Rigorous phase-modulus correlations for forward scattering amplitude at high energies

Jan Fischer and Pavel Kolář
*Institute of Physics, Czechoslovak Academy of Sciences, Prague, Czechoslovakia**

Ivo Vrkoč
Mathematical Institute, Czechoslovak Academy of Sciences, Prague, Czechoslovakia†
(Received 3 March 1975)

Starting from the basic results of axiomatic field theory a series of new asymptotic theorems for the symmetric $F_S(E)$ and the antisymmetric $F_A(E)$ forward scattering amplitudes are derived, E being the laboratory energy. The spins are neglected in the present paper. Assuming that the phase of F_S or F_A is asymptotically bounded within a certain range of values the corresponding bounds on $|F_S|$ or $|F_A|$, respectively, are obtained. Further, we show that certain conditions on the ratio $\mathrm{Re}\,F_S(E)/\mathrm{Im}\,F_S(E)$ and on the signs of $\mathrm{Re}\,F_S(E)$ and $\mathrm{Im}\,F_S(E)$ for large enough energies can be formulated as criteria for the asymptotic behavior of the symmetric total cross section $\sigma_S(E)$. Analogous results are obtained for the antisymmetric forward scattering amplitude. Finally, useful necessary and sufficient conditions for the asymptotic rise or boundedness of $\sigma_S(E)$ and $\sigma_A(E)$ are derived. The results obtained are discussed in light of existing high-energy models and high-energy experiments.

I. INTRODUCTION

Remarkable progress has been made recently in the experimental investigation of the forward scattering amplitude at high energies. Let us mention the discovery of the rise of the total proton-proton cross section between 100 GeV (lab) and 2000 GeV (lab) at CERN,[1] the positivity of the real part in $pp \to pp$ experiments at Fermilab,[2] and the K_S^0-regeneration experiments at Serpukhov giving the phase and the modulus of the antisymmetric part of the kaon-proton forward scattering amplitude up to 50 GeV (lab).[3] From the point of view of theory, it is important that these experiments give independent information on two physical quantities which are correlated by analyticity [e.g., the real and imaginary part of the forward scattering amplitude $F(E)$, its modulus and its phase, the total cross section, and the ratio $\mathrm{Re}F(E)/\mathrm{Im}F(E)$, etc.]. It is therefore worth revising and completing the existing high-energy theorems in order to confront them with the new experimental results.

In the past, theoretical analyses of asymptotic phenomena were performed many times and various mathematical tools were employed.[4-11] Khuri and Kinoshita[4] applied Meiman's theorems[12] in order to derive asymptotic constraints on the modulus of the forward scattering amplitude provided that the ratio of the real to the imaginary part is bounded. Later, a number of authors rederived or strengthened their results using different methods.[5-10] An elegant approach based on phase dispersion relations for the forward scattering amplitude was proposed and developed by Vernov.[11]

In the present paper we use Vernov's approach in order to investigate the asymptotic properties of the forward scattering amplitude. Starting from the basic results of axiomatic field theory as usually adopted in the S-matrix framework, we obtain correlations between the bounds on the modulus and those on the phase of the symmetric and antisymmetric forward scattering amplitude in the asymptotic region. Simultaneously, we examine the method itself from the mathematical point of view.

It is assumed throughout the paper that the internal symmetries of elementary particles related to spin, isospin, hypercharge, etc., may be neglected. Nevertheless, this does not exclude an internal structure of particles like electric or baryonic charge, retaining a distinction between a particle and its antiparticle.

The contents of the paper can be characterized as follows. Using the notion of forward scattering amplitude $F_{S,A}(E)$ (S denoting the symmetric and A the antisymmetric amplitude) we define a function $f(E)$ possessing certain analyticity and symmetry properties. We prove in Theorem 1 that the phase of $f(E)$ is bounded in some complex neighborhood of $E=\infty$ (Sec. II, Appendix A). Then we derive our main Theorem 2 (Sec. III, Appendix B), which relates bounds on the modulus to those on the phase of $f(E)$ at asymptotic energies. This is a generalization of a theorem obtained by Vernov.[11] Finally, the results obtained for $f(E)$ are applied to the forward scattering amplitudes $F_S(E)$ and $F_A(E)$, thus generating a series of asymptotic theorems for the amplitudes. As a by-product of the basic constraints (Theorems 3 and 6), we obtain criteria for different types of the asymptotic behavior of the real and imaginary parts of

$F_{S,A}(E)$; further, we derive correlations among various asymptotic conditions, etc. (Theorems 4, 5, 7, and 8 and the subsequent discussions). In particular, necessary and sufficient conditions for the unbounded rise, for the boundedness, and for the asymptotic vanishing of the symmetric and antisymmetric total cross sections are derived (Corollaries 1 and 2). In Sec. IV, the symmetric amplitude is studied in detail, whereas Sec. V is devoted to the antisymmetric amplitude.

We systematically remove the assumption of analyticity in the cut energy plane and replace it by the weaker assumption of analyticity around $E=\infty$ in the upper half plane. Thus, our results can be applied also to processes for which the complex energy plane possesses a finite central region of possible nonanalyticity.[13]

In general, some of the results obtained are valid only for certain sequences of energies E_k tending to infinity; if, however, an extra assumption [see assumption (F7)] is made forbidding certain violent oscillations of $F_{S,A}(E)$ at infinity, the validity extends to whole continuous energy intervals around $E=\infty$.

We remind the reader that our results are asymptotic and can be applied to experimental data only if the data are assumed to determine the asymptotic behavior. Then, our constraints and criteria constitute a basis for the analysis of the recent high-energy data. They also serve as a classification tool for hadron-hadron scattering models at high energies. There are some recent examples[14–19] to illustrate this in Secs. IV and V.

The reader who is interested mainly in applications can dispense with Secs. II and III and focus on Theorems 4, 5, 7, and 8 and on the subsequent discussion of consequences. We would like to draw the reader's attention particularly to Corollaries 1 and 2. An example of applying Theorems 3 and 6 to physical data is given in Sec. VI.

II. ASYMPTOTIC BOUNDEDNESS OF THE PHASE OF FORWARD SCATTERING AMPLITUDE

We shall establish conditions under which the forward scattering amplitude has no zeros and a bounded phase in the asymptotic region.

Consider the forward scattering of a particle a by another particle b, $a+b\to a+b$, together with the corresponding crossing process, $\bar{a}+b\to\bar{a}+b$, where $\bar{a}$ is the antiparticle to particle a. Let $F_S(E)$ and $F_A(E)$ denote, respectively, the sum and the difference of the scattering amplitudes for $a+b\to a+b$ and $\bar{a}+b\to\bar{a}+b$, E being the laboratory energy of the incident particle a. If complex values of E are considered, we shall use the symbol z. In formulas and statements which are valid both for $F_S(z)$ and for $F_A(z)$, we shall use simply the symbol $F(z)$. $F(z)$ has the following properties:

(F1) There exists $r_0\geq 0$ such that $F(z)$ is analytic in the upper half of the z plane excluding the semicircular disk of radius r_0 around the origin (this region $\text{Im}z>0$, $|z|>r_0$ will be denoted by $\mathfrak{D}$). This was proved by Bros, Epstein, and Glaser.[13]

(F2) For every $z\in\mathfrak{D}$, we have

$$F_S(z)=F_S^*(-z^*)\,,$$

$$F_A(z)=-F_A^*(-z^*)\,.$$

(F3) We assume throughout the paper that $F(z)$ has a well-defined value at every large enough energy and is continuous on the closure of $\mathfrak{D}$ [not including $z=\infty$; $F(E+i0)$ will be denoted $F(E)$]. Strictly speaking, this assumption is not necessarily satisfied because the scattering amplitude on the boundary is a distribution in energy. We assume therefore that $F(E)$ can be obtained by a regularization procedure of the scattering amplitude and that such $F(E)$ is continuous on the real axis.

(F4) The scattering amplitude $F(z)$ is polynomially bounded in $\mathfrak{D}$ for $|z|\to\infty$. The rigorous proof is due to Epstein, Glaser, and Martin.[20]

(F5) For real values of E, $F(E)$ satisfies the Froissart-Martin bound: there exist $r_1>0$ and $C>0$ such that

$$\left|\frac{F(E)}{E\ln^2E}\right|<C$$

for any $E>r_1$. (See Ref. 21.)

(F6) The imaginary part $\text{Im}F_S(E)$ of $F_S(E)$ is non-negative on the real axis for every $E>r_2$, r_2 being some positive number.

We define, in terms of the scattering amplitude, the function $f(z)$ by the formula

$$f(z)=\eta(-iz)^aF_S(z)\quad(z\in\mathfrak{D}) \tag{2.1}$$

in the case of the symmetric amplitude and

$$f(z)=i\eta(-iz)^aF_A(z)\quad(z\in\mathfrak{D}) \tag{2.2}$$

in the case of the antisymmetric amplitude, a being real and η being +1 or −1. $f(z)$ has the following properties:

(f1) $f(z)$ is analytic in $\mathfrak{D}$.

(f2) $f(-z^*)=f^*(z)$ in $\mathfrak{D}$. This property follows from (F2).

(f3) The condition (f3) has the same form for $f(z)$ as (F3) has for $F(z)$.

(f4) $f(z)$ is polynomially bounded in $\mathfrak{D}$ for $|z|\to\infty$.

(f5) (F5) implies for $f(E)$ that

$$|E^{-a-1}f(E)(\ln E)^{-2}|<C$$

for any $E>r_1$.

First, we establish the following theorem.

Theorem 1. Let $f(z)$ be a function of complex z satisfying the conditions (f1) to (f4). Let further the following conditions be satisfied:

(f5′) $$\lim_{E\to\infty} \frac{f(E)}{E^2} = 0 \,.$$

(f6′) There exists a positive number r_2' such that $\mathrm{Im} f(E) \geq 0$ for every $E > r_2'$ and

$$\int_{r_2'}^{\infty} \frac{\mathrm{Im} f(E)}{E} dE = +\infty \,.$$

Then there exists some $E_H > 0$ such that $\mathrm{Im}[f(z)/z] > 0$ for every z in $\mathfrak{D}$, $|z| > E_H$.

This theorem is a special case of Theorem A, which is proved in Appendix A.

III. ASYMPTOTIC PHASE-MODULUS CORRELATION

We shall exploit now the analyticity of $\ln f(z)$ for obtaining relations between the phase and the modulus of $f(z)$ in the asymptotic region. Obviously, at every zero of $f(z)$ inside or on the integration contour a singularity of $\ln f(z)$ arises producing an unknown term in the Cauchy theorem. If, however, $f(z)$ satisfies the conditions of Theorem 1, no zeros in $\mathfrak{D}$, $|z| > E_H$, occur. (We mention that if analyticity is assumed in the whole twice-cut plane, the absence of zeros was proved by Jin and Martin.[22])

Concerning possible zeros on the real axis, no additional assumption has to be made. Indeed, zeros on the real axis are allowed provided that the assumptions of Theorem 1 hold (see Lemma, Appendix B). These assumptions imply, nevertheless, that the Lebesgue measure of the set of zeros is zero and that the set is nowhere dense.

We present now the following theorem.

Theorem 2. Let $f(z)$ be a function of complex z satisfying the conditions (f1), (f2), (f3), (f4), (f5′), and (f6′). Let further $\mathrm{Re}\, f(E)$ not change sign above some E_R, $E_R > 0$. Denote $\mu(E)$ and $\nu(E)$ two functions which are integrable on the interval (E_0, E) for every $E > E_0$ and fulfill the constraints

$$0 \leq \mu(E) \leq \nu(E) \leq \tfrac{1}{2} \,. \tag{3.1}$$

If the inequalities

$$\tan[\pi\mu(E)] \leq \left|\frac{\mathrm{Im} f(E)}{\mathrm{Re} f(E)}\right| \leq \tan[\pi\nu(E)] \tag{3.2}$$

are satisfied for every $E > E_0$, there exists an infinite sequence of points E_k ($k = 1, 2, 3, \ldots$) such that $E_k \to \infty$ as $k \to \infty$ and that the inequalities

$$C_1 \exp\left[2\int_{E_0}^{E_k} \frac{\mu(E')}{E'} dE'\right] \leq |f(E_k)| \leq C_2 \exp\left[2\int_{E_0}^{E_k} \frac{\nu(E')}{E'} dE'\right] \tag{3.3}$$

and

$$D_1 \exp\left[2\int_{E_0}^{E_k} \frac{1-\nu(E')}{E'} dE'\right] \leq |f(E_k)| \leq D_2 \exp\left[2\int_{E_0}^{E_k} \frac{1-\mu(E')}{E'} dE'\right] \tag{3.4}$$

hold for every $k = 1, 2, 3, \ldots$ provided that $\mathrm{Re} f(E) \leq 0$ and $\mathrm{Re}\, f(E) \geq 0$ for $E > E_0$, respectively. C_1, C_2, D_1, D_2 are positive and independent of E_k. Moreover, in every interval $(E, \alpha E)$, $E > \max(E_0, E_R, E_H)$, $\alpha > 1$, there exists a subinterval I such that (3.3) and (3.4) hold for every $E \in I$. In this case, C_1, C_2, D_1, and D_2 depend on α.

The proof of the theorem is given in Appendix B.

Under the assumptions made, the bounds obtained are the best possible ones in the sense that there are functions which saturate them.

Remark 1. If the condition on the sign of $\mathrm{Re} f(E)$ is suppressed, the upper bound in (3.2) cannot be used and (3.3) and (3.4) reduce to

$$C_1 \exp\left[2\int_{E_0}^{E_k} \frac{\mu(E')}{E'} dE'\right] \leq |f(E_k)| \leq D_2 \exp\left[2\int_{E_0}^{E_k} \frac{1-\mu(E')}{E'} dE'\right] . \tag{3.5}$$

Remark 2. Analogous bounds can be easily obtained if (3.2) is replaced by a more general condition which is no longer symmetric under the change of sign of $\mathrm{Re}\, f(E)$; for instance,

$$-\tan[\pi\mu_1(E)] \leq \frac{\mathrm{Im}\, f(E)}{\mathrm{Re} f(E)} \leq \tan[\pi\mu_2(E)] .$$

Remark 3. (This remark pertains also to the subsequent Theorems 3 and 6.) One cannot extend the validity of (3.3) and (3.4) to every E larger than some positive number, unless an extra assumption is made concerning oscillations of $f(E)$. The form of Theorem 2 suggests that it is convenient to have the limitation on oscillations, for instance, in the following form:

(f7) There exist three positive numbers $\beta_0 > 1$,

N, and $E_{\rm osc}$ such that the inequality

$$\left|\frac{f(\beta E)}{f(E)}\right| \leq N \tag{3.6}$$

holds for all $\beta \in (1, \beta_0)$ and for all $E > E_{\rm osc}$. Assumption (F7) has the same form for $F(E)$.

If (3.6) is fulfilled, the constants C_1, C_2, D_1, and D_2 occurring in (3.3) and (3.4) become dependent on β_0. Further, the validity of (3.3) and (3.4) can be extended to a whole continuous interval $E > E_B$, E_B being positive, the constants being replaced by $C_1N^{-1}, C_2N, D_1N^{-1}, D_2N$, respectively. The case when N in (3.6) is multiplied by a known function of energy can be treated analogously and the corresponding generalization is straightforward.

IV. APPLICATIONS TO THE SYMMETRIC FORWARD SCATTERING AMPLITUDE

Writing (2.1) separarately for the real part and the imaginary part of $f(E)$ we obtain, taking the principal value of the power $(-iE)^a$,

$$\mathrm{Re}f(E) = \eta E^a \left[\cos\left(\frac{\pi a}{2}\right) \mathrm{Re}F_S(E) + \sin\left(\frac{\pi a}{2}\right) \mathrm{Im}F_S(E)\right],$$
$$\mathrm{Im}f(E) = \eta E^a \left[\cos\left(\frac{\pi a}{2}\right) \mathrm{Im}F_S(E) - \sin\left(\frac{\pi a}{2}\right) \mathrm{Re}F_S(E)\right]. \tag{4.1}$$

Choosing various values of η and a in (4.1), we obtain a number of interesting asymptotic theorems for the symmetric forward scattering amplitude. In the present paper, we shall restrict ourselves to the case of integral values of a.

Firstly, let us take a even, $a = 2n$, n being an integer. We get from (4.1)

$$\mathrm{Re}f(E) = \eta E^{2n}(-1)^n \mathrm{Re}F_S(E),$$
$$\mathrm{Im}f(E) = \eta E^{2n}(-1)^n \mathrm{Im}F_S(E). \tag{4.2}$$

We take $\eta = (-1)^n$ because then condition (F6) is satisfied also for $\mathrm{Im}f(E)$. Applying Theorem 2 to this case, we obtain the following result.

Theorem 3. Let $F_S(z)$ satisfy (F1), (F2), (F3), (F4), (F6), and

$$\lim_{E\to\infty} F_S(E)E^{2n-2} = 0. \tag{4.3}$$

Let further

$$\int^{\infty} \mathrm{Im}F_S(E)^{2n-1} dE = +\infty \tag{4.4}$$

(in the following, we shall suppress the lower integration limit in such expressions). Let $\rho_S(E) = \mathrm{Re}F_S(E)/\mathrm{Im}F_S(E)$ satsify the constraints

$$\tan[\pi\mu(E)] \leq |\rho_S(E)| \leq \tan[\pi\nu(E)] \tag{4.5}$$

for all $E > E_0$ [with $\mu(E)$ and $\nu(E)$ integrable and obeying (3.1)] and let $\mathrm{Re}F_S(E)$ not change sign above E_0. Then there exists an infinite sequence of points E_k $(k = 1, 2, 3, \ldots)$ such that $\lim_{k\to\infty} E_k = +\infty$ and that the inequalities

$$C_1 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{-\nu(E')}{E'} dE'\right] \leq |F_S(E_k)| \leq C_2 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{-\mu(E')}{E'} dE'\right] \tag{4.6}$$

and

$$D_1 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{\mu(E')}{E'} dE'\right] \leq |F_S(E_k)| \leq D_2 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{\nu(E')}{E'} dE'\right] \tag{4.7}$$

hold for every $k = 1, 2, 3, \ldots$ provided that

$$\mathrm{Re}F_S(E) \leq 0 \tag{4.8}$$

and

$$\mathrm{Re}F_S(E) \geq 0 \tag{4.9}$$

for $E > E_R$, respectively. C_1, C_2, D_1, D_2 are positive constants independent of E_k. Moreover, in any interval $(E, \alpha E)$, $\alpha > 1$, $E > \max(E_0, E_R, E_H)$, there exists a subinterval I such that (4.6) and (4.7) hold for every $E \in I$ with C_1, C_2, D_1, D_2, depending on α.

The theorem is a direct consequence of Theorem 2.

We remind the reader of Remark 3.

Remark 4. If the condition on the sign of $\mathrm{Re}F_S(E)$ is suppressed, $F_S(E)$ is asymptotically bounded by the left-hand side of (4.6) from below and by the right-hand side of (4.7) from above.

Remark 5. Assuming a special energy dependence of $\mu(E)$ and $\nu(E)$ we obtain a more lucid form of the bounds (4.6) and (4.7). Taking, for instance, $\mu(E)$ and $\nu(E)$ constant in energy we get $F_S(E)$ to be bounded by certain powers of energy. From the point of view of applications to high-energy models and experimental data, the useful choice of $\mu(E)$, $\nu(E)$ appears to be

$$\mu(E)=\mu_1+\mu_2/\ln E,$$
$$\nu(E)=\nu_1+\nu_2/\ln E, \tag{4.10}$$

where μ_1, μ_2, ν_1, and ν_2 are constants. Omitting scale factors in logarithmic expressions, we obtain from (4.6) and (4.7)

$$C_1E_k^{-2n+1-2\nu_1}(\ln E_k)^{-2\nu_2}\leq |F_S(E_k)|\leq C_2E_k^{-2n+1-2\mu_1}(\ln E_k)^{-2\mu_2}, \tag{4.11}$$

$$D_1E_k^{-2n+1+2\mu_1}(\ln E_k)^{2\mu_2}\leq |F_S(E_k)|\leq D_2E_k^{-2n+1+2\nu_1}(\ln E_k)^{2\nu_2} \tag{4.12}$$

for the case (4.8) and (4.9), respectively. Remark 4 refers to this case too.

Remark 6. The case of a odd can be treated analogously. Putting $a=2n+1$ we get, instead of (4.2),

$$\mathrm{Re}f(E)=\eta E^{2n+1}(-1)^n\mathrm{Im}F_S(E),$$
$$\mathrm{Im}f(E)=-\eta E^{2n+1}(-1)^n\mathrm{Re}F_S(E) \tag{4.2'}$$

and obtain a theorem which has exactly the same form as Theorem 3, the only change being that conditions (4.3), (4.4), (4.8), and (4.9) are replaced by

$$\lim_{E\to\infty}F_S(E)E^{2n-1}=0, \tag{4.3'}$$

$$(-1)^n\eta\mathrm{Re}F_S(E)\leq 0,\quad E>E_R$$
$$(-1)^n\eta\int^{\infty}\mathrm{Re}F_S(E)E^{2n}dE=-\infty, \tag{4.4'}$$

$$\eta=(-1)^n, \tag{4.8'}$$

$$\eta=-(-1)^n, \tag{4.9'}$$

respectively, and $\max(E_0, E_R, E_H)$ changes into $\max(E_0, r_2, E_H)$. Further, n in formulas (4.7) and (4.12) has to be replaced by $n+1$. Note, however, that Remark 4 is not directly transferred to this theorem because $\mathrm{Im}F_S(E)$ is non-negative because of (F6). Remark 5 can be easily extended to this case.

Taking n negative, we see that the condition (4.4) cannot be fulfilled by any scattering amplitude, because it leads to an asymptotic behavior which is in contradiction with (F5). On the other hand, the positive values of n lead, because of (4.3), to a very fast vanishing of the symmetric total cross section $\sigma_S(E)$, which is (because of the optical theorem) asymptotically proportional to $\mathrm{Im}F_S(E)/E$. The choice $n=0$ is of particular physical interest.

We shall therefore discuss now the case $n=0$ in some detail. Notice that (4.4) represents by itself a very weak restriction, permitting $\sigma_S(E)$ to decrease asymptotically as

$$[E\ln E(\ln\ln E)\cdots(\ln\cdots\ln E)]^{-1}.$$

Thus, although going beyond the properties which have been proved from axiomatic field theory, (4.4) is quite plausible.

A closer discussion of Theorem 3 allows a number of general consequences to be drawn for the asymptotic behavior of the forward scattering amplitude. It is remarkable that most of them follow without assuming (4.4) and even independently of (4.5). We shall mention some of them here.

Theorem 4. Assume that $F_S(z)$ fulfills (F1)–(F6). If either (i) $\mathrm{Re}F_S(E)\leq 0$ for all $E>E_R$, E_R being some positive number, or (ii)

$$\int_{E_0}^{\infty}\tan^{-1}\rho_S(E)\frac{dE}{E} \tag{4.13}$$

converges for some E_0, then

$$\liminf_{E\to\infty}\sigma_S(E)<+\infty. \tag{4.14}$$

Proof. It is obvious that if

$$\int^{\infty}\sigma_S(E)dE \tag{4.15}$$

converges, then

$$\liminf_{E\to\infty}\sigma_S(E)=0$$

and the theorem is valid. Thus, we may suppose that (4.15) diverges and that, consequently, (4.4) is satisfied with $n=0$. Applying Theorem 3 we obtain that if $\mathrm{Re}F_S(E)\leq 0$, then

$$\liminf_{E\to\infty}\sigma_S(E)<\infty.$$

On the other hand, if the sign of $\mathrm{Re}F_S(E)$ is not asymptotically nonpositive we can use Remark 4. Putting

$$\nu(E)=\frac{1}{\pi}\tan^{-1}\rho_S(E),$$

we get

$$\left|\frac{F_S(E_k)}{E_k}\right|\leq D_2\exp\left[\frac{2}{\pi}\int_{E_0}^{E_k}\tan^{-1}\rho_S(E')\frac{dE'}{E'}\right].$$

Since (4.13) converges we conclude that the integral above is finite and the theorem follows.

Thus Theorem 4 allows us to establish conditions under which $\sigma_S(E)$ cannot tend to infinity. Reversing the statement and assuming that

$$\lim_{E\to\infty}\sigma_S(E)=\infty,$$

we find that

$$\int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E} = +\infty \tag{4.16}$$

and, further, $\mathrm{Re}F_S(E)$ cannot stay nonpositive at all energies. The latter part of this statement represents the well-known result of Khuri and Kinoshita,[5] which has been obtained here by other means.

The next theorem gives a useful condition under which $\sigma_S(E)$ cannot tend to finite value.

Theorem 5. Assume that $F_S(z)$ satisfies (F1)–(F6). Let the condition

$$\int^{\infty} \sigma_S(E)dE = +\infty \tag{4.17}$$

be satisfied and let $\rho_S(E)$ be bounded for all sufficiently large E

$$0 \le \rho_S(E) \le M, \tag{4.18}$$

M being some positive number. Further, let

$$\int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E} = +\infty. \tag{4.19}$$

Then

$$\limsup_{E\to\infty} \sigma_S(E) = +\infty.$$

Proof. According to Theorem 3 there is a sequence E_k, $E_k \to \infty$ for $k \to \infty$ such that

$$|F_S(E_k)| \ge D_1 E_k \exp\left[\frac{2}{\pi}\int_{E_0}^{E_k} \tan^{-1}\rho_S(E')\frac{dE'}{E'}\right], \tag{4.20}$$

where we have chosen

$$\mu(E) = \frac{1}{\pi}\tan^{-1}\rho_S(E).$$

The theorem immediately follows from (4.18), (4.19), and (4.20).

Theorems 4 and 5 suggest establishing a necessary and sufficient condition for $\sigma_S(E)$ to be bounded. First of all, we observe that if the limit

$$\lim_{E\to\infty} \rho_S(E) \tag{4.21}$$

exists, then the condition (4.19) can be replaced by

$$\int^{\infty} \rho_S(E)\frac{dE}{E} = +\infty. \tag{4.22}$$

Indeed, (4.7) implies that (4.21) must be zero because of Froissart-Martin bound (F5). Then, evidently, (4.19) and (4.22) are equivalent.

We are now in a position to obtain the following corollary.

Corollary 1. Let $F_S(E)$ satisfy (F1)–(F6) and let $\mathrm{Re}F_S(E)$ not change sign above some energy. Assume further that (4.17) is satisfied and that the limits (finite or infinite)

$$\lim_{E\to\infty} \rho_S(E), \quad \lim_{E\to\infty} \sigma_S(E)$$

exist. A necessary and sufficient condition for

$$\lim_{E\to\infty} \sigma_S(E) = +\infty$$

is

$$\int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E} = +\infty.$$

This corollary is a consequence of Theorems 4 and 5.

Remark 7. If, instead of (4.17), a stronger condition on $\sigma_S(E)$ is imposed, for instance,

$$\lim_{E\to\infty} \sigma_S(E)E^{\epsilon} = +\infty \tag{4.23}$$

for every $\epsilon > 0$, then one can distinguish between the vanishing and the nonvanishing of $\sigma_S(E)$. A necessary and sufficient condition for

$$0 < \lim_{E\to\infty} \sigma_S(E) < +\infty,$$

$$\lim_{E\to\infty} \sigma_S(E) = 0$$

is then

$$-\infty < \int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E} < \infty,$$

$$\int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E} = -\infty,$$

respectively. Indeed, (4.23) implies the violation of the upper bound in (4.6) with $n=0$ for all positive (constant) values of μ and for all E above some energy. Thus, reversing Theorem 3 we see from (4.5) that $\rho_S(E)$ must tend to zero. Then, replacing $|F_S(E)|$ by $E\sigma_S(E)$ in (4.6) and (4.7), the remark follows immediately from Theorem 3.

Remark 8. One can replace the integral

$$\int^{\infty} \tan^{-1}\rho_S(E)\frac{dE}{E}$$

by

$$\int_{E_0}^{\infty} \rho_S(E)\frac{dE}{E}$$

everywhere in Corollary 1 and Remark 7, provided that $\rho_S(E)$ is integrable over the interval (E_0, E) for all $E > E_0$.

The remainder of the present section will be devoted to some new results related to the real part of the symmetric scattering amplitude.

In the first place, it is easy to prove the following statement: If (F1)–(F6) are fulfilled and $\mathrm{Re}F_S(E) \le 0$ for large enough energies, then the integral

$$\int^{\infty} \mathrm{Re}F_S(E)\frac{dE}{E^2} \tag{4.24}$$

converges. Indeed, because of (F5), (4.3) for $n=-1$ is fulfilled. Writing down (4.6) for $n=-1$, $\mu(E)=0$, $\nu(E)=\frac{1}{2}$,

$$C_1 E_k{}^2 \leq |F_S(E_k)| \leq C_2 E_k{}^3,$$

we get a contradiction with (F5). Thus, (4.4′) must be violated, i.e., (4.24) converges.

Another interesting result can be formulated as follows: Let (F1)–(F6) be fulfilled and let

$$\lim_{E\to\infty}\int_{E_0}^{\infty}\left[\frac{1}{\pi}\tan^{-1}|\rho_S(E')|-\frac{1}{\ln E'}\right]\frac{dE'}{E'}=+\infty. \tag{4.25}$$

Then either both of the following integrals

$$\int^{\infty}\mathrm{Re}F_S(E)\frac{dE}{E^2}, \quad \int^{\infty}\mathrm{Im}F_S(E)\frac{dE}{E} \tag{4.26}$$

converge or $\mathrm{Re}F_S(E)$ cannot be asymptotically non-negative.

Indeed, applying Theorem 3 we see that all conditions except (4.4) [or (4.4′)] are fulfilled, the upper bound in (4.5) not being specified. Assume that at least one of the integrals (4.26) diverges and that $\mathrm{Re}F_S(E)\geq 0$ for large enough energies. Then, using (4.7) we see that condition (4.25) contradicts the bound (F5).

To give a better insight into the condition (4.25) we remark that it is satisfied by functions such as

$$|\rho_S(E)|=\frac{\pi\beta}{\ln E} \text{ with } \beta>1,$$

$$|\rho_S(E)|=\frac{\pi}{\ln E}\left(1+\frac{b}{\ln\ln E}\right) \text{ with } b>0$$

and by most models giving $\sigma_S(E)$ bounded. On the other hand, models giving an unbounded rise of $\sigma_S(E)$ may satisfy (4.25) only if $\mathrm{Re}F_S(E)$ changes sign infinitely many times. Of course, this is not the case of the existing models.[14–19]

Concluding the present section, let us mention an interesting result which follows from (4.12) provided that

$$\lim_{E\to\infty}\rho_S(E)\ln E$$

exists. Then, confronting (4.12) with (F5) we find that

$$0\leq\lim_{E\to\infty}\rho_S(E)\ln E\leq\pi$$

for $\mathrm{Re}F_S(E)$ asymptotically non-negative. Hence $\mathrm{Re}F_S(E)$ may grow, but considerably slower than $\mathrm{Im}F_S(E)$. In particular, the bound on $\mathrm{Re}F_S(E)$ here is a factor $\ln E$ lower than the Froissart-Martin bound (F5).

V. APPLICATIONS TO THE ANTISYMMETRIC FORWARD SCATTERING AMPLITUDE

In this section, we shall apply Theorem 2 to the function $f(z)$ as defined by (2.2). Choosing various values of η and a, this will imply a number of asymptotic theorems for the antisymmetric amplitude $F_A(E)$.

From (2.2) it follows that the real parts and the imaginary parts of $f(E)$ and $F_A(E)$ are connected by the following linear relations:

$$\mathrm{Re}f(E)=\eta E^a\left[\sin\left(\frac{\pi a}{2}\right)\mathrm{Re}F_A(E)-\cos\left(\frac{\pi a}{2}\right)\mathrm{Im}F_A(E)\right]$$
$$\mathrm{Im}f(E)=\eta E^a\left[\cos\left(\frac{\pi a}{2}\right)\mathrm{Re}F_A(E)+\sin\left(\frac{\pi a}{2}\right)\mathrm{Im}F_A(E)\right]. \tag{5.1}$$

We shall again restrict ourselves to the case of integral values of a.

Choosing $a=2n$, n being an integer, we obtain from (5.1)

$$\mathrm{Re}f(E)=(-1)^{n+1}\eta E^{2n}\mathrm{Im}F_A(E),$$
$$\mathrm{Im}f(E)=(-1)^n\eta E^{2n}\mathrm{Re}F_A(E), \tag{5.2}$$

and have from Theorem 2 the following consequences.

Theorem 6. Assume that $F_A(z)$ satisfies (F1)–(F4) and

$$\lim_{E\to\infty}F_A(E)E^{2n-2}=0, \tag{5.3}$$

and that

$$\eta(-1)^n\mathrm{Re}F_A(E)\geq 0 \text{ for all } E>E_R,$$
$$\eta(-1)^n\int^{\infty}\mathrm{Re}F_A(E)E^{2n-1}dE=+\infty. \tag{5.4}$$

Let $\rho_A(E)=\mathrm{Re}F_A(E)/\mathrm{Im}F_A(E)$ satisfy the constraints

$$\tan[\pi\bar{\mu}(E)]\leq|\rho_A(E)|^{-1}\leq\tan[\pi\bar{\nu}(E)] \tag{5.5}$$

for all $E>E_0$ [with $\bar{\mu}(E)$, $\bar{\nu}(E)$ integrable and satisfying (3.1)] and let $\mathrm{Im}F_A(E)$ not change sign above some E_I. Then there exists an infinite sequence of points $E_k (k=1,2,3,\ldots)$ such that $\lim_{k\to\infty}E_k=+\infty$ and that the following inequalities

$$C_1E_k{}^{-2n+1}\exp\left[2\int_{E_0}^{E_k}\frac{-\bar{\nu}(E')}{E'}dE'\right]\leq|F_A(E_k)|\leq C_2E_k{}^{-2n+1}\exp\left[2\int_{E_0}^{E_k}\frac{-\bar{\mu}(E')}{E'}dE'\right] \tag{5.6}$$

and

$$D_1 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{\overline{\mu}(E')}{E'} dE'\right] \le |F_A(E_k)| \le D_2 E_k^{-2n+1} \exp\left[2\int_{E_0}^{E_k} \frac{\overline{\nu}(E')}{E'} dE'\right] \quad (5.7)$$

hold for all $k=1,2,3,\ldots$ provided that

$$\eta(-1)^n \operatorname{Im} F_A(E) \ge 0 \quad (5.8)$$

and

$$\eta(-1)^n \operatorname{Im} F_A(E) \le 0 \quad (5.9)$$

for $E > E_I$, respectively. C_1, C_2, D_1, D_2 are positive and independent of E_k. Moreover, in every interval $(E, \alpha E)$, $\alpha > 1$, $E > \max(E_H, E_0, E_I, E_R)$ there exists a subinterval I such that (5.6) and (5.7) hold for all $E \in I$ with C_1, C_2, D_1, D_2, depending on α.

We remind the reader of Remark 3.

Remark 9. If the condition on the sign of $\operatorname{Im} F_A(E)$ is suppressed, $F_A(E)$ is asymptotically bounded by the left-hand side of (5.6) from below and the right-hand side of (5.7) from above.

Remark 10. Assuming a special energy dependence of $\overline{\mu}(E)$, $\overline{\nu}(E)$ one can obtain a simplified form of the bounds (5.6) and (5.7) in complete analogy with Remark 5.

Remark 11. The case of a odd can be treated analogously. Putting $a=2n+1$ we get instead of (5.2)

$$\operatorname{Re} f(E) = \eta(-1)^n E^{2n+1} \operatorname{Re} F_A(E), \quad \operatorname{Im} f(E) = \eta(-1)^n E^{2n+1} \operatorname{Im} F_A(E), \quad (5.2')$$

and obtain a new theorem which differs from Theorem 6 in that conditions (5.3), (5.4), (5.8), and (5.9) are replaced by

$$\lim_{E\to\infty} F_A(E) E^{2n-1} = 0, \quad (5.3')$$

$$\eta(-1)^n \operatorname{Im} F_A(E) \ge 0 \text{ for all } E > E_I, \qquad \eta(-1)^n \int^{\infty} \operatorname{Im} F_A(E) E^{2n} dE = +\infty, \quad (5.4')$$

$$\eta(-1)^n \operatorname{Re} F_A(E) \ge 0, \quad (5.8')$$

$$\eta(-1)^n \operatorname{Re} F_A(E) \le 0, \quad (5.9')$$

for $E > E_R$, respectively. Besides this, n in (5.7) has to be replaced by $n+1$ and the condition "let $\operatorname{Im} F_A(E)$ not change sign above E_I" transforms into "let $\operatorname{Re} F_A(E)$ not change sign above E_R." Remark 9 is valid with $\operatorname{Im} F_A(E)$ replaced by $\operatorname{Re} F_A(E)$.

A closer discussion of Theorem 6 allows us to draw a number of general consequences for the asymptotic behavior of the antisymmetric forward scattering amplitude. In order not to contradict the Froissart-Martin bound, we have to choose $n \ge 0$ in (5.4) and $n \le -1$ in (5.4′). With regard to physical applications, the following cases appear to be of particular interest: $n=0$ in Theorem 6 [both for (5.8) and for (5.9)], $n=0$ in Remark 11 [case (5.8′)], and $n=-1$ in Remark 11 [case (5.9′)]. We shall now discuss these cases in more detail. First, let us mention that the following theorem can easily be proved.

Theorem 7. Let $F_A(z)$ satisfy (F1)–(F5) and let $\operatorname{Re} F_A(E)$ and $\operatorname{Im} F_A(E)$ not change sign above some energy. Assume that

$$\int^{\infty} |\operatorname{Re} F_A(E)| \frac{dE}{E} = +\infty \quad (5.10)$$

and either (i) $\operatorname{Re} F_A(E)$ have the same sign as $\operatorname{Im} F_A(E)$, or (ii) $\int^{\infty} \tan^{-1}|\rho_A(E)|^{-1}(dE/E)$ converges. Then

$$\liminf_{E\to\infty} \left|\frac{F_A(E)}{E}\right| < \infty. \quad (5.11)$$

In case (ii) we have, moreover,

$$\limsup_{E\to\infty} \left|\frac{F_A(E)}{E}\right| > 0. \quad (5.12)$$

Proof. Relation (5.11) follows from Theorem 6 in the same manner as (4.14) from Theorem 3. The formula (5.12) follows from the lower bound in (5.7).

Remark 12. Theorem 7 remains valid if (5.10) is replaced by

$$\int^{\infty} |\operatorname{Im} F_A(E)| E^{-2} dE = +\infty. \quad (5.13)$$

Note, however, that in this case condition (i) is, because of (F5), satisfied by no physical amplitude. Thus, we arrive at the following consequence.

Let $F_A(z)$ satisfy (F1)–(F5) and let $\operatorname{Re} F_A(E)$ and $\operatorname{Im} F_A(E)$ not change sign above some energy. If (5.13) holds, then the signs of $\operatorname{Re} F_A(E)$ must be asymptotically different.

An appropriate illustration of this result is given by the amplitude proposed by Łukaszuk and Nicolescu,[19] in which $\sigma_A(E)$ increases like $\ln E$ and, consequently, (5.13) is satisfied. Then, necessarily, $\operatorname{Im} F_A(E)$ and $\operatorname{Re} F_A(E)$ have opposite signs, as seen in formulas (2) and (3) of their paper.

We consider now the analog of Theorem 5 to give conditions under which $\sigma_A(E)$ must tend, on an infinite sequence of energies, to zero or to infinity.

Theorem 8. Assume that $F_A(E)$ satisfies (F1)–(F5) and that $\mathrm{Re}F_A(E)$ and $\mathrm{Im}F_A(E)$ do not change signs for large enough energies. Let (5.10) be satisfied and let the integral

$$\int^{\infty} \tan^{-1}|\rho_A(E)|^{-1}\frac{dE}{E} \tag{5.14}$$

be divergent. If $\mathrm{Re}F_A(E)$ and $\mathrm{Im}\,F_A(E)$ have equal signs for large enough energies, then

$$\liminf_{E\to\infty}\left|\frac{F_A(E)}{E}\right| = 0\,. \tag{5.15}$$

If, however, $\mathrm{Re}F_A(E)$ and $\mathrm{Im}\,F_A(E)$ have different signs, then

$$\limsup_{E\to\infty}\left|\frac{F_A(E)}{E}\right| = \infty\,. \tag{5.16}$$

The proof is entirely analogous to the proof of Theorem 5 given previously.

Remark 13. Remark 12 holds unchanged for Theorem 8, too.

Remark 14. Theorems 7 and 8 can be conveniently used as criteria for the asymptotic behavior of the antisymmetric total cross section $\sigma_A(E)$. For instance, (5.11) and (5.15) immediately imply that

$$\liminf_{E\to\infty}|\sigma_A(E)| < \infty \tag{5.17}$$

and

$$\liminf_{E\to\infty}|\sigma_A(E)| = 0\,, \tag{5.18}$$

respectively. If, moreover, $[\rho_A(E)]^{-1}$ is bounded from below by a positive constant, (5.12) and (5.16) imply that

$$\limsup_{E\to\infty}|\sigma_A(E)| > 0 \tag{5.19}$$

and

$$\limsup_{E\to\infty}|\sigma_A(E)| = \infty\,, \tag{5.20}$$

respectively.

Let us discuss several implications of Theorem 6. Inserting $n=0$ into (5.7) we easily derive the following statement: If $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ have opposite signs and do not change them for large enough energies and if, further,

$$\int^{\infty} |\mathrm{Re}F_A(E)|\frac{dE}{E} \tag{5.21}$$

diverges, then

$$\liminf_{E\to\infty}|\rho_A(E)|^{-1} = 0\,. \tag{5.22}$$

If, in addition, the limit

$$\lim_{E\to\infty}\left|\frac{\ln E}{\rho_A(E)}\right|$$

exists, we find

$$\lim_{E\to\infty}\left|\frac{\ln E}{\rho_A(E)}\right| \le \pi\,. \tag{5.23}$$

This result deserves a closer discussion in connection with physical applications. Assuming that $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ do not change signs beyond some energy (but not assuming any correlation between the signs) we conclude that if the inequality

$$|\rho_A(E)|^{-1} \ge \frac{\pi\gamma}{\ln E}\,,\quad \gamma > 1 \tag{5.24}$$

holds for large enough energies, then either the integral (5.21) converges or the signs of $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ must be equal in the asymptotic region. We mention that condition (5.24) can be generalized in complete analogy with (4.25).

A typical illustration of this is made by the phenomenological model in Ref. 16, in which $\rho_A(E)$ is asymptotically constant so that (5.24) is satisfied; since, however, (5.21) diverges, the signs of $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ must be equal. On the other hand, if we violate (5.24) (this was done by adding a term linear in energy in this model[17]), there is no correlation between the signs. Indeed, the fit to kaon-proton scattering gives equal signs, whereas the fit to pion-proton scattering leads to opposite signs in this model.[17]

Another application of Theorem 6 yields a criterion of convergence of the integral

$$\int^{\infty} \sigma_A(E)\frac{dE}{E}\,, \tag{5.25}$$

which is frequently studied in the literature.[9,23,24] Assume that $\sigma_A(E)$ does not change sign for large enough energy and that at least one of the following conditions is satisfied: (i) The signs of $\sigma_A(E)$ and $\mathrm{Re}F_A(E)$ are equal; (ii) there exists $\gamma>1$ such that (5.24) holds for E sufficiently large. Then (5.25) must converge.

Comparing this result with the analogous theorems of Weinberg,[23] Grunberg and Truong,[9] and others, we see that assumptions concerning the asymptotic behavior of $F_A(E)$, which are usually required for the convegence of (5.25), are replaced here by (i) or (ii).

To give an example of the application of Theorem 7, let us mention that it yields another criterion indicating when the signs of $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ must be different (cf. Remark 12). To show this, let us assume that $F_A(E)/E$ tends to infinity. Then, (5.11) is violated and we draw from Theorem 7 the following consequence.

If $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ do not change signs above some energy and if $\lim_{E\to\infty}|F_A(E)/E|=\infty$, then either (5.21) converges or the signs are asymptotically different and (5.14) is divergent.

An appropriate illustration of this result is given, again, by the amplitude proprosed by Łukaszuk and Nicolescu,[19] in which $|F_A(E)/E|$ increases like $(\ln E)^2$ and (5.21) diverges. Then $\mathrm{Im}F_A(E)$ and $\mathrm{Re}F_A(E)$ must have opposite signs. Also, (5.14) diverges. On the other hand, (5.24) is not fulfilled since $|\rho_A(E)|^{-1}$ behaves like $\pi/\ln E$.

Theorems 6, 7, and 8 suggest a way to establish necessary and sufficient conditions for different types of asymptotic behavior of $\sigma_A(E)$. We shall summarize them in the following corollary.

Corollary 2. Let $F_A(z)$ satisfy (F1)–(F5) and let $\mathrm{Re}F_A(E)$ and $\mathrm{Im}F_A(E)$ not change signs beyond some energy. If (5.10) is satisfied and the limits $\lim_{E\to\infty}\sigma_A(E)$, $\lim_{E\to\infty}\tau_A(E)$ exist, $\tau_A(E)$ being defined as

$$\tau_A(E)=\frac{2}{\pi}\int_{E_0}^{E}\tan^{-1}[1/\rho_A(E')]\frac{dE'}{E'} + \tfrac{1}{2}\ln\{1+[\rho_A(E)]^2\}, \tag{5.26}$$

then a necessary and sufficient condition for

$$\lim_{E\to\infty}|\sigma_A(E)|=+\infty\,,$$

$$0<\lim_{E\to\infty}|\sigma_A(E)|<\infty\,, \tag{5.27}$$

$$\lim_{E\to\infty}\sigma_A(E)=0$$

is

$$\lim_{E\to\infty}\tau_A(E)=-\infty\,,$$

$$-\infty<\lim_{E\to\infty}|\tau_A(E)|<+\infty\,, \tag{5.28}$$

$$\lim_{E\to\infty}\tau_A(E)=+\infty\,,$$

respectively.

Proof. Since $\lim_{E\to\infty}\tau_A(E)$ exists and formulas (5.28) exhaust all its possible values, it is enough to prove that conditions (5.28) are sufficient for (5.27). According to Theorem 6, there exists a sequence $E_k, E_k\to\infty$ for $k\to\infty$, so that the inequalities

$$A_1\{1+[\rho_A(E_k)]^2\}^{-1/2}\exp\left[-\frac{2}{\pi}\int_{E_0}^{E_k}\tan^{-1}[1/\rho_A(E')]\frac{dE'}{E'}\right] \le \left|\frac{\mathrm{Im}F_A(E_k)}{E_k}\right| \le A_2\{1+[\rho_A(E_k)]^2\}^{-1/2}\exp\left\{-\frac{2}{\pi}\int_{E_0}^{E_k}\tan^{-1}[1/\rho_A(E')]\frac{dE'}{E'}\right\} \tag{5.29}$$

hold. Note that (5.29) is independent of the signs of $\mathrm{Re}F_A(E)$ and $\mathrm{Im}F_A(E)$. Taking the logarithm of (5.29) we have

$$A_1'-\tau_A(E_k)\le\ln|\sigma_A(E_k)|\le A_2'-\tau_A(E_k)\,.$$

Hence, we easily obtain the individual statements of the corollary.

It is worth discussing several high-energy models in light of this corollary. In the model proposed by Bourrely and Fischer[16] both $\mathrm{Re}F_A(E)$ and $\mathrm{Im}F_A(E)$ behave asymptotically like $\sqrt{E}$. Thus, $\lim_{E\to\infty}\sigma_A(E)=0$. As seen from (5.26), the first term in $\tau_A(E)$ tends to plus infinity, whereas the second one is bounded. Thus, $\lim_{E\to\infty}\tau_A(E)=+\infty$. Further, in Ref. 17 we have $\mathrm{Re}F_A(E)\sim E$, $\mathrm{Im}F_A(E)\sim\sqrt{E}$ asymptotically and, again, $\lim_{E\to\infty}\sigma_A(E)=0$. However, the first term on the right-hand side of (5.26) is finite, while the second one is rising as $\ln E$. Thus, again, $\lim_{E\to\infty}\tau_A(E)=+\infty$. Finally, the model of Łukaszuk and Nicolescu[19] gives $\sigma_A(E)$ behaving as $\ln E$, whereas $\tau_A(E)$ consists of two competing terms,

$$-\frac{2}{\pi}\int^{E}\tan^{-1}\frac{\pi}{\ln E'}\frac{dE'}{E'}+\ln\ln E\,,$$

thus resulting in $\lim_{E\to\infty}\tau_A(E)=-\infty$.

Remark 15. The corollary remains valid if (5.10) is replaced by (5.13).

VI. CONCLUDING REMARKS

The consequences of the most general properties of the S matrix for asymptotic energies have been investigated. The method has been based on the analyticity properties of the logarithm of the forward scattering amplitude. The method has turned out to be able to give a number of new, physically interesting asymptotic theorems.

It is to be noticed that all results contained in Sec. IV and V (Theorems 3–8, the two corollaries, and other consequences) correlate measurable quantities like total cross sections, the real and the imaginary part of the forward scattering amplitude, their ratio, their signs, etc., at asymptotic energies. It is typical for the theorems obtained that their assumptions may be widely changed without changing the final statements. In particular, the role of the real and the imaginary part can be interchanged as seen, for instance, from Remarks 6 and 11.

Taking into account only the most general features of the forward scattering amplitude, the results are suited for various analyses and consistency tests of high-energy data with analyticity. The high-energy models are usually classified according to the asymptotic behavior of $\sigma_S(E)$, $\sigma_A(E)$. We have shown that this is equivalent to a classification according to $\int^E \rho_S(E')\,d(\ln E')$, $\tau_A(E)$, respectively. Indeed, the results allow us to formulate, in terms of $\rho_S(E)$, $\rho_A(E)$, respectively, necessary and sufficient conditions for every given asymptotic behavior of the total cross sections $\sigma_S(E), \sigma_A(E)$. The basic classification has been presented in Corollaries 1 and 2 and Remarks 7, 8, and 15, and the application of the procedure to a specific asymptotic behavior is straightforward.

To show one of the possible applications of our theorems to experimental data, let us discuss the results of Sec. V in connection with the existing data on K_s^0 regeneration.[3] Note that these measurements give direct information on the antisymmetric $K^0p - \bar{K}^0p$ forward scattering amplitude. Let us consider Theorem 6 and Remark 11. Take $\bar{\mu}(E), \bar{\nu}(E)$ constant in energy and put $n=0$. The conditions

$$\lim_{E\to\infty} F_A(E)E^{-1}=0\,,$$

$$\int^\infty \mathrm{Im}F_A(E)dE=\infty\,,$$

$$\mathrm{Re}F_A(E)\geq 0$$

are fulfilled by any reasonable fit. Their violation is possible only in very "unphysical" situations like a very fast vanishing of $\sigma_A(E)$ or a very fast increase of $|F_A(E)|$. From Ref. 3 we get the input for the phase of $F_A(E)$.

$$\varphi_{F_A}\in(41°,57°),\quad F_A(E)=F_{\bar{K}^0p}(E)-F_{K^0p}(E)$$

so that $\bar{\mu}(E)=0.23$ and $\bar{\nu}(E)=0.32$. Then, (5.6) gives

$$C_1E^{0.36}\leq|F_A(E)|\leq C_2E^{0.54}\,.$$

The experimental fit from Ref. 3 is

$$F_A(E)=CE^{0.47\pm0.13}\,.$$

Thus, we see that the experimental values of φ_{F_A} and $|F_A(E)|$ between 10 and 50 GeV exhibit the correlation which is required by Theorem 6 for the asymptotic region.

It is interesting to compare our results for the symmetric amplitude with the classical paper of Khuri and Kinoshita.[4] Our assumptions differ from those made by Khuri and Kinoshita only in (F1) and in the fact that most of our results have been derived under the additional condition (f6′). (F1) is weaker than the corresponding assumption in Ref. 4, whereas (f6′) is more restrictive.

The requirement (f6′) ensures the Herglotz property of $f(z)/z$ and, thus, excludes the zeros of $f(z)$ in the asymptotic region of $\mathfrak{D}$. The absence of zeros was proved by Jin and Martin[22] for $f(z)$ analytic in the twice-cut plane, but under condition (f6′) it follows even if the analyticity of $f(z)$ in the central part of the complex energy plane is not assumed [assumption (f1)]. As shown in our Lemma of Appendix B, one of the consequences of (f6′) is that our results follow without additional assumptions about zeros on the cut, which are made, e.g., in Meiman's theorems (see Refs. 4 and 5 and Appendix A).

To sum up we can say that assumption (f6′), which is very plausible from the physical point of view, enables us to make other assumptions weaker and, besides, to obtain results which are stronger than those contained in Khuri and Kinoshita's Theorems 1 and 2. Theorem 3 is reproduced in our approach completely without additional assumptions. Moreover, we obtain information on the sign of $\mathrm{Re}F_S(E)$. These results follow from Theorem 3 of the present paper.

We would like to mention that relations (2.1) and (2.2) do not exhaust all possibilities of constructing the auxiliary function $f(z)$ in terms of $F_S(E)$, $F_A(E)$. In particular, $f(z)$ may be defined as a weighted integral of $F(z)$ in analogy with the approach of Khuri and Kinoshita[5] or Grunberg and Truong.[9] In this way, the use of Theorem 2 allows a number of results of different approaches to be obtained from a unified point of view.

ACKNOWLEDGMENTS

We are indebted to Professor H. Cornille, Professor N. N. Khuri, and Professor A. Martin for interesting discussions with one of us (J.F.) and for valuable comments. We also thank Dr. S. Borenstein for carefully reading the manuscript and for useful comments.

APPENDIX A

This section will be devoted to the proof of Theorem A, which is slightly more general than Theorem 1. Assumptions (f2) and (f6′) will be considerably weakened. Comments on relations among assumptions of the two theorems will be given in remarks.

Theorem A. Let $f(z)$ fulfill (f1) and (f3). If

$$\begin{aligned}&\mathrm{Im}f(z)\geq 0 \text{ for } \mathrm{Im}z=0\,,\quad \mathrm{Re}z\geq r_2'\,,\\&\mathrm{Im}f(z)\leq 0 \text{ for } \mathrm{Im}z=0\,,\quad \mathrm{Re}z\leq -r_2'\,,\end{aligned}\tag{A1}$$

$$\lim_{|z|\to\infty}\frac{f(z)}{z^2}=0\,,\tag{A2}$$

$$\int_{-\infty}^{-r} \frac{\mathrm{Im} f(E)}{E} dE + \int_{r}^{\infty} \frac{\mathrm{Im} f(E)}{E} dE > \int_0^{\pi} \mathrm{Re} f(re^{i\varphi}) d\varphi , \tag{A3}$$

$$r \geq \max(r_0, r_2') ,$$

then there exist such $E_H > 0$ that $\mathrm{Im}[f(z)/z] > 0$ for $|z| > E_H$, $z \in \mathfrak{D}$. Here $\mathfrak{D}$ is the region defined in Sec. II.

Proof. Let $\mathfrak{D}^*$ be a region of the complex plane which consists of the points z fulfilling $\mathrm{Im} z < 0$ and $|z| > r$. For our purposes it is convenient to define the function f also in the region $\mathfrak{D}^*$. Put

$$f(z) = f^*(z^*) \text{ for } z \in \mathfrak{D}^* . \tag{A4}$$

The function f is now analytic in the union $\mathfrak{D} \cup \mathfrak{D}^*$ and its real part is continuous on the boundary of $\mathfrak{D} \cup \mathfrak{D}^*$.

Using (A4) we can write down the Cauchy theorem for $f(z)/z$ along the curve $\mathfrak{L}$ depicted in Fig. 1

$$\frac{f(z)}{z} = \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{f(t)}{t(t-z)} dt .$$

Hence,

$$\frac{f(z)}{z} = \frac{1}{\pi} \int_r^E \frac{\mathrm{Im} f(t)}{t(t-z)} dt + \frac{1}{\pi} \int_{-E}^{-r} \frac{\mathrm{Im} f(t)}{t(t-z)} dt + \frac{1}{2\pi i} \int_{|t|=E} \frac{f(t)}{t(t-z)} dt - \frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)}{t(t-z)} dt .$$

Consider only the imaginary part of this equation:

$$\mathrm{Im} \frac{f(z)}{z} = \frac{\mathrm{Im} z}{\pi} \left\{ \int_r^E \frac{\mathrm{Im} f(t) dt}{t[(t-\mathrm{Re} z)^2 + (\mathrm{Im} z)^2]} + \int_{-E}^{-r} \frac{\mathrm{Im} f(t) dt}{t[(t-\mathrm{Re} z)^2 + (\mathrm{Im} z)^2]} \right\}$$
$$+ \mathrm{Im} \left[\frac{1}{2\pi i} \int_{|t|=E} \frac{f(t) dt}{t(t-z)} \right] - \mathrm{Im} \left[\frac{1}{2\pi i} \int_{|t|=r} \frac{f(t) dt}{t(t-z)} \right], \quad z \in \mathfrak{D} . \tag{A5}$$

The last integral on the right-hand side of (A5) has the form

$$\mathrm{Im} \left[\frac{1}{2\pi i} \int_{|t|=r} \frac{f(t) dt}{t(t-z)} \right] = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} f(re^{i\varphi}) \frac{(\mathrm{Im} z - r \sin\varphi) d\varphi}{r^2 + |z|^2 - 2r(\mathrm{Re} z \cos\varphi + \mathrm{Im} z \sin\varphi)}$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Im} f(re^{i\varphi}) \frac{(r\cos\varphi - \mathrm{Re} z) d\varphi}{r^2 + |z|^2 - 2r(\mathrm{Re} z \cos\varphi + \mathrm{Im} z \sin\varphi)} .$$

In order to estimate this expression we mention that

$$\left| \frac{|z|^2}{\mathrm{Im} z} \left[\frac{\mathrm{Im} z - r \sin\varphi}{r^2 + |z|^2 - 2r(\mathrm{Re} z \cos\varphi + \mathrm{Im} z \sin\varphi)} + \frac{r \sin\varphi}{r^2 + |z|^2 - 2r \mathrm{Re} z \cos\varphi} \right] - 1 \right|$$
$$\leq \left(2\frac{r}{|z|} + 5\frac{r^2}{|z|^2} + 4\frac{r^3}{|z|^3} + \frac{r^4}{|z|^4} \right) \left(1 - \frac{r}{|z|} \right)^{-4} , \tag{A6}$$

$$\left| \frac{|z|^2}{\mathrm{Im} z} \left[\frac{r\cos\varphi - \mathrm{Re} z}{r^2 + |z|^2 - 2r(\mathrm{Re} z \cos\varphi + \mathrm{Im} z \sin\varphi)} - \frac{r\cos\varphi - \mathrm{Re} z}{r^2 + |z|^2 - 2r \mathrm{Re} z \cos\varphi} \right] \right| \leq \left(2\frac{r}{|z|} + \frac{r^2}{|z|^2} \right) \left(1 - \frac{r}{|z|} \right)^{-4} \tag{A7}$$

and that [according to (A4)]

$$\int_0^{2\pi} \mathrm{Re} f(re^{i\varphi}) \frac{r \sin\varphi \, d\varphi}{r^2 + |z|^2 - 2r \mathrm{Re} z \cos\varphi} = 0 ,$$

$$\int_0^{2\pi} \mathrm{Im} f(re^{i\varphi}) \frac{r\cos\varphi - \mathrm{Re} z}{r^2 + |z|^2 - 2r \mathrm{Re} z \cos\varphi} d\varphi = 0 .$$

Thus we have

$$\frac{|z|^2}{\mathrm{Im} z} \mathrm{Im} \left[\frac{1}{2\pi i} \int_{|t|=r} \frac{f(t)}{t(t-z)} dt \right] \leq \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} f(re^{i\varphi}) d\varphi + \frac{M}{2\pi} \gamma(z) , \quad z \in \mathfrak{D} \tag{A8}$$

where $M = \max_\varphi |f(re^{i\varphi})|$ and $\gamma(z)$ is the sum of the right-hand sides of (A6) and (A7). Evidently $\gamma(z) \to 0$ for $|z| \to \infty$.

Due to (A2) and (A4), estimates analogous to (A6) and (A7) can be found showing that the expression

$$\operatorname{Im}\left[\frac{1}{2\pi i}\int_{|t|=E}\frac{f(t)}{t(t-z)}dt\right]$$

on the right-hand side of (A5) converges to zero for $E\to\infty$. (A5) and (A8) imply

$$\frac{|z|^2}{\operatorname{Im}z}\operatorname{Im}\frac{f(z)}{z}\geq\frac{1}{\pi}\int_r^\infty\frac{|z|^2\operatorname{Im}f(t)dt}{t[(t-\operatorname{Re}z)^2+(\operatorname{Im}z)^2]}$$
$$+\frac{1}{\pi}\int_{-\infty}^{-r}\frac{|z|^2\operatorname{Im}f(t)dt}{t[(t-\operatorname{Re}z)^2+(\operatorname{Im}z)^2]}-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\varphi})d\varphi-\frac{M}{2\pi}\gamma(z)\,,\quad z\in\mathfrak{D}\,.$$

Let $|z|\to\infty$; we obtain

$$\liminf_{|z|\to\infty}\left[\frac{|z|^2}{\operatorname{Im}z}\operatorname{Im}\frac{f(z)}{z}\right]\geq\frac{1}{\pi}\liminf_{|z|\to\infty}\left(\int_r^\infty\frac{\operatorname{Im}f(t)dt}{t\{[t/|z|-(\operatorname{Re}z/|z|)]^2+[(\operatorname{Im}z/|z|)]^2\}}\right.$$
$$\left.+\int_{-\infty}^{-r}\frac{\operatorname{Im}f(t)dt}{t\{[t/|z|-(\operatorname{Re}z/|z|)]^2+[(\operatorname{Im}z/|z|)]^2\}}-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\varphi})d\varphi\right).$$

With respect to (A1) we can use Fatou's lemma and since

$$\lim_{|z|\to\infty}\left[\left(\frac{t}{|z|}-\frac{\operatorname{Re}z}{|z|}\right)^2+\left(\frac{\operatorname{Im}z}{|z|}\right)^2\right]=\lim_{|z|\to\infty}\left(\frac{t^2}{|z|^2}-\frac{2t}{|z|}\frac{\operatorname{Re}z}{|z|}+1\right)=1\,,$$

we have

$$\liminf_{|z|\to\infty}\left[\frac{|z|^2}{\operatorname{Im}z}\operatorname{Im}\frac{f(z)}{z}\right]\geq\frac{1}{\pi}\left(\int_r^\infty\frac{\operatorname{Im}f(t)}{t}dt+\int_{-\infty}^{-r}\frac{\operatorname{Im}f(t)dt}{t}\right)-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\varphi})d\varphi\,.$$

If we use (A4) once more we see that

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\varphi})d\varphi=\frac{1}{\pi}\int_0^{\pi}\operatorname{Re}f(re^{i\varphi})d\varphi$$

and the statement of Theorem A follows immediately from (A3).

Remark 16. If (f2) is assumed, then $\operatorname{Im}f(-E)=-\operatorname{Im}f(E)$ and condition (A3) can be simplified to

$$\int_r^\infty\frac{\operatorname{Im}f(E)}{E}dE>\tfrac{1}{2}\int_0^\pi\operatorname{Re}f(re^{i\varphi})d\varphi\,.$$

Since $f(z)$ is assumed to be continuous up to the boundary, $\operatorname{Re}f(re^{i\varphi})$ is continuous and finite so that $\int_0^\pi\operatorname{Re}f(re^{i\varphi})d\varphi$ is a finite number. It means that (f2) and (f6′) imply (A3).

Remark 17. Because of the Phragmén-Lindelöf theorem (see, e.g., Ref. 25), (f4) and (f5′) imply (A2).

APPENDIX B

In this Appendix we present the proof of Theorem 2.

If $f(z)$ fulfills (f2), then also $-f(z)$ fulfills (f2). In the following, we shall need $\ln[-f(z)]$. To avoid ambiguity we shall use the principal value of the logarithm: $\ln w=\ln|w|+i\arg w$ with $-\pi<\arg w\leq\pi$. In this case $(\ln w)^*=\ln w^*$ for any complex number w. Condition (f2) for $-f(z)$ implies

$$\{\ln[-f(z)]\}^*=\ln[-f(-z^*)]\,. \qquad \text{(B1)}$$

Since $f(z)$ is supposed to satisfy the conditions of Theorem 1 there exists E_H such that

$$\operatorname{Im}\left(\frac{f(z)}{z}\right)>0\quad\text{for }|z|>E_H,\quad\operatorname{Im}z>0. \qquad \text{(B2)}$$

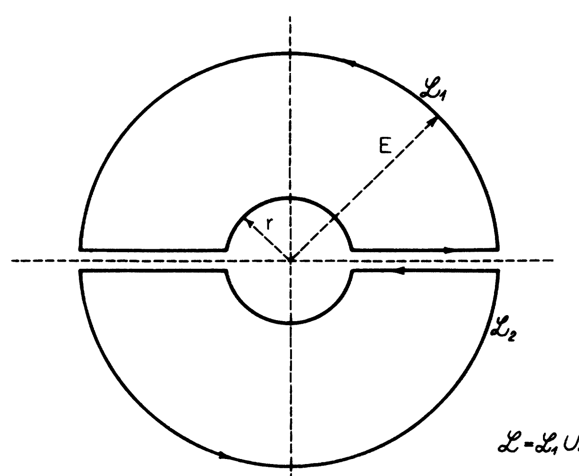

FIG. 1. Integration contour $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2$ in the z plane (see Appendix A).

This yields

$$-\pi < \arg[-f(z)] < \pi \tag{B3}$$

so that $\ln[-f(z)]$ is analytic in the intersection of $\mathfrak{D}$ with $|z| > E_H$. This region will be denoted $\mathfrak{D}_H$ in the following.

We define the function $l_\alpha(z)$ by

$$l_\alpha(z) = \int_z^{\alpha z} \frac{\ln[-f(z')]}{z'}\, dz', \quad \alpha > 1, \quad z, z' \in \mathfrak{D}_H .$$

This function is also analytic in $\mathfrak{D}_H$. It follows immediately from (B.1) that

$$l_\alpha(z) = l_\alpha^*(-z^*), \quad z \in \mathfrak{D}_H . \tag{B4}$$

We shall prove now the following lemma.

Lemma. Let a function $f(z)$ fulfill the conditions (f1) and (f3) and let there exist a number $E_H \geq r_0$ such that (B.2) holds. Then the function $l_\alpha(z)$ is analytic in $\mathfrak{D}_H$ and

$$\lim_{z\to E} l_\alpha(z) = \int_E^{\alpha E} \frac{\ln[-f(t)]}{t}\, dt = l_\alpha(E) .$$

Proof. It is clear that $f(E)$ is different from zero in any interval of the real axis. In the opposite case we should obtain $f(z) = 0$, which contradicts (B2). Thus we can choose numbers $r, R, E_H < r < R$ such that $f(r) \neq 0$, $f(-r) \neq 0$, $f(R) \neq 0$, $f(-R) \neq 0$. Further, we choose a number β, $1 < \beta < R/r$ such that there is a positive integer n with the property $\beta^n = \alpha$ and that

$$f(E) \neq 0$$

for

$$E \in [r, r\beta] \cup [R/\beta, R] \cup [-r\beta, -r] \cup [-R, -R/\beta] . \tag{B5}$$

Denote $\mathfrak{D}_{r,R} = \{z\colon\ \mathrm{Im} z > 0,\ |z| > r,\ |z| < R\}$. Being continuous, f is bounded on $\bar{\mathfrak{D}}_{r,R}$, i.e., $|f(z)| \leq M_1$ for $z \in \bar{\mathfrak{D}}_{r,R}$. In the following we shall use the function $\tilde{f}(z) = f(z)/M_1$. Obviously,

$$|\tilde{f}(z)| \leq 1 \quad \text{for } z \in \bar{\mathfrak{D}}_{r,R}. \tag{B6}$$

Let further $g(w)$ be a conformal mapping of the unit circle K, $K = \{w\colon\ |w| < 1\}$ onto $\mathfrak{D}_{r,R/\beta}$. Denote

$$h_t(w) = \frac{\ln[-\tilde{f}(g(w)t)]}{t}$$

for $|w| < 1$ and every $t \in [1, \beta]$. Because of (B3), $h_t(w)$ is an analytic function of w in K for every $t \in [1, \beta]$.

In the following, we shall need the well-known class H_1 of analytic functions. We remind the reader that a function $u(w)$ belongs to H_1 if it is analytic on K and

$$\sup_{\rho<1} \int_0^{2\pi} |u(\rho e^{i\varphi})|\, d\varphi$$

is finite[26,27]. Let us prove now the following statement.

Statement 1. The function $h_t(w)$ belongs to H_1 for every $t \in [1, \beta]$ and

$$\int_0^{2\pi} |h_t(\rho e^{i\varphi})|\, d\varphi \leq M_2, \quad \rho < 1$$

where M_2 depends neither on ρ nor on t.

Proof. Since $\mathrm{Im} h_t(w) = (1/t)\arg[-f(g(w)t)]$ and (B3) holds we see that

$$|\mathrm{Im} h_t(w)| \leq \pi . \tag{B7}$$

Let γ be a positive number. According to Theorem 6 of Ref. 28 there exists a number M_3 such that

$$\frac{1}{2\gamma} \left| \int_{\varphi-\gamma}^{\varphi+\gamma} h_t(\rho e^{i\varphi}) d\varphi \right| \leq M_3.$$

Since $\mathrm{Re}\, h_t(w) = (1/t) \ln f(g(w)t)$ and (B6) holds we have $\int_{\varphi-\gamma}^{\varphi+\gamma} |\mathrm{Re} h_t(\rho e^{i\varphi})|\, d\varphi \leq 2M_3\gamma$.
This implies $\int_0^{2\pi} |\mathrm{Re} h_t(\rho e^{i\varphi})|\, d\varphi \leq 2M_3(\pi+\gamma)$.
Inequality (B7) yields $\int_0^{2\pi} |\mathrm{Im} h_t(\rho e^{i\varphi})|\, d\varphi \leq 2\pi^2$.
Statement 1 is proved.

A conclusion of the Nevanlinna theorem[27] states that there exist finite values of limits $\lim_{\rho\to 1} h_t(\rho e^{i\varphi})$ for almost all φ. They will be denoted by $h_t(e^{i\varphi})$. Let F_t be the set of φ, $-\pi < \varphi \leq \pi$, for which $\tilde{f}(g(e^{i\varphi})t) = 0$. Certainly,

$$\lim_{\rho\to 1} |\mathrm{Re} h_t(\rho e^{i\varphi})| = \frac{1}{t} \lim_{\rho\to 1} |\ln|\tilde{f}(g(\rho e^{i\varphi})t)||$$

$$= \frac{1}{t}\, |\ln|\tilde{f}(g(e^{i\varphi})t)|| \geq 0$$

for $\varphi \notin F_t$ and

$$\lim_{\rho\to 1} |\mathrm{Re} h_t(\rho e^{i\varphi})| = \frac{1}{t} \lim_{\rho\to 1} |\ln|\tilde{f}(g(\rho e^{i\varphi})t)|| = \infty$$

for $\varphi \in F_t$. The existence of finite values of $h_t(e^{i\varphi})$ almost everywhere means that

$$\mu(F_t) = 0 \quad \text{for every } t \in [1, \beta], \tag{B8}$$

where μ is the Lebesgue measure. The continuity of f, g, and $\ln z$ (at the nonzero points) yields

$$h_t(e^{i\varphi}) = \frac{1}{t} \ln[-\tilde{f}(g(e^{i\varphi})t)] \quad \text{for } \varphi \in F_t. \tag{B9}$$

Using Fatou's lemma and Statement 1 we obtain

$$\int_0^{2\pi} |h_t(e^{i\varphi})|\, d\varphi \leq M_2. \tag{B10}$$

Considering (B9) and applying Fubini's theorem we conclude that

$$\int_1^{\beta} h_t(e^{i\varphi})dt = \int_1^{\beta} \ln[-\tilde{f}(g(e^{i\varphi})t)]\, \frac{dt}{t} \tag{B11}$$

for almost all φ. Analogously, considering (B10) and applying Fubini's theorem once more we see that

$$\int_1^{\beta} h_t(e^{i\varphi})dt$$

converges for almost all φ. (B12)

Statement 2. The function

$$(1/t)\ln[-\tilde{f}(g(\sigma)t)]\, dt$$

is a continuous function of σ on the circle $|\sigma| = 1$. The *proof* will be divided into three parts.

(i) Let $g(e^{i\varphi})$ not be real; then, because of (B2), $\tilde{f}(g(e^{i\varphi})t) \neq 0$ for all $t \in [1, \beta]$. Since it is a continuous function the statement is proved.

(ii) Let $g(e^{i\varphi}) = r$. Since $g(e^{i\varphi})t \in [r, r\beta]$ for all $t \in [1, \beta]$ and since f is nonzero there [see (B5)] and continuous the statement is proved, too. The cases $g(e^{i\varphi}) = -r$, $g(e^{i\varphi}) = R$, $g(e^{i\varphi}) = -R$ can be treated in the same manner.

(iii) Assume, finally, $g(e^{i\varphi}) \in (r, R/\beta)$. Denote

$$Q = \{\varphi:\ g(e^{i\varphi}) \in (r, R/\beta)\}\ .$$

From (B11) and (B12) we see that $\int_1^{\beta}(1/t)\ln[-\tilde{f}(g(e^{i\varphi})t)]\, dt$ is convergent for almost all φ. Let Q' be the set of such φ. We have

$$\int_1^{\beta}(1/t)\ln[-\tilde{f}(g(e^{i\varphi})t)]\, dt = \int_{\xi}^{\beta\xi}(1/\tau)\ln[-\tilde{f}(\tau)]\, d\tau,$$

where ξ stands for $g(e^{i\varphi})$. The function $g(e^{i\varphi})$ for $\varphi \in Q$ maps an analytic part of the boundary onto an analytic part of the boundary. This yields[29] that $g(w)$ has nonvanishing and continuous derivatives with respect to φ on Q. This means that the set

$$\Gamma = \{\xi:\ \xi = g(e^{i\varphi}),\ \varphi \in Q'\}$$

has the full Lebesgue measure, i.e., $\mu(\Gamma) = R/\beta - r$. Thus, we have proved that

$\int_{\xi}^{\beta\xi}(1/\tau)\ln[-\tilde{f}(\tau)]\, d\tau$ converges for almost all ξ, $\xi \in (r, R/\beta)$. Bearing in mind that $\beta > 1$ we deduce that this integral is convergent for all ξ, $\xi \in (r, R/\beta)$. Now the well-known theorem can be used stating that any integral is a continuous function of the upper limit. Statement 2 follows now from the fact that ξ is a continuous function of φ. If $g(e^{i\varphi}) \in (-R/\beta, -r)$ the proof is the same.

Now, using Fichtenholz's theorem[27] we obtain

$$h_t(\rho e^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} h_t(e^{i\psi})\frac{1-\rho^2}{1-2\rho\cos(\rho-\psi)+\rho^2}\, d\psi,$$

$$0 \leq \rho < 1.$$

Owing to (B10), we can integrate this relation over t

$$h(\rho e^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\psi})\frac{1-\rho^2}{1-2\rho\cos(\rho-\psi)+\rho^2}\, d\psi\ , \tag{B13}$$

where $h(w) = \int_1^{\beta} h_t(w)dt$. Because of (B11) we can substitute $\int_1^{\beta}(1/t)\ln[-\tilde{f}(g(e^{i\varphi})t)]\, dt$ instead of $\int_1^{\beta} h_t(e^{i\varphi})dt$ on the right-hand side of (B13). Using Statement 2 and (B13) we obtain that $h(w)$ is continuous up to the boundary of K. With respect to the definition of h we conclude that $l_{\beta}(z) = \int_1^{\beta}(1/t)\ln[-f(tz)]\, dt$ is continuous up to the boundary $\mathfrak{D}_{r,R/\beta}$ and that $l_{\beta}(E) = \int_1^{\beta}(1/E)\ln[-f(tE)]dt$. As the number r can be chosen arbitrarily close to E_H and the number R arbitrarily large we obtain that $l_{\beta}(z)$ is continuous up to the boundary of $\mathfrak{D}_H$. Since

$$l_{\alpha}(z) = \sum_{k=0}^{n-1} l_{\beta}(\beta^k z)$$

the lemma is proved.

We apply now the Cauchy theorem to the function $z^{-1}l_{\alpha}(z)$. Applying the lemma, we can shift the integration contour up to the real axis. We have

$$\int_{\mathfrak{C}} \frac{l_{\alpha}(z)dz}{z} = 0. \tag{B14}$$

The contour $\mathfrak{C}$ is depicted in Fig. 2, E_M satisfying

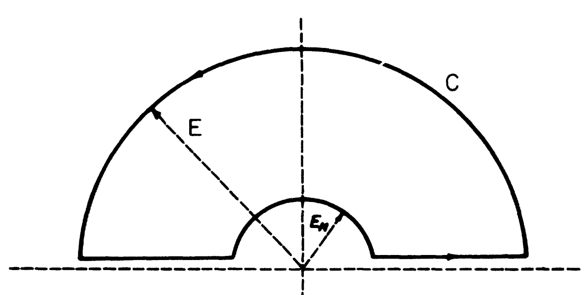

FIG. 2. Integration contour $\mathfrak{C}$ in the z plane [see Appendix B, formula (B14)].

the inequality $E_M \geq \max(E_H, E_0, E_R)$. Making use of (B3), (B14) can be rewritten as

$$\int_0^\pi \mathrm{Re}\, l_\alpha(Ee^{i\varphi})d\varphi + 2\int_{E_M}^{E} \frac{\mathrm{Im}\, l_\alpha(E')}{E'}\, dE' + \int_\pi^0 \mathrm{Re}\, l_\alpha(E_M e^{i\varphi})d\varphi = 0 \quad \text{(B15)}$$

with some positive $E > E_M$. The mean-value theorem allows us to express the first integral in the form

$$\pi\, \mathrm{Re}\, l_\alpha(Ee^{i\bar{\varphi}}),$$

$\bar{\varphi}$ being a number between 0 and π depending on E. We shall, further, define the function $\lambda_\alpha(E,\varphi)$ by formula

$$\lambda_\alpha(E,\varphi) = \mathrm{Re}\, l_\alpha(Ee^{i\varphi}) - \mathrm{Re}\, l_\alpha(E)\,. \quad \text{(B16)}$$

Then, using (B15) and (B16) we find

$$\mathrm{Re}\, l_\alpha(E) = -\frac{2}{\pi}\int_{E_M}^{E} \frac{\mathrm{Im}\, l_\alpha(E')}{E'}dE' + \tilde{\lambda}_\alpha(E,\bar{\varphi}), \quad \text{(B17)}$$

where

$$\tilde{\lambda}_\alpha(E,\bar{\varphi}) = -\lambda_\alpha(E,\bar{\varphi}) - \frac{1}{\pi}\int_\pi^0 \mathrm{Re}\, l_\alpha(E_M e^{i\varphi})d\varphi\,. \quad \text{(B18)}$$

Substituting the imaginary part of $l_\alpha(z)$ into (B17) we obtain

$$\mathrm{Re}\, l_\alpha(E) = -\frac{2}{\pi}\int_{E_M}^{E}\frac{dE'}{E'}\int_{E_M}^{E}\frac{\mathrm{Im}\ln[-f(E'')]}{E''}dE'' + \tilde{\lambda}_\alpha(E,\bar{\varphi})\,. \quad \text{(B19)}$$

We recall that, according to (3.2),

$$\tan[\pi\mu(E)] \leq |\mathrm{Im} f(E)/\mathrm{Re} f(E)| \leq \tan\pi\nu(E), \quad E > E_0 \quad \text{(B20)}$$

where $\mu(E)$ and $\nu(E)$ are integrable over the interval (E_0, E) for all $E > E_0$ and that (3.1) holds for $E > E_0$. Hence, we obtain the following restrictions on the phase $\arg[-f(E)]$:

$$-\pi\nu(E) \leq \arg[-f(E)] \leq -\pi\mu(E), \quad \text{for } \mathrm{Re} f \leq 0 \quad \text{(B21)}$$

$$-\pi + \pi\mu(E) \leq \arg[-f(E)] \leq -\pi + \pi\nu(E), \quad \text{for } \mathrm{Re} f \geq 0 \quad \text{(B22)}$$

and $-\pi + \pi\mu(E) \leq \arg[-f(E)] \leq -\pi\mu(E)$ if the sign of $\mathrm{Re} f(E)$ is not specified. (B21), (B22), and (B19) imply

$$2\int_{E_M}^{E}\frac{dE'}{E'}\int_{E'}^{\alpha E'}\frac{\mu(E'')}{E''}dE'' + \tilde{\lambda}_\alpha(E,\bar{\varphi}(E)) \leq \mathrm{Re}\, l_\alpha(E) \leq 2\int_{E_M}^{E}\frac{dE'}{E'}\int_{E'}^{\alpha E'}\frac{\nu(E'')}{E''}dE'' + \tilde{\lambda}_\alpha(E,\bar{\varphi}(E)) \quad \text{(B23)}$$

for $\mathrm{Re} f(E) \leq 0$ and

$$2\int_{E_M}^{E}\frac{dE'}{E'}\int_{E'}^{\alpha E'}\frac{1-\nu(E'')}{E''}dE'' + \tilde{\lambda}_\alpha(E,\bar{\varphi}(E)) \leq \mathrm{Re}\, l_\alpha(E) \leq 2\int_{E_M}^{E}\frac{dE'}{E'}\int_{E'}^{\alpha E'}\frac{1-\mu(E'')}{E''}dE'' + \tilde{\lambda}_\alpha(E,\bar{\varphi}(E)) \quad \text{(B24)}$$

for $\mathrm{Re} f(E) \geq 0$ above some energy.

Further, we show that the function $\tilde{\lambda}_\alpha(E,\bar{\varphi}(E))$ is uniformly bounded as a function of E and $\bar{\varphi}$. We have

$$|\mathrm{Re}\, l_\alpha(Ee^{i\varphi}) - \mathrm{Re}\, l_\alpha(E)| \leq \int_0^\varphi \left|\frac{\partial \mathrm{Re}\, l_\alpha(Ee^{i\varphi'})}{\partial\varphi'}\right| d\varphi'$$

for every φ in $(0,\pi)$. Using the Cauchy-Riemann conditions for $l_\alpha(z)$ we find

$$\left|\frac{\partial \mathrm{Re}\, l_\alpha(Ee^{i\varphi})}{\partial\varphi}\right| = |\mathrm{Im}\ln[-f(\alpha Ee^{i\varphi})] - \mathrm{Im}\ln[-f(Ee^{i\varphi})]| \leq 2\pi.$$

Hence

$$|\lambda_\alpha(E,\varphi)| \leq 2\pi^2$$

uniformly for $E > E_M$, $0 \leq \varphi \leq \pi$. Thus, there exists a constant K such that

$$|\tilde{\lambda}_\alpha(E,\varphi)| \leq K. \quad \text{(B25)}$$

To prove Theorem 2 we suppose that the conclusion is false and obtain a contradiction. Suppose that (3.3) is not true; then there is a number $\alpha_0 > 1$ such that for any pair of positive numbers C_1' and C_2' one can find energy E_1 so that at least one of the inequalities

$$\ln|f(E)| > \ln C_2' + 2\int_{E_M}^{E}\frac{\nu(E')}{E'}dE', \quad \text{(B26)}$$

$$\ln|f(E)| < \ln C_1' + 2\int_{E_M}^{E}\frac{\mu(E')}{E'}dE' \quad \text{(B27)}$$

holds for every value of E in the interval $(E_1, \alpha_0 E_1)$. Note that we suppose $\mathrm{Re} f(E) \leq 0$ asymptotically.

Keeping α_0 fixed, we choose the numbers C_1' and C_2' so that

$$C_1' < \exp[-(K+2A)/\ln\alpha_0], \quad C_2' > \exp(K/\ln\alpha_0), \tag{B28}$$

where

$$A = \int_{E_M}^{\alpha_0 E_M} \frac{dE'}{E'} \int_{E_M}^{E'} \frac{\mu(E'')dE''}{E''} \geq 0. \tag{B29}$$

Since (B23) holds, we obtain that at least one of the relations

$$2\int_{E_M}^{E} \frac{dE'}{E'} \int_{E'}^{\alpha_0 E'} \frac{\mu(E'')}{E''} dE'' - K \leq \mathrm{Re}\, l_{\alpha_0}(E_1) < \ln C_1' \ln\alpha_0 + 2\int_{E_1}^{\alpha_0 E_1} \frac{dE'}{E'} \int_{E_M}^{E'} \frac{\mu(E'')}{E''} dE'', \tag{B30}$$

$$\ln C_2' \ln\alpha_0 + 2\int_{E_1}^{\alpha_0 E_1} \frac{dE'}{E'} \int_{E_M}^{E'} \frac{\nu(E'')}{E''} dE'' < \mathrm{Re}\, l_{\alpha_0}(E_1) \leq K + 2\int_{E_M}^{E_1} \frac{dE'}{E'} \int_{E'}^{\alpha_0 E'} \frac{\nu(E'')}{E''} dE'' \tag{B31}$$

is satisfied, E_1 depending on C_1' and C_2'. Further, we have

$$\int_{E}^{\alpha_0 E} \frac{dE'}{E'} \int_{E_M}^{E'} \frac{\mu(E'')}{E''} dE'' = \int_{E_M}^{E} \frac{dE'}{E'} \int_{E^1}^{\alpha_0 E'} \frac{\mu(E'')}{E''} dE'' + A \tag{B32}$$

for all $E \geq E_M$. To prove this, we observe that the derivatives of the integrals are identical in the interval (E_M, ∞). Obviously, an analogous result holds for $\nu(E)$, too.

Combining (B30) and (B31) with (B28) we easily obtain a contradiction. By this, we have proved that, in every interval $(E, \alpha E)$, $E > E_M$, $\alpha > 1$, there exists a subinterval I such that (3.3) holds for all $E \in I$. Hence, (3.3) holds for a sequence E_k.

Relation (3.4) can be proved in a completely analogous way.

Note added in proof. Corollary 1 admits interesting generalizations. For instance, Remarks 7 and 8 remain valid if (4.23) is replaced by the less restrictive condition (4.17). Details will be published later.

---

*Present address: Fyzikální ústav ČSAV, Na Slovance 2, 18040 Praha 8, Czechoslovakia.
†Present address: Matematický ústav ČSAV, Žitná 25, 11567 Praha 1, Czechoslovakia.

[1]U. Amaldi *et al.*, Phys. Phys. Lett. 43B, 231 (1973); 44B, 112 (1973); S. R. Amendolia *et al.*, *ibid.* 44B, 119 (1973).
[2]V. Bartenev *et al.*, Phys. Rev. Lett. 31, 1367 (1973); W. F. Baker *et al.*, in *Proceedings of the XVII International Conference on High Energy Physics, London, 1974,* edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974).
[3]V. K. Birulev *et al.*, JINR Dubna Report No. E 1-6851, 1972 (unpublished).
[4]N. N. Khuri and T. Kinoshita, Phys. Rev. 137, B720 (1965).
[5]N. N. Khuri, T. Kinoshita, Phys. Rev. 140, B706 (1965).
[6]R. Wit, Acta Phys. Pol. 29, 771 (1966).
[7]J. S. Jin and S. W. McDowell, Phys. Rev. 138, B1279 (1965).
[8]H. Cornille, Nuovo Cimento 66A, 217 (1970).
[9]G. Grunberg and Tran N. Truong, Phys. Rev. D 9, 2874 (1974).
[10]Yu. S. Vernov, Zh. Eksp. Teor. Fiz. 50, 672 (1966); 53, 191 (1967).
[11]Yu. S. Vernov, Yad. Fiz. 10, 176 (1969).
[12]M. Meiman, Zh. Eksp. Teor. Fiz. 43, 2277 (1962).
[13]J. Bros, H. Epstein, and V. Glaser, Nuovo Cimento 31, 1265 (1964); Commun. Math. Phys. 1, 240 (1965).
[14]Hung Cheng, J. K. Walker, and Tai Tsun Wu, Phys. Lett. 44B, 97 (1973); 44B, 283 (1973).
[15]P. Kroll, Lett. Nuovo Cimento 7, 745 (1973).
[16]C. Bourrely and J. Fischer, Nucl. Phys. B61, 513 (1973).
[17]C. Bourrely, J. Fischer, and Z. Sekera, Nucl. Phys. B67, 452 (1973).
[18]W. Bartel and A. N. Diddens, CERN Internal Report No. NP 73-4, 1973 (unpublished).
[19]L. Łukaszuk and B. Nicolescu, Lett. Nuovo Cimento 8, 405 (1973).
[20]H. Epstein, V. Glaser, A. Martin, Commun. Math. Phys. 13, 257 (1969).
[21]M. Froissart, Phys. Rev. 123, 1053 (1961); A. Martin, *ibid.* 129, 1432 (1963); Nuovo Cimento 42, 930 (1966).
[22]Y. S. Jin and A. Martin, Phys. Rev. 135, 1369 (1964).
[23]S. Weinberg, Phys. Rev. 124, 2049 (1961).
[24]Tran N. Truong and W. S. Lam, Phys. Rev. D 6, 2875 (1972).
[25]R. Boas, *Entire Functions* (Academic, New York, 1954), p. 3.
[26]K. Hoffman, Banach Spaces of Analytic Functions (Prentice-Hall, Englewood Cliffs, N. J., 1962).
[27]I. I. Privalov, *Granichnye svoystva analiticheskih funktsii,* (Gostehizdat, Moskva-Leningrad, 1950) (German translation: VEB Deutscher Verlag der Wissenschaften, Berlin, 1956).
[28]I. Vrkoč, Czech. Math. J. 98, 455 (1973).
[29]G. M. Goluzin, *Geometricheskaya teoria funktsii kompleksnogo peremennogo* (Nauka, Moskva, 1966).