# Dual currents in arbitrary space-time dimension*

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Schwarz has shown a procedure for describing off-mass-shell states in dual models. The construction of such states is an important preliminary to constructing physical currents. The basic requirement motivating the construction of such states is that amplitudes with off-mass-shell lines should contain exactly the same spectrum of hadrons as the corresponding on-mass-shell model. This condition, which one must necessarily impose if one hopes to describe physical currents, led Schwarz to construct a dual current for a particular choice of space-time dimension $D$, namely $D = 16$. Since this is not the critical dimension of the Veneziano model, a modification of the model appears to be necessary in order to bring the two dimensions into coincidence. In this paper we construct a method whereby it may be possible to find solutions which hold for any choice of space-time dimension $D$. We construct this current perturbatively, and find a unique solution up to the 6th level, given a few assumptions.

In the $\alpha(0)=1$ Veneziano model[1] it is desired to construct an off-mass-shell scalar state of momentum $q$. We are motivated by the fact that the lack of a realistic dual current has been one of the chief shortcomings of the dual model.[2] We write the state in the form[3] $S\,|0q\rangle$, where $S$ is an operator constructed out of the harmonic-oscillator raising operators $\{a^{\mu}_{-m}\}$ and the momentum operator $a_0^{\mu}$ ($\mu=1,2,\ldots,D;\ m=1,2,\ldots,\infty$).

In order to implement the fundamental spectrum condition we require the state to satisfy the Virasoro conditions[3]:

$$(L_n - L_0 + 1 - n)S\,|0q\rangle = 0, \quad n=1,2,\ldots. \tag{1}$$

It is sufficient to just consider $n=1$ and $n=2$ because $L_1$ and $L_2$ generate all the additional Virasoro conditions through the algebra. Schwarz showed that a solution to (1) is

$$S = e^T, \tag{2}$$

where

$$T \equiv \tfrac{1}{2}\sum_{l,m} A_{lm} a_{-l}\cdot a_{-m} \tag{3}$$

and

$$A_{lm} \equiv \frac{(-1)^{l+m}}{l+m}\binom{-\frac{1}{2}}{l}\binom{-\frac{1}{2}}{m}. \tag{4}$$

However, solution (2) will satisfy (1) only for the case $D=16$. In order to arrive at a solution which does not require a particular choice of $D$ it is now proposed that we try a solution of the form

$$S = S(D) = e^T B(D), \tag{5}$$

where $B$ is an operator such that

$$(L_n - L_0 + 1 - n)e^T B\,|0q\rangle = 0, \quad n=1,2. \tag{6}$$

Using the commutation relations between $L_n$ and $a_{-l}\cdot a_{-m}$ the Virasoro conditions can then be put in the form

$$(L_1 - L_0)e^T B\,|0q\rangle = e^T[L_1 - L_0, B]\,|0q\rangle = 0, \tag{7}$$

$$\begin{aligned}(L_2 - L_0 - 1)e^T B\,|0q\rangle &= e^T\left\{\left[L_2 + \sum_{r=0}^{\infty} A_{1r} a_{-r}\cdot a_1 - L_0, B\right] + (\tfrac{1}{16}D - 1)B\right\}|0q\rangle \\ &= 0.\end{aligned} \tag{8}$$

We now attempt to construct an operator $B$ which will satisfy (7) and (8). To convince oneself that such an operator may in fact exist, consider the following: Let

$$S(D) = 1 + S_1 + S_2 + S_3 + \cdots, \tag{9}$$

where

$$S_1 = K_1 a_0\cdot a_{-1},$$

$$S_2 = K_{21} a_0\cdot a_{-2} + K_{22} a_{-1}\cdot a_{-1} + K_{23} a_0\cdot a_{-1} a_0\cdot a_{-1},$$

i.e., $S_n$ is a linear sum of all products

$$\prod_{i=1}^{r} a_{-x_i}\cdot a_{-y_i}$$

such that

$$\sum_{i=1}^{r}(x_i + y_i) = n$$

(products $a_0\cdot a_0$ are not included). One substitutes (9) into the equations $(L_1 - L_0)S\,|0q\rangle = 0$ and $(L_2 - L_0 - 1)S\,|0q\rangle = 0$ subject only to the requirement that the coefficients be of the form $K = a + bq^2$, where $a$ and $b$ are constraints (recall that $a_0^2 = 2q^2$). It turns out that the resulting simultaneous equations can then be solved uniquely, first for the coefficient on the $S_1$ level, then for those on the $S_2$ level, and so on. Below are shown the results up to the $S_6$ level (with $e^T$ factored out):

$$
\begin{aligned}
S(D) = e^T\Big[1 &+ \frac{u}{2^4}\frac{a(02)}{D} - \frac{u}{2^4}\frac{a(11)}{D} - \frac{u}{2^4}\frac{a(0011)}{D(D-1)} - \frac{u}{2^4}\frac{a(12)}{D} - \frac{u}{2^4}\frac{a(0012)}{D(D-1)} \\
&+ \frac{u}{2^4}\frac{a(03)}{D} + \left(\frac{-3u}{2^6} + \frac{uv}{3\times 2^8}\right)\frac{a(13)}{D} + \left(\frac{-3u}{2^6} + \frac{uv}{3^2\times 2^7}\right)\frac{a(0013)}{D(D-1)} \\
&+ \left(\frac{15u}{2^8} - \frac{uv}{3\times 2^{10}}\right)\frac{a(04)}{D} + \left(\frac{-3u}{2^8} - \frac{uv}{2^{10}}\right)\frac{a(22)}{D} + \left(\frac{-3u}{2^8} - \frac{uv}{2^{11}}\right)\frac{a(0022)}{D(D-1)} \\
&+ \frac{uv}{3\times 2^9}\frac{a(1102)}{D(D-1)} + \left(\frac{-u}{2^6} - \frac{uv}{3\times 2^8}\right)\frac{a(23)}{D} + \left(\frac{-u}{2^6} - \frac{5uv}{3^2\times 2^{10}}\right)\frac{a(0023)}{D(D-1)} \\
&+ \left(\frac{-5u}{2^7} + \frac{uv}{2^9}\right)\frac{a(14)}{D} + \left(\frac{-5u}{2^7} + \frac{uv}{3\times 2^8}\right)\frac{a(0014)}{D(D-1)} + \left(\frac{7u}{2^7} - \frac{uv}{3\times 2^9}\right)\frac{a(05)}{D} \\
&+ \frac{uv}{3\times 2^9}\frac{a(1103)}{D(D-1)} - \frac{uv}{3\times 2^{10}}\frac{a(2201)}{D(D-1)} + \left(\frac{5\times 3\times 7u}{2^{11}} - \frac{19uv}{5\times 2^{12}} + \frac{uv^2}{5\times 3^3\times 2^{13}}\right)\frac{a(06)}{D} \\
&+ \left(\frac{-5^2u}{2^{11}} - \frac{11uv}{3\times 2^{12}} + \frac{uv^2}{3^2\times 2^{13}}\right)\frac{a(24)}{D} + \left(\frac{-5^2u}{2^{11}} - \frac{5uv}{2^{14}} + \frac{uv^2}{3\times 2^{16}}\right)\frac{a(0024)}{D(D-1)} \\
&+ \left(\frac{-5\times 7u}{2^{10}} + \frac{71uv}{5\times 3\times 2^{11}} - \frac{uv^2}{5\times 3^2\times 2^{12}}\right)\frac{a(15)}{D} + \left(\frac{-5\times 7u}{2^{10}} + \frac{7\times 17uv}{5^2\times 3\times 2^{10}} - \frac{uv^2}{5^2\times 3^2\times 2^9}\right)\frac{a(0015)}{D(D-1)} \\
&+ \left(\frac{-5u}{2^{10}} - \frac{uv}{2^{11}} - \frac{uv^2}{3^3\times 2^{12}}\right)\frac{a(33)}{D} + \left(\frac{-5u}{2^{10}} - \frac{uv}{3\times 2^{11}} - \frac{uv^2}{3^4\times 2^{12}}\right)\frac{a(0033)}{D(D-1)} \\
&+ \left(\frac{11\times 7uv}{5\times 3\times 2^{13}} - \frac{uv^2}{5\times 2^{15}}\right)\frac{a(1104)}{D(D-1)} + \left(\frac{23uv}{5\times 3^2\times 2^{11}} + \frac{uv^2}{5\times 3\times 2^{13}}\right)\frac{a(1203)}{D(D-1)} \\
&+ \left(\frac{-19uv}{3^3\times 2^{11}} - \frac{uv^2}{3^4\times 2^{13}}\right)\frac{a(2301)}{D(D-1)} + \left(\frac{13uv}{5\times 3^3\times 2^{10}} - \frac{11uv^2}{5\times 3^4\times 2^{12}}\right)\frac{a(1302)}{D(D-1)} \\
&+ \left(\frac{-uv}{3\times 2^{13}} + \frac{uv^2}{3^2\times 2^{15}}\right)\frac{a(1122)}{D(D-1)} + \left(\frac{-uv}{3\times 2^{13}} + \frac{uv^2}{3\times 2^{16}}\right)\frac{a(001122)}{D(D-1)(D-2)} \\
&+ \cdots\Big], \qquad (10)
\end{aligned}
$$

where

$$u \equiv D - 16, \quad v \equiv D - 8,$$

and

$$a(xy) \equiv a_{-x}\cdot a_{-y}, \qquad (11)$$

$$a(x_1y_1x_2y_2) \equiv 2a(x_1y_1)a(x_2y_2) - a(x_1y_2)a(x_2y_1) - a(x_1x_2)a(y_1y_2), \qquad (12)$$

$$a(001122) \equiv 2[a(00)a(1122) - a(01)a(0122) - a(02)a(0211)]. \qquad (13)$$

In the calculations (10), note the presence of factors $1/D$, $1/(D-1)$, and $1/(D-2)$. These factors may cause $S(D)$ not to be finite. We will see presently a possible way of deleting these factors. Note also the presence in calculations (10) of groupings of harmonic oscillators $a(x_1y_1x_2y_2)$ and $a(00\,11\,22)$. In general, let us define $a(x_1y_1\cdots x_Ny_N)$ in the following way:

$$a(x_1y_1\cdots x_Ny_Nx_{N+1}y_{N+1}) = \tfrac{1}{2}\left[4 - \sum_{j=1}^{N}(S_{x_{N+1},x_j} + S_{x_{N+1},y_j} + S_{y_{N+1},x_j} + S_{y_{N+1},y_j})\right]a(x_{N+1}y_{N+1})a(x_1y_1\cdots x_Ny_N), \qquad (14)$$

where $S_{k,a}A_aB_b = A_kB_b$, $S_{a,b}A_aB_b = A_bB_a$, i.e., $S_{k,a}$ changes indices. One can easily show that (12) and (13) agree with this definition. Also let us define

$$\nabla a(x_1y_1\cdots x_{N+1}\,y_{N+1}) = \tfrac{1}{2}\left(1 + \sum_{j=1}^{N} S_{x_{N+1},x_j}S_{y_{N+1},y_j}\right)\left[4 - \sum_{i=1}^{N}(S_{x_{N+1},x_i} + S_{x_{N+1},y_i} + S_{y_{N+1},x_i} + S_{y_{N+1},y_i})\right]\delta_{1x_{N+1}}\delta_{1y_{N+1}}a(x_1y_1\cdots x_Ny_N). \qquad (15)$$

Then by induction one can show the following to be true:

$$a(x_1y_1\cdots x_jy_j\cdots x_Ny_N)=a(x_1y_1\cdots y_jx_j\cdots x_Ny_N),\quad j=1,2,\ldots,N \tag{16}$$

$$a(x_1y_1\cdots x_jy_j\cdots x_Ky_K\cdots x_Ny_N)=a(x_1y_1\cdots x_Ky_K\cdots x_jy_j\cdots x_Ny_N),\quad j,K=1,2,\ldots,N \tag{17}$$

$$a(x_1y_1x_2y_2\cdots x_Ny_N)+a(x_1y_2x_2y_1\cdots x_Ny_N)+a(x_1x_2y_1y_2\cdots x_Ny_N)=0. \tag{18}$$

It follows from (16), (17), and (18) that $a(x_1y_1\cdots x_Ny_N)$ vanishes if any three indices are equal. The commutation relations turn out to be as follows:

$$[L_1,a(x_1y_1\cdots x_Ny_N)]=\sum_{j=1}^{N}(x_jS_{x_j,x_j-1}+y_jS_{y_j,y_j-1})a(x_1y_1\cdots x_Ny_N), \tag{19}$$

$$[L_2,a(x_1y_1\cdots x_Ny_N)]=\left\{\sum_{j=1}^{N}[\theta(x_j-2)x_jS_{x_jx_j-2}+\theta(y_j-2)y_jS_{y_j,y_j-2}]+\sum_{j=1}^{N}(\delta_{1x_j}S_{x_j,-1}+\delta_{1y_j}S_{y_j,-1})+(D-N+1)\nabla\right\}\times a(x_1y_1\cdots x_Ny_N) \tag{20}$$

(here it is to be understood that the destruction operator $a_1$, resulting from the application of $S_{x,-1}$ to a product, is to stand to the right of all other operators in the product), and

$$[a_{-n}\cdot a_1,a(x_1y_1\cdots x_Ny_N)]=\sum_{j=1}^{N}(\delta_{1x_j}S_{n,x_j}+\delta_{1y_j}S_{n,y_j})a(x_1y_1\cdots x_Ny_N). \tag{21}$$

Suppose we now write $S(D)=e^{T}B(D)$ and express $B$ as a linear combination of the operators $a(x_1y_1\cdots x_Ny_N)$, $N=1,2,3,\ldots$. It seems reasonable to take coefficients having a structure similar to that of the operators $a(x_1y_1\cdots x_Ny_N)$, so let us take coefficients $H(x_1y_1\cdots x_Ny_N)$ having the following properties:

$$H(x_1y_1\cdots x_jy_j\cdots x_Ky_K\cdots x_Ny_N)=H(x_1y_1\cdots x_Ky_K\cdots x_jy_j\cdots x_Ny_N)=H(x_1y_1\cdots y_jx_j\cdots x_Ky_K\cdots x_Ny_N),$$

$$H(x_1y_1x_2y_2\cdots x_Ny_N)+H(x_1y_2x_2y_1\cdots x_Ny_N)+H(x_1x_2y_1y_2\cdots x_Ny_N)=0.$$

Let

$$B(D)=1+\sum_{N=1}^{\infty}\sum_{xy}H(x_1y_1\cdots x_Ny_N)\frac{a(x_1y_1\cdots x_Ny_N)}{D(D-1)\cdots(D-N+1)}, \tag{22}$$

where

$$\sum_{xy}\equiv\sum_{x_1=0}^{\infty}\sum_{y_1=0}^{\infty}\cdots\sum_{x_N=0}^{\infty}\sum_{y_N=0}^{\infty}.$$

Using (19), (20), and (21) we see that (7) is satisfied provided

$$\sum_{j=1}^{N}S_{x_1x_j}S_{y_1y_j}[(x_1+1)H(x_1+1,y_1,x_2y_2\cdots x_Ny_N)+(y_1+1)H(x_1,y_1+1,x_2y_2\cdots x_Ny_N)-(x_1+y_1)H(x_1y_1x_2y_2\cdots x_Ny_N)]=0,$$
$$x,y\geq 0,\quad N=1,2,3,\ldots \tag{23}$$

and (8) is satisfied provided that

$$(\tfrac{1}{16}D-1)+H(1,1)=0 \tag{24}$$

and that

$$\sum_{j=1}^{N}S_{x_1x_j}S_{y_1y_j}[(x_1+2)H(x_1+2,y_1,x_2y_2\cdots x_Ny_N)+(y_1+2)H(x_1,y_1+2,x_2y_2\cdots x_Ny_N)-(x_1+y_1)H(x_1y_1x_2y_2\cdots x_Ny_N)$$
$$+A_{1x_1}H(1,y_1,x_2y_2\cdots x_Ny_N)+A_{1y_1}H(1,x_1,x_2y_2\cdots x_Ny_N)]$$
$$+(\tfrac{1}{16}D-1)H(x_1y_1\cdots x_Ny_N)+(N+1)(N+2)H(1,1,x_1y_1\cdots x_Ny_N)=0,$$
$$x,y\geq 0,\quad N=1,2,3,\ldots. \tag{25}$$

Using (24) and the sets of equations corresponding to $N=1$ and $N=2$ in (23) and (25) one can easily calculate all the $H$ coefficients up to the $S_6$ level and so duplicate the calculations (10) exactly.

The $H$ coefficients can be shown in general to be linear sums of $uv^n$ ($u\equiv D-16$, $v\equiv D-8$, $n=0,1,2,\ldots$). The presence of factors $1/(D-1)$, $1/(D-2)$, etc. will thus cause $B(D)$ to be not finite when $D$ takes on positive integer values. To bypass this difficulty, consider the following procedure: Define

$$\overline{B}(D) \equiv d(d-1)\cdots(d-D)B(d)|_{d=D}, \qquad (26)$$

where $D$ is now taken to be specific positive integer corresponding to the number of dimensions of the space-time. One finds

$$\overline{B}(D)=\sum_{N=D}^{\infty}\sum_{xy}\overline{H}(x_1y_1\cdots x_{N+1}y_{N+1})a(x_1y_1\cdots x_{N+1}y_{N+1}), \qquad (27)$$

where

$$\overline{H}(x_1y_1\cdots x_{N+1}y_{N+1}) \equiv \frac{(-1)^{N-D}}{(N-D)!}H(x_1y_1\cdots x_{N+1}y_{N+1}).$$

On the other hand, upon substituting (27) directly into (7) and (8) one of course finds that the coefficients $H(x_1y_1\cdots x_Ny_N)$, $N=D+1, D+2, D+3, \ldots$, still satisfy the set of equations $N=D+1, D+2, D+3, \ldots$ of (23) and (25).

Thus, assuming that the sets of equations (23) and (25) are solvable, we can take $S(D)=e^T\overline{B}(D)$ as the off-mass-shell operator; i.e., $e^T\overline{B}(D)|0q\rangle$ will be an off-mass-shell scalar state of momentum $q$ in a model in which the dimension of the space-time is $D$.

## CONCLUSION

Given the original assumptions concerning the coefficients in $S$, we have obtained a unique solution to a dual current in an arbitrary number of space-time dimensions up to the 6th level, and a prescription for carrying the calculations to all orders.

Unfortunately, we have not been able to find a closed analytic expression for such a current. However, the uniqueness of our solution is a strong indication that such an expression exists. Also, it is not clear how our solution relates to the theorem of Collins and Friedman.[4] Our solution may be an evasion of their theorem.

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