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Relativistic partial-wave analysis for three-meson systems. II*

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We extend our relativistic partial-wave analysis to the $I > 0$ channels of the three-pion system. For the sake of definiteness, we apply our general procedure to the minimal-dynamics K-matrix model. In particular, we first generate properly symmetrized three-pion states using a group-theoretical approach. Using these, we then construct symmetrized scattering amplitudes and develop the minimal-dynamics K-matrix equations satisfied by the operators which enter into the symmetrized amplitudes. We find that for the $I = 1$ channel of the three-pion system, the $i_{ij} = 0,2$ subsystem isospin channels contribute to the scattering amplitude on the same footing as does the $i_{ij} = 1$ subsystem isospin channel. Thus the calculated properties of $I = 1$ three-pion resonances may be as dependent on the $i_{ij} = 0$ phase parameters as on the $i_{ij} = 1$ phase parameters which were assumed to be dominant in a number of previous calculations.

I. INTRODUCTION

In a previous paper,[1] we proposed a partial-wave analysis of the type used by Omnès[2] for relativistic three-particle systems and applied it to the minimal-dynamics K-matrix three-to-three equations[3] for the $I=0$ channel of the three-pion system. In particular we employed the relativistic definition of the two-body internal momentum rather than the Galilean definition as had been used in previous three-body approaches.[4-8] Further, we found that owing to Bose statistics, the three coupled K-matrix equations of Eq. (3.2) of Ref. 3 for the three-pion $I=0$ channel could be replaced by a single integral equation for an operator χ related to the T^i by Eqs. (5.4) and (5.6) of Ref. 1 and by Eq. (3.2) of the third paper in Ref. 5. It is the purpose of the present study to extend this treatment to the three-pion $I>0$ channels. We will find that since more than one two-body isospin subsystem enters into each of the $I=1,2$ channels, the properly symmetrized amplitudes will be more complicated than those for the $I=0,3$ channels where only a single two-body isospin channel contributes.

This difficulty, as well as our procedure for handling it may be clarified in the following way. We recall that the two-particle permutation group or symmetric group $\mathcal{S}_2$ has two one-dimensional irreducible representations commonly called the symmetric, $\mathcal{S}$, and antisymmetric, $\mathcal{A}$, representations. However for the three-particle symmetric group $\mathcal{S}_3$, besides these two one-dimensional irreducible representations, there is also a two-dimensional irreducible representation, $\mathfrak{M}$, of mixed symmetry. This mixed representation is specified up to a unitary transformation and for the remainder of this treatment we will employ the particular matrix realization given in Table I.

Our three-pion states may be taken to be direct products of three-particle momentum-space states and three-pion isospin states. In constructing totally symmetric three-pion states, one convenient method[9] is to first construct three-particle momentum-space states and three-pion isospin states each of which transforms under permutations according to the symmetric or antisymmetric representations or as the vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ under the mixed representation as given in Table I. Then the linear combinations of the direct product of these states which give rise to totally symmetric outer product states $|\mathcal{S}\psi\rangle$ are formed. Specifically we have

$$|\mathcal{S}\psi\rangle = \begin{cases} |\mathcal{S}(\vec{p}_1\vec{p}_2\vec{p}_3)\rangle|\mathcal{S}(\tau_1\tau_2\tau_3)\rangle\,, \\ |\mathcal{A}(\vec{p}_1\vec{p}_2\vec{p}_3)\rangle|\mathcal{A}(\tau_1\tau_2\tau_3)\rangle\,, \\ |\mathfrak{M}^1(\vec{p}_1\vec{p}_2\vec{p}_3)\rangle|\mathfrak{M}^1(\tau_1\tau_2\tau_3)\rangle \\ \quad+|\mathfrak{M}^2(\vec{p}_1\vec{p}_2\vec{p}_3)\rangle|\mathfrak{M}^2(\tau_1\tau_2\tau_3)\rangle\,, \end{cases} \tag{1.1}$$

where $|\mathfrak{M}^1\rangle$ and $|\mathfrak{M}^2\rangle$ transform as the vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, respectively, under the mixed representation. A similar method has long been employed in the construction of totally antisymmetric three-nucleon states.[10]

It is easily verified that the $I=0$ three-pion isospin state transforms according to the antisymmetric representation and that the $I=3$ isospin state transforms according to the symmetric representation. Thus in Ref. 1 we were able to construct a totally symmetric $I=0$ three-pion state by antisymmetrizing the momentum-space portion of the state. However, it is well known that the $I=2$ and some of the $I=1$ (see Table II) three-pion isospin states transform under permutations according to the mixed representation. Thus in constructing totally symmetrized three-pion $I=1,2$ states, a simple symmetrization or antisymmetrization of the momentum-space por-

TABLE I. Matrix realizations of the irreducible representations of the three-particle symmetric group $\mathcal{S}_3$.

Permutation \ Representation	$\mathcal{S}$	$\mathcal{A}$	$\mathcal{M}$
$\mathcal{P}_{123\to123}$	[1]	[1]	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
$\mathcal{P}_{123\to231}$	[1]	[1]	$\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}$
$\mathcal{P}_{123\to312}$	[1]	[1]	$\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}$
$\mathcal{P}_{123\to132}$	[1]	[−1]	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
$\mathcal{P}_{123\to213}$	[1]	[−1]	$\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}$
$\mathcal{P}_{123\to321}$	[1]	[−1]	$\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}$

tion of the state will not suffice, as seen in Eq. (1.1).

We will generate properly symmetrized amplitudes from states symmetrized in the above way, while carrying out a decomposition of our dynamical equations with respect to both three-body total isospin I and two-body subsystem isospin i_{ij}. Such a decomposition is pertinent to the study of the resonant behavior of the strongly interacting three-pion system both because of total isospin conservation and because of the fact that the observed resonances of this system appear to decay primarily via a single subsystem isospin channel, i.e., the $(i_{ij}=1)$ $\rho+\pi$ channel.

In Sec. II, we construct two sets of states of definite total isospin, one set having definite two-body subsystem isospin and the other having definite permutation symmetry. We will see that while the former states are convenient for the expansion of the two-body t matrices, the latter states are convenient in obtaining symmetrized three-body states as in Eq. (1.1). In Sec. III we generalize our method of generating states of definite total angular momentum and parity and in Sec. IV we use the results of the previous two sections to construct symmetric three-pion states. In Sec. V, we apply these results to the minimal-dynamics K-matrix equations, generate integral equations for suitably defined operators $[X]_{\nu n}$, and express the symmetrized transition amplitudes in terms of these operators. Finally in Sec. VI, we discuss these integral equations and compare our approach to those previous calculations which included only the contribution from the $i_{ij}=1$ subsystem isospin channel[5,7,8,11] and to the recent calculations of Ascoli and Wyld[12] and of Brayshaw[13] which include the contributions of the $i_{ij}=0,1$ systems to the $I=1$ channel analysis of the reaction $\pi^-p\to\pi^-\pi^+\pi^-p$ and to the search for three-body resonant poles in the A_1 channel using the boundary-condition model, respectively. If one, as in previous calculations, is interested only in the positions of the three-pion resonance poles rather than in the construction of Dalitz plots, the use of totally symmetrized amplitudes is unnecessary and Sec. IV and the last half of Sec. V may be omitted. For such calculations, the use of our Eq. (6.7) without the final symmetrizations is sufficient.

II. CONSTRUCTION OF THREE-PION ISOSPIN STATES

The standard procedure for constructing three-body states of total isospin I with a third component M_I from the single-particle isospin states $|\tau_i m_i\rangle$ consists of coupling particles i and j to form two-body isospin i_{ij} and then coupling this to particle k for each of the three cyclic coupling orderings $(ijk)=(123), (231), (312)$. This coupling scheme gives rise to the states

$$|IM_I i_{ij}\rangle^k \equiv \sum_{m_1 m_2 m_3} \langle \tau_i m_i \tau_j m_j | i_{ij} m_{ij}\rangle \times \langle i_{ij} m_{ij} \tau_k m_k | IM_I\rangle |\tau_1 m_1\rangle |\tau_2 m_2\rangle |\tau_3 m_3\rangle, \tag{2.1}$$

with the normalization

$${}^k\langle I' M_I' i_{ij}' | IM_I i_{ij}\rangle^k = \delta_{I'I}\delta_{M_I' M_I}\delta_{i_{ij}' i_{ij}}. \tag{2.2}$$

This three-particle basis is useful for expanding operators which conserve two-body isospin, such as the t matrix for pion-pion scattering. However, these states in general have no special transformation properties under permutations.

We find it convenient to consider also another basis of three-body isospin states whose elements transform under permutations according to the irreducible representations of $\mathcal{S}_3$. We denote such states by $|IM_In\rangle_k$ where, roughly speaking, n specifies the representation under which the state vector $|IM_In\rangle_k$ transforms under permutations. The exact interpretation and range of n for $0 \leq I \leq 3$ is given in Table II. These two bases are related by the unitary transformation

$$|IM_I n\rangle_k = \sum_{i_{ij}} W^I_{ni_{ij}} |IM_I i_{ij}\rangle^k , \tag{2.3}$$

where

$$W^0 = [1],$$

$$W^1 = \begin{bmatrix} \frac{1}{3}\sqrt{5} & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & -\frac{1}{3}\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix},$$

$$W^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.4}$$

and

$$W^3 = [1].$$

The elements of this new isospin basis are normalized as

$$_k\langle I'M_I'n' | IM_In\rangle_k = \delta_{I'I}\,\delta_{M_I'M_I}\,\delta_{n'n} \tag{2.5}$$

and the states of different coupling order have the overlap

$$_k\langle I'M_I'n' |IM_In\rangle_1 = \delta_{I'I}\,\delta_{M_I'M_I'}\,[\mathcal{P}_{123\to kij}]_{nn'} , \tag{2.6}$$

with i, j, and k cyclic and where $\mathcal{P}_{123\to kij}$ are the matrix realizations of the irreducible representations (for $I=1$, the outer product of the $\mathcal{S}$ and $\mathfrak{M}$ representations) given in Table I. Then using Eqs. (2.3)–(2.6) we may evaluate the matrix elements

$$[\hat{\theta}^k]^I_{n'n} \equiv {}_1\langle I'M_I'n' | \hat{\theta}^k |IM_In\rangle_1 \tag{2.7}$$

for an arbitrary operator $\hat{\theta}^k$ which conserves I, M_I, and i_{ij}. The results for $0 \leq I' = I \leq 3$ and $M_I' = M_I$ are given in Table III with the notation

$$\theta^k_{i_{ij}} \equiv {}^k\langle IM_I i_{ij} | \hat{\theta}^k | IM_I i_{ij}\rangle^k . \tag{2.8}$$

TABLE II Transformation properties under permutations of the isospin basis elements $|IM_In\rangle_k$.

Total isospin I	n	Transformation property
3	0	$\mathcal{S}$
2	1	$\mathfrak{M}^1$
	2	$\mathfrak{M}^2$
1	0	$\mathcal{S}$
	1	$\mathfrak{M}^1$
	2	$\mathfrak{M}^2$
0	0	$\mathcal{A}$

III. CONSTRUCTION OF STATES OF DEFINITE TOTAL ANGULAR MOMENTUM

Following the notation of Ref. 1, we define the α frame to be an arbitrarily chosen three-body center-of-mass frame and we consider only those two-body center-of-mass frames γ_{ij} obtained from the α frame by the pure Lorentz boosts of velocity $-[\vec{p}_i(\alpha)+\vec{p}_j(\alpha)]/[e_i(\alpha)+e_j(\alpha)]$. We again label the three initial-state four-momenta in an arbitrary frame η by $(\vec{p}_i(\eta), e_i(\eta))$, where $i=1, 2, 3$, and the three final-state four-momenta in the η frame by $(\vec{p}_i'(\eta), e_i'(\eta))$. By the definitions of the α frame and γ_{ij} frames, we have

$$\sum_i \vec{p}_i(\alpha) = \sum \vec{p}_i'(\alpha) = 0,$$

$$\sum_i e_i(\alpha) \equiv \mathfrak{M}, \tag{3.1}$$

and

$$e_i(\gamma_{ij}) + e_j(\gamma_{ij}) \equiv M_{ij} .$$

We wish to explicitly make the distinction that the momenta subscripts (i.e., b, β, or i) serve to label the single-particle states and the position of that state in the three-particle state vector denotes whether particle number 1, 2, or 3 is in that single-particle state. We consider the three-particle momentum states $|\vec{p}_b'\vec{p}_c'\vec{p}_d'\rangle_\alpha$ and $|\vec{p}_\beta\vec{p}_\gamma\vec{p}_\delta\rangle_\alpha$, where the sets of indices (bcd) and $(\beta\gamma\delta)$ are permutations of (123). We take r and ρ to be the kth elements of the sets (bcd) and $(\beta\gamma\delta)$, respectively, and take the ith and jth elements to be (s,t) and (δ,τ), respectively, where i, j, and k are cyclic. Then as in Eqs. (2.24) and (3.2) of Ref. 1, we have for the on-shell two-body t matrix in the three-body space

$$
{}_{\alpha}\langle \vec{p}_b'\vec{p}_c'\vec{p}_d'|t^k|\vec{p}_\beta\vec{p}_\gamma\vec{p}_\delta\rangle_\alpha = {}_{\alpha}\langle \vec{p}_r'|\vec{p}_\rho\rangle_\alpha \; {}_{\alpha}\langle \vec{p}_s'\vec{p}_t'|\hat{t}\,|\vec{p}_\sigma\vec{p}_\tau\rangle_\alpha
$$

$$
= 8(2\pi)^6 e_r'(\alpha) e_s'(\gamma_{st}) e_t'(\gamma_{st})\, \delta(\vec{p}_r'(\alpha) - \vec{p}_\rho(\alpha))\, \delta((\vec{p}_s'(\alpha) + \vec{p}_t'(\alpha) - \vec{p}_\sigma(\alpha) - \vec{p}_\tau(\alpha))
$$

$$
\times f(M_{st}', \theta_{st,\sigma\tau}), \tag{3.2}
$$

where $\theta_{st,\sigma\tau}$ is the angle between the initial two-body relative momentum $\vec{k}_{\sigma\tau}$ and the final two-body relative momentum $\vec{k}_{st}'$ as defined in Eq. (2.13) of Ref. 1. The overlap of the three-particle momentum-space states with the states of total angular momentum J, third component M along a space-fixed axis, and third component μ_ρ along the direction of $\vec{p}_\rho(\alpha)$ is[14]

$$
{}_{\alpha}\langle e_\beta e_\gamma e_\delta J M \mu_\rho|\vec{p}_\beta\vec{p}_\gamma\vec{p}_\delta\rangle_\alpha = \left[\frac{2J+1}{\pi^2}(2\pi)^9\right]^{1/2} \delta\left(\sum_j \vec{p}_j(\alpha)\right) \delta(e_\beta - (p_\beta{}^2 + m_1{}^2)^{1/2})
$$

$$
\times\, \delta(e_\gamma - (p_\gamma{}^2 + m_2{}^2)^{1/2})\delta(e_\delta - (p_\delta{}^2 + m_3{}^2)^{1/2})\, \mathfrak{D}^{*J}_{M\mu_\rho}(A_\rho B_\rho C_\rho), \tag{3.3}
$$

where the Euler angles A_ρ, B_ρ, C_ρ align the $\hat{x}_\rho(\alpha)\hat{z}_\rho(\alpha)$ plane coincident with the momentum plane with $\hat{z}_\rho(\alpha)$ parallel to $\vec{p}_\rho(\alpha)$ and with $\hat{y}_\rho(\alpha)$ parallel to $\vec{p}_\rho(\alpha)\times\vec{p}_\sigma(\alpha)$. Using Eqs. (3.2) and (3.3) and employing the projections of the total angular momentum in the direction of the momentum of the spectator particle we obtain

$$
{}_{\alpha}\langle e_b' e_c' e_d' J' M' \mu' \lambda'|t^k|e_\beta e_\gamma e_\delta J M \mu \lambda\rangle_\alpha = \delta_{J'J}\,\delta_{M'M}\,\delta_{\lambda'\lambda}\,\frac{\delta(e_r'(\alpha) - e_\rho(\alpha))}{2(2\pi)^3 p_r'(\alpha)}\, e_s'(\gamma_{st}) e_t'(\gamma_{st})
$$

$$
\times \sum_{\mu_r'} \left[d^J_{\mu'\mu_r'}(-D_r') + \lambda d^J_{\mu'\mu_r'}(\pi - D_r')\right] d^J_{\mu_r'\mu}(D_\rho) \int_0^{2\pi} dU_{st,\sigma\tau} \cos(\mu_r' U_{st,\sigma\tau})
$$

$$
\times f(M_{st}', \theta_{st,\sigma\tau}(\xi_{st}', \xi_{\sigma\tau}, U_{st,\sigma\tau})), \tag{3.4}
$$

where the angles ξ, ξ', D, and D' as well as $\lambda = \pm 1$ (which is related to the parity of the three-pion system) are defined in Ref. 1. We note that for the special case $(bcd) = (\beta\gamma\delta) = (123)$, we have $(rst) = (\rho\sigma\tau) = (kij)$ and Eq. (3.4) reduces to Eq. (4.5) of Ref. 1.

TABLE III. Matrix elements in the isospin basis $|IM_I n\rangle_1$ of an operator $\hat{\theta}^k$ which conserves I, M_I, and i_{ij}. We employ the notation $\theta^k_{i_{ij}} \equiv {}^k\langle IM_I i_{ij}|\hat{\theta}^k|IM_I i_{ij}\rangle^k$.

Total isospin I	$[\hat{\theta}^k]^I_{n'n}$
0	$[\theta_1^k]$ for $k=1,2,3$
1	$\begin{pmatrix} \frac{5}{9}\theta_0^1 + \frac{4}{9}\theta_2^1 & \frac{2}{9}\sqrt{5}(\theta_0^1 - \theta_2^1) & 0 \\ \frac{2}{9}\sqrt{5}(\theta_0^1 - \theta_2^1) & \frac{4}{9}\theta_0^1 + \frac{5}{9}\theta_2^1 & 0 \\ 0 & 0 & \theta_1^1 \end{pmatrix}$ for $k=1$
	$\begin{pmatrix} \frac{5}{9}\theta_0^k + \frac{4}{9}\theta_2^k & -\frac{1}{9}\sqrt{5}(\theta_0^k - \theta_2^k) & \pm\frac{1}{9}\sqrt{15}(\theta_0^k - \theta_2^k) \\ -\frac{1}{9}\sqrt{5}(\theta_0^k - \theta_2^k) & \frac{1}{9}\theta_0^k + \frac{3}{4}\theta_1^k + \frac{5}{36}\theta_2^k & \mp\frac{1}{36}\sqrt{3}(4\theta_0^k - 9\theta_1^k + 5\theta_2^k) \\ \pm\frac{1}{9}\sqrt{15}(\theta_0^k - \theta_2^k) & \mp\frac{1}{36}\sqrt{3}(4\theta_0^k - 9\theta_1^k + 5\theta_2^k) & \frac{1}{3}\theta_0^k + \frac{1}{4}\theta_1^k + \frac{5}{12}\theta_2^k \end{pmatrix}$ for $k=2,3$
2	$\begin{pmatrix} \theta_2^1 & 0 \\ 0 & \theta_1^1 \end{pmatrix}$ for $k=1$
	$\begin{pmatrix} \frac{3}{4}\theta_1^k + \frac{1}{4}\theta_2^k & \pm\frac{1}{4}\sqrt{3}(\theta_1^k - \theta_2^k) \\ \pm\frac{1}{4}\sqrt{3}(\theta_1^k - \theta_2^k) & \frac{1}{4}\theta_1^k + \frac{3}{4}\theta_2^k \end{pmatrix}$ for $k=2,3$
3	$[\theta_2^k]$ for $k=1,2,3$

IV. CONSTRUCTION OF SYMMETRIZED THREE-PION STATES

As mentioned in Sec. I, we consider the space of three-pion states to be the direct product of the space of the three-particle energy–angular-momentum states and the space of the three-pion isospin states; explicitly we take

$$|\psi_n^{IM_I}(e_\beta e_\gamma e_\delta; JM\mu\lambda)\rangle_\alpha \equiv |e_\beta e_\gamma e_\delta JM\mu\lambda\rangle_\alpha |IM_I n\rangle_1 . \qquad (4.1)$$

The energy–angular-momentum states, and therefore the states of Eq. (4.1), do not have definite transformation properties under permutations. However, we may construct energy–angular-momentum states which transform according to the irreducible representations of $\mathcal{S}_3$:

$$|\mathcal{S}(e_1e_2e_3); JM\mu\lambda\rangle_\alpha = \frac{1}{\sqrt{6}}(|e_1e_2e_3JM\mu\lambda\rangle_\alpha + |e_2e_3e_1JM\mu\lambda\rangle_\alpha + |e_3e_1e_2JM\mu\lambda\rangle_\alpha + |e_1e_3e_2JM\mu\lambda\rangle_\alpha + |e_2e_1e_3JM\mu\lambda\rangle_\alpha + |e_3e_2e_1JM\mu\lambda\rangle_\alpha), \qquad (4.2)$$

$$|\mathcal{A}(e_1e_2e_3); JM\mu\lambda\rangle_\alpha = \frac{1}{\sqrt{6}}(|e_1e_2e_3JM\mu\lambda\rangle_\alpha + |e_2e_3e_1JM\mu\lambda\rangle_\alpha + |e_3e_1e_2JM\mu\lambda\rangle_\alpha - |e_1e_3e_2JM\mu\lambda\rangle_\alpha - |e_2e_1e_3JM\mu\lambda\rangle_\alpha - |e_3e_2e_1JM\mu\lambda\rangle_\alpha) \qquad (4.3)$$

$$|\mathfrak{M}^1_{x,y}(e_1e_2e_3); JM\mu\lambda\rangle_\alpha = \frac{1}{\sqrt{6}}[x|e_1e_2e_3JM\mu\lambda\rangle_\alpha + y|e_2e_3e_1JM\mu\lambda\rangle_\alpha - (x+y)|e_3e_1e_2JM\mu\lambda\rangle_\alpha + x|e_1e_3e_2JM\mu\lambda\rangle_\alpha + y|e_2e_1e_3JM\mu\lambda\rangle_\alpha - (x+y)|e_3e_2e_1JM\mu\lambda\rangle_\alpha], \qquad (4.4)$$

and

$$|\mathfrak{M}^2_{x,y}(e_1e_2e_3); JM\mu\lambda\rangle_\alpha = \frac{1}{\sqrt{6}}\left[\left(\frac{x+2y}{\sqrt{3}}\right)|e_1e_2e_3JM\mu\lambda\rangle_\alpha - \left(\frac{2x+y}{\sqrt{3}}\right)|e_2e_3e_1JM\mu\lambda\rangle_\alpha + \left(\frac{x-y}{\sqrt{3}}\right)|e_3e_1e_2JM\mu\lambda\rangle_\alpha - \left(\frac{x+2y}{\sqrt{3}}\right)|e_1e_3e_2JM\mu\lambda\rangle_\alpha + \left(\frac{2x+y}{\sqrt{3}}\right)|e_2e_1e_3JM\mu\lambda\rangle_\alpha - \left(\frac{x-y}{\sqrt{3}}\right)|e_3e_2e_1JM\mu\lambda\rangle_\alpha\right], \qquad (4.5)$$

where x and y are arbitrary complex coefficients.

For the $|IM_I n=0\rangle_1$ states of Table I, the construction of totally symmetrized three-pion states is easily accomplished using Eq. (1.1). For the $|IM_I n=1,2\rangle_1$ states, a complete description of the totally symmetrized states includes an evaluation of the coefficients x and y. Writing the symmetrization operator $\mathcal{S}$ (Ref. 15) as

$$\mathcal{S} \equiv \frac{1}{\sqrt{6}}\sum_{(ijk)} \mathcal{P}_{123\to ijk} = \frac{1}{\sqrt{6}}\sum_{\text{cyclic}} \mathcal{P}_{\text{cyclic}}(\mathcal{I} + \mathcal{P}_{123\to 132}), \qquad (4.6)$$

we have for $n=1,2$

$$|\mathcal{S}\psi_n^{IM_I}(e_1e_2e_3; JM\mu\lambda)\rangle_\alpha = \frac{1}{\sqrt{6}}\{[|e_1e_2e_3JM\mu\lambda\rangle_\alpha + (-1)^{n+1}|e_1e_3e_2JM\mu\lambda\rangle_\alpha]|IM_I n\rangle_1 + [|e_2e_3e_1JM\mu\lambda\rangle_\alpha + (-1)^{n+1}|e_3e_2e_1JM\mu\lambda\rangle_\alpha]|IM_I n\rangle_3 + [|e_3e_1e_2JM\mu\lambda\rangle_\alpha + (-1)^{n+1}|e_2e_1e_3JM\mu\lambda\rangle_\alpha]|IM_I n\rangle_2\} \qquad (4.7)$$

Using this along with Eq. (2.6) to transform the isospin states to the $|IM_I n\rangle_1$ basis, we obtain $(x,y)=(2,-1)$ for $n=1$ and $(x,y)=(0,1)$ for $n=2$.

The three-pion states then become

$$|\mathcal{S}\psi_n^{IM_I}(e_1e_2e_3;JM\mu\lambda)\rangle_\alpha=\begin{cases}\frac{1}{2}\sum_{n'}|\mathfrak{M}_{2,-1}^{n'}(e_1e_2e_3);JM\mu\lambda\rangle_\alpha|IM_In'\rangle_1 \text{ for } n=1\,,\\ \frac{\sqrt{3}}{2}\sum_{n'}|\mathfrak{M}_{0,1}^{n'}(e_1e_2e_3);JM\mu\lambda\rangle_\alpha|IM_In'\rangle_1 \text{ for } n=2\,,\end{cases}\tag{4.8}$$

in agreement with the general forms given in Eq. (1.1) and in Ref. 9.

V. MINIMAL-DYNAMICS K-MATRIX AMPLITUDES

Using the notation

$$[T^k_{JM\lambda IM_I}(e_b'e_c'e_d';e_\beta e_\gamma e_\delta;\mu'\mu)]_{n'n}\equiv{}_\alpha\langle\psi_{n'}^{IM_I}(e_b'e_c'e_d';JM\mu'\lambda)|T^k|\psi_n^{IM_I}(e_\beta e_\gamma e_\delta;JM\mu\lambda)\rangle_\alpha\,,\tag{5.1}$$

we expand Eq. (3.2) of Ref. 3 in our unsymmetrized direct-product basis of Eq. (4.1) as

$$\begin{aligned}[T^k_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{n'n}={}&{}_\alpha\langle e_1'e_2'e_3'\,JM\mu'\lambda|[t^k]^I_{n'n}|e_1e_2e_3JM\mu\lambda\rangle_\alpha\\&-\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\\&\quad\times\sum_{\mu''n''}{}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|[t^k]^I_{n'n''}|e_1''e_2''e_3''JM\mu''\lambda\rangle_\alpha i\pi\delta(\mathfrak{M}''-\mathfrak{M})\\&\quad\times\{[T^i_{JM\lambda IM_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{n''n}+[T^j_{JM\lambda IM_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{n''n}\}\,,\end{aligned}\tag{5.2}$$

where we have used the completeness relation

$$\mathcal{I}=\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\sum_{\substack{I''M_I''n''\\J''M''\lambda''\mu''}}|\psi_{n''}^{I''M_I''}(e_1''e_2''e_3'';J''M''\mu''\lambda'')\rangle_{\alpha\,\alpha}\langle\psi_{n''}^{I''M_I''}(e_1''e_2''e_3'';J''M''\mu''\lambda'')|\tag{5.3}$$

and have assumed that T^k conserves total isospin, its third component, and parity, and that t^k conserves the two-body isospin i_{ij}. Mindful that many two-body isospin system are allowed in some of the total I channels, we follow the general procedure of Ref. 5 and decompose the T^k operators by

$$[T^k]_{n'n}=\sum_{i_{ij}}[T^k_{i_{ij}}]_{n'n}\,,\tag{5.4}$$

where $[T^k_{i_{ij}}]$ is the amplitude for particles i and j interacting last with two-body isospin i_{ij}. For $I=0,3$ we have only $i_{ij}=1$ and $i_{ij}=2$, respectively, but for $I=1$ we have $i_{ij}=0,1,2$ and for $I=2$, $i_{ij}=1,2$.

Substituting Eq. (5.4) into Eq. (5.2) and using Table III, we find that for fixed I, i_{ij}, k, and n, the matrix elements $[T^k_{i_{ij}}]_{n'n}$ are not independent for all values of n'. In particular, the relationships among them and the definition of the independent amplitudes

$$[W^k_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n}$$

in terms of the $[T^k_{i_{ij}}]_{n'n}$ matrix elements are given in Table IV. Rewriting Eq. (5.2) in terms of these independent amplitudes we have

$$\begin{aligned}[W^k_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n}={}&[J^k_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n}\\&-\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\sum_{\mu''\nu'}{}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^k_\nu|e_1''e_2''e_3''JM\mu''\lambda\rangle_\alpha\\&\quad\times i\pi\delta(\mathfrak{M}''-\mathfrak{M})\mathcal{Y}^I_{\nu\nu'}\{[W^i_{JM\lambda IM_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{\nu'n}\\&\quad+(-1)^{\nu+\nu'}[W^j_{JM\lambda IM_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{\nu'n}\}\,,\end{aligned}\tag{5.5}$$

where

$$\mathcal{Y}^0=\mathcal{Y}^3=[1]\,,\quad\mathcal{Y}^1=\begin{bmatrix}\frac{1}{3}&-\frac{2}{9}\sqrt{3}&-\frac{2}{3}\\\frac{1}{2}\sqrt{3}&-\frac{1}{2}&\frac{1}{2}\sqrt{3}\\-\frac{5}{6}&-\frac{5}{18}\sqrt{3}&\frac{1}{6}\end{bmatrix},\quad\text{and}\quad\mathcal{Y}^2=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}\sqrt{3}\\-\frac{1}{2}\sqrt{3}&-\frac{1}{2}\end{bmatrix},\tag{5.6}$$

and where $[J^k]_{\nu n}$ is the inhomogeneous term defined by

$$[J^k_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n} \equiv {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^k_\nu|e_1e_2e_3JM\mu\lambda\rangle_\alpha G^{I;k}_{\nu n}\,, \tag{5.7}$$

with the matrix $G^{I;k}_{\nu n}$ given in Table V. Employing the operators

$$\begin{aligned}[][X_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n} \equiv [W^1_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n} + [W^2_{JM\lambda IM_I}(e_3'e_1'e_2';e_1e_2e_3;\mu'\mu)]_{\nu n}\\ + [W^3_{JM\lambda IM_I}(e_2'e_3'e_1';e_1e_2e_3;\mu'\mu)]_{\nu n}\end{aligned} \tag{5.8}$$

and permuting the final-state and intermediate-state variables of Eq. (5.5), we obtain

$$\begin{aligned}[][X_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n} = [L_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\nu n}\\ -\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\\ \times\sum_{\mu''\nu'}{}'[E_\nu+(-1)^{\nu+\nu'}F_\nu]\,i\pi\delta(\mathfrak{M}''-\mathfrak{M})\mathcal{Y}^I_{\nu\nu'}[X_{JM\lambda IM_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{\nu' n}\,,\end{aligned} \tag{5.9}$$

where

$$E_\nu = {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^1_\nu|e_3''e_1''e_2''JM\mu''\lambda\rangle_\alpha$$

and (5.10)

$$F_\nu = {}_\alpha\langle e_1'e_2'e_3'JM\mu'\lambda|t^1_\nu|e_2''e_3''e_1''JM\mu''\lambda\rangle_\alpha$$

and where $[L]_{\nu n}$ is defined as in Eq. (5.8) with $[J^k]_{\nu n}$ replacing $[W^k]_{\nu n}$.

Essentially what has been accomplished through the introduction of the operators $T^k_{i_{ij}}$, W^k, and $[X]_{\nu n}$ is that for a given total I we are able to replace a set of integral equations coupling all possible final subsystem isospin channels to all possible initial subsystem isospin channels by another set of integral equations coupling a single final subsystem isospin channel to a single initial subsystem isospin channel. Also, while the initial equations of Eq. (5.2) are coupled both in inter-

TABLE IV The relationships among the matrix elements of the operators $T^k_{i_{ij}}$ in the $|IM_In\rangle_1$ isospin basis and the definition of the independent matrix elements $[W^k]_{\nu n}$ in terms of the $T^k_{i_{ij}}$ matrix elements.

Total isospin	Subsystem isospin	Particle index k	Matrix elements
0	1	1, 2, 3	$[W^k]_{\nu=1,n=0} \equiv [T^k_1]_{n'=0,n=0}$
1	0	1	$[W^1]_{\nu=0,n} \equiv -\frac{1}{2}[T^1_0]_{n'=1,n} = -(1/\sqrt{5})[T^1_0]_{n'=0,n}$; $[T^1_0]_{n'=2,n}=0$
		2	$[W^2]_{\nu=0,n} \equiv [T^2_0]_{n'=1,n} = -(1/\sqrt{5})[T^2_0]_{n'=0,n} = -(1/\sqrt{3})[T^2_0]_{n'=2,n}$
		3	$[W^3]_{\nu=0,n} \equiv [T^3_0]_{n'=1,n} = -(1/\sqrt{5})[T^3_0]_{n'=0,n} = (1/\sqrt{3})[T^3_0]_{n'=2,n}$
	1	1	$[W^1]_{\nu=1,n} \equiv -\frac{1}{2}[T^1_1]_{n'=2,n}$; $[T^1_1]_{n'=0,n} = [T^1_1]_{n'=1,n}=0$
		2	$[W^2]_{\nu=1,n} \equiv [T^2_1]_{n'=2,n} = (1/\sqrt{3})[T^2_1]_{n'=1,n}$; $[T^2_1]_{n'=0,n}=0$
		3	$[W^3]_{\nu=1,n} \equiv [T^3_1]_{n'=2,n} = -(1/\sqrt{3})[T^3_1]_{n'=1,n}$; $[T^3_1]_{n'=0,n}=0$
	2	1	$[W^1]_{\nu=2,n} \equiv -\frac{1}{2}[T^1_2]_{n'=1,n} = \frac{1}{4}\sqrt{5}[T^1_2]_{n'=0,n}$; $[T^1_2]_{n'=2,n}=0$
		2	$[W^2]_{\nu=2,n} \equiv [T^2_2]_{n'=1,n} = \frac{1}{4}\sqrt{5}[T^2_2]_{n'=0,n} = -(1/\sqrt{3})[T^2_2]_{n'=2,n}$
		3	$[W^3]_{\nu=2,n} \equiv [T^3_2]_{n'=1,n} = \frac{1}{4}\sqrt{5}[T^3_2]_{n'=0,n} = (1/\sqrt{3})[T^3_2]_{n'=2,n}$
2	1	1	$[W^1]_{\nu=1,n} \equiv -\frac{1}{2}[T^1_1]_{n'=2,n}$; $[T^1_1]_{n'=1,n}=0$
		2	$[W^2]_{\nu=1,n} \equiv [T^2_1]_{n'=2,n} = (1/\sqrt{3})[T^2_1]_{n'=1,n}$
		3	$[W^3]_{\nu=1,n} \equiv [T^3_1]_{n'=2,n} = -(1/\sqrt{3})[T^3_1]_{n'=1,n}$
	2	1	$[W^1]_{\nu=2,n} \equiv -\frac{1}{2}[T^1_2]_{n'=1,n}$; $[T^1_2]_{n'=2,n}=0$
		2	$[W^2]_{\nu=2,n} \equiv [T^2_2]_{n'=1,n} = -(1/\sqrt{3})[T^2_2]_{n'=2,n}$
		3	$[W^3]_{\nu=2,n} \equiv [T^3_2]_{n'=1,n} = (1/\sqrt{3})[T^3_2]_{n'=2,n}$
3	2	1, 2, 3	$[W^k]_{\nu=2,n=0} \equiv [T^k_2]_{n'=0,n=0}$

TABLE V. The matrix $G_{\nu n}^{I,k}$ which enters into the inhomogeneous term in the minimal-dynamics K-matrix equations of Eq. (5.5).

Total isospin	Particle index k	Inhomogeneous matrix $G_{\nu n}^{I,k}$
0	1,2,3	[1]
1	1	$\begin{pmatrix} -\frac{1}{9}\sqrt{5} & -\frac{2}{9} & 0 \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{9}\sqrt{5} & -\frac{5}{18} & 0 \end{pmatrix}$
	2, 3	$\begin{pmatrix} -\frac{1}{9}\sqrt{5} & \frac{1}{9} & \mp\frac{1}{9}\sqrt{3} \\ 0 & \pm\frac{1}{4}\sqrt{3} & \frac{1}{4} \\ \frac{1}{9}\sqrt{5} & \frac{5}{36} & \mp\frac{5}{36}\sqrt{3} \end{pmatrix}$
2	1	$\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$
	2, 3	$\begin{pmatrix} \pm\frac{1}{4}\sqrt{3} & \frac{1}{4} \\ \frac{1}{4} & \mp\frac{1}{4}\sqrt{3} \end{pmatrix}$
3	1,2,3	[1]

mediate-state subsystem isospin i''_{ij} and particle index k, our final equations are coupled only in i''_{ij}. However, the price we pay for uncoupling the particle index is that the $[X]_{\nu n}$, by themselves, do not have an intuitive physical interpretation as do the operators T^k. But, as we will see below, Bose symmetry allows us to equate the symmetrized T matrix with suitably summetrized matrix elements of the $[X]_{\nu n}$ operators. For $I=0$, using Eqs. (5.4) and (5.8) and using Table IV we find that

$$_\alpha\langle \mathcal{S}\psi_0^{00}(e_1'e_2'e_3'; JM\mu'\lambda)|T|\mathcal{S}\psi_0^{00}(e_1e_2e_3; JM\mu\lambda)\rangle_\alpha$$

$$=\sqrt{6}\,[X_{JM\lambda 00}(\mathcal{A}(e_1'e_2'e_3'); e_1e_2e_3; \mu'\mu)]_{10}\,, \quad (5.11)$$

as in Ref. 1. Also with $\nu=\nu'=1$ and $n=n'=0$, Eq. (5.9) reduces to Eq. (5.5) of Ref. 1. A similar situation occurs for the $I=3$ channel where we have

$$_\alpha\langle \mathcal{S}\psi_0^{3M_I}(e_1'e_2'e_3'; JM\mu'\lambda)|T|\mathcal{S}\psi_0^{3M_I}(e_1e_2e_3; JM\mu\lambda)\rangle_\alpha$$

$$=\sqrt{6}\,[X_{JM\lambda 3M_I}(\mathcal{S}(e_1'e_2'e_3'); e_1e_2e_3; \mu'\mu)]_{20}\,. \quad (5.12)$$

In each of these cases, we have a single integral equation for X as opposed to the original three coupled integral equations for the T^k. For the $I=1,2$ channels, the situation is somewhat more complicated in that, as may be seen from Eq. (4.8), the symmetrized amplitude $_\alpha\langle \mathcal{S}\psi_{n'}^I|T|\mathcal{S}\psi_n^I\rangle_\alpha$ is an appropriate sum of the amplitudes for all the allowed subsystem isospin channels. Explicitly for $I=2$ we find that

$$_\alpha\langle \mathcal{S}\psi_{n'=1}^{2M_I}(e_1'e_2'e_3'; JM\mu'\lambda)|T|\mathcal{S}\psi_n^{2M_I}(e_1e_2e_3; JM\mu\lambda)\rangle_\alpha$$

$$=-\sqrt{6}\sum_{\nu=1}^{2}[X_{JM\lambda 2M_I}(\mathfrak{M}_{2,-1}^{3-\nu}(e_1'e_2'e_3'); e_1e_2e_3; \mu'\mu)]_{\nu n} \quad (5.13)$$

and

$$_\alpha\langle \mathcal{S}\psi_{n'=2}^{2M_I}(e_1'e_2'e_3'; JM\mu'\lambda)|T|\mathcal{S}\psi_n^{2M_I}(e_1e_2e_3; JM\mu\lambda)\rangle_\alpha$$

$$=-3\sqrt{2}\sum_{\nu=1}^{2}[X_{JM\lambda 2M_I}(\mathfrak{M}_{0,1}^{3-\nu}(e_1'e_2'e_3'); e_1e_2e_3; \mu'\mu)]_{\nu n}\,. \quad (5.14)$$

The symmetrized amplitudes for the $I=1$ channel may be further reduced and the $\mathcal{S}\otimes\mathfrak{M}$ direct-product nature of the $I=1$ isospin system may be seen more clearly by employing the transformation

$$[\overline{X}]_{0n} \equiv -\sqrt{5}\,[X]_{0n} + \frac{4}{\sqrt{5}}[X]_{2n}\,,$$

$$[\overline{X}]_{1n} \equiv [X]_{1n}\,, \quad (5.15)$$

and

$$[\overline{X}]_{2n} \equiv [X]_{0n} + [X]_{2n}\,.$$

Then Eq. (5.9) becomes

$$[\overline{X}_{JM\lambda 1M_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\overline{\nu}n}=[\overline{L}_{JM\lambda 1M_I}(e_1'e_2'e_3';e_1e_2e_3;\mu'\mu)]_{\overline{\nu}n}$$
$$-\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\sum_{\mu''\overline{\nu}'}i\pi\delta(\mathfrak{M}''-\mathfrak{M})\overline{\mathcal{Y}}^1_{\overline{\nu}\,\overline{\nu}'}[\overline{X}_{JM\lambda 1M_I}(e_1''e_2''e_3'';e_1e_2e_3;\mu''\mu)]_{\overline{\nu}'n}, \tag{5.16}$$

with

$$\overline{\mathcal{Y}}^1_{\overline{\nu}\,\overline{\nu}'}=\begin{bmatrix} \frac{5}{9}(E_0+F_0)+\frac{4}{9}(E_2+F_2) & \frac{2}{9}\sqrt{15}(E_0-F_0)-\frac{2}{9}\sqrt{15}(E_2-F_2) & \frac{2}{9}\sqrt{5}\,(E_0+F_0)-\frac{2}{9}\sqrt{5}\,(E_2+F_2) \\ 0 & -\frac{1}{2}(E_1+F_1) & \frac{1}{2}\sqrt{3}\,(E_1-F_1) \\ -\frac{1}{9}\sqrt{5}\,(E_0+F_0)+\frac{1}{9}\sqrt{5}\,(E_2+F_2) & -\frac{1}{2}\sqrt{3}\,[\frac{4}{9}(E_0-F_0)+\frac{5}{9}(E_2-F_2)] & -\frac{1}{2}[\frac{4}{9}(E_0+F_0)+\frac{5}{9}(E_2+F_2)] \end{bmatrix}. \tag{5.17}$$

In terms of these $\overline{X}$ operators, the symmetrized amplitude are

$${}_\alpha\langle\, \mathcal{S}\psi^{1M_I}_{n' \underline{=} 0}(e_1'e_2'e_3';\,JM\mu'\lambda)|T|\mathcal{S}\psi^{1M_I}_{n}(e_1e_2e_3;JM\mu\lambda)\rangle_\alpha=\sqrt{6}\,[\overline{X}_{JM\lambda 1M_I}(\mathcal{S}(e_1'e_2'e_3');e_1e_2e_3;\mu'\mu)]_{0n}, \tag{5.18}$$

$${}_\alpha\langle\, \mathcal{S}\psi^{1M_I}_{n' \underline{=} 1}(e_1'e_2'e_3';JM\mu'\lambda)|\,T\,|\,\mathcal{S}\psi^{1M_I}_{n}(e_1e_2e_3;JM\mu\lambda)\rangle_\alpha=-\sqrt{6}\sum_{\overline{\nu}=1}^{2}[\overline{X}_{JM\lambda 1M_I}(M^{3-\overline{\nu}}_{2,-1}(e_1'e_2'e_3');e_1e_2e_3;\mu'\mu)]_{\overline{\nu}n}, \tag{5.19}$$

and

$${}_\alpha\langle\, \mathcal{S}\psi^{1M_I}_{n' \underline{=} 2}(e_1'e_2'e_3';JM\mu'\lambda)|T|\mathcal{S}\psi^{1M_I}_{n}(e_1e_2e_3;JM\mu\lambda)\rangle_\alpha=-3\sqrt{2}\sum_{\overline{\nu}=1}^{2}[\overline{X}_{JM\lambda 1M_I}(\mathfrak{M}^{3-\overline{\nu}}_{0,1}(e_1'e_2'e_3');e_1;e_1e_2e_3;\mu'\mu)]_{\overline{\nu}n}. \tag{5.20}$$

in analogy with Eqs. (5.11)–(5.14).

VI. DISCUSSION AND CONCLUSIONS

The method that we have used in Sec. V for constructing the integral equations satisfied by the $[X]_{\nu n}$ operators is similar to the method used by Basdevant and Kreps.[5] The great difference however from their treatment is that they, as do Mennessier *et al.*,[7] in order to justify the pole approximation to the off-shell t matrix neglect the contributions from the $i_{ij}\neq 1$ isospin subsystems at the outset while we have included these contributions exactly. For example, the $I=1$ A_1 meson decays predominantly via the $(i_{ij}=1)$ $\rho+\pi$ channel. The symmetrized amplitude for this process is given in Eq. (5.20). Although the $\overline{\nu}=1$ term of Eq. (5.20) is a function of the $i_{ij}=1$ scattering parameters to first order and the $i_{ij}=0,2$ parameters to second order, the $\overline{\nu}=2$ term is a function of the $i_{ij}=0,2$ scattering parameters to first order and the $i_{ij}=1$ parameters to second order. Then the neglect of the $i_{ij}=0,2$ contribution is effectively a neglect of half of the scattering amplitude. However, since the π-π phase shift δ_0^2 remains smaller than 30° for $M_{ij}\lesssim 1.4$ GeV,[16] the neglect of the $i_{ij}=2$ channels is defensible. But since the $i_{ij}=0$ S-wave phase shift δ_0^0 goes through 90° at $M_{ij}=0.86$ GeV,[17] the $i_{ij}=0$ channels cannot be neglected at the outset. Thus the statement that the decay of a three-pion resonance is ρ dominated does not of itself imply that the symmetrized decay amplitude is $i_{ij}=1$ dominated. The inclusion of the $i_{ij}=0$ subsystem isospin channel for the $I=1$ system has been studied recently by Ascoli and Wyld[12] in the analysis of the $\pi^-p\to\pi^-\pi^+\pi^-p$ reaction. They employed the minimal-dynamics K-matrix model neglecting the $i_{ij}=2$ phases. However, their calculation was carried out in a charge-state basis rather than an isospin basis, thus simplifying the symmetrization problem to that of symmetrizing with respect to the two π^- mesons for the $|--+\rangle$ system. With these assumptions they did find the $i_{ij}=0$ contribution to the transition amplitude to be significant in the region of the A_1 mass. Similar conclusions concerning the importance of the $i_{ij}=0$ phases were reached by Brayshaw,[13] who examined the resonant structure of the A_1 channel using the $i_{ij}=0,1$ phase parameters with the boundary-condition model.

It is of interest to consider the inclusion of only the $i_{ij}=1$ isospin subsystem as a zeroth-order approximation to our equations with the purpose of seeing whether simplifications occur in Eq. (5.9). For $I=1,2$, this amounts to setting $T^k\approx T^k_{i_{ij}=1}$ with the result

$$[X_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} = \delta_{\nu,1}\delta_{n,2}X_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu), \tag{6.1}$$

with a similar expression resulting for $[L]_{\nu n}$. Then Eq. (5.9) becomes for $I=1,2$

$$X_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\mu'\mu) = L_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\,\mu'\mu) - \tfrac{1}{2}\int de_1''(\alpha)de_2''(\alpha)de_3''(\alpha)\sum_{\mu''}(E_1+F_1)i\pi\delta(\mathfrak{M}''-\mathfrak{M})X_{JM\lambda IM_I}(e_1''e_2''e_3'';\, e_1e_2e_3;\, \mu''\mu). \tag{6.2}$$

This serves to uncouple the integral equations with respect to subsystem isospin and causes the $I=1$ and $I=2$ systems to become degenerate. These zeroth-order equations, Eq. (6.2), are in agreement with Basdevant and Kreps[5] except for the details of their partial-wave decomposition and the use of two-body relativistic kinematics as explained in Ref. 1.

Returning to our original formalism of Eq. (5.9), after performing a partial-wave expansion of E_ν and F_ν as in Eq. (5.2) of Ref. 1 and performing two of the energy integrations, we obtain

$$\begin{aligned}[X_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} &= [L_{JM\lambda IM_I}(e_1'e_2'e_3';e_1e_2e_3;\, \mu'\mu)]_{\nu n} \\ &- \frac{4\pi i M_{23}'}{[e_1'(\alpha)^2-m_\pi{}^2]^{1/2}(M_{23}'^2-4m_\pi{}^2)^{1/2}}\sum_{l+\nu=\text{even}} e^{i\delta_l^\nu}\sin\delta_l^\nu\, Y_l^{\mu'}(\xi_1',0) \\ &\times\sum_{\mu''\nu'}\mathcal{Y}_{\nu\nu'}^I\int de_1''(\alpha)Y_l^{\mu'}(\xi'',0)[d^J_{\mu'\mu''}(D'')+\lambda d^J_{\mu'\mu''}(\pi+D'')] \\ &\times\{[X_{JM\lambda IM_I}(e_1'',e_2'+e_3'-e_1'',e_1';\, e_1e_2e_3;\, \mu''\mu)]_{\nu'n} \\ &+(-1)^{\nu'}[X_{JM\lambda IM_I}(e_1'',e_1',e_2'+e_3'-e_1'';\, e_1e_2e_3;\, \mu''\mu)]_{\nu'n}\},\end{aligned} \tag{6.3}$$

where ξ_1' is the angle between $\vec{p}_1'(\alpha)$ and the relative momentum $\vec{k}_{23}'(\alpha)$ as defined in Eq. (2.13) of Ref. 1, where

$$\cos\xi'' = \frac{M_{23}'[e_2'(\alpha)+e_3'(\alpha)-2e_1''(\alpha)]}{(M_{23}'^2-4m_\pi{}^2)^{1/2}[e_1'(\alpha)^2-m_\pi{}^2]^{1/2}} \tag{6.4}$$

and where

$$\cos D'' = \frac{M_{23}'^2-2e_1''(\alpha)[e_2'(\alpha)+e_3'(\alpha)]}{2[e_1''(\alpha)^2-m_\pi{}^2]^{1/2}[e_1'(\alpha)^2-m_\pi{}^2]^{1/2}}. \tag{6.5}$$

Defining the operators

$$[\chi_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} \equiv [X_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} + (-1)^\nu[\chi_{JM\lambda IM_I}(e_1'e_3'e_2';e_1e_2e_3;\, \mu'\mu)]_{\nu n}, \tag{6.6}$$

Eq. (6.3) may be simplified to

$$\begin{aligned}[\chi_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} &= [\mathfrak{L}_{JM\lambda IM_I}(e_1'e_2'e_3';\, e_1e_2e_3;\, \mu'\mu)]_{\nu n} - \frac{4\pi i M_{23}'}{[e_1'(\alpha)^2-m_\pi{}^2]^{1/2}(M_{23}'^2-4m_\pi{}^2)^{1/2}}\sum_{l+\nu=\text{even}} e^{i\delta_l^\nu}\sin\delta_l^\nu Y_l^{\mu'}(\xi_1',0) \\ &\times\sum_{\mu''\nu'}\mathcal{Y}_{\nu\nu'}^I\int de_1''(\alpha)Y_l^{\mu'}(\xi'',0)\,[d^J_{\mu'\mu''}(D'')+\lambda d^J_{\mu'\mu''}(\pi+D'')] \\ &\times[\chi_{JM\lambda IM_I}(e_1'',e_2'+e_3'-e_1'';e_1e_2e_3;\, \mu''\mu)]_{\nu'n},\end{aligned} \tag{6.7}$$

where $[\mathfrak{L}]_{\nu n}$ is defined as in Eq. (6.6) with $[L]_{\nu n}$ replacing $[X]_{\nu n}$ and where the limits of the $de_1''(\alpha)$ integration are

$$\frac{e_2'(\alpha)+e_3'(\alpha)}{2} \pm \frac{[e_1'(\alpha)^2-m_\pi{}^2]^{1/2}(M_{23}'^2-4m_\pi{}^2)^{1/2}}{2M_{23}'}.$$

Since the $[X]_{\nu n}$ operators are symmetric under the simultaneous interchange of the momenta and isospins of the second and third particles, the symmetrized T-matrix elements may be written in terms of the $[\chi]_{\nu n}$ operators in exactly the same way as they were for the $[X]_{\nu n}$ operators in Eqs.

(5.11)–(5.14) and (5.18)–(5.20) except for the retention of only those terms representing even permutations of $(e_1'e_2'e_3')$.

Although the present isospin decomposition and partial-wave analysis were formulated in terms of the minimal-dynamics K-matrix model, the same approach may be applied to the dynamical models of three-pion systems considered in previous treatments.[4,5,7] Calculations using the minimal-dynamics model of Eq. (6.7) to examine the resonant structure of the various J^π channels of the three-pion system are now underway.

As an alternative to the construction of three-particle wave functions with proper symmetry by group-theoretic methods,[10] Harper, Kim, and Tubis[18] have proposed a method by which one obtains properly antisymmetrized three-nucleon amplitudes by first constructing states $|1,23\rangle$ antisymmetric under the interchange of the dynamical variables of particles 2 and 3, then reducing the Faddeev equations to a single integral equation for the matrix element $\langle 1,23|T^1|\mathcal{A}\psi\rangle$, where $|\mathcal{A}\psi\rangle$ is totally antisymmetric, and then constructing the totally antisymmetrized amplitude by the prescription

$$\langle \mathcal{A}\psi'|T|\mathcal{A}\psi\rangle = \sum_{\text{cyclic}} \mathcal{P}_{\text{cyclic}} \langle 1,23|T|\mathcal{A}\psi\rangle, \qquad (6.8)$$

where $\sum \mathcal{P}_{\text{cyclic}}$ acts on the final-state dynamical variables. Our approach of Sec. V is similar to the Bose version of the Harper-Kim-Tubis method. While in Eqs. (4.6) and (4.7) we construct wave functions symmetric under the interchange of the energies and isospins of particles 2 and 3, we then form the totally symmetrized wave functions by applying the operator $\sum_{\text{cyclic}} \mathcal{P}_{\text{cyclic}}$ to these states *before* reducing the dynamical equations rather than after as was done in Ref. 18. This allows us to perform a subsystem isospin decomposition of the symmetrized wave function and thus make use of the simplifications arising from the subsystem isospin decomposition of the operators T^i as in Eq. (5.4) and in Table III.

In summary then, we have given a procedure based on group theory for the construction of properly symmetrized T-matrix elements for three-pion systems and have given the minimal-dynamics K-matrix version of the integral equations satisfied by the operators which enter into the symmetrized amplitudes. We have employed proper one-body and two-body relativistic kinematics throughout and have used a consistent construction of states of definite total angular momentum as described fully in Ref. 1. Further, our equations suggest that for the case of $I=1$ three-pion resonances which decay via the $\rho+\pi$ channel, the contributions to the scattering amplitude from the $i_{ij}=0,2$ channels may be as important[12-13] as the contribution from the $i_{ij}=1$ channel which was assumed to be dominant by other authors.[5,7]

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