except for the trivial case $\mathrm{Im}h_2=0$.

It remains to determine the region of variation of x for each solution y in (103). The conditions to impose are clearly the following:

(a) $D\geq 0$ so that $y=\mathrm{Im}\rho$ is real;

(b) $\mathrm{Re}^2\rho=|\rho|^2-(\mathrm{Im}\rho)^2\geq 0$, i.e., $x-y^2\geq 0$;

(c) $(\mathrm{Im}h_1)^2=|h_1|^2-(\mathrm{Re}h_1)^2\geq 0$.

A detailed numerical analysis of all these conditions yields the following result. The amplitudes $\{f,g\}$ and $\{f',g'\}$ will satisfy all unitarity conditions, if we take either

$$\text{(i)}\quad y_-=\frac{-B-\sqrt{D}}{2A} \text{ for } 0.1483\leq x\leq 4.6051 \tag{104}$$

or

$$\text{(ii)}\quad y_+=\frac{-B+\sqrt{D}}{2A} \text{ for } 0.1008\leq x\leq 1.0683. \tag{105}$$

(5) ST_1'. We use expressions (93) and (94) for the amplitudes and experimental quantities. Writing ζ_i in the form (91) and performing the transformation $a\to -a^*$, $h_1\to h_1^*$, $h_2\to h_2$, we see that the $(\zeta_i)_{ST_1'}$ obtained from ζ_i by the transformation ST_1' obey $(\zeta_i)_{ST_1'}=(\zeta_i)^*_{T_2'}$, where $(\zeta_i)_{T_2'}$ are the phase factors obtained by performing transformation T_2'. Thus we see that the amplitudes f and g (or $\{a, h_1, h_2\}$) must obey the same constraints as for the case T_2', i.e., the same as for T_1', with h_1 and h_2 interchanged. Since the conditions on a, i.e., on ρ, are unchanged, the allowed regions are the same as in (104) and (105). The effect of the transformation ST_1' is

$$f\to (f)_{ST_1'}=-(f)^*_{T_2'},$$
$$g\to (g)_{ST_1'}=(g)^*_{T_2'},$$

i.e., ST_1' boils down to T_2' supplemented by the "global" transformation.

To summarize, the only nontrivial ambiguity that occurs when I and D_{ml} are measured is due to the transformation T_1' (or T_2') and is analyzed in formulas (93) to (105). It can always be resolved by a measurement of either P or D_{ll}.

V. TRANSFORMATIONS OF AMPLITUDES LEAVING I AND D_{ll} INVARIANT

The case when I and D_{ll} are known is somewhat simpler to analyze than the previous cases, since now two moduli are known. We write the amplitudes as

$$f(t)=\frac{a}{2k(1+t^2)}\,\frac{(t-F)(t+F)}{-F^2}, \quad g(t)=\frac{ic}{k}\,\frac{t}{1+t^2}. \tag{106}$$

Transformations conserving $|f|^2$, $|g|^2$, and $\mathrm{Im}f(0)$ are listed in (24). In our case they reduce to

$$S:\ a \to -a^*,\quad F \to F,\quad c \to c;$$
$$T:\ a \to a,\quad F \to F^*,\quad c \to c; \tag{107}$$
$$S^\psi:\ a \to a,\quad F \to F,\quad c \to ce^{i\psi},$$

and all conbinations of these.

In this case we found a different parametrization to be more useful. It gives an explicit solution of the unitarity equations in terms of three real angles, namely

$$\begin{aligned}
&\zeta_0 = 1 + 2i\sin\phi\, e^{i\phi},\\
&\zeta_+ = |A|e^{i\alpha} + i|B|e^{i\beta},\\
&\zeta_- = |A|e^{i\alpha} - 2i|B|e^{i\beta},\\
&\quad |A| = \frac{1}{[1 + 8\sin^2(\alpha-\beta)]^{1/2}},\\
&\quad |B| = \frac{2\sin(\alpha-\beta)}{[1 + 8\sin^2(\alpha-\beta)]^{1/2}},\\
&\quad 0 \le \phi < \pi,\quad 0 \le \alpha - \beta < \pi,\\
&\quad -\pi < \alpha \le \pi,\quad -\pi < \beta \le \pi,
\end{aligned} \tag{108}$$

so that

$$\begin{aligned}
&a = 2\sin\phi\, e^{i\phi} - 3i(|A|e^{i\alpha} - 1),\\
&b = 2\sin\phi\, e^{i\phi} + 3i(|A|e^{i\alpha} - 1),\\
&c = 3|B|e^{i\beta}.
\end{aligned} \tag{109}$$

Consider individual transformations.

(1) STS^ψ. The effect of this transformation is

$$f \to -f^*,\quad g \to -g^* e^{i(2\beta+\psi)}.$$

We must now rewrite the transformed quantities a', b', and c' in the form (109), using new parameters α', β', and ϕ'. We obtain $\phi' = \pi - \phi$, $\alpha' = -\alpha$, $\beta' = \beta + \psi$, and the constraint $\sin(\alpha' - \beta') = \sin(\alpha - \beta)$. This has two solutions:

(a) $\psi = -\pi - 2\beta$. Then $f \to -f^*$, $g \to g^*$, which is the global ambiguity.

(b) $\psi = -2\alpha$. We have

$$f \to -f^*,\quad g \to -g^* e^{2i(\beta-\alpha)}. \tag{110}$$

This ambiguity exists for all values of α, β, and ϕ in (108). The experimental quantities transform as follows:

$$\begin{aligned}
&I' = I,\quad D'_{ll} = D_{ll},\\
&P' = P\cos2(\alpha-\beta) + D_{ml}\sin2(\alpha-\beta),\\
&D'_{ml} = P\sin2(\alpha-\beta) - D_{ml}\cos2(\alpha-\beta).
\end{aligned} \tag{111}$$

We see that if we require in addition that $P' = P$ we find that $\alpha - \beta = 0$, i.e., $g = 0$ and hence $P' = P = 0$ (and also $D'_{ml} = -D_{ml} = 0$). Similarly, if we require instead that $D'_{ml} = D_{ml}$ we find $\alpha - \beta = \pi/2$, i.e., the global ambiguity $f \to -f^*$, $g \to g^*$.

(2) ST. This is a special case of STS^ψ with $\psi = 0$ and the results are obtained from (110) and (111) by putting $\alpha = 0$ (since $\psi = -2\alpha$).

(3) S. The condition $|\zeta_0|^2 = |\zeta_0'|^2$ with $\zeta_0 = 1 + (ia/2)(1 - 1/F^2)$ implies

$$(a + a^*)(F^{*2} - F^2) = 0. \tag{112}$$

If $a = -a^*$ then S is the identity transformation; if $F^2 = F^{*2}$, then F is either real or pure imaginary and S is equivalent to ST.

(4) S^ψ. The transformation is

$$f \to f,\quad g \to ge^{i\psi}, \tag{113}$$

i.e., $a' = a$, $b' = b$, $c' = ce^{i\psi}$. The condition that the transformed quantities can be written in the form (109) implies $\sin(\alpha - \beta - \psi) = \sin(\alpha - \beta)$. The case $\psi = 0$ corresponds to the identity transformation, hence we only consider $\psi = -\pi + 2(\alpha - \beta)$. The transformation S^ψ then is

$$f \to f,\quad g \to -ge^{2i(\alpha-\beta)}. \tag{114}$$

This ambiguity exists for all values of α, β, and ϕ in (108) and we have

$$\begin{aligned}
&I' = I,\quad D'_{ll} = D_{ll},\\
&P' = -P\cos2(\alpha-\beta) - D_{ml}\sin2(\alpha-\beta),\\
&D'_{ml} = P\sin2(\alpha-\beta) - D_{ml}\cos2(\alpha-\beta).
\end{aligned} \tag{115}$$

Clearly, if we ask in addition that $P' = P$ and/or $D'_{ml} = D_{ml}$ we find $2(\alpha - \beta) = \pi$. However, in this case S^ψ is the identity transformation. In other words, a measurement of P or D_{ml} will always resolve the ambiguity (114).

(5) T. We write ζ_0 as $\zeta_0 = 1 + ia/2 - ia/2F^2$ and require $|\zeta_0|^2 = |\zeta_0'|^2$. This implies equation (112). If F^2 is real, then T is an identity transformation; if $a = -a^*$, then T is equivalent to ST.

(6) SS^ψ. The condition $|\zeta_0|^2 = |\zeta_0'|^2$ again implies equation (112). If $a = -a^*$ then SS^ψ is equivalent to S^ψ, if $F^2 = F^{*2}$, then SS^ψ is equivalent to STS^ψ.

(7) TS^ψ. As above, $|\zeta_0|^2 = |\zeta_0'|^2$ implies equation (112). If $a = -a^*$, then TS^ψ is equivalent to STS^ψ. If $F^2 = F^{*2}$, then TS^ψ is equivalent to S^ψ.

To summarize this section we note that nontrivial ambiguities occur for the STS^ψ and the S^ψ transformations which can be viewed as the two basic transformations. Both rotate the quantities P and D_{ml} among each other and hence a measurement of either of these quantities will resolve the ambiguity.

VI. CONCLUSIONS

The results of this investigation are summarized in Tables I–IV. Table I lists all cases when the polarization is $P = 0$. We have seen in Sec. II that in this case we have a "global" ambiguity, independent of any partial-wave expansion, namely

TABLE II. Amplitude ambiguities for I and P measured [see Eq. (54)].

Type	Ambiguity	Parameter constraints for $f=\frac{a(1+f_1f_2t^2)}{2k(1+t^2)}$, $g=\frac{-at(f_1+f_2)}{2k(1+t^2)}$	Possible resolution of ambiguity through a measurement of D_{ll} and/or D_{ml}
T_1T_2	$\phi\equiv\arg(f_1+f_2)$, $\phi\to-\phi$	$\|a\|^2=\kappa\,\mathrm{Im}\,a$ ($\kappa=3$ or 5), $f_1f_2=1-6/\kappa$, $\|f_1+f_2\|=3\sqrt{2}/\kappa$ $\tan\phi=\pm[(8\kappa^2/\|a\|^2)-9]^{1/2}$, $8\kappa^2\geq 9\|a\|^2$.	D_{ll}: no (left unchanged) D_{ml}: yes, unless $D_{ml}=0$ $[t=0, t=\kappa/(6-\kappa)]$
$T_1(a)$	$\phi\equiv\arg(f_1)$, $\phi\to-\phi$	$\|f_1\|=1$, $f_2=-i+e^{i\eta}$, $a=6i/(2+ie^{i\eta})$, $\cos\phi=(4\sin\eta-5)/4\cos\eta$, $\frac{5-\sqrt{7}}{8}\leq\sin\eta\leq\frac{5+\sqrt{7}}{8}$.	D_{ll}: yes, unless $t=0$ or $\mathrm{Im}f_2=0$(case T_1T_2) D_{ml}: yes, unless $t=0$ or $1-\|f_2\|^2t=0$
$T_1(b)$	$\phi\equiv\arg(f_1)$, $\phi\to-\phi$	$\|f_1\|^2=\frac{33\sin^2\rho+30\sin\rho+1}{2(1+3\sin\rho)(5+3\sin\rho)}$, $\|f_1\|\cos\phi=\frac{3\cos\rho(1-\sin\rho)}{2(1+3\sin\rho)(5+3\sin\rho)}$, $f_2=(i+3e^{i\rho})/4$, $a=16i\cos\rho[\cos\rho(5+3\sin\rho)-i(1+3\sin\rho)(1-\sin\rho)]^{-1}$, $-0.8253\leq\sin\rho\leq-0.4520$, or $-0.0029\leq\sin\rho\leq 1$.	idem
ST_1	$\mathrm{Im}f_1\to-\mathrm{Im}f_1$, $\mathrm{Re}a\to-\mathrm{Re}a$	$\mathrm{Im}f_2\equiv x$, $\|f_2\|^2=y_\pm=\{-22x^2+6x-8\pm[108x(3x^3+4x^2-2x-2)]^{1/2}\}(10x-16)^{-1}$, $\mathrm{Re}f_1=-\mathrm{Re}f_2[4x+y-5]\,[2(4x^2+xy+5x+y-2)]^{-1}$, $\|f_1\|^2=[8x^2+2xy+2x-3y][4x^2+xy+5x+y-2]^{-1}$, $a=6i\,\mathrm{Re}f_2[3\,\mathrm{Re}f_2+f_1(2-\mathrm{Im}f_2-\|f_2\|^2)]^{-1}$, for y_+: $-0.6022\leq x\leq-0.0243$, or $0.7483\leq x\leq 1$, for y_-: $-0.6022\leq x\leq-0.0486$, or $0.7483\leq x\leq 1.0170$.	D_{ll}: yes, unless $t=0$ or $\mathrm{Im}f_2=0$(case ST_1T_2, i.e., global ambiguity) D_{ml}: yes, unless $t=0$ or $1-\|f_2\|^2t^2=0$

TABLE III. Amplitude ambiguities for I and D_{ml} measured [see Eq. (94)].

Type	Ambiguity	Parameter constraints for $f=\frac{a(1+h_1h_2t^2)}{2k(1+t^2)}$, $g=\frac{ia(h_1+h_2)t}{2k(1+t^2)}$	Possible resolution of ambiguity through a measurement of P and/or D_{ll}
T_1'	$h_1 \to h_1^*$	$\mathrm{Im}a=y+2$, $\lvert a\rvert^2=x+4y+4$, $h_2=3(1-2i/a)$, $\lvert h_1\rvert^2=\frac{1}{3}+\frac{2}{3x}-\frac{18}{25x^2+40y+16}$, $\mathrm{Re}h_1=-\frac{2}{3}-\frac{1}{3x}+\frac{27}{25x^2+40y+16}$, where $y_\pm=\frac{35x^2+36x-8\pm 9[1225x^4+280x^3-544x^2+64x+64]^{1/2}}{-80(7x+2)}$, and for y_+: $0.1008\le x\le 1.0683$, for y_-: $0.1483\le x\le 4.6051$.	P: yes, unless $t=0$ or $t^2\lvert h_2\rvert^2=1$; D_{ll}: yes, unless $\mathrm{Im}h_2=0$(case $T_1'T_2'$, i.e., global ambiguity)
ST_1'	$h_1\to h_1^*$, $a\to -a^*$	Equivalent to T_2' and the global ambiguity.	

$$\{f,g\} \text{ and } \{f'=-f^*, g'=g^*\}$$

are indistinguishable (this corresponds to the ambiguity $\delta_l \to -\delta_l$ for the phase shifts). In Tables II III, and IV we summarize all ambiguities in the amplitudes f and g, occurring when two experimental quantities (I,P), (I,D_{ml}), or (I,D_{ll}) are measured. The global ambiguity and the Minami ones have been excluded. We also show how the ambiguity can be resolved by the measurement of a suitably chosen third quantity. We see that "unremovable" ambiguities only occur for very specific values of t (i.e., of the scattering angle). This points to the relevance of studies of the type performed in this paper to the planning of experiments. When performing an experiment to distinguish between two sets of amplitudes it is obviously necessary to avoid values of t close to the ones where nonremovable ambiguities occur.

The over-all situation for πN scattering thus seems to be better than for spinless scattering, in that Crichton-like ambiguities[10] can be resolved by further experiments. Generally speaking, the

TABLE IV. Amplitude ambiguities for I and D_{ll} measured.

$f(t)=\frac{1}{2k(1+t^2)}(a+bt^2)$, $g(t)=\frac{ic}{k}\frac{t}{1+t^2}$

$a=2\sin\phi e^{i\phi}-3i(e^{i\alpha}[1+8\sin^2(\alpha-\beta)]^{-1/2}-1)$, $-\pi<\alpha\le\pi$

$b=2\sin\phi e^{i\phi}+3i(e^{i\alpha}[1+8\sin^2(\alpha-\beta)]^{-1/2}-1)$, $-\pi<\beta\le\pi$

$c=6\sin(\alpha-\beta)e^{i\beta}[1+8\sin^2(\alpha-\beta)]^{-1/2}$, $0\le\phi<\pi$, $0\le\alpha-\beta<\pi$

Quantity \ Transformation	STS^ψ $\psi=-2\alpha$	ST $(\psi=-2\alpha\equiv 0)$	S^ψ $\psi=-\pi+2(\alpha-\beta)$
f'	$-f^*$	$-f^*$	f
g'	$-g^*e^{2i(\beta-\alpha)}$	g	$-ge^{2i(\alpha-\beta)}$
I'	I	I	I
D_{ll}'	D_{ll}	D_{ll}	D_{ll}
P'	$P\cos2(\alpha-\beta)+D_{ml}\sin2(\alpha-\beta)$	$P\cos2\beta-D_{ml}\sin2\beta$	$-P\cos2(\alpha-\beta)-D_{ml}\sin2(\alpha-\beta)$
D_{ml}'	$P\sin2(\alpha-\beta)-D_{ml}\cos2(\alpha-\beta)$	$-P\sin2\beta-D_{ml}\cos2\beta$	$P\sin2(\alpha-\beta)-D_{ml}\cos2(\alpha-\beta)$
Resolution of ambiguity	P or D_{ml}	P or D_{ml}	P or D_{ml}

higher the spins involved, the more overdetermined the amplitudes will be. Thus for spinless scattering we have one amplitude, i.e., two real functions to determine from one experimental quantity (the cross-section) and from one unitarity equation. In πN scattering we wish to determine four real functions, in principle from the four quantities I, P, D_{ll}, and D_{ml} [see, however, Eq. (4)] and from two unitarity equations [coupling $f(k, \theta)$ and $g(k, \theta)$ together]. The amplitudes are thus overdetermined and it can be expected that this overdetermination can be used to eliminate ambiguities due to bifurcation points in the nonlinear unitarity equations. Clearly, in nucleon-nucleon scattering or in any scattering involving higher spins the amplitudes should be even more overdetermined.

In the near future we plan to return to the problem of amplitude analysis from several points of view. The obvious ones are to look at πN scattering using a larger number of partial waves (in particular $l=0$, 1, and 2) and to look at NN scattering from the same point of view. An equally important problem is that of ambiguities in other parametrizations of amplitudes, like Regge-pole fits,[26] eikonal representations, two variable expansions,[27] etc. A different problem presently under consideration is the use of inelastic unitarity, say above the first inelastic threshold but below the second one, to reconstruct simultaneously the amplitudes of different processes, related by unitarity (e.g., $\pi N \to \pi N$ and $\pi N \to \pi\pi N$).

A related problem to be investigated is the use of other general principles, in particular, analyticity, to remove continuous ambiguities in phase shifts (due to the unknown over-all phase of the total amplitudes above the region where unitarity can be effectively used). For some relevant results on the connection between the analyticity properties of phase shifts and time delay in scattering we refer to the literature.[28,29]

All conclusions about the uniqueness of amplitude reconstruction and phase-shift analysis in particular are, of course, modified by the existence of experimental errors. The ambiguities in the reconstruction of the parameters in amplitudes from data and the question of the stability of results with respect to a variation of the data then arises, in addition to the problem of the existence and uniqueness of a solution.[30,31] We plan to return to some aspects of this problem in connection with πN and NN scattering.

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