PHYSICAL REVIEW D VOLUME 13, NUMBER 5 1 MARCH 1976

Discrete two-variable expansions of physical scattering amplitudes

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(Received 13 October 1975)

A model-independent method of performing energy-dependent scattering-amplitude analysis is presented. It makes use of previously developed two-variable expansions of scattering amplitudes but involves only summations and no integrals. The energy and angle dependence are displayed in known functions. The angle, as usual, figures in Legendre polynomials; the energy is contained in Gegenbauer polynomials, if the data analysis is performed over a finite energy region and in specifically constructed basis functions when the analysis concerns the entire energy region. The purpose of the expansions is to make it possible to analyze all data for a given two-body reaction simultaneously (for all energies and angles) and to store the obtained information in the expansion coefficients. These then characterize the dynamics of a specific reaction, rather than a certain kinematic situation.

I. INTRODUCTION

The aim of this paper is to propose and discuss a new method for treating two-body scattering processes. The method is a generalization of ordinary phase-shift analysis and a modification of relativistic two-variable expansions presented earlier.[1-4] The main point of the method is that the scattering amplitudes are considered simultaneously as functions of two independent kinematic parameters, in the present case these being the center-of-mass-system energy and scattering angle. The amplitudes are then expanded into sums over known functions of the energy and angle. These functions, having their origin in a group-theoretical treatment of Lorentz invariance, already possess some of the kinematical properties that follow from general principles of scattering theory.

Generally speaking, the most complete physical information that can be extracted from scattering experiments is a reconstruction of the scattering amplitudes as functions of energy and angle. Below the threshold of inelastic processes this can be done completely (with the exception of possible discrete ambiguities due to the nonlinear character of the unitarity equations), since elastic unitarity will supply the over-all phase that cannot be directly observed experimentally. Above the inelastic threshold a "complete experiment" makes it possible to reconstruct all amplitudes (except for the over-all phase) directly from scattering data, independently for each energy and angle[5] (information on the over-all phase can be obtained via unitarity only if all open inelastic channels are studied simultaneously). For pion-nucleon scattering the complete experiment involves three measurements (say, the differential cross section, recoil nucleon polarization, and one of the Wolfenstein polarization rotation parameters). For nucleon-nucleon scattering the complete experiment involves nine measurements[5-7] (for each energy and angle). Since some of the measurements involved in the complete experiment are difficult to perform, other methods of reconstructing the amplitudes are very useful. Conventional phase-shift analysis is a classical example of such a method. It has the advantage that it provides a discrete parametrization of angular dependencies, by introducing a discrete (and physically meaningful) variable, namely the angular momentum. Below the inelastic threshold it automatically satisfies unitarity by taking the phase shifts to be real; above this threshold the phase shifts are complex. A problem with phase-shift analysis is that it only parametrizes amplitudes for one fixed energy. Data analyses must thus either be performed separately for each given energy or an energy dependence can be introduced using certain models or assumptions.[8-10] For, e.g., nucleon-nucleon scattering, an additional problem is that different physical quantities needed in the phase-shift analysis are often measured at close, but different, energies. The data must then be interpolated (or extrapolated) to a chosen energy, before they can be used in the analysis, thus introducing further errors.

Similar comments hold for other methods of analysis, like Regge-pole expansions, impact parameter analyses, etc.

In this article we provide a model-independent method for performing energy-dependent phase-shift analysis. The starting point is provided by the two-variable expansions, mentioned above.[1-4]

The essence of the method, as reviewed in Ref. 1, is that two-body scattering amplitudes were considered as functions over a Lorentz group (or Galilei group in the nonrelativistic case) manifold and then expanded in terms of the basis functions of irreducible representations of the corresponding group. The dependence on both kinematic parameters (energy and angle, energy and momentum transfer, transverse and longitudinal momentum, etc.) is displayed explicitly in known functions, whereas the dynamics of the process are carried by the expansion coefficients, i.e., the Lorentz (or Galilei) amplitudes. The advantages of the method are that a large degree of separation of kinematics and dynamics is achieved, that many general features of scattering theory are incorporated automatically, that dynamical assumptions can be formulated in terms of the Lorentz (Galilei) amplitudes, and that, in principle, all data from kinematically accessible regions can be treated simultaneously.

For three-body decays, when the kinematic region is finite (i.e., the Mandelstam variables s, t, and u vary over a finite region, namely the Dalitz plot) the method is quite simple and leads to a double sum expansion in terms of basis functions of the rotation group O(4).[3] For scattering, on the other hand, the physical region (say, of the s channel) is infinite and expansions are performed in terms of basis functions of the noncompact group O(3, 1) (in the relativistic case). This implies that the expansions will involve at least one integral, in addition to a sum, and possibly two integrals (depending on the specific basis that we choose). This feature is a definite drawback in phenomenological fits to data. In the present article we show how this drawback can be overcome. More specifically, we present two principally different methods by means of which the integrals in the expansions can be replaced by sums. We consider expansions based on the $O(3,1)\supset O(3)\supset O(2)$ reduction, where the compact group O(3) provides the usual partial-wave expansion into a sum over angular momenta whereas the Lorentz group O(3, 1) provides an integral expansion of the partial-wave amplitude. The energy dependence is then contained in specific Legendre functions and the integration is over the "four-dimensional angular momentum" σ associated with the Lorentz group (see below).

The method of replacing the integral by a sum (discretizing the expansions) depends on whether the expansion is performed for a finite or infinite energy region. If we only wish to treat scattering over a finite energy region, say, from the elastic threshold to the first inelastic one, we can project the relevant section of the O(3, 1) manifold onto an O(4) one and then expand in terms of O(4) basis functions, thus immediately obtaining a sum (this is like a transition from a Fourier integral to a Fourier sum).

When treating scattering over an infinite energy region (e.g., from threshold energy to infinitely large energies) we shall make use of specific properties of the O(3, 1) basis functions. These can be written as finite sums of hyperbolic functions, thus making it possible to separate the dependence on the energy variable from the integration variable σ and to actually symbolically perform the integration, introducing new expansion coefficients and functions.

In the present article we restrict ourselves to the scattering of spinless particles, but a generalization to reactions involving particles with spin is immediate.

In Sec. II we present expansions for a finite energy region and in Sec. III we consider expansions valid for all energies. The conclusions and future outlook are summarized in Sec. IV.

II. DISCRETE TWO VARIABLE EXPANSIONS FOR FINITE ENERGY REGIONS

The usual partial-wave expansion of a spinless scattering amplitude for the reaction $1+2\rightarrow 3+4$ can be written as

$$f(E,\theta)=\sum_{l=0}^{\infty}(2l+1)a_l(E)P_l(\cos\theta), \tag{1}$$

where E and θ are the center-of-mass-system energy and scattering angle. The form of this expansion is identical in the relativistic and nonrelativistic case and the rotation group O(3), providing the Legendre polynomials $P_l(z)$ figures as the "little group," leaving the total energy-momentum vector p_1+p_2 invariant. The variables E and θ are treated asymmetrically, in that the energy is contained in the unknown partial-wave amplitude $a_l(E)$, whereas the dependence on θ is displayed explicitly. A more symmetric treatment, as well as a greater separation of kinematics and dynamics, is provided by the Lorentz-group expansion using the $O(3,1)\supset O(3)\supset O(2)$ basis and supplementing expansion (1) as follows[1-4]:

$$f(E,\theta)=\sum_{l=0}^{\infty}(2l+1)\int_{\delta-i\infty}^{\delta+i\infty}d\sigma(\sigma+1)^2\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}A_l(\sigma)\frac{1}{(\sinh a)^{1/2}}P_{1/2+\sigma}^{-l-1/2}(\cosh a)P_l(\cos\theta), \tag{2}$$

where

$$\cosh a = \frac{s + m_1^2 - m_2^2}{2m_1\sqrt{s}},$$

$$\sinh a = \frac{[s-(m_1+m_2)^2]^{1/2}[s-(m_1-m_2)^2]^{1/2}}{2m_1\sqrt{s}}, \quad (3)$$

$$s = (p_1 + p_2)^2 = E^2.$$

Here m_1 and m_2 are the masses of particles 1 and 2 (they could be replaced by m_3 and m_4), $P_\nu^\mu(z)$ is a Legendre function, $\Gamma(z)$ are Γ functions (providing a normalization), and $A_l(\sigma)$ are the expansion coefficients, or Lorentz amplitudes. Expansion (2) can be interpreted as being the partial-wave expansion (1) supplemented by an integral expansion of the partial-wave amplitude:

$$a_l(E) = \int_{\delta - i\infty}^{\delta + i\infty} d\sigma(\sigma+1)^2 \frac{\Gamma(\sigma+1)}{\Gamma(\sigma - l + 1)} A_l(\sigma) \times \frac{1}{(\sinh a)^{1/2}} P_{1/2+\sigma}^{-l-1/2}(\cosh a). \quad (4)$$

The parameter δ determining the integration path is related to the asymptotic behavior of $a_l(E)$ [expansion (4) allows $a_l(E)$ to increase polynomially for $E \to \infty$ as E^6]. The elastic threshold $s = (m_1 + m_2)^2$ corresponds to $a \to 0$ in (4), and since $(\sinh a)^{-1/2} P_{1/2+\sigma}^{-l-1/2}(\cosh a)$ behaves as $(\sinh a)^l$ for $a \to 0$, the partial-wave amplitudes have good threshold behavior built in:

$$a_l(E) \underset{a\to 0}{\propto} (\sinh a)^l. \quad (5)$$

To facilitate phenomenological applications of the expansion (2) we wish to replace the integral over σ in (4) [and (2)] by a sum. The representation holds in the entire physical region $0 \le a < \infty$, $0 \le \theta \le \pi$, but in this section (contrary to the following one), we consider the case when we wish to apply the expansion in a finite energy region only, for $(m_1+m_2)^2 \le s \le s_{\max}$, i.e., $0 \le a \le a_{\max}$.

The variables a and θ [together with an additional cyclic variable ϕ $(0 \le \phi < 2\pi)$] parametrize a "cup" on an O(3,1) hyperboloid, rather than an entire hyperboloid.[1] The situation then becomes very similar to that of three-body decays, treated in Ref. 3. The finite section of the hyperboloid can be parallely mapped onto an O(4) semisphere of radius $R = \sinh a_{\max}$. We then obtain a correspondence between points on the semisphere

$$v_s = R(\cos\beta, \sin\beta\sin\theta\cos\phi, \sin\beta\sin\theta\sin\phi, \sin\beta\cos\theta) \quad (6)$$

and the kinematic parameters of the reaction. Indeed, θ is the c.m. system scattering angle, we have

$$\sin\beta = \frac{\sinh a}{\sinh a_{\max}}, \quad 0 \le \beta \le \pi/2 \quad (7)$$

and ϕ is an azimuthal angle on which the scattering amplitude does not depend. The scattering amplitude can then be considered to be a function of a point on this semisphere and expanded in terms of O(4) basis functions, just as in the case of three-body decays.[3] The obtained expansion is

$$f(\beta,\theta) = \sum_{l=0}^{\infty}(2l+1)\sum_{n=l}^{\infty} a_{nl}N_{nl}\sin^l\beta \times C_{n-l}^{l+1}(\cos\beta)P_l(\cos\theta). \quad (8)$$

In (8) $C_{n-l}^{l+1}(\cos\alpha)$ is a Gegenbauer polynomial, directly related to the Legendre functions in (1):

$$(\sin\alpha)^l C_{n-l}^{l+1}(\cos\alpha) = \frac{1}{2^l}\left(\frac{\pi}{2}\right)^{1/2} \frac{\Gamma(n+l+2)}{\Gamma(n-l+1)\Gamma(l+1)} \times \frac{1}{(\sin\alpha)^{1/2}} P_{1/2+n}^{-l-1/2}(\cos\alpha), \quad (9)$$

a_{nl} are the expansion coefficients, and

$$N_{nl} = e^{-il\pi/2}\frac{2^{l+1/2}\Gamma(l+1)}{2\pi} \times \left[(2l+1)\frac{(n+1)\Gamma(n-l+1)}{\Gamma(n+l+2)}\right]^{1/2} \quad (10)$$

is a normalization coefficient.

Thus, for energies satisfying

$$E_0 \le E \le E_{\max}, \quad (11)$$

where E_0 is the elastic threshold and $E_{\max}$ an arbitrary fixed energy, the partial-wave amplitude is represented by a sum

$$a_l(E) = \sum_{n=l}^{\infty} a_{nl}N_{nl}\sin^l\beta\, C_{n-l}^{l+1}(\cos\beta). \quad (12)$$

Note that the factor $(\sin\beta)^l$ assures the correct threshold behavior for each partial-wave amplitude. Note also that the energy variable used for decays[3] was $\alpha = 2\beta$, rather than β, because then $0 \le \alpha \le \pi$, and $\alpha = 0$ and $\alpha = \pi$ correspond to a threshold and pseudothreshold, both of which lie on the boundary of the decay region. In the present case $\beta=\pi/2$ corresponds to $a=a_{max}$ and there is no reason for $a_l(E)$ to vanish for $l\neq 0$ at $E=E_{max}$.

Expansion (12) can now be directly used to fit scattering data measured at all angles $0\leq\theta\leq\pi$ and all energies $E_0\leq E\leq E_{max}$. It would be of great interest to investigate the stability of the coefficients a_{nl} with respect to the choice of E_{max}. A very natural choice would be $E_{max}=E_1$, where E_1 is the first inelastic threshold (say, that of 1-pion production in NN scattering). The amplitude $f(E,\theta)$ must then satisfy elastic unitarity, which for the partial-wave amplitude implies

$$a_l(E)=\frac{e^{2i\delta_l(E)}-1}{2ik}, \qquad (13)$$

where the phase shift $\delta_l(E)$ is real and $k^2=E^2/4-m^2$.

Elastic unitarity for the O(4) amplitudes a_{nl} is somewhat more complicated. Indeed, (12) and (13) imply

$$e^{2i\delta_l}=1+2ik\sum_{n=l}^{\infty}a_{nl}N_{nl}\sin^l\beta C_{n-l}^{l+1}(\cos\beta). \qquad (14)$$

Taking the square modulus of both sides of (14), we obtain

$$\sum_{n=l}^{\infty}(a_{nl}e^{i\pi l}-a_{nl}^*)\chi_{nl}(\beta)=2ik\sum_{n'=l}^{\infty}\sum_{n''=l}^{\infty}a_{n'l}a_{n''l}^*\chi_{n'l}(\beta)\chi_{n''l}(\beta), \qquad (15)$$

where

$$\chi_{nl}(\beta)=N_{nl}\sin^l\beta C_{n-l}^{l+1}(\cos\beta). \qquad (16)$$

Making use of the O(4) Clebsch-Gordan coefficients[3,11] we find

$$\chi_{n'l}(\beta)\chi_{n''l}(\beta)(l0l0|L0)=\frac{1}{\pi\sqrt{2}}[n'+1)(n''+1)]^{1/2}(2l+1)\sum_N(N+1)^{1/2}\begin{Bmatrix}\frac{n'}{2}&\frac{n'}{2}&l\\\frac{n''}{2}&\frac{n''}{2}&l\\\frac{N}{2}&\frac{N}{2}&L\end{Bmatrix}\chi_{NL}, \qquad (17)$$

where the curly-bracket object is an O(3) $9j$ symbol.[12] Substituting (17) back into (15) we obtain a set of nonlinear constraints for the O(4) coefficients a_{nl}, following from elastic unitarity,

$$(a_{nl}e^{i\pi l}-a_{nl}^*)(l0l0|L0)=4\sqrt{2}\,ik\sum_{n',n''=L}^{\infty}a_{n'l}a_{n''l}^*(n'+1)^{1/2}(n''+1)^{1/2}\times\sum_N(N+1)^{1/2}\begin{Bmatrix}\frac{n'}{2}&\frac{n'}{2}&l\\\frac{n''}{2}&\frac{n''}{2}&l\\\frac{N}{2}&\frac{N}{2}&L\end{Bmatrix}\int_0^\pi\chi_{NL}(\beta)\chi_{nl}^*(\beta)\sin^2\beta\,d\beta, \qquad (18)$$

valid for all L satisfying $0\leq L\leq 2l$. Both sides of (18) vanish if $2l+L$ is odd. For even l (18) simplifies, since we can choose $L=l$. Then

$$(a_{nl}e^{i\pi l}-a_{nl}^*)=\left(\frac{2}{\pi}\right)^{1/2}ik(-1)^{l/2}\left(\frac{(3l+1)!\,(2l+1)(n+1)}{l!}\right)^{1/2}\frac{[(l/2)!]^3}{l!\,(3l/2)!}\times\sum_{n',n''=l}^{\infty}a_{n'l}a_{n''l}^*[(n'+1)(n''+1)]^{1/2}\begin{Bmatrix}\frac{n'}{2}&\frac{n'}{2}&l\\\frac{n''}{2}&\frac{n''}{2}&l\\\frac{n}{2}&\frac{n}{2}&l\end{Bmatrix} \qquad (19)$$

(l = even).

To summarize, Eq. (8) can be directly applied to analyze scattering data for all angles and energies in a finite region from the elastic threshold to any chosen E_{max}. If E_{max} is chosen to be the first inelastic threshold, then elastic unitarity provides certain constraints on the expansion coefficients a_{nl}, exhibited in (18) and (19). The relations (18) and (19) are, of course, nonlinear. However, they involve only sums and no integrals and are thus simpler than the unitarity relation for the total amplitude $f(E,\theta)$.

III. DISCRETE TWO-VARIABLE EXPANSIONS FOR ALL ENERGIES

If we wish to analyze data in the entire physical scattering region, the variable a in (4) varies in the region $0 \leq a < \infty$ and no natural mapping onto a sphere is possible. Instead, we shall "discretize" expansion (4) by making use of specific properties of the expansion functions. This will be performed in two different manners.

A. Basis functions as elementary functions

The Lorentz-group basis functions $(\sinh a)^{-1/2}P^{-l-1/2}_{1/2+\sigma}(\cosh a)$ can be written as finite sums of elementary functions. Indeed, using formula 8.777.1 of Ref. 13, representing the terminating hypergeometric series by the corresponding sums and reorganizing terms, we find

$$\frac{1}{(\sinh a)^{1/2}}P^{-l-1/2}_{1/2+\sigma}(\cosh a)=\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{2^l}\frac{1}{(\sinh a)^{l+1}}\sum_{n=0}^{\infty}(-1)^n\binom{l}{n}\frac{\Gamma(\sigma-n+1)}{\Gamma(\sigma+l-n+2)}\sinh(\sigma+l+1-2n)a. \tag{20}$$

In particular, for $l=0$ and $l=1$ we have

$$\frac{1}{(\sinh a)^{1/2}}P^{-1/2}_{1/2+\sigma}(\cosh a)=\left(\frac{2}{\pi}\right)^{1/2}\frac{\sinh(\sigma+1)a}{(\sigma+1)\sinh a},$$

$$\frac{1}{(\sinh a)^{1/2}}P^{-3/2}_{1/2+\sigma}(\cosh a)=\frac{1}{(2\pi)^{1/2}}\frac{1}{\sigma(\sigma+1)(\sigma+2)}\frac{1}{(\sinh a)^2}[-(\sigma+2)\sinh\sigma a+\sigma\sinh(\sigma+2)a].$$

We now wish to substitute (20) into (4) and formally integrate over σ. To do this we expand $\sinh(\sigma+l+1-2n)a$ into a power series and obtain

$$a_l(E)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{2^l}(-1)^n\binom{l}{n}$$
$$\times\int_{\delta-i\infty}^{\delta+i\infty}d\sigma A_l(\sigma)\frac{\Gamma(\sigma-n+1)}{\Gamma(\sigma+l-n+2)}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}(\sigma+1)^2(\sigma+l+1-2n)^{2m+1}\frac{a^{2m+1}}{(\sinh a)^{l+1}}. \tag{21}$$

We interpret (21) as a new expansion in terms of the functions

$$T_{lm}(a)=\frac{a^{2m+1}}{(\sinh a)^{l+1}} \tag{22}$$

with new expansion coefficients equal to

$$B_{lm}=\frac{(m-l)!}{(2m+1)!}\left(\frac{2}{\pi}\right)\frac{1}{2^l}\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)^2A_l(\sigma)\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}\left[\sum_{n=0}^{l}(-1)^n\binom{l}{n}\frac{\Gamma(\sigma+1-n)}{\Gamma(\sigma+l-n+2)}(\sigma+l+1-2n)^{2m+1}\right]d\sigma. \tag{23}$$

The coefficient $(m-l)!$ in (23) was introduced for further convenience.

The sum in (23) is evaluated in the Appendix in terms of elementary symmetric functions. Using (A7) an and (A8) we find

$$B_{lm}=0 \text{ for } 0\leq m<l,$$

$$B_{lm}=\frac{(m-l)!\,l!}{(2m+1)!}2^{2m-l}\left(\frac{2}{\pi}\right)^{1/2}\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)^2A_l(\sigma)\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}P_{2(m-l)}(\alpha)d\sigma, \tag{24}$$

where $P_{2(m-l)}(\alpha)$ is defined in the Appendix.

Finally, we write the expansion of the partial-wave amplitude as

$$a_l(E)=\frac{a^{2l+1}}{(\sinh a)^{l+1}}\sum_{k=0}^{\infty}\frac{1}{k!}B_{l,l+k}a^{2k}, \tag{25}$$

where $B_{l,l+k}$ are the new phenomenological amplitudes, related to the Lorentz amplitudes $A_l(\sigma)$ by (24). Comparing (25) with a Taylor expansion we can express the coefficients as

$$B_{l,l+k}=\frac{k!}{(2k)!}\frac{d^{2k}}{da^{2k}}\left[\frac{\sinh a^{l+1}}{a^{2l+1}}\;a_l(E)\right]_{a=0}. \tag{26}$$

Expansion (25) has the correct threshold behavior of the partial-wave amplitudes built in. Since the expansion should be used for all energies and since the coefficients $B_{l,l+k}$ do not depend on energy, there is no point in enforcing elastic unitarity. Notice that the sum in (25) will converge, e.g., if all $B_{l,l+k}$ are of the same order of magnitude.

B. Basis functions as definite integrals

An alternative method of replacing the integration (4) by a sum is based on an integral representation for the Legendre functions. Indeed, formula 8.715.1 of Ref. 13 gives

$$\frac{1}{(\sinh a)^{1/2}}P_{1/2+\sigma}^{-l-1/2}(\cosh a)=\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{l!\,(\sinh a)^{l+1}}\int_0^a(\cosh a-\cosh t)^l\cosh(\sigma+1)t\,dt. \tag{27}$$

Substituting (27) into (4) and expanding $\cosh(\sigma+1)t$ we obtain

$$a_l(E)=\sum_{m=0}^{\infty}D_{lm}R_{lm}(a), \tag{28}$$

where

$$R_{lm}(a)=\frac{1}{(\sinh a)^{l+1}}\int_0^a(\cosh a-\cosh t)^l t^{2m}dt \tag{29}$$

are the new expansion functions and

$$D_{lm}=\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{l!\,(2m)!}\int_{\delta-i\infty}^{\delta+i\infty}A_l(\sigma)\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}(\sigma+1)^{2m+2}d\sigma \tag{30}$$

are the new coefficients.

The function $R_{lm}(a)$ can either be tabulated directly using integral (29), or it can be expressed in terms of finite sums. To do this we expand $(\cosh a-\cosh t)^l$ into a binomial series and use the relation

$$(\cosh t)^p=\frac{1}{2^p}\sum_{q=0}^{p}\binom{p}{q}\cosh(p-2q)t.$$

The integration then gives

$$\begin{aligned}R_{lm}(a)=\frac{(2m)!}{(\sinh a)^{l+1}}\sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^p\binom{l}{p}\frac{1}{2^p}(\cosh a)^{l-p}\\ \times\Bigg\{(1-\delta_{p,2q})\binom{p}{q}(p-2q)^{-2m-1}\Bigg[\sum_{k=0}^{m}\frac{(p-2q)^{2k}a^{2k}}{(2k)!}\sinh(p-2q)a\\ -\sum_{k=1}^{m}\frac{(p-2q)^{2k-1}a^{2k-1}}{(2k-1)!}\cosh(p-2q)a\Bigg]\\ +\delta_{p,2q}\binom{p}{p/2}\frac{a^{2m+1}}{(2m+1)!}\Bigg\}.\end{aligned} \tag{31}$$

We should again note that the expansion functions $R_{lm}(a)$ have proper threshold behavior. To see this, consider (29) for $a\to 0$. Expanding the hyperbolic functions in the integrand and keeping the lowest nonvanishing terms, we have

$$R_{lm}(a) \underset{a\to 0}{\sim} \frac{1}{(\sinh a)^{l+1}}\frac{1}{2^l}\int_0^a (a^2-t^2)^l t^{2m}$$

$$= \frac{a^{2l+2m+1}}{(\sinh a)^{l+1}}\frac{1}{2^l}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\frac{1}{2m+2k+1}. \tag{32}$$

The sum in (32) can be performed (see Appendix) and we obtain

$$R_{lm}(a) \underset{a\to 0}{\sim} \frac{l!}{2^{l+1}}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+l+\frac{3}{2})}a^{l+2m}. \tag{33}$$

A further relevant property of these functions is that

$$0 \le R_{lm}(a) \le \left(\frac{\cosh a - 1}{\sinh a}\right)^l \frac{a^{2m+1}}{(2m+1)\sinh a}, \tag{34}$$

so that

$$\lim_{a\to\infty} R_{lm}(a) = 0. \tag{35}$$

The expansion coefficients D_{lm} are independent of energy (and angle) and the expansion (28) holds in the entire energy region. There is thus no point in enforcing elastic unitarity. Expansions (25) and (28) should be used directly to fit scattering data for all angles and energies simultaneously.

IV. CONCLUSIONS

The main result of this paper is a method by means of which it is possible to perform energy-dependent amplitude analysis in a model-independent way. Fits to data can be performed over a finite energy region [expansion (12)] or an infinite one [expansions (25) or (28)]. The expansions involve sums only and the expansion functions demonstrate proper threshold behavior. All data for a given process should be taken at the energy and angle where the measurement was performed, without any further interpolation (or extrapolation).

In this paper we concentrated on the scattering of spinless particles only; however, the generalization to arbitrary spins should be straightforward. The O(3,1) and O(4) expansions underlying the methods of this article have been generalized to arbitrary spins.[4] Work in this direction is in progress; in particular, discrete two-variable expansions of nucleon-nucleon scattering amplitudes will be presented in the near future and applied to analyze nucleon-nucleon scattering data.

The expansion functions are smooth functions of the energy and are hence not particularly appropriate for a study of narrow resonances, bound states, and similar phenomena. In data analysis in the resonance region (e.g., for $\pi\pi$ or πN scattering) Breit-Wigner-type resonances should best be considered separately.

Other two-variable expansions have been considered earlier,[1,2] corresponding to the reductions $O(3,1)\supset O(2,1)\supset O(2)$, $O(3,1)\supset E(2)\supset O(2)$ [$E(2)$ is the Euclidean group of the plane] or to "nonsubgroup bases."[14] The first of these physically corresponds to a generalization of Regge-pole theory ("momentum-transfer-dependent Regge-pole expansions"). These expansions, in general, involve double integrals and we plan to discuss methods of "discretizing" these expansions in the near future.

ACKNOWLEDGMENT

The authors are much indebted to Professor R. T. Sharp for very helpful discussions, especially concerning the material contained in the Appendix.

One of the authors (J. P.) acknowledges the hospitality of the D.Ph.P.E., CEN Saclay and the financial support of the France-Quebec Scientific Exchange Program during his stay at Saclay.

APPENDIX. EVALUATION OF CERTAIN SUMS

1. We wish to evaluate the sum

$$\chi(l,m) = \sum_{n=0}^{l}(-1)^n\binom{l}{n}\frac{\Gamma(\sigma+1-n)}{\Gamma(\sigma+l+2-n)}(\sigma+l+1-2n)^{2m+1}$$

figuring in (23). To do this, consider the path integral

$$I = \oint \frac{z^{2m+1}\Gamma(z-\frac{1}{2}(\sigma+l+1))\Gamma(z+\frac{1}{2}(\sigma-l+1))}{\Gamma(z-\frac{1}{2}(\sigma+l+1)+l+1)\Gamma(z+\frac{1}{2}(\sigma-l+1)+l+1)}dz \tag{A1}$$

along a circle in the z plane containing all the $l+1$ poles of the ratio of the first two Γ functions and the $l+1$ poles of the ratio of the second two Γ functions. The integral is equal to $2\pi i$ times the sum of the residues of all poles enclosed, i.e.,

$$I = 2\pi i\left[\sum_{n=0}^{l}\frac{(-1)^n\Gamma(-n+\sigma+1)}{n!\,(l-n)!\,\Gamma(-n+\sigma+l+2)}\left(\frac{\sigma+l+1}{2}-n\right)^{2m+1} + \sum_{N=0}^{l}\frac{(-1)^N\Gamma(-N-\sigma-1)}{N!\,(l-N)!\,\Gamma(-N-\sigma+l)}\left(\frac{-\sigma+l-1}{2}-N\right)^{2m+1}\right]. \tag{A2}$$

Transforming the second sum we show that it is equal to the first one. Hence

$$I = 4\pi i \frac{\chi(l,m)}{l!\,2^{2m+1}}. \tag{A3}$$

It is now necessary to evaluate the integral (A1) directly. Using well-known properties of the Γ functions we find

$$I = \oint z^{2m+1-2l-2} \frac{dz}{\prod_{k=1}^{2l+2}(1-\alpha_k/z)}, \tag{A4}$$

where

$$\alpha_k = -\alpha_{2l+3-k} = \frac{\sigma - l - 1 + 2k}{2}, \quad k = 1, 2, \ldots, l+1.$$

The fraction in (A4) can be expressed as a product of geometrical series in terms of elementary symmetric functions[15]:

$$\left[\prod_{k=1}^{2l+2}\left(1-\frac{\alpha_k}{z}\right)\right]^{-1} = \sum_{s=0}^{\infty} P_s(\alpha) z^{-s}, \tag{A5}$$

where

$$\begin{aligned} &P_s(\alpha) = 0 \text{ for } s<0, \\ &P_0(\alpha) = 1, \\ &P_1(\alpha) = \sum_{i=1}^{2l+2} \alpha_i, \\ &P_2(\alpha) = \sum_{i\le j} \alpha_i\alpha_j, \\ &P_s(\alpha) = \sum_{i_1\le i_2\le\cdots\le i_s} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_s} \text{ for } s>0. \end{aligned} \tag{A6}$$

Substituting (A5) into (A4) we find

$$\begin{aligned} I &= \oint \sum_{s=0}^{\infty} P_s(\alpha) z^{2m-1-2l-s} \\ &= 2\pi i P_{2(m-l)}(\alpha), \end{aligned}$$

since the contour integral is nonzero only if $s = 2(m-l)$. Finally, we obtain

$$\chi(l,m) = l!\,2^{2m} P_{2(m-l)}(\alpha). \tag{A7}$$

In particular, we have

$$\chi(l,m) = 0 \text{ for } m<l. \tag{A8}$$

2. The sum

$$\tilde{\chi}(l,m) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} \frac{1}{2m+2k+1} \tag{A9}$$

in (32) can be calculated analogously. Indeed, the integral

$$\tilde{I} = \oint \frac{1}{z} \frac{\Gamma(-z+m+\frac{1}{2})}{\Gamma(-z+m+\frac{1}{2}+l+1)} dz \tag{A10}$$

calculated around a circle encompassing all poles of the integrand is according to the above method equal to zero. Using the Cauchy theorem we obtain

$$0 = \tilde{I} = 2\pi i \left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+l+\frac{3}{2})} - \sum_{k=0}^{l} \frac{(-1)^k}{k!\,(l-k)!\,(m+k+\frac{1}{2})} \right). \tag{A11}$$

Hence

$$\sum_{k=0}^{l} (-1) \binom{l}{k} \frac{1}{2m+2k+1} = \frac{l!}{2} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+l+\frac{3}{2})}. \tag{A12}$$

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