Critical look at f_K/f_π in the $(3,\bar{3})+(\bar{3},3)$ model

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We discuss the prediction of f_K/f_π that follows from chiral symmetry in the $(3,\bar{3})+(\bar{3},3)$ model from a general point of view. We present the results of the one-loop corrections in the renormalizable SU_3 σ model. These results are compared with a number of other approaches, including chiral perturbation theory. All approaches discussed here have uncertainties in f_K/f_π at least as large as, often much larger than, the experimental error in f_K/f_π. This is traced to the appearance of the mass difference $m_\eta{}^2-m_K{}^2$ which is uncertain to $\pm 4\%$ owing to isospin splitting.

I. INTRODUCTION

By now it is widely accepted that the hadronic Hamiltonian is nearly invariant under a Goldstone-realized symmetry, i.e., $SU_2\times SU_2$. Because the states have large spontaneous breaking of this symmetry the sorts of predictions are quite different from SU_3, for example, which has a normal limit. Another symmetry of this type is $SU_3\times SU_3$, which is on less firm footing since it surely has larger explicit breaking than either of the above two. Nevertheless, the view of it as an approximate Goldstone symmetry with the pseudoscalar octet as the Goldstone bosons and breaking belonging dominantly to the $(3,\bar{3})+(\bar{3},3)$ representation gives a good account of low-energy dynamics.[1]

The $(3,\bar{3})+(\bar{3},3)$ picture of symmetry breaking treated by Gell-Mann, Oakes, and Renner[2] is limited to the case in which the pseudoscalar masses satisfy the Gell-Mann–Okubo formula and the ratio of decay constants $f_K/f_\pi=1$. This paper is concerned with the relation between f_K/f_π and the pseudoscalar masses when both the above restrictions are relaxed. Through the use of the Ward identities and smoothness assumptions, it is possible to relate the above quantities as[2a]

$$\left(\frac{f_\pi}{f_K}-2A\right)\left(\frac{f_\pi}{f_K}-2A'\right)+\frac{1}{2}\left(\frac{f_\pi}{f_K}\right)^2=0, \tag{1.1}$$

where

$$A=\frac{m_\eta{}^2-m_K{}^2}{m_\eta{}^2-m_\pi{}^2},\quad A'=\frac{m_{\eta'}{}^2-m_K{}^2}{m_{\eta'}{}^2-m_\pi{}^2}. \tag{1.2}$$

It is this relation we wish to discuss and offer corrections to. Equation (1.1) has appeared either explicitly or implicitly in many formulations of this model. It appears in papers by Schechter and Ueda and by Cicogna, Strocchi, and Caffarelli.[3] It is implicit in a paper by Kenney[4] and in SU_3 σ-model determinations of f_K/f_π in tree order.[5] Equation (1.1) is a reasonably clear prediction of the $(3,\bar{3})+(\bar{3},3)$ model involving well-determined quantities. Further, the corrections to these quantities can be calculated in the SU_3 model reliably in the one-loop order of ordinary perturbation theory.

The purpose of this paper is to clarify some confusion in the literature of predictions based on Eq. (1.1) (Sec. II), present corrections to Eq. (1.1) calculated in the renormalizable SU_3 σ model (Sec. III), and make a comparison of these and other predictions with chiral perturbation theory[1,6] (Sec. IV). The goals of this work and chiral perturbation theory overlap. They both start from similar postulates but have quite different operating assumptions and approximations. For small symmetry breaking they should agree, but for physical values of parameters there is a fairly large disagreement which we discuss. Section II A outlines a derivation of Eq. (1.1).

II. ZEROTH-ORDER RELATION BETWEEN f_K/f_π AND PSEUDOSCALAR MASSES

A. Derivation

To derive Eq. (1.1) we start with the assumption that the Hamiltonian consists of a piece that is invariant under $SU_3\times SU_3$, denoted $\mathcal{H}_{\rm sym}$, and a piece that breaks this symmetry in a simple way:

$$\mathcal{H}=\mathcal{H}_{\rm sym}+\epsilon_0 u_0+\epsilon_8 u_8. \tag{2.1}$$

The nonet of scalar densities, u_i, is taken to be a member of the $(3,\bar{3})+(\bar{3},3)$ representation of $SU_3\times SU_3$ along with a nonet of pseudoscalar densities v_i. There are certain operator relations and the Ward identities that follow from these assumptions which are in no sense new. We do not wish to give a derivation but rather will state the needed results and guide the reader through a published derivation, Ref. 5, Sec. III. We first weaken our assumption, Eq. (2.1), to identify u_i and v_i as the nonets of scalar and pseudoscalar fields σ_i and ϕ_i, and we employ the Lagrangian machinery

$$\mathcal{L}=\mathcal{L}_{\rm sym}+\mathcal{L}_{\rm sb},$$
$$\mathcal{L}_{\rm sb}=-\epsilon_0\sigma_0-\epsilon_8\sigma_8. \tag{2.1'}$$

If one sticks to the u's and v's as unspecified densities it is necessary at some point along the way to assume that matrix elements of the v's are dominated by the quantities one would get if they were in fact pseudoscalar fields. Under these weaker assumptions, Eq. (2.1'), the following relations are exact consequences:

$$f_{ia}m_a^2=-\sum_{j,k}d_{ijk}\epsilon_j Z_{ka}^{1/2}, \tag{2.2a}$$

$$f_{ia}m_a^2=-\sum_{j,k,b}d_{ijk}Z_{jb}^{-1/2}\langle\sigma_k\rangle D_{ab}^{R-1}(0), \tag{2.2b}$$

where

$a=\pi,K,\eta,\eta'$, $i=1,\ldots,8$,

$j,k=0,\ldots,8$, d_{ijk} are standard.[7]

The f_{ia} are decay constants defined by

$$\partial_\mu A_i^\mu=\sum_a f_{ia}m_a^2\phi_a^R, \tag{2.3}$$

where A_i^μ are the axial currents—normalized as in Ref. 5—and ϕ_a^R are the renormalized pseudoscalar fields. The relation between renormalized and unrenormalized propagators is

$$D_{ab}^R=\sum_{j,k}Z_{ja}^{-1/2}D_{jk}Z_{kb}^{-1/2}. \tag{2.4}$$

The indices a,j,k run over the same values as noted above, with b the same as a. The $Z_{ja}^{-1/2}$ are defined such that D_{ab}^R has poles of unit residue in the 11 component at the η mass and the 22 component at the η' mass, and no pole in the off-diagonal elements. The symbol $Z_{ja}^{1/2}$ is defined by $\sum_a Z_{ia}^{1/2}Z_{ja}^{-1/2}=\delta_{ij}$.[8] The vacuum expectation values of the scalar fields, $\langle\sigma_k\rangle$, are nonzero only for $k=0,8$.

Equation (2.2a) follows from an operator relation. To see this, multiply Eq. (2.2a) by ϕ_a^R and sum over a. Invoking the definition of f_{ia} via Eq. (2.3) we get

$$\partial_\mu A_i^\mu=-\sum_{j,k}d_{ijk}\epsilon_j\{Z_{ka}^{1/2}\phi_a^R\}. \tag{2.5}$$

The quantity in curly brackets is the unrenormalized ϕ field and this equation is just the operator relation [Ref. 5, Eq. (3.4)]

$$\partial_\mu A_i^\mu=i[Q_i^A,\mathcal{L}_{\rm SB}].$$

Equation (2.2b) is a Ward identity that follows from taking a zero-momentum limit of $\partial_\mu\langle T(A_i^\mu(x),\phi_a^R(0))\rangle$ as in Ref. 5, Eq. (3.12). In the tree approximation, Eq. (2.2b) gives the well-known relations between decay constants and $\langle\sigma_0\rangle$ and $\langle\sigma_8\rangle$.

In order to get predictions from Eq. (2.2) assumptions must be made, and arguments ensue. We take the following point of view: *Let us make the weakest assumption that will give us the relation between f_K/f_π and the pseudoscalar masses which one would obtain in tree solutions of the most general SU_3 (nonrenormalizable) σ model, Eq. (1.1), and give no a priori justification for it.* The assumptions are simple and not dependent on detailed dynamics. The justification lies in the comparison with experiment together with a demonstration that corrections to this relation are indeed small when calculated in the most general version of the renormalizable SU_3 σ model.

Our approximations are

$$Z_{ja}^{1/2}=Z^{1/2}R_{ja}, \tag{2.6a}$$
$$Z_{ja}^{-1/2}=Z^{-1/2}R_{ja}, \tag{2.6b}$$
$$D_{ab}^{R-1}(0)=-\lambda\delta_{ab}m_b^2, \tag{2.6c}$$

where

$$R_{ja}=\begin{array}{c}\\8\\0\end{array}\begin{array}{c}\begin{array}{cc}\eta & \eta'\end{array}\\\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\end{array}$$
$$=\delta_{ja},\quad j=a=1,\ldots,7.$$

Equation (2.6a) states that we allow only one mixing angle—a mass mixing—and the normalizations of all the pseudoscalars are the same—but need not be equal to unity. Equation (2.6b) clarifies our notation.[8]

Using these approximations—Eq. (2.6)—Eqs. (2.2) can be written

$$\frac{f_K m_K^2}{f_\pi m_\pi^2}=\frac{1-a/2}{1+a},$$
$$\frac{f_{8\eta}m_\eta^2}{f_\pi m_\pi^2}=\frac{(1-a)\cos\theta-\sqrt{2}\,a\sin\theta}{1+a}, \tag{2.7a}$$
$$\frac{f_{8\eta'}m_{\eta'}^2}{f_\pi m_\pi^2}=\frac{(1-a)\sin\theta+\sqrt{2}a\cos\theta}{1+a},$$

$$\frac{f_K}{f_\pi}=\frac{1-b/2}{1+b},$$
$$\frac{f_{8\eta}}{f_\pi}=\frac{(1-b)\cos\theta-\sqrt{2}b\sin\theta}{1+b}, \tag{2.7b}$$
$$\frac{f_{8\eta'}}{f_\pi}=\frac{(1-b)\sin\theta+\sqrt{2}b\cos\theta}{1+b},$$

where $a=\epsilon_8/\sqrt{2}\epsilon_0$, $b=\langle\sigma_8\rangle/\sqrt{2}\langle\sigma_0\rangle$. Eliminating $f_{8\eta}$, $f_{8\eta'}$, a, b, and θ gives the desired relations—Eq. (1.1).

B. Discussion

Let us suppose we wish to use Eq. (1.1) to predict f_K/f_π. It turns out that f_K/f_π is extremely sensitive to the values of masses. We wish to plot this in a way that shows this clearly and hence we pick as the variables f_π/f_K and m_K^2. Equation (1.1) is a conic section in these variables (hence the choice of f_π/f_K rather than the inverse). For physical values of m_π^2, m_η^2, and $m_{\eta'}^2$ the conic section is an ellipse as shown in Fig. 1, curve I. The first thing to note is that a vertical tangent to this curve lies between the charged K and neutral K masses. Since we neglect isospin splitting, the central K mass has an uncertainty of the order of the isospin splitting. For horizontal shifts of this curve of the order of 1 or 2% (for whatever reason) f_K/f_π can easily vary from 1.25 to 2.5. However, given the physical f_K/f_π, one gets a respectable K mass.

The point with error bars in Fig. 1 gives the physical parameters. The "error" bars on the K mass value are simply from the isospin splitting. We take

$$\frac{f_K}{f_\pi f_+(0)} = 1.27 \pm 0.03$$

from Chounet *et al.*[9] We take $f_+(0) = 0.98 \pm 0.01$. According to the Ademollo-Gatto theorem[10] this should be close to unity. This value is supported by two calculations.[5,11] Thus $f_K/f_\pi = 1.25 \pm 0.04$.

We concur with Cicogna *et al.*[3] that m_K^2 and f_K/f_π

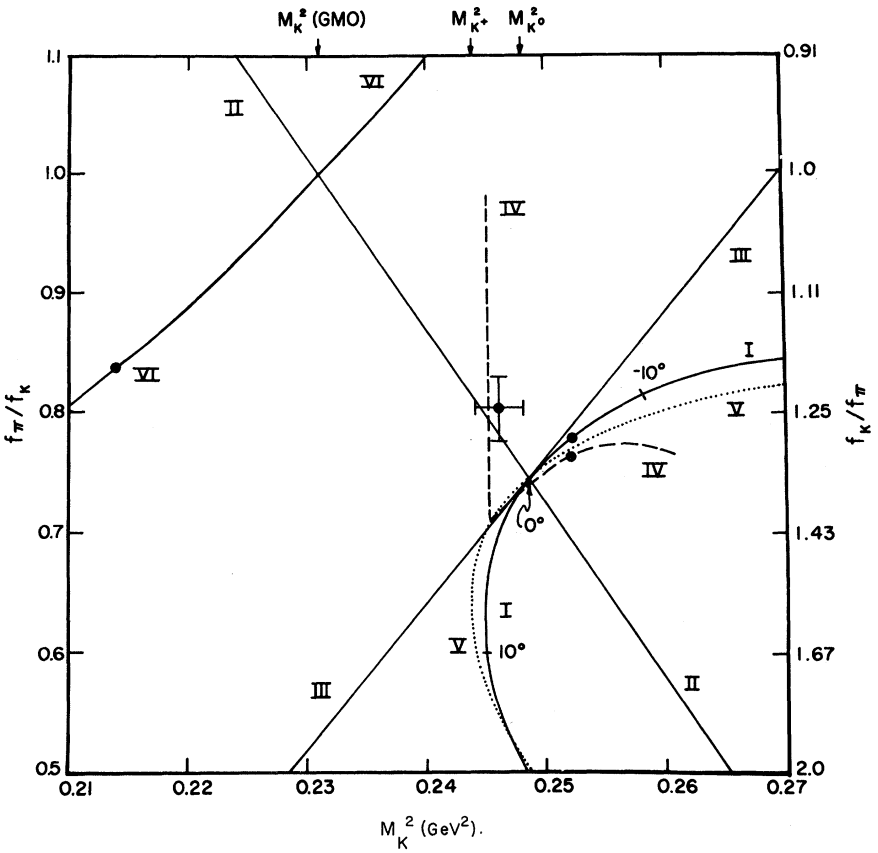

FIG. 1. Plot of f_π/f_K as a function of m_K^2 in units of GeV^2 for various schemes. Note the expanded scale for m_K^2. The charged and neutral K masses are indicated at the top. m_K^2 (GMO) is the value for octet breaking assuming a pure octet η. The error bars on the experimental point are discussed in the text. The various curves are as follows: I is the conic section, Eq. (1.1); II is a zero-mixing prediction, Eq. (2.9a); III is obtained from curve II by linearizing in the SU_3 breaking; IV is tree+loop corrections in the SU_3 σ model for the case in which we renormalize at m_K^2; V is the same as IV except that we renormalized at f_K in lieu of m_K^2; curve IV is the prediction from "nonanalytic" terms in chiral perturbation theory. The numerical values of the fixed masses were taken as $m_\pi = 138.1$ MeV, $m_\eta = 548.8$ MeV, $m_{\eta'} = 958.1$ MeV.

are *consistent* with Eq. (1.1) to a few percent. Kenney[4] rejects Eq. (1.1) on the basis of a large discrepancy with experiment which can be traced to his determination of f_K/f_π from the K mass. In a paper of ours—Ref. 12—we recognized this sensitivity but did not recognize that the result was independent of the detailed form of $\mathcal{L}_{\text{sym}}$. We still got a bad value of f_K/f_π (≈ 1.45), but that can be traced to our inadequate treatment of scalar mesons. This is discussed in a later paper, Ref. 5, where the problem is resolved; the values taken are marked with a dot on curve I in Fig. 1. Other papers on related work include those of Schechter and Ueda, and Geddes and Graham[13] and references therein.

The mixing angle can be found from Eq. (2.7)

$$\tan\theta = -\frac{1}{2\sqrt{2}}\frac{f_\pi/f_K - 4A}{f_\pi/f_K - A} \tag{2.8a}$$

$$= 2\sqrt{2}\,\frac{f_\pi/f_K - A'}{f_\pi/f_K - 4A'}\,. \tag{2.8b}$$

The equality of these two forms follows from Eq. (1.1), i.e., it is understood that Eq. (1.1) must be satisfied by the values inserted in Eq. (2.8). This angle is marked on curve I, Fig. 1. Eliminating f_π/f_K gives a formula for $\tan\theta$ that differs from the standard form based on pure octet mass splitting since higher powers of ϵ_8 generate 27 mass splittings. For $\theta = 0$, Eq. (2.8) reduces to

$$\frac{f_K}{f_\pi} = \frac{1}{4A} = \frac{1}{4}\,\frac{m_\eta^2 - m_\pi^2}{m_\eta^2 - m_K^2}\,, \tag{2.9a}$$

$$\frac{f_K}{f_\pi} = \frac{1}{A'} = \frac{m_{\eta'}^2 - m_\pi^2}{m_{\eta'}^2 - m_K^2}\,. \tag{2.9b}$$

This corresponds to the point marked $\theta = 0$ on curve I, Fig. 1. If we evaluate Eqs. (2.9a) and (2.9b) with physical masses, f_K/f_π is overdetermined—Eq. (2.9a)—giving a good value: 1.28, Eq. (2.9b); a fair value: 1.34. Alternatively Kenney[4] uses Eq. (2.9b) to calculate $m_{\eta'}$ giving 1030 MeV compared to physical 960.

Auvil and Deshpande[14] arrived at Eq. (2.9a) from a different and interesting route. They consider spectral function sum rules and arrive at essentially the same equations as our Eq. (2.2). (They also have corresponding equations for the κ, but those separate completely from the determination of f_K/f_π in their final analysis and we can ignore them. When we go to second order in the σ model, the κ becomes a wide resonance and Z_κ is ill defined.) The new aspect in their paper is a derivation of relations among the Z's from consideration of the Wilson operator-product expansion.[15] Their result replaces our Eq. (2.6) with less restrictive conditions that allow for symmetry breaking in the Z's. However, in fitting their results to experiment they are led back to zero mixing and $Z_\pi = Z = Z_\eta \neq Z_{\eta'}$, which is now only slightly weaker than our assumptions, Eq. (2.6), when mixing is zero. They then obtain Eq. (2.9a) and a modified Eq. (2.9b) with the replacement of $m_{\eta'}^2 \to m_{\eta'}^2(Z_\pi/Z_{\eta'})$.[16] Hence they do not overdetermine f_K/f_π in terms of experimentally known quantities but rather obtain a value for $Z_{\eta'}$. We plotted their prediction of f_π/f_K, Eq. (2.9a), in Fig. 1, curve II. We note that their prediction for f_K/f_π, Eq. (2.6a), is also sensitive to the K mass; taking charged and neutral masses gives

$$f_K/f_\pi = 1.28 \pm 0.04,$$

and so these "errors" are of the order of the experimental error in f_K/f_π. In the next section we will compare their operator-product prediction of the Z's with our loop calculation.

It would be misleading to consider the close fit of curve I to experiment as a strong support of the operator-product expansion. Auvil and Deshpande[14] showed the f_K/f_π is determined from a set of 11 coupled equations in 11 unknown which would determine a curve on our plot, Fig. 1, and it is that curve that would provide a meaningful comparison. Through a numerical calculation they conclude that their mixing angles were small so they set them equal to zero. This had the effect of decoupling the equations rather than overconstraining them (as would be true in general). It is not clear that their 11 coupled equations would produce a curve close to curve II in Fig. 1.

Returning now to our discussion of Eq. (1.1), since SU_3 is a good symmetry, we should check to see that if we linearize Eq. (1.1) in SU_3 breaking, the prediction will not change drastically. To linearize, we note that the quantities $A - \frac{1}{4}$ and $A' - 1$ are of the order ϵ_8:

$$A - \tfrac{1}{4} = \tfrac{1}{4}\,\frac{3m_\eta^2 + m_\pi^2 - 4m^2}{m_\eta^2 - m_\pi^2}\,, \tag{2.10a}$$

$$A' - 1 = \frac{m_\pi^2 - m_K^2}{m_{\eta'}^2 - m_\pi^2}\,. \tag{2.10b}$$

Expanding $f_\pi/f_K - 1$ in $(A - \frac{1}{4})$ and $(A' - 1)$ gives the leading correction to the Gell-Mann–Okubo formula for finite $m_{\eta'}$ and $f_K/f_\pi \neq 1$

$$\tfrac{1}{4}(3m_\eta^2 + m_\pi^2 - 4m_K^2) = -\frac{1}{2}\,\frac{(m_K^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)}{m_{\eta'}^2 - m_\pi^2} + \frac{1}{4}\left(1 - \frac{f_\pi}{f_K}\right)(m_\eta^2 - m_\pi^2). \tag{2.11}$$

This is linear in the variables f_π/f_K, and m_K^2 and is the straight line plotted in Fig. 1, denoted curve

III. This line is tangent to the ellipse at the point $\theta = 0$. In the neighborhood of the physical point the curves are indeed close.

III. LOOP CALCULATION

Up to now we have described model-independent results, i.e., those that do not depend on the details of the dynamical model. For example, Eq. (1.1) will hold, given the approximations Eq. (2.6), in any SU_3 σ model, independent of the specific form of $\mathcal{L}_{\text{sym}}$. In addition, we have noted that Eq. (1.1) is in fact the tree approximation which is the first term in the loop expansion of perturbation theory. We have calculated the one loop term in Ref. 5. This is possible to do only for renormalizable forms of $\mathcal{L}_{\text{sym}}$. (Of course one can use an unrenormalizable theory if one is content to introduce new parameters at each order—which we are not.) In this section we restrict $\mathcal{L}_{\text{sym}}$ to contain at most the fourth power of fields and in this sense the loop calculation is model-dependent.

The loop calculation in Ref. 5 is lengthy and involved. We attempted a global fit and determined parameters accordingly. We wish to draw on that calculation to present the loop correction to f_π/f_K as a function of m_K^2. The essential features are spelled out in Secs. II and V A of Ref. 5. For the most part this is a straightforward application of Feynman rules. However, there are conventions that must be made in determining the finite part of counterterms and we will restate them here.

The Lagrangian contains six parameters. In second order we can introduce six counterterms corresponding to these parameters (four of which are divergent). We chose the counterterms such that m_π, m_K, m_η, $m_{\eta'}$, and f_π contain no second-order corrections. There is a sixth counterterm that must be fixed to fully define the loop calculation. However, all the results in this paper are independent of its choice and since there are a number of technical problems associated with its determination we will skip the discussion here.[17]

In Fig. 1 we have marked with a dot on curves I (tree), and IV (tree + loop) the parameters reported in Ref. 5 determined from a global fit. The input parameters are summarized in Ref. 5, Table II. f_π enters simply as a factor $1/f_\pi^2$ multiplying the loop part. The dots lie at the same m_K^2 because m_K^2 was chosen to have no loop correction. Moving away from the dot requires one of two procedures: Either attempt a global fit at each value of m_K^2, or place a reasonable condition on one free parameter. The former requires prohibitive labor of little value. The free parameter is m_σ (tree)—the tree value of the σ mass. If we hold it fixed $m_{\sigma'}$ (tree)—the upper σ mass—can go to ∞ as we vary m_K^2 causing divergences in the perturbation expansion. Hence we hold $m_{\sigma'}$ (tree) $= 1.25$ GeV fixed, and let m_σ (tree) vary. This gives curve IV. Note that it follows the tree curve closely and then abruptly approaches a vertical asymptote. The beautiful agreement with experiment is fortuitous and due to a pathology in our renormalization procedure. The vertical asymptote arises because perturbation theory is diverging at the turning point of curve I since it is unable to produce the double-valued nature of the function there.

To verify that the double-valued nature is the culprit we changed the renormalization procedure by dropping m_K as a constant parameter (no loop corrections) and instead replaced it by f_K. In this sense curve I is single-valued in the region of interest. Since f_π has no loop correction, f_K/f_π will have none and the corrections will be along horizontal lines. This procedure is less pathological in this neighborhood and gives a smaller correction. That these two procedures cross the tree curve at the same point is necessary; that they cross at $\theta = 0$ is accidental. Our conclusion from all this is that if one eliminates known pathologies from the loop calculation then the corrections to curve I are small but not necessarily better than the tree predictions.

Finally we report the value of wave-function renormalization constants to compare with the Auvil and Deshpande[14] result. The following two relations[18] were derived by them as approximate relations from the Wilson operator-product expansion:

$$T_{jk} \equiv R_{ja} Z_{a\alpha}{}^{1/2} Z_{b\beta}{}^{1/2} R_{k\beta}, \tag{3.1}$$

$$4(Z_K/Z_\pi - 1) = 3(T_{88}/Z_\pi - 1) \qquad (0.048) = (0.054), \tag{3.2a}$$

$$Z_K/Z_\pi - 1 = \frac{3}{\sqrt{2}} T_{80}/Z_\pi \qquad (0.012) = (-0.049). \tag{3.2b}$$

The values calculated from our Ref. 5, Table IV, are shown below in parentheses. Equation (3.2a) is the Gell-Mann–Okubo octet-breaking relation for the Z's and is well satisfied. We see this agreement as accidental in our calculation since small changes in the parameters destroy this agreement. Equation (3.2b) depends in detail on the mixing and is not satisfied. These equations, Eqs. (3.2a) and (3.2b), are derived by pole-dominating spectral-function sum rules.[14,18] The sum rules themselves are exactly satisfied in our model since we specify that the $(3, \bar{3}) + (\bar{3}, 3)$ objects are in fact canonical fields. Hence, Eqs. (3.2a)

and (3.2b) are tests of pole dominance of exact sum rules in our model.

We have emphasized above that the loop calculation of the relation between f_K/f_π and the pseudoscalar masses depends on two new parameters compared to the tree calculation. One is f_π which is known experimentally (95 MeV) and enters only as a factor $1/f_\pi^2$ multiplying the loop. The second parameter is m_σ^2 or $m_{\sigma'}^2$ and is a free parameter in the sense that there is no unique way to fix it in the absence of a global fit.[5] This is a reflection of the model dependence of the loop calculation. That m_σ or $m_{\sigma'}$ reflects model dependence is well known in the tree-order determination of scattering amplitudes. For example, in $\pi\pi$ scattering the tree graphs in the SU_2 σ model depend on m_σ. If one takes m_σ to ∞, the amplitudes are finite and approach the model-independent Weinberg[19] amplitude. In a future paper we will discuss the significance of taking a large $m_{\sigma'}$ in an attempt to get model-independent results. For the moment though we wish only to present the results of taking $m_{\sigma'}$ large. This is shown in Fig. 2. The shaded area is bounded by two curves. One, marked $m_{\sigma'}^2=1.57$, is curve V in Fig. 1. The other is obtained by increasing $m_{\sigma'}^2$ to $m_{\sigma'}^2=8$. This shows the sensitivity to $m_{\sigma'}$ over a fairly large range.

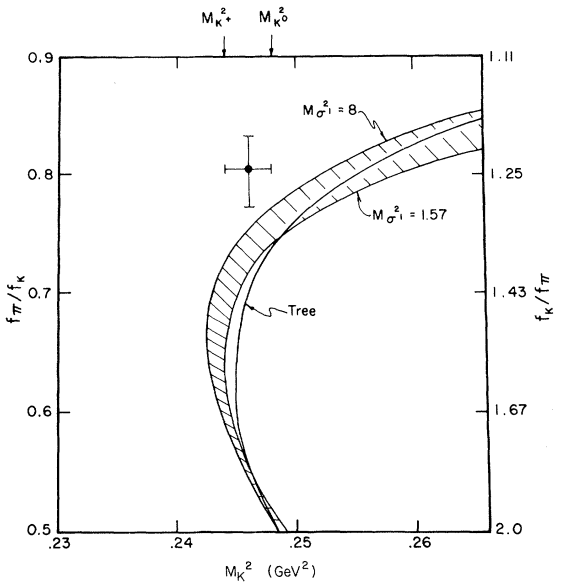

FIG. 2. Plot of f_π/f_K as a function of m_K^2 in units of GeV^2. The curves marked tree and $m_{\sigma'}^2=1.57$ are the same as curves I and V in Fig. 1. The curves marked $m_{\sigma'}^2=8$ is obtained by simply increasing $m_{\sigma'}^2$. The shaded area represents the range of predictions for this range in $m_{\sigma'}^2$.

IV. COMPARISON WITH CHIRAL PERTURBATION THEORY

We would now like to compare these loop corrected results with the results of chiral perturbation theory which is based on the evaluation of "nonanalytic" terms in expansions about Goldstone realized symmetries. A good review of this work is given by Pagels.[20] There are a number of similarities and differences between this approach and ours which are worth pointing out.

Li and Pagels[21] pointed out that the approach to symmetry limits in general is not analytic in the symmetry-breaking parameters but gives terms such as $\epsilon\ln\epsilon$ and $\epsilon^{1/2}$. This nonanalyticity can be traced to the vanishing of the Goldstone boson masses as the symmetry limit is approached. For quantities in which the leading term is nonanalytic it is possible to calculate the coefficient of such a term.

Let us first look at the prediction of f_K/f_π, Ref. 1, Eq. (7.11),

$$\frac{f_K}{f_\pi}-1=\frac{3(m_K^2-m_\pi^2)}{64\pi^2 f_\pi^2}\ln\frac{\Lambda^2}{\mu^2}. \tag{4.1}$$

Λ is an arbitrary cutoff, μ^2 is the mean pseudoscalar mass$^2\approx 0.17$ GeV^2. For Λ equal to the nucleon mass, this gives a reasonable value $f_K/f_\pi-1=0.2$. What about our calculation? Our value of $(f_K/f_\pi-1)$ comes from two sources. First there is a large tree contribution, 0.29, plus a small loop correction, 0.02 to f_K/f_π. We know that the tree solution does *not* have a logarithmically nonanalytic piece. We also know that the loop contribution does. Very close to the $SU_3\times SU_3$ limit our loop part should agree with Eq. (4.1) since the mechanism for generating the nonanalytic behavior is present in the one-loop graphs. In addition the cutoff will be determined in a chiral-invariant way. Now, does all this mean that we are in violent disagreement with chiral perturbation theory? Should we compare our loop piece 0.02 with their "nonanalytic" piece 0.2? The answer to both questions is no; these are the wrong things to compare. The point is that the loop contribution contains analytic pieces as well as nonanalytic pieces and we are only reporting the sum. Since our result is numerical it is hard to separate them and we have not attempted to separate them analytically.

We can make a more meaningful comparison if we bring in another prediction of chiral perturbation theory, i.e., the Gell-Mann–Okubo mass difference, Ref. 1, Eq. (7.10),

$$\tfrac{3}{4}m_\eta^2+\tfrac{1}{4}m_\pi^2-m_K^2=\frac{(m_K^2-m_\pi^2)^2}{48\pi^2 f_\pi^2}\ln\frac{\Lambda^2}{\mu^2}. \tag{4.2}$$

This formula gives the right magnitude of the

mass splitting if Λ/μ^2 takes the same value as above but gives the wrong sign as noted in Ref. 1. If we combine the two predictions, Eqs. (4.1) and (4.2), we get the point marked on curve IV in Fig. 1. Curve IV itself is obtained by eliminating the undetermined cutoff from Eqs. (4.1) and (4.2) giving

$$\tfrac{3}{4}m_\eta^2+\tfrac{1}{4}m_\pi^2-m_K^2=\frac{4}{9}\left(\frac{f_K}{f_\pi}-1\right)(m_K^2-m_\pi^2). \quad (4.3)$$

It is this equation we should compare with the results of the σ model. The discrepancy between curve V and curves III or IV represents the effect of contributions to these quantities from terms other than the leading nonanalytic pieces, albeit a model-dependent determination. It is gratifying that this gives an improved fit.

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[2a]*Note added in proof.* In the limit $f_K/f_\pi=1$, this equation reduces to Schwinger's mass formula [J. Schwinger, Phys. Rev. 135, B816 (1964)]. This formula was recently rederived by S. Caser and M. Testa [Caltech Report No. CALT-68-536 (unpublished)].

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[8]This is admittedly perverse if one regards $Z^{1/2}$ and $Z^{-1/2}$ as matrices but welcome when one tries to remember which index has which range. It also agrees with Ref. 5.

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[16]There is a typographical error which gives $2\sqrt{2}$ in this replacement which is inconsistent with their numerical values.

[17]This counter term is denoted Z_{f_1} in Ref. 5, Sec. V.A.

[18]We have translated their mixing angles into our notation. Eq. (3.2a), (3.2b) correspond to their first and second Eq. (6).

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[20]See Ref. 1, Sec. VI, VIII.

[21]L-F. Li and H. Pagels, Phys. Rev. Lett. 27, 1089 (1971); Phys. Rev. D 5, 1509 (1972).