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Mass formula for nonets

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The relation between the masses and mixing angle in a nonet of mesons is studied in a simple quark-gluon picture. We find that both the splitting of μ_η^2 from μ_π^2 and the departure of θ_P from $\theta_{\rm ideal}$ are due to annihilation forces. Mass formulas which describe the masses in a nonet in terms of two parameters are derived and found to be in good agreement with observed masses for the 0^{-+}, 1^{--}, and 2^{++} nonets.

It has been appreciated for some time now that the quark model can offer guidance in the problem of understanding the masses and mixing angle in a nonet of mesons.[1] Simply stated, the $I_3=Y=0$ combinations

$$|M(I=1,I_3=0)\rangle=-\frac{1}{\sqrt{2}}(|u\bar{u}\rangle-|d\bar{d}\rangle), \quad (1a)$$

$$|M_{ns}\rangle=\frac{1}{\sqrt{2}}(|u\bar{u}\rangle+|d\bar{d}\rangle), \quad (1b)$$

and

$$|M_s\rangle=|s\bar{s}\rangle \quad (1c)$$

are eigenstates of the mass operator in the absence of any interactions. Since $|M_{ns}\rangle=(\frac{2}{3})^{1/2}|M_0\rangle-(\frac{1}{3})^{1/2}|M_8\rangle$ and $|M_s\rangle=(\frac{2}{3})^{1/2}|M_8\rangle+(\frac{1}{3})^{1/2}|M_0\rangle$, where $|M_8\rangle$ and $|M_0\rangle$ are the octet and singlet $I=0$ mesons, we see that the states (1) correspond to a mixing angle $\theta_{\rm ideal}=\arctan 1/\sqrt{2}\simeq 35°$. This is very nearly the mixing angle observed for the vector (1^{--}) and tensor (2^{++}) meson nonets; moreover, $\mu_\omega\simeq\mu_\rho$ and $\mu_f\simeq\mu_{A_2}$ as suggested by (1). The pseudoscalar (0^{-+}) mesons, however, seem to have nothing to do with this simple picture: They blatantly display a mixing angle of about $-10°$, and μ_η is far from μ_π.

A comforting phenomenological explanation of this unruly behavior of the pseudoscalar mesons can be given.[2] When interactions are turned on, they can both change the energies of the states (1) and mix them. If the interactions should result in a near degeneracy of μ_{ns} and μ_s on a scale set by the mixing strength, then the eigenstates of the mass matrix would become

$$|M_{m8}\rangle=\frac{1}{\sqrt{2}}(|M_s\rangle-|M_{ns}\rangle) \quad (2a)$$

and

$$|M_{m0}\rangle=\frac{1}{\sqrt{2}}(|M_{ns}\rangle+|M_s\rangle), \quad (2b)$$

where "$m8$" and "$m0$" stand for "mostly octet" and "mostly singlet," respectively; it is these combinations which correspond to a mixing angle of $\theta=\theta_{\rm ideal}-45°\simeq-10°$. This observation, while offering no understanding of how such a degeneracy might occur, at least sheds some light on the significance of a mixing angle of $-10°$.

Working by this light, we have found a partial explanation for the anomalous behavior of the pseudoscalar mesons which is simple and, in our opinion, quite attractive. The explanation is based on a quark model with low-mass quarks interacting via gluons which are singlets under ordinary SU(3). Deep-inelastic data indicate quite clearly that the quarks are light; extractions of the three-quark component of the proton's wave function[3] indeed indicate that $m_u=m_d\simeq\frac{1}{3}M_N$. These observations are strongly supported by the fact that if the quarks are pointlike (as they appear to be) then in order to explain the proton's magnetic moment one must conclude that

$$m_u=m_d\simeq\frac{M_N}{2.79}=337\text{ MeV}. \quad (3)$$

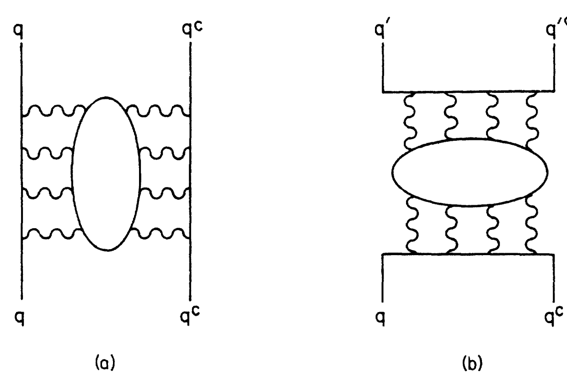

FIG. 1. Contributions to meson masses in a quark-gluon picture.

TABLE I. Predictions of the masses in the established nonets. The format here is M: theoretical value of μ_M in GeV (experimental value ± (experimental uncertainty + 0.01 for the unknown electromagnetic correction)). The parameters are $A_P = 0.27$, $S_P = -0.43$; $A_V = 0.01$, $S_V = 0.14$; and $A_T = -0.06$, $S_T = 1.31$.

π: 0.14 (0.14 ± 0.01)	ρ: 0.77 (0.77 ± 0.02)	A_2: 1.32 (1.31 ± 0.02)
K: 0.49 (0.50 ± 0.01)	K^*: 0.90 (0.89 ± 0.01)	K_N: 1.41 (1.42 ± 0.02)
η: 0.52 (0.55 ± 0.01)	ϕ: 1.03 (1.02 ± 0.01)	f': 1.48 (1.52 ± 0.01)
η': 1.03 (0.96 ± 0.01)	ω: 0.78 (0.78 ± 0.01)	f: 1.27 (1.27 ± 0.02)

This conclusion is in turn strongly supported by noting that with $m_s - m_d \simeq 140$ MeV as is indicated by the spacing in the decuplet, one predicts that $\mu_\Lambda = -\frac{1}{3}(m_d/m_s)\mu_p \simeq -0.66$ as is observed. On these grounds as well as others[4] we therefore take $m_u \simeq m_d \simeq 0.34$ GeV and $m_s \simeq 0.48$ GeV in what follows.

As discussed in Ref. 2, there are three possible contributions in our picture to the (mass)^2 matrix[5] of a meson nonet: the (mass)^2 of the quark-antiquark pair, a "scattering" term S arising from diagrams such as Fig. 1(a), and an annihilation term A arising from diagrams such as Fig. 1(b).[6] The (mass)^2 matrix is therefore diagonal in the 6×6 submatrix of $u\bar{d}$, $u\bar{s}$, $d\bar{u}$, $d\bar{s}$, $s\bar{u}$, and $s\bar{d}$ with

$$\mu^2(I=\tfrac{1}{2}, I_3=\pm\tfrac{1}{2}) = 0.67 + S, \tag{4}$$

$$\mu^2(I=1, I_3=\pm 1) = 0.45 + S, \tag{5}$$

while the 3×3 submatrix of $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$ is just

$$\mu^2 = \begin{vmatrix} 0.45+S+A & A & A \\ A & 0.45+S+A & A \\ A & A & 0.92+S+A \end{vmatrix} . \tag{6}$$

It is instructive to diagonalize μ^2 in two stages. First, since $|u\bar{u}\rangle$ and $|d\bar{d}\rangle$ are degenerate, $|M(I=1, I_3=0)\rangle$ and $|M_{ns}\rangle$ diagonalize the $u\bar{u} - d\bar{d}$ subspace giving

$$\hat{\mu}^2 = \begin{vmatrix} 0.45+S & 0 & 0 \\ 0 & 0.45+S+2A & \sqrt{2}A \\ 0 & \sqrt{2}A & 0.92+S+A \end{vmatrix} , \tag{7}$$

with $\mu^2(I=1, I_3=0) = \mu^2(I=1, I_3=\pm 1)$ as it must. In this form, we can clearly see that *the strong splitting of μ_η^2 from μ_π^2 and the strong mixing of $|M_{ns}\rangle$ and $|M_s\rangle$ in the pseudoscalars are due to the same mechanism:* the existence of a substantial annihilation amplitude of positive sign. Solving (7), we find

$$\mu_{m8}^2 = (0.92+S+A)\cos^2\phi + (0.45+S+2A)\sin^2\phi + 2\sqrt{2}A\cos\phi\sin\phi, \tag{8}$$

$$\mu_{m0}^2 = (0.45+S+2A)\cos^2\phi + (0.92+S+A)\sin^2\phi - 2\sqrt{2}A\cos\phi\sin\phi, \tag{9}$$

and

$$\theta = \theta_{\text{ideal}} + \phi, \tag{10}$$

where

$$\tan 2\phi = \frac{2\sqrt{2}A}{0.47 - A}. \tag{11}$$

In (4), (5), (8), and (9) we now have predictions for the four masses of a nonet in terms of only the two parameters S and A. The results of a fit to these masses are given in Table I for the 0^{-+}, 1^{--}, and 2^{++} nonets. The agreement is clearly very good (and can be made even better, of course, if allowance is made for symmetry-breaking in S and A due to the inequality of m_s and m_d). Equations (10) and (11) further predict that $\theta_P \simeq -17°$, $\theta_V \simeq 37°$, and $\theta_T \simeq 27°$, values which are supported by scattering experiments.[7] As a result, we feel some justification in claiming that the two mysteries of the pseudoscalar mesons have been distilled into one: The same force which makes $\mu_\eta^2 \gg \mu_\pi^2$ rotates the pseudoscalar meson nonet sharply away from ideality.

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[1]R. H. Dalitz, in *High Energy Physics*, 1965 Les Houches Lectures, edited by C. DeWitt and M. Jacob (Gordon and Breach, New York, 1966).

[2]N. Isgur, Phys. Rev. D 12, 3770 (1975).

[3]J. F. Gunion, Phys. Rev. D 1, 242 (1974) and references therein.

[4]These quark masses have also been successfully applied in studying the two-photon decays of the pseudoscalar mesons [N. Isgur, Phys. Rev. D 13, 129 (1976)] and K_{l3} decay [N. Isgur, Phys. Rev. D 12, 3666 (1975)].

[5]Although the conventional use of the $(\text{mass})^2$ matrix for mesons is disquieting in a quark model, it is necessary and to some extent justifiable.

[6]After completing this work we discovered that a linear mass formula similar to our $(\text{mass})^2$ formula has been proposed by A. De Rújula and S. L. Glashow, Phys. Rev. Lett. 34, 46 (1975). The use of a linear mass formula, however, leads directly to a number of difficulties, among which are a failure to adequately describe the pseudoscalar meson nonet, and quark masses which are nonet-dependent [see J. Pasupathy and G. Rajasekaran, Phys. Rev. Lett. 34, 1250 (1975)].

[7]For example, the ratio of the differential cross sections for $\pi^- p \to \eta n$ and $\pi^- p \to \eta' n$ at $p_{\text{lab}} = 40$ GeV/c [W. D. Apel *et al.*, in *Proceedings of the XVII International Conference on High Energy Physics, London, 1974*, edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. I-41] implies that $\theta_P = -19 \pm 2°$.