Matrix Padé approximants and the Bethe-Salpeter equation of the N-N interaction

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We discuss the problem of which diagrams should be taken for the matrix Padé analysis and for the Bethe-Salpeter equation. We relate the "potential" of the Bethe-Salpeter equation with bare intermediate nucleons to the "potential" of the Bethe-Salpeter equation with dressed intermediate nucleons. We disucss the origin of the differences in recent calculations of the nucleon-nucleon phase shifts based on matrix Padé approximants.

Recently several papers appeared in which the N-N phase shifts were calculated by using the matrix Padé approximants.[1-6] The calculations give similar but not exactly the same results. In order to clarify the origin of these differences we would like to present a critical evaluation of the formalisms used in the above-mentioned papers. All of these papers base their considerations on amputated four-point functions. In some of these papers the considerations are based directly on these amputated four-point functions,[2,5,6] while in other papers the Bethe-Salpeter equation (BSE), which generates all the amputated four-point functions, is employed.[1,3,4] We would expect therefore that all the above-mentioned approaches, which treat the same quantities, should lead to the same results, if no approximations are used.

In the following we shall discuss the problems of amputated diagrams related to the above papers. Next the BSE with intermediate dressed nucleon propagators will be analyzed. Our aim will be to find a proper form, equivalent to the BSE, where the Green's function of the intermediate nucleons will be composed of two bare nucleon propagators. Finally we will inspect the vertex renormalization used in the above papers.

First we treat the problem of amputated diagrams. We will demonstrate that no diagrams with nucleon legs are permitted. Although this should be obvious by definition, we make the demonstration for heuristic purposes. Let us take for example the amputated diagrams of Fig. 1. They are obtained by amputating the diagrams of Fig. 2, where in Fig. 2 the heavy circles denote that the nucleon propagators are fully dressed. Let us examine Fig. 3. If the dressed nucleon propagators of the diagrams (a) and (b) of Fig. 3 are expanded in a perturbation series, one can find that the expansion of diagram (b) is part of diagram (a), i.e., it is contained in it. As a consequence, if the diagrams of Fig. 2 are expanded in a perturbation series, the diagram (b) is contained in the diagram (a). This is an example of double counting of diagrams. Going back to Fig. 1 we infer that diagrams (a) and (b) originated from the same ancestor (before amputation), therefore the diagram (b) should not be included in the list of amputated diagrams from which the amputated four-point Green's function is composed. The same reasoning can be adopted to the other amputated diagrams which still contain dressed legs. The conclusion can be formulated in a short way: The amputated four-point Green's function is decomposed into amputated four-point diagrams without dressed nucleon legs.

The BSE (in a form similar to the Lippmann-

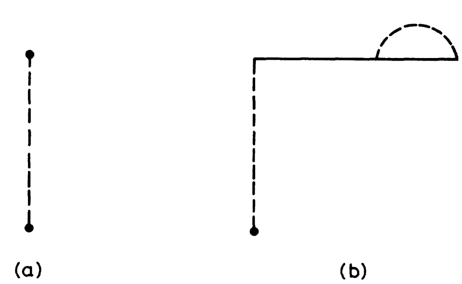

FIG. 1. Amputated diagrams.

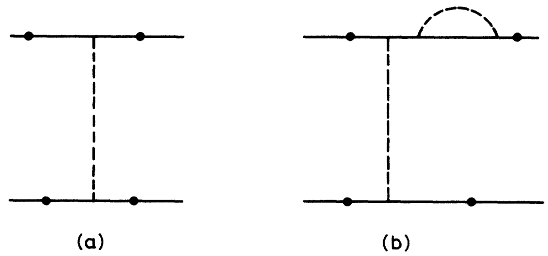

FIG. 2. The diagrams before amputation.

Schwinger equation) can be written in an operational form,

$$M=I+IS_1'S_2'M\,, \tag{1}$$

where M is the sum of all amputated four-point diagrams without dressed legs, I is the sum of all amputated dressed irreducible skeletons, S_1' and S_2' are the completely dressed propagators of the intermediate nucleons. The BSE, for the sake of technical difficulties, is usually handled in approximations in which the dressed nucleon propagator S' is replaced by the bare propagator S and the "potential" I is approximated by the simplest diagrams. The following question arises: What should be the form of the BSE if S' is replaced by S? We shall consider a few possibilities. First,

$$M=I+I'S_1S_2M\,. \tag{2}$$

The above equation can be obtained directly from Eq. (1) if the fully dressed propagator S' is presented in the form

$$S'=(1-S\Sigma)^{-1}S\,, \tag{3}$$

where Σ is the nucleon mass operator. Substituting Eq. (3) into Eq. (1) we obtain

$$M=I+I(1-S_1\Sigma_1)^{-1}(1-S_2\Sigma_2)^{-1}S_1S_2M\,. \tag{4}$$

Hence, comparing Eq. (4) with Eq. (2) we obtain

$$I'=I(1-S_1\Sigma_1)^{-1}(1-S_2\Sigma_2)^{-1}\,. \tag{5}$$

This form of the "potential" is suitable for bound-state problems, for which Eq. (1) should be replaced by

$$M=IS_1'S_2'M\,, \tag{6}$$

or

$$M=I'S_1S_2M\,. \tag{7}$$

I' now has dressed legs from the right-hand side. I' can be regarded as a "potential" only for the bound-state problem, but not as a "potential" for the scattering problem, because in Eq. (2) $I'\neq I$. But if we replace Eq. (1) by

$$M=\tilde I+\tilde IS_1S_2M\,, \tag{8}$$

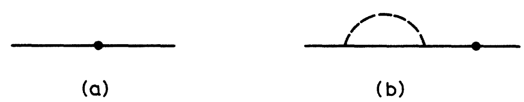

FIG. 3. Dressed nucleon propagators.

the $\tilde I$ can be regarded as the "potential" for the scattering problem. Thus we have come to the second case of the "potential." Let us find the form of $\tilde I$. As a matter of fact Eq. (8) can serve as its definition. It can be obtained as the solution of the equation

$$\tilde I=M-\tilde IS_1S_2M\,, \tag{9}$$

if M, the sum of all amputated diagrams (without nucleon dressed legs), is known.

Formally one can solve Eq. (8) by

$$\tilde I=M(1+S_1S_2M)^{-1}\,. \tag{10}$$

Equation (10) may have the expansion

$$\tilde I=M-MS_1S_2M+MS_1S_2MS_1S_2M-\cdots\,, \tag{11}$$

from which the contributing diagrams to $\tilde I$ can be evaluated. But we can get a more transparent relation. For this purpose we write Eq. (2) as

$$M=(1-I'S_1S_2)^{-1}I\,. \tag{12}$$

Substituting Eq. (12) into Eq. (10) we obtain

$$\begin{aligned}\tilde I&=(1-I'S_1S_2)^{-1}I[1+S_1S_2(1-I'S_1S_2)^{-1}I]^{-1}\\&=\{[1+S_1S_2(1-I'S_1S_2)^{-1}I][(1-I'S_1S_2)^{-1}I]^{-1}\}^{-1}\\&=\{[(1-I'S_1S_2)^{-1}I]^{-1}+S_1S_2\}^{-1}\\&=[I^{-1}(1-I'S_1S_2)+S_1S_2]^{-1}\\&=(1-I'S_1S_2+IS_1S_2)^{-1}I\,.\end{aligned}$$

Hence we have shown that

$$\tilde I=[1-(\Delta I)S_1S_2]^{-1}I\,, \tag{13}$$

where

$$\Delta I=I'-I=I'(S_1\Sigma_1+S_2\Sigma_2-S_1\Sigma_1S_2\Sigma_2)\,, \tag{14}$$

where the last relation was obtained from Eq. (5).

From Eq. (13) we can get the equation

$$\tilde I=I+\Delta IS_1S_2\tilde I\,. \tag{15}$$

From Eq. (13) or Eq. (15) we can get the expansion

$$\tilde I=I+\Delta IS_1S_2I+\Delta IS_1S_2\Delta IS_1S_2I+\cdots\,. \tag{16}$$

The above relation enables us to get a simple diagrammatic interpretation of $\tilde I$. From Eq. (16) and Eq. (14) we can see that the "potential" $\tilde I$ of the BSE with bare intermediate nucleon propagators S_1S_2 has no dressed nucleon legs.

The above are not the only equivalent forms of the Bethe-Salpeter equation. As a matter of fact there exist an infinite number of them which instead of the amputated function of Eq. (1) employ a function related to M and equal to it on the mass shell. Let us consider the possibilities

$$f_2 M f_1 = f_2 I f_1 + f_2 I f_3 S_1 S_2 f_4 M f_1 , \quad (17)$$

where f_1, f_2, f_3, f_4 are operators depending on $S_1\Sigma_1$ and on $S_2\Sigma_2$ and are equal to the identity operator when the external momenta are on the mass shell. Moreover,

$$f_3 S_1 S_2 f_4 = (1 - S_1\Sigma_1)^{-1}(1 - S_2\Sigma_2)^{-1} S_1 S_2 . \quad (18)$$

Equation (17) can be recast in a form which seems to be the most symmetric one by setting

$$f_2 = f_4 = (1 - \Sigma_1 S_1)^{-1/2}(1 - \Sigma_2 S_2)^{-1/2} , \quad (19a)$$

$$f_1 = f_3 = (1 - S_1\Sigma_1)^{-1/2}(1 - S_2\Sigma_2)^{-1/2} . \quad (19b)$$

Above, the following identity is understood to hold:

$$(1 - S\Sigma)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-S\Sigma)^n . \quad (20)$$

Now Eq. (17) will have the form

$$\hat{M} = \hat{I} + \hat{I} S_1 S_2 \hat{M} , \quad (21)$$

where

$$\hat{M} = f_2 M f_1 , \quad (22a)$$

$$\hat{I} = f_2 I f_1 , \quad (22b)$$

and f_1, f_2 are given by Eqs. (19) and (20). The new interaction kernel $\hat{I}$ will contain now diagrams with dressed legs according to the procedure given by Eqs. (22), (19), and (20). The above form was used in Ref. 1. For example, in the above case, the diagram of Fig. 4 will appear in the interaction kernel $\hat{I}$ with a factor $\frac{1}{2}$ which results from the binomial expansion (20).

All the above-presented forms should lead to the same solution of the complete Bethe-Salpeter equation. However, if approximations are used, different results might be obtained by using different schemes. This follows since the matrix Padé approximants might be applied not only to the amputated function M (as in Refs. 3 and 4) but also to the four-point function $f_2 M f_1$ of Eq. (17). In Ref. 1 the form given by Eqs. (19) and (20) is chosen. In Ref. 5 the form

$$f_2 = (1 - \Sigma_1 S_1)^{-1}(1 - \Sigma_2 S_2)^{-1} , \quad (23a)$$

$$f_1 = (1 - S_1\Sigma_1)^{-1}(1 - S_2\Sigma_2)^{-1} \quad (23b)$$

is chosen. The different forms may lead to different matrix Padé approximants, although in the limit of taking into account all diagrams the results should coincide.

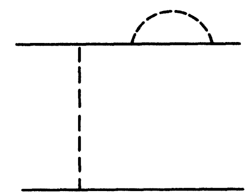

FIG. 4. The fourth-order diagram in dispute.

We will now find the above discussion useful in analyzing the previously mentioned papers. There are two differences between the work of Refs. 1, 2, 5, and 6 and Refs. 3 and 4:

(a) In Refs. 1, 2, 5, and 6 the wave-function renormalization diagram b of Fig. 1 is incorporated in the calculations, while in Refs. 3 and 4 it is not. In Ref. 1 the above diagram is incorporated in the BSE.

(b) There is a difference in treating the vertex diagram. In Refs. 3 and 4 the contribution of the fourth-order vertex diagram is based on the derivation of Wortman,[7,8] while in Refs. 1, 2, 5, and 6 a more recent and more complete derivation is presented. We found that in Wortman's derivation[7,8] terms, which vanish after being sandwiched between positive-energy spinors, were dropped out, because in his work he was concerned with positive-energy spinors only. On the other hand, these missing terms in Refs. 3 and 4 were used in Refs. 1, 2, 5, and 6.

In conclusion we can summarize our findings. Our main derived results are Eq. (13) and Eq. (17) which relates the "potential" of the BSE with bare intermediate nucleons to the "potential" of the BSE with dressed intermediate nucleons. These derivations might be helpful in the future in establishing which diagrams should be taken for the matrix Padé analysis and the BSE. We have pointed out the inconsistencies existing in Refs. 1–6 which result from the use of different four-point functions to which different matrix Padé approximants correspond. However, all the calculations of Refs. 1–6 can be consistent within the framework of their four-point function schemes. The deviations observed by using matrix Padé approximants corresponding to the different schemes may serve as indication of the convergence properties.

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