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Propagation of "heat" in hadronic matter*

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We investigate the space-time evolution of a local excitation in hadronic matter (h.m.) in connection with the establishment of local thermodynamical equilibrium as assumed in statistical and hydrodynamical models. After a critical discussion of the concept of instantaneous equilibrium, we point out that peripheral reactions are a particularly useful source of information for the study of the fate of an excitation in h.m. We consider a local excitation ("hot spot") corresponding to a pre-equilibrium phase and which is created in peripheral inelastic reactions with $m_\pi \ll q \ll p_i$, where p_i is the incoming momentum of the projectile and q the momentum transfer. By solving the diffusion equation we obtain the distribution of the temperature field in the excited target (projectile) and compute all the relevant physical quantities such as average momenta of secondaries, multiplicities, and mass and energy distributions in semi-inclusive peripheral reactions. It turns out that these quantities have a pronounced angular dependence leading to an asymmetry in these observables. The measurement of this asymmetry can provide information on the constants of h.m. We also discuss how large-$p_\perp$ events observed in the CERN ISR energy range might be due to pre-equilibrium emission in close analogy to pre-equilibrium nuclear decay.

I. INTRODUCTION

The most characteristic phenomenological feature of strong interactions is probably the fact that provided the center-of-mass energy of a reaction is sufficiently high, it is at least as easy to produce many hadrons as few hadrons. This high-multiplicity effect which is not seen in reactions which are governed by electromagnetic or weak interactions has suggested to some physicists the fruitful idea that high-energy strong interactions are describable by statistical methods.[1-4] One bonus of such an approach is the possibility of explaining the cutoff in the transverse-momentum ($p_\perp$) distribution of secondaries produced in strong reactions, a phenomenon which is intimately related to the high-multiplicity effect and probably as characteristic of strong interactions as the first effect. (It is probably fair to say that statistical models are at present the only models which can explain in a natural way the cutoff at small $p_\perp$.) In connection with these theoretical developments the new concept of hadronic matter (h.m.) was introduced by Hagedorn, and it is the determination of the equation of state of h.m. which some physicists believe is the main task of strong-interaction physics.[5]

As a matter of fact the statistical bootstrap model, e.g.,[2,3] provides such an equation of state, at least in the asymptotic limit $E\to\infty$. On the other hand, the hydrodynamical model in its initial version[4] starts from a blackbody-type equation of state and in later developments a somewhat more general equation is assumed [$p = c_0^2\epsilon$, where p is the pressure, c_0 the speed of sound assumed to be a free (but constant) parameter, and ϵ the energy density].

Statistical methods in general and an equation of state in particular assume the existence, at least at a certain stage, of thermodynamical equilibrium, and it is this so far unprovable assumption which makes it difficult for many physicists to accept the statistical approach to strong interactions. We will explore how essential, indeed, is this assumption in the development of statistical methods in the theory of strong interactions.

II. THERMODYNAMICAL EQUILIBRIUM IN STATISTICAL (AND HYDRODYNAMICAL) MODELS AND THE PRESENT APPROACH

A. Role of equilibrium in statistical models

The first application of statistical methods to a strongly interacting system was done in nuclear physics a long time ago[6] and the extensions of these methods to h.m. are in some sense influenced by this early development. The applicability of statistical methods to nuclear systems has been justified by the high density of nuclear states. It is assumed that a nuclear target bombarded by an energetic projectile is excited to a compound state which corresponds to thermodynamical equilibrium; this compound nucleus then decays with an isotropic angular distribution of "evaporation" nucleons and with an energy distribution characterized by a Boltzmann distribution and a temperature T related to the excitation energy E by the equation of state of nuclear matter

$$E = E(T). \tag{2.1}$$

A characteristic element in such a statistical model is the assumption that the formation of the compound system and its decay are separated processes so that the equilibrium state does not "remember" how it was formed. This point of view remained unchallenged for almost 30 years until more recently deviations from equilibrium behavior have been observed (pre-equilibrium decay), which will be discussed later on.

A rather straightforward, but bold extension of these ideas to particle physics is due to Fermi[7] who considered central collisions of two hadrons and postulated that the kinetic energy available in the small Lorentz-contracted volume of the two colliding particles as seen in the center-of-mass system is instantaneously transformed into internal energy of a system in thermodynamical equilibrium. This system then decays according to the laws of its equilibrium state.

Fermi does not try to explain how this "instantaneous" equilibrium state is reached, and as we shall see, all the other more sophisticated and more successful statistical models are deficient in this respect. Moreover, Fermi's simple model could not explain the cutoff of $p_\perp$ and the mass distribution of secondaries (dominance of pions), unless the argument was invoked that Fermi's model does not apply to the energy regime where observational data exist, but to much higher energies. On the other hand, physicists were interested in a model which could explain available data. That is why Pomeranchuk[8] put forward the idea that the equilibrium state is not formed instantaneously in the Lorentz-contracted volume but after a certain time has elapsed, during which the system has expanded into a volume the radius of which exceeds the range of strong-interaction forces. During this time the system has cooled down to a temperature $T_0 \sim m_\pi$, and the decay from this equilibrium state explains the cutoff of $p_\perp$ and the dominance of light secondaries.

Pomeranchuk did not elaborate on the expansion phase of the initial fireball. This was done by Landau[4] who treated the expanding system of h.m. as an ideal fluid with an equation of state

$$p = \epsilon/3, \tag{2.2}$$

according to the laws of relativistic hydrodynamics

$$\partial T_{ik}/\partial x_k = 0, \tag{2.3}$$

where T_{ik} is the energy-momentum tensor of the (ideal) system and x_k the space-time coordinates;

$$T_{ik} = (\epsilon + p)u_i u_k + p\,\delta_{ik}, \tag{2.4}$$

where u_i is the four-velocity of the fluid.

Although Landau's theory represents a huge step forward in comparison with Fermi's model, it is still unsatisfactory for many reasons, some of which are discussed below:

(1) It applies only to central collisions.

(2) It assumes that h.m. is an ideal fluid with viscosity ν and heat conductivity $\kappa = \nu = 0$.

(3) It neglects particle emission during the expansion phase when the energy density is still very high. Large-$p_\perp$ events are thus essentially ignored.

(4) It does not elaborate on the first stage of the collision when the kinetic energy is transformed into internal energy within the Lorentz-contracted volume.

Deficiency (1) is eliminated in some sense in the statistical bootstrap model[2] by the introduction of an *empirical* velocity weight function

$$F(\lambda_1\gamma_0) = \int d^3x\,dt\,2\pi b\,db\,u(\lambda, \vec{x}, t, b, \gamma_0), \tag{2.5}$$

where

$$\lambda = \text{sign}(v)\frac{\gamma - 1}{\gamma_0 - 1}, \qquad \gamma = (1 - v^2)^{-1/2}, \quad \gamma_0 = (1 - v_0^2)^{-1/2}, \tag{2.6}$$

b is the impact parameter, v the local *collective* velocity at space-time point $\vec{x}, t$, v_0 the c.m. system velocity of incoming particles and $u(\lambda, \vec{x}, t, b, \gamma_0)$ the probability density of finding the collective velocity λ at the place x at time t in a collision with impact parameter b. With the help of $F(\lambda, \gamma_0)$ the momentum spectrum of particles of mass m takes the form

$$W^R(p)d^3p = \int_{-1}^{1} F(\lambda)L^R(\lambda,\gamma_0)F_m(p', \epsilon)d^3p'\,d\lambda, \tag{2.7}$$

where $f_m(p', \epsilon)$ is the isotropic momentum distribution of particles m produced in a local λ frame at local energy density ϵ, and $L^R(\lambda, \gamma_0)$ the Lorentz boost which transforms the spectrum from the λ frame to the arbitrary frame R.

In this approach, too, the assumption is made that there is instantaneous local thermodynamic equilibrium, leading to $f_m(p', \epsilon)$ at a temperature T. The new element of this theory is the bootstrap equation for the mass distribution of hadrons which limits the temperature to values $T \leq T_0$, where T_0 is a universal maximum temperature $T_0 \simeq m_\pi$. It is clear that in this way the cutoff in $p_\perp$ is obtained. On the other hand, the statement $T \leq T_0$ is a very strong one with far-reaching implications in thermodynamics, astrophysics, and, of course, elementary-particle physics. Taken at face value, this prediction would imply that the $p_\perp$ cutoff would be a permanent characteristic of strong interactions at any value of center-of-mass energy. However, it is known[9] that for laboratory momenta

$p^{lab} = s/2m_N \geq 100$ GeV/c, where m_N is the nucleon mass, deviations from this cutoff are observed which become very striking with the increase of s and $p_\perp$ (see Fig. 1). They are very similar to those observed in nuclear pre-equilibrium decay (cf. Fig. 2) and the natural question arises whether they are not due to the same physical mechanism. This possibility is intimately linked with the problem of instantaneous local equilibrium, as will be shown below.

B. Problem of local instantaneous equilibrium; pre-equilibrium in h.m.

Local instantaneous equilibrium is a macroscopic concept, but in some sense it does contradict our macroscopic experience. In order to have local equilibrium it is necessary to assume that no heat (energy) is transferred from the local volume to the rest of the system. This means that the heat conductivity of the system $\kappa = 0$. On the other hand, to have instantaneous equilibrium it is necessary that the relaxation time

$$\tau_0 \sim l^2/\chi \tag{2.8}$$

be zero, where

$$\chi = \frac{\kappa}{\rho C_p} \tag{2.9}$$

[C_p is the specific heat (at fixed pressure), ρ the density, and l the dimension of the system]. This condition can thus be fulfilled only if either $l \simeq 0$, which is in contradiction with the uncertainty principle $\Delta l \Delta q \geq 1$, where q is the momentum (transfer) involved or if $\chi = \infty$, which is in contradiction with the *local* equilibrium condition ($\kappa = 0$). In some sense one would have to assume that at first to have an instantaneous equilibrium one must have $\kappa = \infty$, and then to keep it localized κ must drop to 0.

Such a change in κ can occur only during a finite time interval in which the system cannot be in equilibrium, in contradiction with the starting assumption. We must conclude, therefore, that before thermodynamic equilibrium is reached there exists a stage of pre-equilibrium.

C. Purpose and content of this paper

In this paper we would like to suggest a way of getting some direct information about the pre-equilibrium phase in a "yes-no" experiment through which effects of κ could be detected.[10] The essential idea of our approach is to consider a local excitation in a target, induced by the transfer of momentum $\vec{q}$ and energy q_0 from a fast-moving projectile and to follow the "diffusion" of this excitation through the target. We shall treat the problem via the diffusion equation, which will be solved with suitable initial and boundary conditions.

It is remarkable that this equation can be solved in our case exactly. The diffusion process depends on a diffusion constant, and this dependence is re-

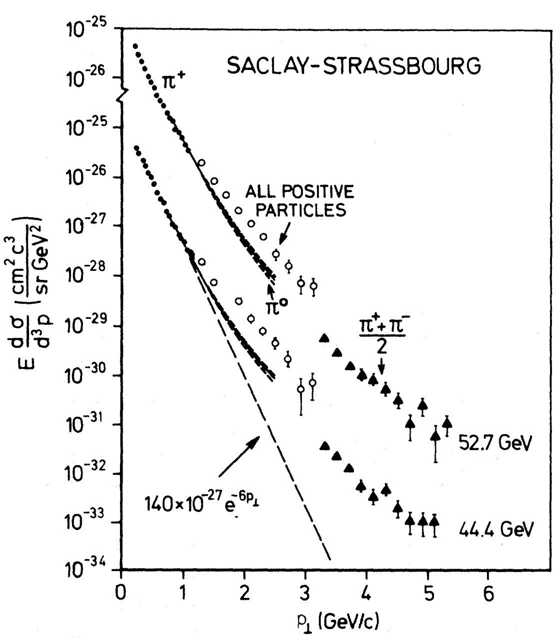

FIG. 1. A typical transverse momentum distribution in particle physics. It is seen that at small $p_\perp$ the slope corresponds to $T \sim m_\pi$, while at large $p_\perp$ the slope increases. The reaction is $p + p$ inclusive, $p_\perp$ is measured at $\theta_{cm} = 90°$. [From S. D. Ellis and R. Thun, in *Proceedings of the IX Rencontre de Moriond*, edited by J. Tran Thanh Van (Université de Paris-Sud, Orsay, 1974), Vol. I, p. 37.]

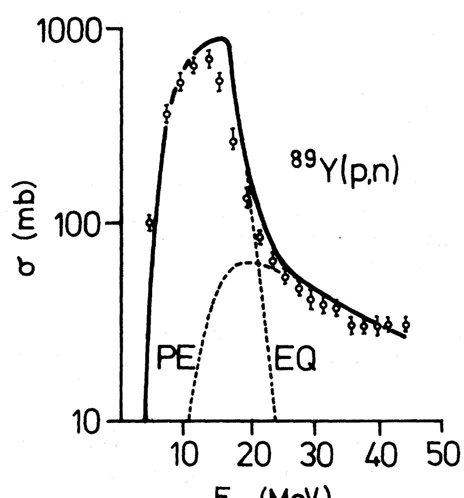

FIG. 2. A typical momentum distribution (excitation function) in nuclear physics. Points represent experimental yields. EQ and PE are the theoretical equilibrium and pre-equilibrium contribution, respectively [from M. Blann, Annu. Rev. Nucl. Sci. 25, 123 (1975)].

flected in the solution of the corresponding equation. Under certain circumstances the local excitation can be assimilated with a "hot" spot, and then our problem is synonymous with the problem of propagation of heat in hadronic matter, and the diffusion constant corresponds to the heat conductivity of h.m. For reasons which will be clarified below it is more convenient to consider peripheral collisions.

The remainder of this paper will thus be organized as follows. In Sec. III we shall discuss the problem of local excitation and of peripheral collisions. In Sec. IV we shall solve the diffusion equation. Section V is devoted to some numerical applications and to a discussion of the physical implications of our approach. It will become evident that a new effect can be predicted which consists in an asymmetry along the direction of $\vec{q}_\perp$ of secondaries emitted in high-energy collisions. The size of this effect is a well-defined function of κ, and, thus, by looking into this effect, direct information about heat conductivity of h.m. can be obtained. Furthermore, it is to be expected that large-$p_\perp$ events originate through this "hot-spot" mechanism, and the experimental investigation of the asymmetry effect might lead to new insight into this problem. The conclusions of the paper are contained in Sec. VI. Appendix A contains mathematical details concerning the solution of the diffusion equation.

III. LOCAL EXCITATION AND PERIPHERAL COLLISIONS

By local excitation we understand the concentration of a certain energy q_0 and momentum $\vec{q}$ in a certain volume in configuration space at a certain time Δt. Quantum mechanics imposes a necessary condition on this localization through Heisenberg's uncertainty relations. Let Δz be the uncertainty of the linear coordinate z. Then the uncertainty Δq_z of q_z must satisfy the inequality

$$\Delta q_z \Delta z \geq 1. \tag{3.1}$$

An analogous relation holds between energy and time,

$$\Delta q_0 \Delta t \geq 1. \tag{3.2}$$

Obviously, we must have $q > \Delta q$, $q_0 > \Delta q_0$. Since the radius of a hadron $R \sim m_\pi^{-1}$, it follows that a localization within the volume of this hadron is possible if

$$q \gg m_\pi. \tag{3.3}$$

For nuclei the corresponding condition is

$$q \gg m_\pi A^{-1/3}. \tag{3.4}$$

Conditions (3.3), (3.4) are independent of the nature of the projectile which transfers the momentum $\vec{q}$. From this it follows that hadrons are, in principle, as efficient in localizing excitations as leptons. A difference between these two classes of probes is yet expected because hadrons are known to induce also peripheral reactions, while leptons, for which the absorption within hadronic matter is negligible, scatter in the whole volume of the target. This is of importance for the effect to be discussed in this paper since we would like to consider excitations localized on the surface of a hadron. We assume that this can be accomplished in a peripheral collision, i.e., a collision with impact parameter $b \sim R$, when the momentum transfer satisfies condition (3.3).

A necessary condition[11] for peripheral reactions is

$$p_i \gg q,$$

where p_i is the incoming momentum. Combined with (3.3) this yields as a necessary condition for localization on the surface

$$p_i \gg q \gg m_\pi. \tag{3.5}$$

Let us consider now a reaction

$$a + b \to c + X \\ \searrow d + e + \cdots, \tag{3.6}$$

which is assumed to be peripheral. (If peripheralism means indeed exchange of quantum numbers,[11,12] examples of reactions of this type could be

$$p + p \to \pi + X,$$

$$p + p \to K + X,$$

$$p + p \to n + X, \text{ etc.}) \tag{3.7}$$

Let $\vec{q}$ be the momentum transfer between a and c. If condition (3.3) is fulfilled, a localization along the $\vec{q}$ axis is possible. Since the reaction is assumed to be peripheral, it is to be expected that the excitation $(q_0, \vec{q})$ will be localized on one of the poles of the sphere by which we represent our target (cf. Fig. 3).

As long as the direction of $\vec{q}$ is not measured, there is apparently no way of distinguishing between the half-space above the tangent plane at N in Fig. 3 and the half-space below. Once, however, $\vec{q}$ is measured, there exists a preferential direction which is taken as the z axis of our problem, and one can talk about an up and a down. If (q_0, q) is now localized at the N pole, the following situation might occur: The energy will be concentrated in a "hot" spot at N from which it will propagate in the upper and lower half-spaces. Since in the upper half-space there is vacuum, while in the lower one there is matter, the propagation will be asymmetric with respect to the plane separating the two spaces, and this asymmetry will manifest itself in the emission products of the excited target.

It is to be expected that emission will not start immediately from the hot spot, but after a certain time delay τ. This can be seen either by applying Pomeranchuk's argument that the number of particles becomes defined only after the initial fireball has expanded to a volume of the order of the range of interaction forces R, or by applying the uncertainty relation (3.2) which can be interpreted in the sense that the energy q_0 cannot be measured before a certain time $\tau \sim q_0^{-1}$ has elapsed. Of course, the exact value of τ cannot be determined by these qualitative arguments, and it will enter in some way as a free parameter in the estimates to be given below. It is convenient to express this propagation of excitation in terms of propagation of "heat," and then in the initial moment the north pole is hot, while in the rest of the target the temperature T is zero as it is in vacuum. There is hence an initial temperature gradient between N and S. Depending on the constants of the problem a situation might occur when emission starts before this temperature gradient has vanished, i.e.,

$$\tau < \tau_0, \tag{3.8}$$

where τ_0 is the relaxation time after which the temperature is essentially uniform in the whole body of the excited target.[13] In this case an asymmetry effect in the emission spectrum is to be expected. A first quantitative estimate of this effect will be given in the subsequent paragraphs.

IV. DIFFUSION EQUATION AND ITS SOLUTION

One of the main postulates of the hydrodynamical model[4,14,15] is the fact that in a first approximation the longitudinal motion is hydrodynamical while the transverse motion is of thermal nature.

The direction in which diffusion takes place is thus given by $\vec{q}_\perp$ rather than by $\vec{q}$. Independent

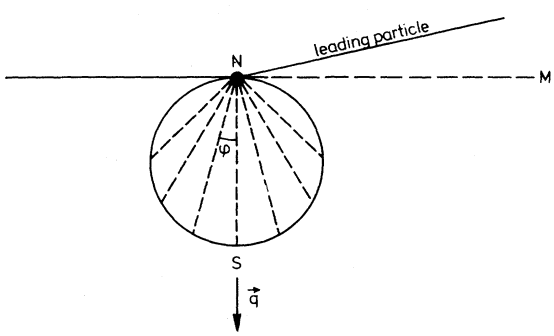

FIG. 3. Local excitation of a spherical target in a peripheral collision.

evidence for a diffusion process in the transverse direction comes from the multiperipheral parton model, where it can be shown [see J. Kogut and L. Susskind, Phys. Rep. C8, 75 (1975)] that in the infinite-momentum frame partons walk randomly in the transverse plane away from the excitation. Accordingly we shall assume that once a hot spot on the surface of the target is created the propagation of this excitation in the body of the target will be controlled essentially by a thermal diffusion process.[16] If the velocity of the diffusion process v is small in comparison with the velocity of light c,

$$v \ll c, \tag{4.1}$$

the problem can be treated by the nonrelativistic diffusion equation

$$\frac{\partial \mathcal{T}}{\partial t} = \chi \Delta \mathcal{T}, \tag{4.2}$$

where $\mathcal{T}$ is the temperature density related to the temperature T by $T = \int_{\Delta V} \mathcal{T} dV$, where ΔV is the volume in which $\mathcal{T}$ is measured, t the time variable, and χ the thermal conductivity defined in Eq. (2.9). The velocity v is determined by χ and the characteristic length l:

$$v \sim (\chi / l)^{1/2}. \tag{4.3}$$

As stressed before, as long as we do not care about the creation of the initial fireball, χ is assumed both in the hydrodynamical and statistical models to be very small, so that condition (4.1) can be considered as a not-too-bad first approximation. An improvement of this treatment can be obtained by taking into account explicitly in Eq. (4.2) the finite velocity with which heat propagates. Usually[17] this is done by adding a term $(1/c_0^2)(\partial^2 \mathcal{T}/\partial t^2)$ on the left-hand side of the equation (4.2)

$$\frac{1}{c_0^2}\frac{\partial^2 \mathcal{T}}{\partial t^2} + \chi^{-1}\frac{\partial \mathcal{T}}{\partial t} = \Delta \mathcal{T} \tag{4.4}$$

In order to solve Eq. (4.4) one needs a boundary condition and two initial conditions. For Eq. (4.2) one initial condition is sufficient. There are two types of boundary conditions used in heat propagation problems depending on whether one considers radiation or not. In the last case one specifies in general the value of T on a certain surface, in the first case the value of the gradient of T.

In our problem the first case applies. Before specifying the concrete form of boundary and initial conditions we have to choose a reference system. This is done in Fig. 4, where it is seen that the positive direction of the z axis points towards the hot spot N. Since the problem has φ symmetry (cf. Fig. 3) only two space coordinates r and θ are needed.

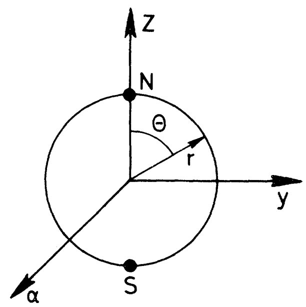

FIG. 4. Coordinate system for the problem of propagation of heat in a sphere.

Our boundary condition now reads

$$\frac{\partial \mathcal{T}}{\partial r}=\lambda'(\mathcal{T}_0^n-\mathcal{T}^n) \text{ at } r=R, \tag{4.5}$$

where λ' is a so-called radiation constant, $\mathcal{T}_0$ the temperature density outside the target (in our case $\mathcal{T}_0=0$), $\mathcal{T}$ the temperature density in the sphere and n a constant which enters the equation of state:

$$\epsilon \propto T^n. \tag{4.6}$$

Fermi and Landau used a blackbody-type equation of state ($n=4$), and this choice is still very fashionable. In the statistical bootstrap model on the other hand $n=0$ (for $T \simeq T_{\max}$). Essentially the equation of state of hadronic matter, i.e., n is not known at present. Furthermore, if one would consider the case $n>1$, our problem would become nonlinear, and no exact analytic solution could be found. This difficulty is known in heat-propagation problems[18] and is handled in general by linearizing Eq. (4.5), i.e., by defining a new effective radiation constant λ so that Eq. (4.5) reads for $\mathcal{T}_0=0$

$$\frac{\partial \mathcal{T}}{\partial r}=-\lambda \mathcal{T}, \quad r=R. \tag{4.7}$$

A similar difficulty arises because of the T dependence of the thermal-conductivity coefficient $\chi=\kappa/C_p$. In the limit

$$T \gg m, \tag{4.8}$$

where m is the highest mass appearing in the problem, dimensional arguments as well as a scalar field theory[19] in which many-particle interactions are replaced by random external forces suggest that

$$\kappa \propto T^3. \tag{4.9}$$

With the energies available so far and which we are interested in, condition (4.8) cannot yet be achieved. For $T \leqslant m$ nothing about $\kappa(T)$ in h.m. is known, and, therefore, we shall assume $\chi=\kappa/C_p=$ constant. This introduces an important mathematical simplification into our problem since in this case Eqs. (4.2), (4.4) are linear and can be solved exactly.

The initial condition for Eq. (4.2) is connected with the localization of the hot spot at the initial moment $t=0$. It is convenient to write it in the form[20]

$$\mathcal{T}=T_i\,\delta(\vec{r}-\vec{a}), \quad t=0 \tag{4.10}$$

where T_i is the initial temperature and a the coordinate of the hot spot. The δ-function-like localization is obviously an idealization, but as long as $q \gg m_\pi$ this approximation is justified. For Eq. (4.4) the supplementary initial condition might specify, e.g., $\partial \mathcal{T}/\partial t$ at $t=0$.

Mathematical solution of Eqs. (4.2), (4.4)

We look for a solution of Eqs. (4.2), (4.4) of the form

$$\mathcal{T}=e^{-At}f(r,\theta), \tag{4.11}$$

where A is a constant. With this ansatz Eq. (4.4) can be separated leading to

$$A^2/c_0^2-A/\chi=\Delta f/f \equiv -\alpha_n^2=\text{const}, \tag{4.12}$$

where the constant α_n will be determined below from the boundary condition. From (4.12) we get two equations:

$$A=\tfrac{1}{2}[c_0^2/\chi \pm (c_0^4/\chi^2-4\alpha_n^2c_0^2)^{1/2}] \tag{4.13}$$

and

$$\Delta f+\alpha_n^2 f=0. \tag{4.14}$$

Equation (4.13) has two solutions $A_\pm$, so that eventually (4.11) becomes

$$\mathcal{T}=(C_+e^{-A_+t}+C_-e^{-A_-t})f, \tag{4.15}$$

where $C_\pm$ have to be determined from the two initial conditions. We are interested in the physical case $c^2\tau_0/\chi \gg 1$ (κ small), and then from (4.13) follows

$$A_+ \simeq c^2/\chi, \quad A_-=\alpha_n^2\chi. \tag{4.16}$$

This shows that the + solution in (4.15) vanishes rapidly in time and only the − solution survives. Since the very small t range is anyway not expected to be described correctly by our approach, we shall neglect the + solution[21] in (4.15). This is equivalent to dealing from the beginning with Eq. (4.2) rather than (4.4). The solution of Eq. (4.2) with boundary condition (4.7) and initial condition (4.10) under the assumption that the hot spot is at the north pole, is (cf. Appendix A and Ref. 18)

$$\mathcal{T}(r,\theta,t)=\frac{T_i}{2\pi(rR)^{1/2}}\sum_{n,\alpha_n}(2n+1)P_n(\mu)\frac{\alpha_n^2e^{-\chi\alpha_n^2t}J_{n+1/2}(\alpha_n r)}{[(R\lambda-\frac{1}{2})^2+\alpha_n^2R^2-(n+\frac{1}{2})^2]J_{n+1/2}(\alpha_n R)}, \text{ with } \mu\equiv\cos\theta. \tag{4.17}$$

V. EXPERIMENTAL IMPLICATIONS

A. "Up-down" asymmetry of particle production as a consequence of temperature field anisotropy

The characteristic feature of (4.17) is the θ dependence of T, which means that the temperature field at a certain time t induced by a hot spot at $t=0$ is anisotropic (isotropy is reached essentially only after a relaxation time $\tau_0 \geq R^2/\chi$, which for small χ is expected to be rather large in comparison with characteristic emission times $\tau \sim m_\pi^{-1}$). This effect is exemplified in Table I where the temperature field is calculated as a function of θ for various values $D=\chi\tau/R^2$ and $\lambda=1$. This anistropy of T leads to an asymmetry (θ dependence) in production processes when an experimental distinction is made between the upper ($q_\perp < 0$) and lower ($q_\perp > 0$) hemispheres (cf. Fig. 3). The asymmetry is referred to a frame in which $\vec{q}+\vec{p}_{\text{target}}=0$. The asymmetry will manifest itself among others through the following:

(i) There is an asymmetry in the average moments of emitted particles (secondaries). Particles produced in the lower hemisphere ($\cos\theta < 0$) will have, on the average, smaller momenta than those in the upper hemisphere ($0 \leq \cos\theta$). This is easy to understand since the upper hemisphere is hotter and $\langle p\rangle$ is an increasing function of T. In order to estimate quantitatively this effect, one has to make an assumption about the emission process. For this purpose we shall adhere to the conventional assumption used in statistical models, namely, that emission takes place from a local equilibrium state, characterized by the *local* temperature T. For meson emission this equilibrium state is described by a Bose-Einstein distribution

$$n=\frac{1}{\exp[(m^2+p^2)^{1/2}/T]-1}. \qquad (5.1)$$

We shall make the further simplifying assumption that emission takes place only from the surface of the excited target (projectile). This is equivalent to the hypothesis that the particles which are created within the interior of the target do not escape because of absorption. This means that in Eq. (5.1) the following expression for the temperature density $\mathcal{T}$ has to be used [cf. Eq. (4.17) with $r=R$]

$$\mathcal{T}(R,\theta,t)=\frac{T_i}{2\pi}\frac{2n+1}{R^3}P_n(\mu)\frac{Z_n^2\exp(-\chi Z_n^2 t/R^2)}{(R\lambda-\frac{1}{2})^2+Z_n^2-(n+\frac{1}{2})^2},$$

$$\text{where } Z_n=\alpha_n R. \qquad (5.2)$$

In order to get from (5.2) a temperature T one has to integrate over the volume from which emission takes place

$$T=\int_{\Delta V}\mathcal{T}\,dV\sim\mathcal{T}\Delta V.$$

The value of ΔV might be expected to be limited from below by p^3, where p is the momentum of the emitted particle. On the other hand, it is $T_i\Delta V$ which enters eventually (5.1). T_i depends on the energy transfer, i.e., on the kinematics of the reaction.

The average momentum of emitted particles $\langle p\rangle$ [it coincides approximately with $\langle p_\perp\rangle$ since we refer to the target (projectile) fragmentation regions and subtract for leading particle effects] is

$$\langle p\rangle=\frac{\int_\tau^\infty dt\int_{-1}^1 d\mu\int np\,d^3p}{\int_\tau^\infty dt\int_{-1}^1 d\mu\int n\,d^3p}, \qquad (5.3)$$

where the integrals on t and $\mu\equiv\cos\theta$ take into account the t and θ dependence of T as given by Eq. (5.2). τ is the minimum time introduced in Sec. III, after which emission starts. The integrals in (5.3) cannot be evaluated in a closed form and have to be estimated numerically. A rough estimate can be made by remembering that for a distribution like (5.1), for $p \gg m, T$ one has in a first approximation

$$\langle p\rangle\simeq T. \qquad (5.4)$$

The error in this estimate can be decreased if one is interested in normalized differences of $\langle p\rangle$, i.e., if one defines an asymmetry parameter

$$\delta_p=\frac{\langle p\rangle_{\text{up}}-\langle p\rangle_{\text{down}}}{\langle p\rangle_{\text{up}}+\langle p\rangle_{\text{down}}}, \qquad (5.5)$$

where

$$\langle p\rangle_{\text{up}}\propto\int_\tau^\infty dt\int_0^1 d\mu\int np\,d^3p \qquad (5.6)$$

and

$$\langle p\rangle_{\text{down}}\propto\int_\tau^\infty dt\int_{-1}^0 d\mu\int np\,d^3p. \qquad (5.7)$$

The introduction of this asymmetry parameter also has the advantage that the result becomes independent of the parameter $T_i\Delta V$ if one simplifies further by using (5.4), i.e.,

TABLE I. Temperature field $T(\theta)$ for $T_i\Delta V=1$, $\lambda=1$, and $0.2\leq D\leq 1$; ($D=\chi\tau/R^2$). We use units such that $m_\pi=1$.

$\cos\theta$	$D=0.2$ T	$D=0.4$ T	$D=0.6$ T	$D=0.8$ T	$D=1$ T
−1	0.030	0.128	0.123	0.086	0.0551
−0.5	0.103	0.18	0.137	0.089	0.0558
0	0.279	0.242	0.151	0.092	0.0565
0.5	0.652	0.314	0.167	0.095	0.0573
1	13.7	0.397	0.182	0.099	0.0580

$$\delta_p \simeq \delta_T = \frac{T(D, \theta=0) - T(D, \theta=\pi)}{T(D, \theta=0) + T(D, \theta=\pi)}. \quad (5.8)$$

Typical values of δ_T for various $D = \chi\tau/R^2$ and λ are given in Tables II and III. It is seen that

(a) δ_T decreases rapidly with D. This is easy to understand since the smaller χ is the more time it takes to reach equilibrium. Also the smaller τ is at a given χ, the faster emission takes place from the target (projectile) which has not yet reached an equilibrium state. Finally, the larger R, the larger the asymmetry because this effect is essentially a finite-size effect. As a consequence of this last feature, the asymmetry is enhanced when nuclei are used as targets (projectiles).

(b) δ_T is a decreasing function of λ. However, this decrease is much slower than the decrease with D and, therefore, in a first approximation the asymmetry depends only on one parameter D.

(ii) Heavier particles (e.g., kaons, nucleons, and antinucleons) will preferably be produced in the upper hemisphere since the larger the mass the higher the temperature necessary for its production [cf. Eq. (5.1)].

(iii) There might be an asymmetry with respect to multiplicities. It is conceivable e.g. that more particles will be emitted in the upper $q_\perp < 0$ hemisphere compared with the number of particles emitted into the lower $q_\perp > 0$ hemisphere. This follows if one assumes, e.g., that emission takes place through an "evaporation" process from the surface of the excited target, an assumption familiar to nuclear physicists.

If emission takes place from a region of local equilibrium described by a blackbody equation of state, we have

$$s \propto T^3 \propto N, \quad (5.9)$$

where s is the entropy density, T the local temperature, and N the number of produced particles.

TABLE II. Asymmetry parameter δ_T [cf. Eq. (6.8)] as a function of D at $\lambda = 1$.

D	δ_T
0.1	1.00
0.2	0.956
0.3	0.761
0.4	0.513
0.5	0.321
0.6	0.195
0.7	0.118
0.8	0.071
0.9	0.043
1.0	0.026

TABLE III. Asymmetry parameter δ_T as a function of λ at $D = 0.5$.

λ	δ_T
0.1	0.404
0.2	0.394
0.3	0.385
0.4	0.374
0.5	0.365
0.6	0.356
0.7	0.347
0.8	0.338
0.9	0.329
1.0	0.321

Since we have always $T_{up} > T_{down}$, it follows[22] that $N_{up} > N_{down}$.

The multiplicity N as a function of $\mu = \cos\theta$ is

$$N(\mu) = \frac{\int n(p; T(\mu, t)) d^3p\, dt}{\int n(p; T(\mu, t)) d^3p\, dt\, d\mu}. \quad (5.10)$$

We have estimated the ratio

$$\delta_N \equiv \frac{N_{up} - N_{down}}{N_{up} + N_{down}}, \quad (5.11)$$

where

$$N_{up} \propto \int_0^1 d\mu \int_\tau^\infty dt \int d^3p \frac{1}{\exp[(M^2 + p^2)^{1/2}/T] - 1} \quad (5.12)$$

and

$$N_{down} \propto \int_{-1}^0 d\mu \int_\tau^\infty dt \int d^3p \frac{1}{\exp[(M^2 + p^2)^{1/2}/T] - 1} \quad (5.13)$$

represent the total number of particles emitted in the upper and lower hemispheres, respectively. The results are represented in Table IV. It is seen that (a) δ_N decreases with $D \equiv \chi\tau/R^2$ and (b) δ_N decreases with $T_i \Delta V$. The first effect has been discussed above. The last effect can be understood in terms of (ii) since a decrease of T_i is equivalent to a corresponding increase in m, and the larger m the larger the asymmetry effect. This effect should disappear if $T_i \gg m$. On the other hand, we have ignored so far the possible T_i dependence of D and λ. This last effect might influence the overall T_i dependence of δ. (For a blackbody equation of state e.g., λ is expected to increase with T_i.) At this stage only an experimental investigation of this feature can answer this question.

TABLE IV. Asymmetry parameter δ_N [cf. Eq. (6.10)] as a function of D and $T_i\Delta V$ (in units $m_\pi=1$), for $\lambda=0.2$.

D	$T_i\Delta V$	δ_N
0.1	1	0.919
0.1	2	0.818
0.1	3	0.733
0.1	4	0.667
0.1	5	0.618
0.5	1	0.382
0.5	2	0.320
0.5	3	0.277
0.5	4	0.242
0.5	5	0.216

(iv) Particles emitted before equilibrium is reached in the *whole* target (projectile) will show a momentum distribution which does not correspond to a single temperature but to a temperature distribution as represented, e.g., by Eq. (4.30). This implies that if one assumes a local equilibrium emission process, as in the thermodynamical bootstrap model, the $p_\perp$ distribution will not be a pure exponential, but will show deviations from the exponential behavior. In particular, if we allow for local temperatures at the hot spot higher than $T_0 \simeq m_\pi$, the large $p_\perp$ tail will be enhanced, as is indeed observed.[9]

Whether this is *the* explanation for the observed deviation from the $T_0=\text{const}=m_\pi$ exponential behavior is, however, yet an open question since we do not know at present the percentage of peripheral reactions which contributes to the observed inclusive cross section represented in Fig. 1. This question is related to the problem of determination of the impact parameter for a given reaction and will be discussed below. On the other hand, the fact that a pre-equilibrium mechanism is indeed responsible for a similar effect in nuclear physics[23] (cf. also Sec. II A) is very suggestive.

B. How and what to measure

It is obvious from the foregoing that the asymmetry could be looked for either in the projectile or target fragmentation regions once the leading particle has been eliminated. It will be very interesting to compare, e.g., propagation of heat in mesons and nucleons. This will tell us something about heat conductivity κ in mesonic matter as compared with κ in nucleonic matter. An important element in the experimental analysis might be the reference frame to be chosen. A possible frame would be the excited target frame. The advantage of this frame versus the target frame lies in the fact that trivial correlations due to momentum conservation are in this way easier to take into account. It is obvious that only an event-by-event analysis is useful in the search for the asymmetry.

At a first look one would be tempted to believe that in this analysis both the measurement of $\vec{q}_\perp$ and the detection of a secondary particle (from the fragmentation region) are necessary. This is, of course, the ideal situation. However, in a first approximation this requirement could be dropped and one could limit the measurement only to secondaries. Indeed in an event-by-event analysis one could define an "up-down" direction just by looking for a *privileged* direction in a plane normal to the incoming momentum $\vec{p}_i$ along which there exists as asymmetry (in multiplicities e.g.). Suppose we define by "up" in the first event the sense on this direction into which the multiplicity is larger and by "down" the opposite sense. Let $N^{(1)}$ and $N_d^{(1)}$ be the corresponding multiplicities ($N \;> N_d$ by definition). In the second event we proceed likewise and obtain $N^{(2)}$ and $N_d^{(2)}$ and so forth. Eventually we compute $\sum_i N_{\text{up}}^{(i)}$ and compare with $\sum_i N_d^{(i)}$. If there is a statistically significant difference, it proves the existence of the asymmetry effect.[24]

C. Impact-parameter analysis and the size of the asymmetry effect

The asymmetry effect exists of course also in reactions which are not peripheral since practically no reaction is totally central. By measuring an asymmetry parameter one substracts automatically the central-collisions events and what is left is due to noncentral collisions. It is clear, however, that the more peripheral a reaction, the higher the asymmetry effect. It would be very desirable to know the weight of different impact parameters for a given reaction in order to relate the measured asymmetry parameters to the theoretical ones as defined in Sec. V. One possibility would be to introduce an empirical distribution as done in Ref. 2 [cf. Eq. (2.5)]. In this way one could get a covariant description using an approach similar to Eq. (2.7). As long as the impact-parameter distribution is not known, one must rely on the information about peripheral reactions quoted in Sec. III. In this order of ideas it should be mentioned that an asymmetry effect is expected to exist also in e-p inelastic scattering, but the size of this asymmetry can hardly be estimated at present as long as the impact-parameter distribution is not determined. Furthermore, a quantitative estimate within the present model can be obtained only if the main parameter of the model $D=\chi\tau/R^2$ is known.[25] (For some physical quantities the knowledge of $T_i\Delta V$ is also necessary.)

VI. CONCLUSIONS AND OUTLOOK

The experimental investigation of the asymmetry effect might provide the possibility of following the space-time evolution of an excitation in hadronic matter. This could bring new insight into the theory of strong interactions especially from the viewpoint of statistical and hydrodynamical models. The knowledge of heat conductivity of hadronic and nuclear matter is important also in astrophysics and cosmology. It is known that the existence of a maximum temperature, e.g., is intimately connected with the understanding of the early stages of the universe. So is obviously also the problem of thermodynamical equilibrium.

One of the most challenging questions in this connection is the possibility of obtaining locally much higher temperatures than $T_0 \simeq m_\pi$.

By studying the asymmetry effect we might also hope to learn new things about peripheral and central collisions, about exchange reactions and last, but not least, about the origin of the large-$p_\perp$ events. There are two main theoretical directions into which further investigation seems necessary:

(i) a theoretical determination of the parameters λ, $\chi\tau/R^2$, and $T_i\Delta V$ as a function of kinematical variables,

(ii) an investigation of the asymmetry effect by a different theoretical approach than the present one. One such possibility (relativistic hydrodynamics) was mentioned in Ref. 16. Another approach could be perhaps the covariant "Boltzmann distribution function" method suggested recently by Carruthers and Zachariasen[26] or the multiperipheral parton model.

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APPENDIX A

A. Integration of the diffusion equation

In Ref. 18 the derivation of the solution of Eq. (4.2) is left as an exercise for the reader. Since we believe that high-energy physicists are not familiar with this type of problem this derivation is sketched in this appendix.

Let us consider Eq. (4.14) in polar variables (we remind the reader that there is no azimuthal dependence)

$$\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right]+\alpha_n{}^2 f=0\,, \qquad \text{(A1)}$$

where $\mu=\cos\theta$. With the ansatz

$$f=f_n(r,\theta)=R_n(r)P_n(\theta)\,, \qquad \text{(A2)}$$

the variables r and θ can again be separated, and Eq. (A1) yields

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n}{d\mu}\right]+n(n+1)P_n=0 \qquad \text{(A3)}$$

and

$$\frac{d^2R_n}{dr^2}+\frac{2}{r}\frac{dR_n}{dr}+\left[\alpha_n{}^2-\frac{n(n+1)}{r^2}\right]R_n=0, \qquad \text{(A4)}$$

where P_n are Legendre polynomials of index n (n integer). We want solutions of Eq. (A4) which are finite at $r=0$. Then

$$R_n=(\alpha_n r)^{-1/2}J_{n+1/2}(\alpha_n r)\,, \qquad \text{(A5)}$$

where $J_{n+1/2}$ is the Bessel function of the first kind of index $n+\frac{1}{2}$. The general solution of Eq. (4.2) for $\lambda\neq 0$ will be

$$\mathcal{T}=\sum_{n,\alpha_n} C_n e^{-\chi\alpha_n{}^2 t}(\alpha_n r)^{-1/2}J_{n+1/2}(\alpha_n r)P_n(\mu)\,, \qquad \text{(A6)}$$

where the constants C_n are to be determined from the initial condition (4.10).

B. Determination of α_n

The expansion (A6) is known in the mathematical literature[27] under the name of Dini series. A remarkable phenomenon characteristic for these series and which does not occur for Fourier-Bessel expansions is the fact that in the case $\lambda=0$ the uniqueness of the Dini expansion can be ensured only by adding to the sum (A6) a constant term which in our case is $3T_i/4\pi R^3$. This effect can be easily understood on physical grounds since for $\lambda=0$, i.e., if there is no radiation, the quantity of heat Q of the system has to be conserved, and, thus $Q=C_pT(t=\infty)=C_pT(t=0)$. The case $\lambda=0$ is interesting for applications in nuclear physics. By substituting (A6) into (4.7) we get

$$(R\lambda-\tfrac{1}{2})J_{n+1/2}(R\alpha_n)+R\alpha_n J'_{n+1/2}(R\alpha_n)=0\,, \qquad \text{(A7)}$$

where

$$J'(x)=\frac{d}{dx}J(x) \qquad \text{(A8)}$$

and α_n are the roots of Eq. (A7).

C. Determination of C_n

We start from the initial condition (4.10) assuming that $\vec{a}=(r',0,0)$, i.e., that the hot spot is located on the Z axis of Fig. 4. Substitution of (A6) into (4.10) gives

$$\frac{T_i\delta(r-r')\delta(\theta)\delta(\psi)}{r^2} = \sum_n C_n(\alpha_n r)^{-1/2}J_{n+1/2}(\alpha_n r)P_n(\mu)\,. \tag{A9}$$

By multiplying with $r^{3/2}J_{n+1/2}(\alpha_n r)P_n(\mu)$ and integrating succesively this equation over ψ, μ, and r within the limits $(0, 2\pi)$, $(-1,1)$, and $(0,R)$, respectively, we get

$$C_n = \frac{2n+1}{2\pi}\,T_i\left(\frac{\alpha_n}{r'}\right)^{1/2} J_{n+1/2}(\alpha_n r')\mathfrak{J}^{-1} \tag{A10}$$

where

$$\mathfrak{J} = \int_0^R r' J_{n+1/2}^2(\alpha_n r')dr'\,. \tag{A11}$$

The value of the integral (A11) is (cf. Ref. 18 Sec. 7.5)

$$\mathfrak{J} = \frac{R^2}{2}\,[J'_{n+1/2}(\alpha_n R)]^2 \quad \text{if } J_n(\alpha_n R)=0 \tag{A11$'$}$$

and

$$\mathfrak{J} = \frac{1}{2\alpha_n^2}\left[R^2\left(\lambda - \frac{1}{2R}\right)^2 + [R^2\alpha_n^2 - (n+\tfrac{1}{2})^2]\right] \times J_{n+1/2}^2(\alpha_n R) \tag{A11$''$}$$

if

$$(R\lambda - \tfrac{1}{2})J_{n+1/2}(R\alpha_n) + R\alpha_n J_{n+1/2}(R\alpha_n) = 0\,. \tag{A12}$$

TABLE V. The first four roots of Eq. (4.23) for $\lambda = 1$ and $0 \le n \le 23$.

n	$Z_{n,1}$	$Z_{n,2}$	$Z_{n,3}$	$Z_{n,4}$
0	1.570	4.712	7.853	10.99
1	2.743	6.116	9.316	12.48
2	3.869	7.443	10.71	13.92
3	4.973	8.721	12.06	15.31
4	6.061	9.967	13.38	16.67
5	7.139	11.18	14.67	18.00
6	8.210	12.39	15.93	19.32
7	9.275	13.57	17.18	20.61
8	10.33	14.75	18.42	21.89
9	11.39	15.91	19.64	23.15
10	12.44	17.07	20.86	24.41
11	13.49	18.21	22.06	25.65
12	14.53	19.35	23.25	26.88
13	15.58	20.49	24.44	28.10
14	16.62	21.62	25.61	29.32
15	17.66	22.74	26.79	30.53
16	18.70	23.86	27.95	31.73
17	19.74	24.97	29.11	32.92
18	20.78	26.08	30.27	34.11
19	21.81	27.19	31.42	35.30
20	22.85	28.30	32.57	36.48
21	23.88	29.40	33.71	37.65
22	24.91	30.50	34.85	38.82
23	25.94	31.59	35.99	39.99

We see thus that here the boundary condition enters the problem since (A12) coincides with (A7). [Eq. (A11$'$) corresponds to the other type of boundary condition, i.e., $T(r=R)=0$]. Introducing (A11$''$) into (A10) we get

$$C_n = \frac{2n+1}{4\pi}\,T_i\left(\frac{\alpha_n}{r'}\right)^{1/2}\alpha_n^2\,\frac{J_{n+1/2}(\alpha_n r')}{J_{n+1/2}^2(\alpha_n R)[R^2(\lambda - 1/2R)^2 + R^2\alpha_n^2 - (n+\frac{1}{2})^2]}\,,$$

and the solution (A6) becomes eventually:

$$\mathcal{T} = \frac{T_i}{2\pi}\,\frac{1}{(rr')^{1/2}}\sum_{n,\alpha_n}(2n+1)P_n(\mu)\,\frac{\alpha_n^2 e^{-\chi\alpha_n^2 t}J_{n+1/2}(\alpha_n r)J_{n+1/2}(\alpha_n r')}{[(R\lambda - \frac{1}{2})^2 + \alpha_n^2R^2 - (n+\frac{1}{2})^2]J_{n+1/2}^2(\alpha_n R)}\,. \tag{A13}$$

In the particular case that $r'=R$, i.e., if the hot spot is at the north pole, we get for the temperature field at a point $P(r,\theta)$ at time t

$$\mathcal{T} = \frac{T_i}{2\pi}\,\frac{1}{(rR)^{1/2}}\sum_{n,\alpha_n}(2n+1)P_n(\mu)\,\frac{\alpha_n^2 e^{-\chi\alpha_n^2 t}J_{n+1/2}(\alpha_n r)}{[(R\lambda - \frac{1}{2})^2 + \alpha_n^2R^2 - (n+\frac{1}{2})^2]J_{n+1/2}(\alpha_n R)}\,. \tag{A14}$$

If the hot spot is not on the Z axis, but at an arbitrary point $P'(r',\theta',\psi')$, the form of solution (A13) is unchanged, but the significance of μ in P_n is now changed into

$$\tilde{\mu} = \cos\gamma\,, \tag{A15}$$

where γ is the angle $P'OP$ and

$$P_n(\cos\gamma) = P_n(\mu)P_n(\mu') + 2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\mu)P_n^m(\mu')\cos m(\psi' - \psi)\,, \tag{A16}$$

with $\mu = \cos\theta$, $\mu' = \cos\theta'$.

We see that the temperature density $\mathcal{T}$ in our problem depends essentially on two dynamical parameters λ and χ. Although λ enters both explicitly and implicitly (via α_n), it turns out that the dependence of T on λ is weak compared with the dependence on χ.

In order to estimate the double sum in (4.14) we have to determine $Z_{n,i} \equiv \alpha_n R$ as a function of λ from the transcendental equation (A7). This can be done by numerical computation. As far as we could gather $Z_{n,i}$ are known in the literature only for $\lambda = 0$ [in this particular case Eq. (A7) reduces to $j_n'(Z_n) = 0$, where $j_n = (\pi/2Z)^{1/2} J_{n+1/2}(Z)$ are the spherical Bessel functions]. We have solved Eq. (A7) on a computer for $0 \leq \lambda \leq 1$ in steps of 0.1 and for $2 \leq \lambda \leq 10$ in steps of 1, for $0 \leq n \leq 23$. There is, of course, an infinity of solutions $Z_{n,i}$ (i is the index which at a fixed n characterizes the ascending order of Z_n), so that both the sums over n and i are infinite. The covergence of these sums is controlled by the exponent in (A6). $Z_{n,i}$ are increasing functions of n and i, but the increase of Z with n is slower than that with i. That is why the sum over α_n (i.e., i) in (A6) converges rapidly, while the sum over n converges very slowly, and that is why in some cases (for χ small) one has to consider in (A6) rather large values of n ($n \geq 20$) so that the use of a computer for the calculation of the double sum (A6) even after $Z_{n,i}$ are determined becomes advisable. In Table V we give as an example the values of $Z_{n,i}$ for $\lambda = 1$ and in Table I we list the values of T for various values of $\cos\theta$ at $\lambda = 1$ for $0.2 \leq D \leq 1$.

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[10]A short account along these lines was given in (a) R. M. Weiner, Phys. Rev. Lett. 32, 630 (1974); and in (b) R. M. Weiner, Contribution No. 523 to the XVII International Conference on High Energy Physics, London, 1974 (unpublished).

[11]A much more difficult task is to find sufficient conditions for peripheral localization. In two-body reactions duality arguments (Ref. 12) suggest that quantum-number exchange reactions are peripheral since the imaginary part of the scattering amplitude for these reactions has a peak at $b \simeq R$. For Pomeron-exchange reactions, e.g., there is no such peak, so that these reactions have been considered to take place in the entire volume of the target. A many-body reaction could in principle be treated in a similar way as two-body reactions by considering the exchange of quantum numbers between the projectile and the leading particle. Another possible criterion for peripheralism might be a symmetry around the t axis of the secondaries. In central collisions a symmetry around the s axis is expected (Ref. 4).

[12]H. Harari, in *Scottish Universities Summer School in Physics, 1970* (Academic, New York, 1971).

[13]In Landau's hydrodynamical model as well as in the statistical bootstrap model (after the establishment of local equilibrium), because of $\kappa = 0$, condition (3.8) is apparently fulfilled.

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[16]As in Landau's treatment we consider the system after shock waves have disappeared. This is probably an acceptable first approximation if the velocity of shock waves is larger than the velocity of heat transfer (cf. below). A more general treatment taking shock waves into account was suggested by E. Feinberg in a correspondence with the author. Work along these lines by N. Masuda and the author is in progress.

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[20]In Ref. 13 we have chosen $\Delta V = 1$, so that $\mathcal{T} = T$.

[21]There are also more practical reasons for this. Indeed, an initial condition on $\partial\mathcal{T}/\partial t$ would introduce at least another free parameter and would complicate

the understanding of the final result.

[22]Another argument in favor of this assumption is the following: Viewed from the hot spot in the lower hemisphere there is matter, and hence the particles emitted in the direction $q_\perp > 0$ will suffer absorption. In the upper hemisphere there is vacuum and hence no absorption, so that the multiplicity for $q_\perp < 0$ will be larger. A very rough estimate of this effect can be given by introducing an absorption coefficient a through the equation

$$n = n_0 \exp(-ad),$$

where n is the number of particles which succeed in getting out from the target in the $q_\perp > 0$ direction, n_0 the initial number of particles emitted (isotropically) by the hot spot, and d the distance over which the particles traveled in the absorbing (spherical) target. We have $d = 2R\cos\varphi$, where φ is the angle defined in Fig. 3 and hence

$$\frac{n_N}{n_S} = \frac{n(\varphi = \pi)}{n(\varphi = 0)} \simeq \exp(2aR) > 1.$$

For $a \sim m_\pi$ this is a strong effect. It is also easy to compare along the same lines the total number of particles emitted in the two hemispheres, respectively. We have

$$\delta = \frac{n(q_\perp < 0)}{n(q_\perp > 0)} \simeq \frac{\int_0^{2R} n_0 r^2 dr}{\int_0^{2R} n_0 e^{-ar} r^2 dr} = \frac{\frac{8}{3}R^3}{2/a^3 - e^{-2aR}(4R^2/a + 4R/a^2 + 2/a^3)}.$$

For $a \sim m_\pi$, we get $\delta \simeq \frac{8}{3}$. An effect which might tend to decrease this ratio δ, on the other hand, could be due to possible multiple production in the target. This might affect especially the multiplicity ratio up/down for slow particles.

[23]Details of the nuclear problem can be found in R. Weiner and M. Weström, Phys. Rev. Lett. 34, 1523 (1975) and a subsequent paper (unpublished).

[24]A similar analysis has been performed by P. Kostka *et al.*, Nucl. Phys. B86, 1 (1975), where a "planarity" effect was found.

[25]One might wonder whether because of the longitudinal momentum transfer $q_\parallel$ the hot spot might not suffer a rotation which could smear out the asymmetry effect. That such a rotation is not expected to affect appreciably the asymmetry can be seen from the following semiqualitative argument. The linear velocity x of the hot spot due to $q_\parallel$ is $x \sim q_\parallel/(m^2 + q_\parallel^2)^{1/2}$, where m is the mass of the hot spot. The rotation period τ_{rot} is then $2\pi/\omega_{rot}$, where $\omega_{rot} = x/R \simeq x m_\pi$. The emission period τ_{em} is $\sim m_\pi^{-1}$, so that

$$\frac{T_{rot}}{T_{em}} \sim 2\pi[(m/q_\parallel)^2 + 1]^{1/2} >> 1.$$

[26]P. Carruthers and F. Zachariasen, presented at Orbis Scientiae II, Univ. of Miami, Coral Gables, Florida, 1975 (unpublished).

[27]G. N. Watson, *Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, New York, 1966).

[28]*Bessel Functions Part III, Zeros and Associated Values, Royal Society Mathematical Tables*, edited by F. W. J. Oliver (Cambridge Univ. Press, Cambridge, England, 1960), Vol. 7.