Two-photon decay of the pseudoscalar mesons*

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The two-photon decays of the pseudoscalar mesons are studied in terms of the fundamental quark-antiquark annihilation amplitude. The similarity of the pseudoscalars to parapositronium is stressed. In particular, it is shown that the atomic physics result for the decay width of parapositronium, modified in certain cases to accommodate strong-binding effects, is applicable to these hadrons and leads to values for their two-photon widths which are in accord with experimental results.

Among all of the successes of the quark model,[1] perhaps its most outstanding dynamical (as opposed to spectroscopic) predictions are those concerning the electromagnetic properties of the hadrons. Correctly predicted by this simple model are, among other things, the magnetic moments of the baryon octet,[2] the radiative widths for processes $H^* \to H\gamma$,[3] and the leptonic decay rates of the neutral vector mesons.[4] Absent from this list of triumphs of the quark model are the two-photon decays of the pseudoscalar mesons, which have been studied[4] only in a hybrid scheme of the quark model and the Gell-Mann–Sharp–Wagner model.[5] The main new result of this article will be to show that the two-photon decays may be understood in terms of the fundamental quark-antiquark annihilation amplitude of Fig. 1.

Our approach is very similar to the treatment of parapositronium.[6] One can easily show by elementary means that the decay rate of parapositronium (P_0) into two photons is

$$\Gamma(P_0 \to \gamma\gamma) = \frac{16\pi\alpha^2}{M_{P_0}^{\;2}} |\psi_{P_0}(0)|^2, \tag{1}$$

where $\psi_{P_0}(r)$ is the wave function of the relative coordinate of the electron-positron system. From the Schrödinger equation we know that $\psi(0) = (\pi a^3)^{-1/2}$ where $a^{-1} = \frac{1}{2}m\alpha$; it follows that $\Gamma(P_0 \to \gamma\gamma) = 0.8 \times 10^{10}\ \mathrm{sec}^{-1}$ as is observed. Two essential assumptions, both of which are undoubtedly justified for positronium, are required in the derivation of this result: (1) A nonrelativistic representation of the bound state is applicable, and (2) the binding energy and kinetic energy are sufficiently small that they may be neglected. The second of these assumptions calls for elaboration: The binding must be weak so its effects on the free-particle annihilation amplitude will be small. In particular, only if the binding energy is small will the free-particle photon energy $m_e c^2$ be approximately equal to $\frac{1}{2}M_{P_0}c^2$.

Of the known two-photon decays $\pi^0 \to \gamma\gamma$, $\eta \to \gamma\gamma$, and $\eta' \to \gamma\gamma$, the decay $\eta \to \gamma\gamma$ comes closest to satisfying the analogy with parapositronium. In particular, $\frac{1}{2}M_\eta$ is sufficiently close to m_d (see Refs. 1 and 7) that we may expect

$$\Gamma(\eta \to \gamma\gamma) = \frac{16\pi\alpha^2}{M_\eta^{\;2}} |\psi_\eta(0)|^2 \left| \sum_{i=u,d,s} a_i \left(\frac{e_i}{e}\right)^2 \right|^2, \tag{2}$$

where the new term simply reflects the fact that the quarks have charges different from the electron (a_i is the amplitude for the η to be a quark-antiquark pair of type i). Of course in this case we have no Schrödinger equation for $\psi(0)$, but a phenomenological analysis of meson wave functions based on the processes $\pi \to l\nu$, $K \to l\nu$, $\rho^0 \to l^+l^-$, $\omega \to l^+l^-$, $\phi \to l^+l^-$, and $\psi \to l^+l^-$, is sufficient to convincingly determine $|\psi(0)|$ to within about 15% (see the Appendix). In this way one finds that

$$\Gamma(\eta \to \gamma\gamma) = (0.6 \pm 0.2)(\cos\theta - 2\sqrt{2}\,\sin\theta)^2\ \mathrm{keV}, \tag{3}$$

where θ is the η-η' mixing angle, and the quoted theoretical error reflects the estimated uncertainty in $|\psi(0)|$. For $\theta = -10°$, the quadratic mixing angle, one finds

$$\Gamma(\eta \to \gamma\gamma) = 1.4 \pm 0.5\ \mathrm{keV}. \tag{4}$$

Although we shall consider refinements of this prediction in what follows, the simple result (2) is already in good agreement with the tabulated width

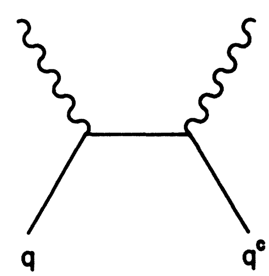

FIG. 1. The quark-antiquark annihilation amplitude.

of 1.0 ± 0.2 keV.[8]

One's immediate inclination is to apply this same method to $\pi^0\to\gamma\gamma$ and $\eta'\to\gamma\gamma$. It is easy to see, however, that it is hopeless to use the positronium approximation for the π^0 decay: The photons being created in this process have energies of less than 70 MeV while the photons from quark-antiquark annihilation at rest have 340 MeV.[1] For $\eta'\to\gamma\gamma$ the situation is not as bad, although somewhat worse than for $\eta\to\gamma\gamma$; the real photons have energies of 480 MeV, which represents a mismatch by a factor of 1.4 instead of a factor of 5. The η' would therefore seem to be a reasonably favorable case and we can tentatively predict, subject to the refinements to be considered shortly, that $\Gamma(\eta'\to\gamma\gamma)\sim 5[\cos\theta+(1/2\sqrt{2})\sin\theta]^2$ keV, which gives a two-photon width of ~ 4 keV for $\theta=-10°$. Experimentally, $\Gamma(\eta'\to\gamma\gamma)<15$ keV at the 95% confidence level.

It is amusing to note that if the $\psi(3105)$ should prove to be a $J^{PC}=1^{--}$ bound state of a charm-anticharm quark pair, then a 0^{-+} bound state, which we can call η_c, must be nearby. If we make the natural assumption that the charm-anticharm binding is similar to the binding of other quark-antiquark systems, then we can expect m_c, the charmed-quark effective mass, to be around 1.5 GeV. The η_c would therefore be an even better candidate for the parapositronium analogy than the η and one predicts that if $M_{\eta_c}\simeq 2.8$ GeV,

$$\Gamma(\eta_c\to\gamma\gamma)=17\pm 5\ \text{keV}. \tag{5}$$

If the η_c has a width comparable to the $\psi(3105)$,[9] a significant fraction of its decays will be via the two-photon channel.

There are two good reasons for turning back at this point to consider the decay $\pi^0\to\gamma\gamma$. One is that although we can appreciate why it is a difficult process with which to contend, it is nevertheless interesting. The second is that if we can make progress in understanding the π^0, we can simultaneously get some idea of the reliability of the predictions make above for η, η', and η_c. We propose, in this vein, to explore the idea that the major flaw in the positronium analogy for the π^0 is the mismatch between the real photon energies and the energies of the photons from quark-antiquark annihilation.

The S-matrix element for the process $0^-(K)\to\gamma(q_1\lambda_1)\gamma(q_2\lambda_2)$ may be cast into the form

$$S(0^-(K)\to\gamma(q_1\lambda_1)\gamma(q_2\lambda_2))=(2\pi)^4\delta^4(q_1+q_2-K)\times\frac{e_\mu(q_1\lambda_1)e_\nu(q_2\lambda_2)}{(2\pi)^3}T^{\mu\nu}, \tag{6}$$

with

$$T^{\mu\nu}=\int d^4z\, e^{i(q_1-q_2)(z/2)}\left\langle 0\left|T\left[j^\mu\left(\frac{z}{2}\right)j^\nu\left(-\frac{z}{2}\right)\right]\right|0^-(K)\right\rangle, \tag{7}$$

and where j^μ is the electromagnetic current. The factor $\delta^4(q_1+q_2-K)$ conserves energy and momentum and ensures that $\omega_1=\omega_2=\frac{1}{2}M_{0^-}$ in the rest frame. *The amplitude (7) is, however, defined even when $q_1+q_2\neq K$.* We shall assume that in the strong-binding realm the relevant amplitude is the one for a quark-antiquark pair with the wave function of the bound state, but with the binding turned off, to annihilate into two photons of energy $\frac{1}{2}M_{0^-}$. This is clearly the simplest and most obviously necessary correction we can make to the naive positronium analogy. One then finds that

$$\Gamma(0^-\to\gamma\gamma)=\frac{16\pi\alpha^2}{M_{0^-}^{2}}|\psi_{0^-}(0)|^2\left|\sum_{i=u,d,s,c}a_i\left(\frac{e_i}{e}\right)^2\left\langle\frac{M_{0^-}}{4p}\ln\left[\frac{m_i^{\,2}+(p+\frac{1}{2}M_{0^-})^2}{m_i^{\,2}+(p-\frac{1}{2}M_{0^-})^2}\right]\right\rangle_{\rm wf}\right|^2, \tag{8}$$

where

$$\langle f(p)\rangle_{\rm wf}\equiv\frac{\int d^3p\,\phi(p)f(p)}{\int d^3p\,\phi(p)}, \tag{9}$$

with $\phi(p)$ the momentum wave function of the quarks. So long as $\phi(p)$ is a rapidly decreasing function of p we need not assume a particular form for it; for our purposes it will be sufficient to make the approximation

$$\langle f(p)\rangle_{\rm wf}\approx f(\langle p^2\rangle_{\rm wf}^{1/2}). \tag{10}$$

If we take the hadrons to have a spatial extension of the order of 0.8 F as indicated by the elastic form factors, then we may expect $\langle p^2\rangle_{\rm wf}^{1/2}\approx 250\pm 100$ MeV/c.

As a consequence of these considerations, the two-photon width of the π^0, which would have been off by three orders of magnitude had we used an unmodified positronium-like formula, becomes

$$\Gamma(\pi^0\to\gamma\gamma)=13\pm 7\ \text{eV}. \tag{11}$$

This result is in good agreement with the experimental value of 7.8 ± 0.9 eV.[10] This drastic improvement in the π^0 rate leads us to reevaluate the η, η', and η_c rates with the results (for $\theta=-10°$)

$$\Gamma(\eta\to\gamma\gamma)=0.7\pm 0.4 \text{ keV}, \tag{12}$$

$$\Gamma(\eta'\to\gamma\gamma)=6\pm 2 \text{ keV}, \tag{13}$$

$$\Gamma(\eta_c\to\gamma\gamma)=15\pm 5 \text{ keV}. \tag{14}$$

The estimated uncertainty in the π^0 rate is almost entirely due to its sensitivity to $\langle p^2\rangle_{\rm wf}^{1/2}$; the uncertainties in the η and η' rates arise in about equal measure from $|\psi(0)|$ and $\langle p^2\rangle_{\rm wf}^{1/2}$, while the uncertainty for η_c is due as before to both M_{η_c} and $|\psi(0)|$. The fact that this procedure effects such a miraculous solution to the π^0 problem while leaving the other decay rates substantially unaffected [the results (12), (13), and (14) are all consistent, within the theoretical uncertainties, with their naive "positronium" values quoted earlier] is very encouraging, and probably indicates that our approach has merit. From any point of view, however, the favorable case of $\eta\to\gamma\gamma$ is sufficiently close to its observed value that the preceding calculation must be considered another impressive confirmation of the atomic structure of the hadrons envisaged in the quark model.

APPENDIX

In the quark model, the processes $\pi\to l\nu$, $K\to l\nu$, $\rho^0\to l^+l^-$, $\omega\to l^+l^-$, $\phi\to l^+l^-$, and $\psi\to l^+l^-$ all depend on $|\psi(0)|$, the wave function of the quarks at zero separation.[4] It is a simple matter to show that if we define the dimensionless parameters f by

$$\langle 0|A^\mu_{1+i2}(0)|\pi^-(K)\rangle=\frac{if_\pi m_\pi K^\mu}{(2\pi)^{3/2}}, \tag{A1}$$

$$\langle 0|A_{4+i5}(0)|K^-(K)\rangle=\frac{if_K m_K K^\mu}{(2\pi)^{3/2}}, \tag{A2}$$

$$\langle 0|j^\mu|V(K\lambda)\rangle=\frac{\epsilon^\mu(K\lambda)}{(2\pi)^{3/2}}f_V m_V{}^2, \tag{A3}$$

where $V=\rho^0$, ω, ϕ, or ψ, the A^μ_i are the octet of axial-vector currents, and j^μ is the electromagnetic current, then the quark model indicates that[11]

$$|\psi_\pi(0)|=\tfrac{1}{2}f_\pi m_\pi{}^{3/2}, \tag{A4}$$

$$|\psi_K(0)|=\tfrac{1}{2}f_K m_K{}^{3/2}, \tag{A5}$$

$$|\psi_\rho(0)|=\frac{1}{\sqrt{2}}f_\rho m_\rho{}^{3/2}, \tag{A6}$$

$$|\psi_\omega(0)|=\frac{3}{\sqrt{2}}f_\omega m_\omega{}^{3/2}, \tag{A7}$$

$$|\psi_\phi(0)|=\tfrac{3}{2}f_\phi m_\phi{}^{3/2}, \tag{A8}$$

$$|\psi_\psi(0)|=\tfrac{3}{4}f_\psi m_\psi{}^{3/2}, \tag{A9}$$

if we interpret the ψ as a charm-anticharm bound state. Since the constants f may be extracted from the experimental decay widths, the wave functions $|\psi(0)|$ can be determined. The results are displayed in Fig. 2; a straight line has been supplied to guide the eye. It is clear that one can with some confidence estimate $|\psi_\eta(0)|=(0.06\pm 0.01)\ \text{GeV}^{3/2}$, $|\psi_{\eta'}(0)|=(0.11\pm 0.01)\ \text{GeV}^{3/2}$, and $|\psi_{\eta_c}(0)|=(0.33\pm 0.04)\ \text{GeV}^{3/2}$. These are the values used in the text.

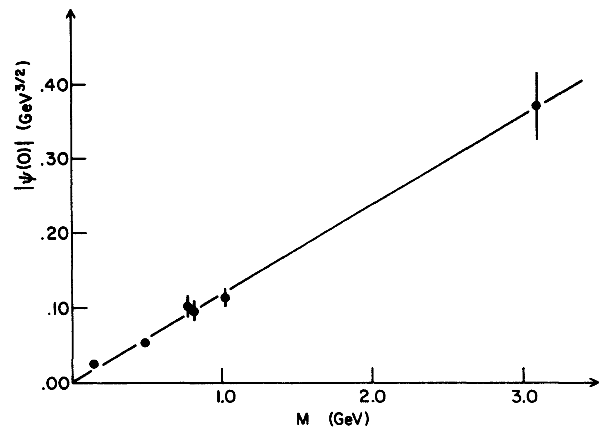

FIG. 2. The quark-antiquark wave function of mesons.

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[1]There is of course no one quark model at this stage. We mean by the term the fractionally-charged-constit-

Mann, Phys. Lett. 8, 214 (1964)] and Zweig [G. Zweig, CERN Reports Nos. TH401, 1964 (unpublished) and 412, 1964 (unpublished)], supplemented by the following "rules of the game": (1) The quarks are pointlike objects, (2) the quark motion inside the hadrons may be treated nonrelativistically, and (3) the doublet quarks u and d have an "effective" mass $m_d=340$ MeV, while the strange quark s has a mass $m_s\equiv m_d+\Delta$, where $\Delta\approx 140$ MeV.

[2]W. Thirring, Acta Phys. Austriaca, Suppl. II, 205 (1965). Note that the quark model described in Ref. 1 builds in some symmetry-breaking since $m_s\neq m_d$. Thus, for example, the symmetric quark-model prediction that $\mu_\Lambda=-\frac{1}{3}\mu_p=-0.93$ is modified to $\mu_\Lambda=-\frac{1}{3}(M_d/M_s)\mu_p\simeq -0.66$ in much better agreement with the experiment value of -0.67 ± 0.06.

[3]C. Becchi and G. Morpurgo, Phys. Rev. 140, 687 (1965).

[4]R. van Royen and V. F. Weisskopf, Nuovo Cimento 50, 617 (1967); 51, 583 (1967).

[5]M. Gell-Mann, D. H. Sharp, and W. Wagner, Phys. Rev. Lett. 8, 261 (1962).

[6]G. Morpurgo, Physics (N.Y.) 2, 95 (1965), mentioned the positronium analogy long ago. van Royen and Weisskopf (Ref. 4) called the π^0 meson "quarkonium."

[7]Since $e_u{}^2 = 4e_d{}^2 = 4e_s{}^2$ we compare m_d rather than m_s (see Ref. 1) to $\frac{1}{2} M_\eta$. This fuzziness will be resolved in what follows.

[8]A recent measurement, however, gives $\Gamma(\eta \to \gamma\gamma) = 0.32 \pm 0.05$ keV. See A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974). Also see Eq. (12) of the present paper.

[9]There are good reasons to believe the width of η_c will be comparable to that of $\psi(3105)$. See, for example, N. Isgur, Phys. Rev. D 12, 3770 (1975).

[10]The experimental situation is actually still somewhat unsettled. The two most recent and precise measurements give 11.7 ± 1.2 eV [G. Bellenttini *et al.*, Nuovo Cimento 66A, 243 (1970)] and 7.3 ± 0.5 eV (V. I. Kryshkin *et al.*, Zh. Eksp. Teor. Fiz. 57, 1917 (1969) [Sov. Phys.—JETP 30, 1037 (1970)]).

[11]There is one ambiguity here regarding the axial-vector coupling constant G_A' of the quarks. If the quarks are pointlike, as we have assumed, then $G_A'=1$. This would then seem to imply that G_A for the nucleons is $\frac{5}{3}$. We have taken the view advocated by, among others, the Dubna group [N. N. Bogoliubov, V. A. Matveyev, and A. N. Tavkhelidze, Nuovo Cimento 48, 132 (1967)] that $\frac{5}{3} \to 1.25$ due to relativistic corrections. One can, on the other hand, show that these same corrections do not apply to f_π and f_K so that one may legitimately use $G_A' = 1$. This interpretation results in a linear rise in $\psi(0)$ versus M as opposed to the behavior postulated in Ref. 4.