Cartesian analysis of angular distributions in $e^+e^- \to \psi' \to \psi\gamma\gamma' \to \mu^+\mu^-\gamma\gamma'$ *

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Angular distributions for the cascade $\psi' \to \chi\gamma' \to \psi\gamma\gamma' \to \mu^+\mu^-\gamma\gamma'$ are calculated for $S_\chi = 0, 1, 2$. In a nonrelativistic quark model the ratio of the partial rates Γ_{M2}/Γ_{E1} can be calculated and is independent of the shape of the central binding potential. This ratio is of the order of (photon energy/quark mass)^2 and is numerically $\lesssim 0.0025$. This supports the use of the dipole approximation for the radiation. The use of Cartesian tensors simplifies the determination of the angular distributions significantly.

Preliminary data[1] indicate that the ψ' (see Ref. 2) [the $\psi(3.7)$] has a radiative decay mode into a new particle (or perhaps several particles) of narrow width which we denote by χ: $\psi' \to \chi\gamma'$. This new χ state itself decays radiatively into a ψ [the $\psi(3.1)$] particle[3]: $\chi \to \psi\gamma$. The important determination of the spin S_χ of the χ is possible from data on angular distributions in the cascade decay $\psi' \to \chi\gamma' \to \psi\gamma'\gamma$. Some of these distributions for $S_\chi = 0, 1, 2$ have been published.[4-8] In this brief note we present the distributions for the complete process $e^+e^- \to \psi' \to \chi\gamma' \to \psi\gamma'\gamma \to \mu^+\mu^-\gamma'\gamma$, taking account of the polarization in the initial e^+e^- collision and of the analysis of the final ψ polarization provided by the $\mu^+\mu^-$ angular correlations. As shown below, dipole radiation is likely to dominate. The separate cases $S_\chi = 0, 1, 2$ are calculated using a Cartesian tensor method, assuming dipole radiation. We should like to emphasize that this method greatly simplifies the calculation of such processes if the particle spins are not excessively large.

We use a simple nonrelativistic quark model to estimate the significance of quadrupole and octupole radiation. We take the ψ' and ψ to be ^3S states and χ to be $^1S, {}^3P_0, {}^3P_1, {}^3P_2$ states. The parity of the ψ', ψ, and $\chi(^1S)$ is odd; the parity of the $\chi(^3P)$ states is even. It follows that the decay $\psi' \to \chi(^1S)\gamma$ must be a pure $M1$ transition. Since there is no ambiguity for the $\chi(^1S)$ we turn to the ^3P states.

In our nonrelativistic model, the ψ's and χ's are composed of bound $q\bar{q}$ pairs described by a Hamiltonian in the center-of-mass frame:

$$H = \frac{p^2}{m_Q} + V(|\vec{r}|), \tag{1}$$

where $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$ and $\vec{r} = (\vec{r}_1 - \vec{r}_2)$ are center-of-mass variables. The potential, V, may depend on the spin state, but is a purely central potential so that $\vec{L}$ and $\vec{S}$ commute with H. *We shall not make any assumption about the actual shape of the potential, V.* In the center-of-mass frame, the interaction of the quarks with the electromagnetic field is taken to be

$$H^{(\gamma)} = \frac{e_Q}{m_Q}\{\vec{p}\cdot[\vec{A}(\vec{r}/2)+\vec{A}(-\vec{r}/2)] + \tfrac{1}{8}g(\vec{\sigma}_1+\vec{\sigma}_2)\cdot[\vec{B}(\vec{r}/2)-\vec{B}(-\vec{r}/2)] + \tfrac{1}{8}g(\vec{\sigma}_1-\vec{\sigma}_2)\cdot[\vec{B}(\vec{r}/2)+\vec{B}(-\vec{r}/2)]\}, \tag{2}$$

where g is the gyromagnetic ratio for the quark whose mass is m_Q and whose charge is e_Q. Since we are considering transitions $^3S \to {}^3P$, with symmetric spin combinations, only $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2$ contributes. The leading nonzero contributions to the transition matrix elements in the static limit are given by

$$H_{fi}^{(\gamma)} = \left\langle \chi(^3P)\left|\frac{e_Q}{m_Q}\left[2\vec{p}\cdot\vec{\epsilon} - \frac{g}{2}\vec{S}\cdot(\vec{k}\times\vec{\epsilon})\vec{k}\cdot\frac{\vec{r}}{2}\right]\right|\psi'\right\rangle, \tag{3}$$

where the photon has momentum $\vec{k}$ and polarization $\vec{\epsilon}$. The first term gives rise to $E1$ radiation. The second contains both $E1$ and $M2$. It is apparent that there is no $E3$ radiation since only $|\Delta L| = 1$ operators can contribute to an S-P transition and the $\vec{\mu}\cdot\vec{B}$ interaction gives at most $|\Delta S| = 1$.

The $M2$ matrix element is isolated by symmetrizing in $\vec{S}$ and $\vec{r}$. The $E1$ radiation arising from $\vec{p}\cdot\vec{\epsilon}$ may be compared directly with the $M2$ radiation since the matrix elements of $\vec{p}$ and $\vec{r}$ are related via Eq. (1): $[H, \vec{r}] = -2i\vec{p}/m_Q$. The ratios of the partial rates for $\psi' \to \chi(^3P_J)\gamma$ [the ratios for $\chi(^3P_J) \to \psi\gamma$ are the same except for the obvious statistical factors] are found to be

$$\frac{\Gamma_{M2}}{\Gamma_{E1}} = \left(\frac{gk}{8m_Q}\right)^2 \times \begin{cases} 0, & S_\chi = 0, \\ 1, & S_\chi = 1, \\ \frac{9}{5}, & S_\chi = 2. \end{cases} \tag{4}$$

Taking the nominal values $g = 2$, $k = 300$ MeV, and $m_Q = 2$ GeV, we find $\Gamma_{M2}/\Gamma_{E1} \lesssim 0.0025$. This analysis suggests that the dipole approximation should suffice, even for amplitudes, where the $M2$ effect should be about 5%.

Although we shall henceforth assume that the radiation is purely dipole, it should be noted that our preceding estimate is based on some assumptions of particular concern. The first is the assumption that the forces in both the spin-singlet and spin-triplet channels are central. The second is that the quarks couple to the electromagnetic current in the naive fashion. The former assumption neglects forces such as the spin-orbit force and the tensor force which might be called for in a detailed phenomenological treatment of the $q\bar{q}$ interaction. The latter assumption is in contrast to the approach of the Melosh transformation, in which the electromagnetic current has a more complex form when written in terms of the (constituent) quarks which are bound by a nonrelativistic potential.[6,7] If the Melosh transformation approach is correct, there should be significant $M2$ amplitudes for $S_\chi = 1, 2$ although $E3$ amplitudes would still be expected to vanish. Finally, we must note that we have approximated phase factors $e^{i\vec{k}\cdot\vec{r}}$ by unity in Eq. (3) since we expect that $kr < 1$.

We have chosen a nonrelativistic quark model for estimating the importance of quadrupole and octupole radiation. We are unwilling to use the detailed predictions of the model literally. For example, it would be dangerous to interpret $(gk/8m_Q)$, which could be determined from the $E1$-$M2$ interference, as actually measuring quark properties. We proceed now to the calculation of angular distributions on the assumption that dipole radiation dominates, as is suggested by our model analysis above. Of course, the angular distributions we obtain depend only on our use of the dipole approximation, and not on the hypothesis that the new particles are nonrelativistic bound states of a quark and an antiquark.

We denote the polarization vector of the 1^- ψ' (or the 1^- ψ) by ψ'_k (or ψ_k). If the χ particle has spin 2, its spin state is described by a symmetrical, traceless spin tensor χ_{kl}. The spin state of a spin-1 χ particle is described by the vector χ_k. We represent the photon spin in the decay $\psi' \to \chi\gamma'$ (or in the subsequent decay $\chi \to \psi\gamma$) by an electric or magnetic field vector E'_k or B'_k (or E_k or B_k). If the χ parity is even, we have the following minimal, gauge-invariant, static ($E1$) couplings for the $\psi' \to \chi\gamma'$:

$$1^- \to 2^+\gamma:\ E'^*_k \chi^*_{kl} \psi'_l , \tag{5}$$

$$1^- \to 1^+\gamma:\ E'^*_k \chi^*_l \psi'_m \epsilon_{klm} , \tag{6}$$

$$1^- \to 0^+\gamma:\ E'^*_k \psi'_k . \tag{7}$$

The couplings for the subsequent $\chi \to \psi\gamma$ decay are similar. If the χ parity is odd, an electric vector E_k should be replaced by a magnetic vector B_k. This has no effect on the final distributions if the photon polarizations are not observed, which we take to be the situation. Since in the ψ' rest frame the recoil velocities of the χ and ψ are much less than c ($v^2/c^2 \ll 1$), the static approximation suffices. We use the polarization sums

$$\sum_{\text{pol}} \chi^*_{kl}\chi_{mn} = \tfrac{1}{2}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) - \tfrac{1}{3}\delta_{kl}\delta_{mn} , \tag{8}$$

$$\sum_{\text{pol}} \chi^*_k \chi_l = \delta_{kl} \tag{9}$$

to compute the angular dependence of the matrix elements for the cascade $\psi' \to \chi\gamma' \to \psi\gamma\gamma'$ in the three cases:

$$1^- \to 2^+\gamma'\gamma:\ (\vec{E}^*\cdot\vec{E}'^*)(\vec{\psi}^*\cdot\vec{\psi}') + (\vec{\psi}^*\cdot\vec{E}'^*)(\vec{\psi}'\cdot\vec{E}^*) - \tfrac{2}{3}(\vec{\psi}^*\cdot\vec{E}^*)(\vec{\psi}'\cdot\vec{E}'^*) , \tag{10}$$

$$1^- \to 1^+\gamma'\gamma:\ (\vec{\psi}^*\cdot\vec{E}'^*)(\vec{E}^*\cdot\vec{\psi}') - (\vec{\psi}^*\cdot\vec{\psi}')(\vec{E}^*\cdot\vec{E}'^*) , \tag{11}$$

$$1^- \to 0^+\gamma'\gamma:\ (\vec{\psi}^*\cdot\vec{E}^*)(\vec{E}'^*\cdot\vec{\psi}') . \tag{12}$$

The decay rate involves the absolute square of one of these quantities. We assume that the photon polarizations are not observed and use

$$\sum_{\text{pol}} E'^*_k E'_l = \delta_{kl} - \hat{k}'_k \hat{k}'_l , \tag{13}$$

$$\sum_{\text{pol}} E^*_k E_l = \delta_{kl} - \hat{k}_k \hat{k}_l . \tag{14}$$

Here $\hat{k}'$ is a unit vector along the direction of the first photon emitted in the decay cascade ($\psi' \to \chi\gamma'$) and $\hat{k}$ is a unit vector along the direction of the final photon ($\chi \to \psi\gamma$). The spin alignment of the final ψ particle is analyzed by its leptonic decay, $\psi \to e^+e^-$ or $\psi \to \mu^+\mu^-$, which provides further angular information. Thus, we work out the decay chain in which the ψ undergoes a leptonic decay involving the couplings $\bar{u}\vec{\gamma}\cdot\vec{\psi}v$, where u (v) is the Dirac spinor of the lepton (antilepton). Since the masses of the leptons are insignificant compared to their energies, we can neglect these masses and secure the replacement

$$\psi^*_k \psi_l \to \sum_{\text{pol}} \bar{u}\gamma_k v \bar{v}\gamma_l u \propto (\delta_{kl} - \hat{l}_k \hat{l}_l) . \tag{15}$$

Here $\hat{l}$ is a unit vector along the direction of one of the leptons. (The angular distribution of other modes where the ψ spin is not analyzed is obtained simply by averaging over the direction of $\hat{l}$.) The colliding e^+e^- producing the original ψ' may be polarized along the direction $\hat{h}$ of the magnetic field in which they move, a direction perpendicular to their line of flight at collision, which we specify by $\hat{n}$. Again we can safely neglect the mass of the electron and derive,[9] in a manner similar to the work leading to Eq. (15),

$$\psi_k'^{*}\psi_l' \to (1-\mathcal{P}^2)(\delta_{kl}-\hat{n}_k\hat{n}_l)+2\mathcal{P}^2\hat{h}_k\hat{h}_l\,, \tag{16}$$

where $\mathcal{P}$ $(-\mathcal{P})$ is the degree of polarization of the incident e^- (e^+), with $0\leqslant\mathcal{P}^2\leqslant 1$. The full angular distributions which follow from Eqs. (10) to (16) are given by the following.

$S_\chi=2$:

$$\begin{aligned}
W_2={}&\tfrac{81}{320}(1-\mathcal{P}^2)\{[1+(\hat{k}'\cdot\hat{k})^2][1+(\hat{n}\cdot\hat{l})^2]+[1+(\hat{k}'\cdot\hat{l})^2][1+(\hat{n}\cdot\hat{k})^2]+\tfrac{4}{9}[1+(\hat{k}\cdot\hat{l})^2][1+(\hat{n}\cdot\hat{k}')^2]\\
&-\tfrac{2}{3}[(\hat{k}'\cdot\hat{k})^2+(\hat{l}\cdot\hat{k})^2+(\hat{l}\cdot\hat{k}')^2-(\hat{l}\cdot\hat{k}')(\hat{l}\cdot\hat{k})(\hat{k}'\cdot\hat{k})-1+(\hat{n}\cdot\hat{l})^2+(\hat{n}\cdot\hat{k}')^2\\
&+(\hat{n}\cdot\hat{k})^2-(\hat{n}\cdot\hat{k}')(\hat{n}\cdot\hat{k})(\hat{k}'\cdot\hat{k})-(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k})(\hat{l}\cdot\hat{k})-(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k}')(\hat{l}\cdot\hat{k}')]\\
&+2(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k})(\hat{l}\cdot\hat{k}')(\hat{k}'\cdot\hat{k})-\tfrac{4}{3}(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k}')(\hat{l}\cdot\hat{k})(\hat{k}'\cdot\hat{k})-\tfrac{4}{3}(\hat{n}\cdot\hat{k}')(\hat{n}\cdot\hat{k})(\hat{l}\cdot\hat{k}')(\hat{l}\cdot\hat{k})\}\\
&+\tfrac{81}{160}\mathcal{P}^2\{[1+(\hat{k}\cdot\hat{k}')^2][1-(\hat{h}\cdot\hat{l})^2]+[1+(\hat{k}'\cdot\hat{l})^2][1-(\hat{h}\cdot\hat{k})^2]+\tfrac{4}{9}[1+(\hat{k}\cdot\hat{l})^2][1-(\hat{h}\cdot\hat{k}')^2]\\
&-\tfrac{2}{3}[1-(\hat{h}\cdot\hat{l})^2-(\hat{h}\cdot\hat{k}')^2-(\hat{h}\cdot\hat{k})^2+(\hat{h}\cdot\hat{k}')(\hat{h}\cdot\hat{k})(\hat{k}'\cdot\hat{k})+(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k})(\hat{l}\cdot\hat{k})+(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k}')(\hat{l}\cdot\hat{k}')]\\
&-2(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k})(\hat{l}\cdot\hat{k}')(\hat{k}'\cdot\hat{k})+\tfrac{4}{3}(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k}')(\hat{l}\cdot\hat{k})(\hat{k}'\cdot\hat{k})+\tfrac{4}{3}(\hat{h}\cdot\hat{k})(\hat{h}\cdot\hat{k}')(\hat{k}'\cdot\hat{l})(\hat{k}\cdot\hat{l})\}\,.
\end{aligned}\tag{17}$$

$S_\chi=1$:

$$\begin{aligned}
W_1={}&\tfrac{27}{64}(1-\mathcal{P}^2)\{[1+(\hat{n}\cdot\hat{k})^2][1+(\hat{l}\cdot\hat{k}')^2]+[1+(\hat{n}\cdot\hat{l})^2][1+(\hat{k}'\cdot\hat{k})^2]\\
&-2[(\hat{k}'\cdot\hat{k})^2+(\hat{l}\cdot\hat{k}')^2+(\hat{l}\cdot\hat{k})^2-(\hat{k}'\cdot\hat{l})(\hat{k}\cdot\hat{l})(\hat{k}'\cdot\hat{k})-1+(\hat{n}\cdot\hat{l})^2+(\hat{n}\cdot\hat{k}')^2\\
&+(\hat{n}\cdot\hat{k})^2-(\hat{n}\cdot\hat{k}')(\hat{n}\cdot\hat{k})(\hat{k}'\cdot\hat{k})-(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k})(\hat{l}\cdot\hat{k})\\
&-(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k}')(\hat{l}\cdot\hat{k}')+(\hat{n}\cdot\hat{l})(\hat{n}\cdot\hat{k})(\hat{l}\cdot\hat{k}')(\hat{k}'\cdot\hat{k})]\}\\
&+\tfrac{27}{32}\mathcal{P}^2\{[1-(\hat{h}\cdot\hat{k})^2][1+(\hat{l}\cdot\hat{k}')^2]+[1-(\hat{h}\cdot\hat{l})^2][1+(\hat{k}'\cdot\hat{k})^2]\\
&-2[1-(\hat{h}\cdot\hat{l})^2-(\hat{h}\cdot\hat{k}')^2-(\hat{h}\cdot\hat{k})^2+(\hat{h}\cdot\hat{k}')(\hat{h}\cdot\hat{k})(\hat{k}'\cdot\hat{k})+(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k})(\hat{l}\cdot\hat{k})\\
&+(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k}')(\hat{l}\cdot\hat{k}')-(\hat{h}\cdot\hat{l})(\hat{h}\cdot\hat{k})(\hat{l}\cdot\hat{k}')(\hat{k}'\cdot\hat{k})]\,.
\end{aligned}\tag{18}$$

$S_\chi=0$:

$$W_0=\tfrac{9}{16}(1-\mathcal{P}^2)\{[1+(\hat{n}\cdot\hat{k}')^2][1+(\hat{l}\cdot\hat{k})^2]\}+\tfrac{9}{8}\mathcal{P}^2\{[1-(\hat{h}\cdot\hat{k}')^2][1+(\hat{l}\cdot\hat{k})^2]\}\,. \tag{19}$$

Here we have normalized the distributions such that their average over all angles is unity.

If the final ψ particle decays into modes which do not analyze its spin or if the leptons in its leptonic decay mode are not observed, then the appropriate angular distributions are obtained by averaging the results above over the direction of $\hat{l}$:

$$\begin{aligned}
\overline{W}_2={}&\tfrac{3}{80}(1-\mathcal{P}^2)[22+6(\hat{k}'\cdot\hat{k})^2+6(\hat{n}\cdot\hat{k})^2+(\hat{n}\cdot\hat{k}')^2+3(\hat{n}\cdot\hat{k}')(\hat{n}\cdot\hat{k})(\hat{k}'\cdot\hat{k})]\\
&+\tfrac{3}{80}\mathcal{P}^2[29+9(\hat{k}'\cdot\hat{k})^2-12(\hat{h}\cdot\hat{k})^2-2(\hat{h}\cdot\hat{k}')^2-6(\hat{h}\cdot\hat{k}')(\hat{h}\cdot\hat{k})(\hat{k}'\cdot\hat{k})]\,,
\end{aligned}\tag{20}$$

$$\overline{W}_1=\tfrac{9}{16}(1-\mathcal{P}^2)[2-(\hat{n}\cdot\hat{k}')^2+(\hat{n}\cdot\hat{k}')(\hat{n}\cdot\hat{k})(\hat{k}'\cdot\hat{k})]+\tfrac{9}{16}\mathcal{P}^2[1+(\hat{k}'\cdot\hat{k})^2+2(\hat{h}\cdot\hat{k}')^2-2(\hat{h}\cdot\hat{k}')(\hat{h}\cdot\hat{k})(\hat{k}'\cdot\hat{k})]\,, \tag{21}$$

$$\overline{W}_0=\tfrac{3}{4}(1-\mathcal{P}^2)[1+(\hat{n}\cdot\hat{k}')^2]+\tfrac{3}{2}\mathcal{P}^2[1-(\hat{h}\cdot\hat{k}')^2]\,. \tag{22}$$

If only the angle between the two photons is observed, then we average over $\hat{n}$ and $\hat{h}$ (the correlation of these two vectors can be disregarded since they never appear together) to obtain these distributions[4,5]:

$$\overline{W}_2=\tfrac{1}{80}[73+21(\hat{k}'\cdot\hat{k})^2]\,, \tag{23}$$

$$\overline{W}_1=\tfrac{3}{16}[5+(\hat{k}'\cdot\hat{k})^2]\,, \tag{24}$$

$$\overline{W}_0=1\,. \tag{25}$$

All of the above results may be obtained with techniques based on spherical rather than Cartesian tensors.[10] For intermediate states with spin greater than 2 the spherical tensors may be more convenient. However, for spins less than 3 the Cartesian tensors provide a significantly more efficient procedure.

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[9]We thank J. L. Rosner for pointing out that the polarized e^+e^- beams provide no additional information beyond that obtainable from unpolarized beams. A purely polarized beam, say one with a density matrix $\hat{z}\hat{z}$, is equivalent to three orientations of an unpolarized beam:

$$\hat{z}\hat{z} = \tfrac{1}{3}[(1 - \hat{x}\hat{x}) + (1 - \hat{y}\hat{y}) - (1 - \hat{z}\hat{z})].$$

Data so far available have $|\mathcal{P}| \ll 1$. For data with significant beam polarization, the $\mathcal{P}$-dependent angular distributions would be required.

[10]See, for example, H. Frauenfelder and R. M. Steffen, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, Vol. 2, edited by K. Siegbahn (North-Holland, Amsterdam, 1966), pp. 1007–1032.