Asymptotic freedom, potential scattering, and high-energy limits*

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The fixed singularities in the complex angular momentum plane of the amplitude are obtained for potentials which correspond to asymptotically free field theories. For potentials of the form $-g^2(r)r^{-2}$, with $g^2(r)$ decreasing logarithmically for $r \to 0$, one obtains an essential singularity with an accumulation of Regge poles. A somewhat faster decrease with the square of the logarithm gives a special branch point. These singularities replace the fixed cuts seen for $g^2(0) > 0$, which corresponds to renormalizable theories with a finite fixed point. The connection of our results with field-theory calculations is discussed. They are also considered from the point of view of the general analytic properties of amplitudes satisfying two-particle and multiparticle t-channel unitarity. In asymptotically free gauge theories, similar singularities may occur at $j = 1$.

I. INTRODUCTION

Within a framework where the structure and interactions of hadrons are described by a theory of elementary fields, it must be possible to also compute the asymptotic behavior of scattering amplitudes on the basis of this field theory. Recent advances in the handling of renormalizable theories give some hope for progress in this direction. The limits of large transverse and longitudinal momenta have been discussed extensively on the basis of asymptotically free field theories, but little has been done, so far, about the more involved problem of high-energy limits with fixed momentum transfer (Regge region).

It is the purpose of this note to report[1] several results about the l-plane singularities of potential scattering models which correspond to asymptotically free field theories in an approximation where multiparticle t-channel unitarity is not completely implemented. Potentials which behave like

$$-r^{-2}\left(\ln\frac{r_0}{r}\right)^{-1} \text{ for } r \to 0 \tag{1.1}$$

are shown to give rise to an essential singularity in the complex l plane which is an accumulation point of Regge poles. They simulate the situation for asymptotically free field theories in the approximation mentioned above.[2] We have a correspondence which is analogous to the one between the fixed l-plane branch points obtained with a potential behaving like r^{-2} for $r \to 0$ (scale-covariant potential)[3] and the fixed cuts found in field theories[4] which are renormalizable with finite fixed points $g_\infty \neq 0$.[5]

Writing the potential in the form

$$-g^2(r)r^{-2}, \tag{1.2}$$

the expression (1.1) corresponds to $g^2(r) \simeq [\ln(r_0/r)]^{-1}$ for $r \to 0$. We show that the character of the fixed l-plane singularity is highly sensitive to the rate of decrease of $g^2(r)$. If

$$g^2(r) \simeq \beta\left(\ln\frac{r_0}{r}\right)^{-2}, \quad \beta > 0 \tag{1.3}$$

then there is no more an essential singularity at $l = -\frac{1}{2}$. For $\beta < \frac{1}{4}$ we find a soft branch point of infinite order, while for $\beta > \frac{1}{4}$ we have a branch point with an accumulation of Regge poles at $l = -\frac{1}{2}$. Furthermore, if $g^2(r)$ vanishes with a power of r,

$$g^2(r) \simeq r^{-\epsilon}, \quad \epsilon > 0 \tag{1.4}$$

the amplitude is known to be meromorphic,[6] and there remains no particular fixed singularity.

We consider these fixed singularities in potential scattering and in field theories also from the more general point of view of the analytic structure of scattering amplitudes in the (l,t) manifold. It is shown that in potential scattering only soft branch points, special essential branch points, or essential singularities are possible as fixed singularities. The same is true for field theories as long as two-particle states are emphasized and there are no shielding cuts[7] present. The latter are only possible in connection with multiparticle t-channel thresholds.[8] They are usually not generated in renormalized Bethe-Salpeter calculations. In the last section we discuss briefly the problem of the influence of many-particle states in the t channel upon the fixed singularities.

The interest in the fixed l-plane singularities of asymptotically free theories lies mainly in their possible importance for the Pomeron and hence for high-energy diffraction scattering. In gauge-theory models of hadrons, such singularities can appear at $l = +1$, but they may be modified by the requirements of multiparticle t-channel states or s-channel unitarity and by the presence of spin.

II. RENORMALIZABLE THEORIES

Superrenormalizable field theories give rise to Regge poles and Regge branch-point surfaces in the complex angular momentum plane of the scattering amplitude. On the other hand, it has been known for a long time that theories which are only renormalizable can have in addition fixed, soft branch points in the l plane.[4,5] At least, these branch points are present within the framework of a scale-invariant ladder approximation. This effect is completely analogous to the situation in Schrödinger theory with a potential which has a short-distance behavior like r^{-2}, and which therefore scales in the same way as the kinetic term in the limit $r\to 0$.[3] For the purpose of our later discussions, it is convenient to give a brief description of this familiar case in a modern language.[9]

Suppose we write

$$V(r)=-g^2(r)r^{-2} \tag{2.1}$$

and assume that for $r\to 0$

$$g^2(r)\simeq g(0)[1+O(r)], \tag{2.2}$$

with $g(0)\neq 0$. In this limit, the effective potential

$$\frac{l(l+1)}{r^2}-\frac{g^2(r)}{r^2} \tag{2.3}$$

can be written in the form

$$\frac{\eta(l)[\eta(l)+1]}{r^2}, \tag{2.4}$$

with

$$\eta_\pm(l)=-\tfrac{1}{2}\pm[(l+\tfrac{1}{2})^2-g^2(0)]^{1/2}. \tag{2.5}$$

The radial wave function behaves like

$$\psi_l(r)\simeq r^{\eta_+(l)} \quad \text{for } r\to 0. \tag{2.6}$$

Since

$$\delta(l)=\eta_+(l)-l \tag{2.7}$$

is negative, we see that ψ_l vanishes less fast in the short-distance limit than the usual power r^l obtained in the absence of the attractive r^{-2} potential. The continued partial-wave amplitude $F(l,k^2)$ has fixed square-root branch points at

$$l=-\tfrac{1}{2}\pm[g^2(0)]^{1/2}. \tag{2.8}$$

They are on the real axis in the case of an attractive limiting potential with $g^2(0)>0$. It is important to note that the branch points depend only upon the limiting value $g^2(0)$. This is the potential analog of a finite *fixed point* $g_\infty{}^2$ which is characteristic for renormalizable field theories. The field-theoretical coupling constant g, defined at a certain subtraction point in the scaling momentum variable, corresponds in potential scattering to $g(r_1)$, where r_1 is an appropriately chosen radius.

In our potential model, the fixed cut is centered around $l=-\frac{1}{2}$, while the analogous branch lines in field theory are in a similar way associated with points like $l=0$ ($l=-2$) for ϕ^4 coupling in four dimensions or $l=1$ for quantum electrodynamics.[4] In the potential theory language, the branch points (8) are transition points where the relevant part[3]

$$\frac{(l+\frac{1}{2})^2-g^2(0)}{r^2} \tag{2.9}$$

of the effective potential for $r\to 0$ is real and changes sign. We have a stable system on one side, and a collapsed one on the other. In field theories, we generally cannot localize arbitrarily close because of the possibility of particle production. Hence we expect that the fixed-cut structure of the amplitude is modified if multiparticle states are properly included. Nevertheless, we presume that there remains some remnant of this fixed-cut structure.

III. ASYMPTOTICALLY FREE THEORIES

Let us now see what happens to the fixed cut described in the previous section if the r^{-2} potential is switched off for $r\to 0$. It all depends upon the rate of decrease of $g^2(r)$. If we assume that

$$g^2(r)=O(r), \tag{3.1}$$

corresponding, for example, to a superposition of Yukawa potentials, it is well known that no trace of the fixed branch cut is left. The scattering amplitude is meromorphic in the l plane. The field theories corresponding to this case are *superrenormalizable*. Because of multiparticle t-channel thresholds, the field-theoretical scattering amplitude is Regge branch-point surfaces in addition to the pole surfaces. In potential scattering, the meromorphic character of the amplitude[6] $F(l,k^2)$ is preserved as long as $g^2(r)$ vanishes with a positive power of r:

$$g^2(r)=O(r^\epsilon), \quad \epsilon>0. \tag{3.2}$$

The situation is quite different if the scale-covariant r^{-2} potential is switched off only logarithmically. First we consider the case

$$g^2(r)\simeq\gamma\left(\ln\frac{r_0}{r}\right)^{-1}, \quad \gamma>0 \tag{3.3}$$

for $r \to 0$. The radius r_0 is an unimportant normalization factor. We will see that this logarithmic breaking of the scale covariance corresponds in field theory to asymptotically free theories.

In order to obtain an exact statement, we choose the explicit potential

$$V(r) = -\left(\frac{\gamma}{r^2[\ln(r_0/r)]} - \frac{2e\gamma}{r_0^2}\right)\theta\left(\frac{r_0}{\sqrt{e}} - r\right), \quad (3.4)$$

which has a continuous derivative at $r = r_0/\sqrt{e}$, where it vanishes. This potential allows an exact solution for $k^2 = 2e\gamma/r_0^2 > 0$. For $r < r_0/\sqrt{e}$, the wave function is given by

$$\psi_l(r) \propto r^l\, U\left(-\frac{\gamma}{2l+1}, 0, (2l+1)\ln\frac{r_0}{r}\right), \quad (3.5)$$

where $U(a,b,z)$ is the Kummer function. Taking the limit $r \to 0$ for $l \neq -\frac{1}{2}$, we obtain the behavior

$$\psi_l(r) \propto \left[(2l+1)\ln\frac{r_0}{r}\right]^{\gamma/(2l+1)} r^l. \quad (3.6)$$

We are only interested in the consequences of the short-distance behavior of the potential (3.4). Comparing the expression (3.6) with Eqs. (2.5) and (2.6), we see that ψ_l still vanishes less fast than the power r^l, but now only by a logarithmic factor. We also find that the limits $l \to -\frac{1}{2}$ and $r \to 0$ are not interchangeable. The wave function has an essential singularity at $l = -\frac{1}{2}$.

Via the Jost function, we can calculate the S matrix for the potential (3.4) at $k^2 = 2e\gamma r_0^{-2}$. It can be written in the form

$$S(\lambda, k^2 = 2e\gamma r_0^{-2}) = -\frac{H^{(2)}_{\lambda+1}(\sqrt{2\gamma}) - (\frac{1}{2}\gamma)^{1/2} H^{(2)}_{\lambda}(\sqrt{2\gamma}) R(\lambda,\gamma)}{H^{(1)}_{\lambda+1}(\sqrt{2\gamma}) - (\frac{1}{2}\gamma)^{1/2} H^{(1)}_{\lambda}(\sqrt{2\gamma}) R(\lambda,\gamma)}, \quad (3.7)$$

where

$$R(\lambda,\gamma) = \frac{U(1-\gamma/2\lambda, 1, \lambda)}{U(-\gamma/2\lambda, 0, \lambda)}, \quad (3.8)$$

and $\lambda = l + \frac{1}{2}$.

The limit $l \to -\frac{1}{2}$ ($\lambda \to 0$) of the ratio R of Kummer functions in Eq. (3.8) is very delicate. We have not been able to find it in the literature, but we can calculate it with the help of a uniformly convergent expansion of the Kummer functions $U(a,b,z)$ around $z = 0$ and for integer values of the parameter b [see, for example, Eq. (9.237) of Ref. 10].

The result is given by

$$R(\lambda,\gamma) \underset{\lambda\to 0}{\simeq} \left(\frac{2}{\gamma}\right)^{1/2} \frac{J_0(\sqrt{2\gamma})\cot(\pi\gamma/2\lambda) + N_0(\sqrt{2\gamma})}{J_1(\sqrt{2\gamma})\cot(\pi\gamma/2\lambda) + N_1(\sqrt{2\gamma})}. \quad (3.9)$$

From these equations we obtain for the S matrix a limiting expression of the form

$$S\left(l, k^2 = \frac{2e\gamma}{r_0^2}\right) \underset{l\to -1/2}{\simeq} A(\sqrt{2\gamma}) \frac{\cot[\pi\gamma/(2l+1)] - C^*(\sqrt{2\gamma})}{\cot[\pi\gamma/(2l+1)] - C(\sqrt{2\gamma})}, \quad (3.10)$$

with

$$A(x) = -\frac{H_0^{(2)}(x)J_0(x) - H_1^{(2)}(x)J_1(x)}{H_0^{(1)}(x)J_0(x) - H_1^{(1)}(x)J_1(x)} \quad (3.11)$$

and

$$C(x) = -\frac{H_0^{(1)}(x)N_0(x) - H_1^{(1)}(x)N_1(x)}{H_0^{(1)}(x)J_0(x) - H_1^{(1)}(x)J_1(x)}. \quad (3.12)$$

Approximate calculations for other values of k^2 have also been done and will be reported elsewhere by one of us (D. H.).[11] They support the results obtained above for $k^2 = 2e\gamma r_0^{-2}$.

We see that there is an infinite number of Regge poles which accumulate at $l = -\frac{1}{2}$. Since we have $k^2 > 0$, the poles are all complex and the elastic unitarity condition is satisfied. Hence we have an essential singularity at $l = -\frac{1}{2}$ which is compatible with two-particle unitarity. The intuitive reason for the existence of this accumulation of Regge poles is to be found in the slowness of the vanishing of $g(r)$ for $r \to 0$. The relevant effective potential[3]

$$\frac{(l+\frac{1}{2})^2 - g^2(r)}{r^2}, \quad g^2(r) \simeq \gamma\left(\ln\frac{r_0}{r}\right)^{-1} \quad (3.13)$$

is attractive down to the small radius

$$r = r_0 \exp\left[-\frac{\gamma}{(l+\frac{1}{2})^2}\right]. \quad (3.14)$$

In contrast, in the case $g^2(0) > 0$, discussed in Sec. I, the system collapses before a large number of resonances or bound states can be formed.

IV. FASTER APPROACH TO ASYMPTOTIC FREEDOM

Before we consider the relevance of these results for asymptotically free field theories, let us briefly discuss the case where $g^2(r)$ in Eq. (1.2) vanishes slightly faster than with the first power of $\ln r$. We assume that

$$g^2(r) \simeq \beta\left(\ln\frac{r_0}{r}\right)^{-2}, \quad \beta > 0 \quad (4.1)$$

for $r\to 0$. Again we can obtain an exact solution with the specific potential

$$V(r)=-\left(\frac{\beta}{r^2[\ln(r_0/r)]^2}-\frac{\beta e^2}{r_0^2}\right)\theta\left(\frac{r_0}{e}-r\right), \quad (4.2)$$

provided $k^2=\beta e^2 r_0^{-2}$. The wave function for $r<r_0/e$ is given by

$$\psi_l(r)\propto r^{-1/2}\left((l+\tfrac{1}{2})\ln\frac{r_0}{r}\right)^{1/2} \times K_{(1-4\beta)^{1/2}/2}\left((l+\tfrac{1}{2})\ln\frac{r_0}{r}\right), \quad (4.3)$$

which does not depend upon the sign of the root $(1-4\beta)^{1/2}$. For $r\to 0$, $l\neq -\frac{1}{2}$, we have

$$\psi_l(r)\propto r^l\left[1-\frac{\beta}{(2l+1)\ln(r_0/r)}+\cdots\right]. \quad (4.4)$$

In contrast in Eq. (3.6) obtained for logarithmically decreasing $g^2(r)$, we find here that the leading term r^l is not modified, but is approached logarithmically. In the superenormalizable case, where $g^2(r)\simeq r^{-\epsilon}$, $\epsilon>0$ for $r\to 0$, the corresponding approach is by a power of r.

The S matrix for $k^2=\beta e^2 r_0^{-2}$ can be calculated as before. It is given by

$$S\left(\lambda,k^2=\frac{\beta e^2}{r_0^2}\right)=-\frac{H_\lambda^{(2)}(\sqrt{\beta})}{H_\lambda^{(1)}(\sqrt{\beta})}\,\frac{\lambda K_\nu'(\lambda)/K_\nu(\lambda)-\lambda+\frac{1}{2}+\sqrt{\beta}H_{\lambda-1}^{(2)}(\sqrt{\beta})/H_\lambda^{(2)}(\sqrt{\beta})}{\lambda K_\nu'(\lambda)/K_\nu(\lambda)-\lambda+\frac{1}{2}+\sqrt{\beta}H_{\lambda-1}^{(1)}(\sqrt{\beta})/H_\lambda^{(1)}(\sqrt{\beta})}, \quad (4.5)$$

where $\lambda=l+\frac{1}{2}$ and

$$\nu=\tfrac{1}{2}(1-4\beta)^{1/2}. \quad (4.6)$$

We note that the amplitude is independent of the sign of the square root in Eq. (4.6) because of the relation $K_\nu=K_{-\nu}$. Choosing the positive sign of the root, we find in the limit $\lambda\to 0$

$$\lambda\frac{K_\nu'(\lambda)}{K_\nu(\lambda)}\underset{\lambda\to 0}{\simeq}-\nu+\frac{2\nu}{1-[\Gamma(1+\nu)/\Gamma(1-\nu)](\frac{1}{2}\lambda)^{-2\nu}}. \quad (4.7)$$

We see that the amplitude generally has a branch point at $\lambda=0$ (corresponding to $l=-\frac{1}{2}$). Unless the parameter 2ν is rational, this branch point is of infinite order. In particular for $\beta>\frac{1}{4}$, we obtain $\nu=i\kappa$, $\kappa=(4\beta-1)^{1/2}>0$, and we can have unlimited oscillations as $\lambda\to 0$. The principal branch, which defines the physical sheet of the amplitude with respect to the branch point at $\lambda=0$, is defined by the principal value of the logarithm in

$$(\lambda)^{-2\nu}=\exp(-2\nu\ln\lambda); \quad (4.8)$$

the corresponding cut runs along the negative real λ axis.

If the positive parameter β is restricted by $\beta\leq\frac{1}{4}$, the index ν is real and we find no poles of the S matrix, at least near $\lambda=0$. However, for $\beta>\frac{1}{4}$ and $\nu=i\kappa$, the denominator in Eq. (4.5) is proportional to

$$1+\frac{2i\kappa}{\frac{1}{2}-i\kappa-\sqrt{\beta}H_1^{(1)}(\sqrt{\beta})/H_0^{(1)}(\sqrt{\beta})}-\frac{\Gamma(1+i\kappa)}{\Gamma(1-i\kappa)}\exp\left(-2i\kappa\ln\frac{\lambda}{2}\right) \quad (4.9)$$

for $\lambda\to 0$. Generally, it gives rise to an unlimited number of poles as λ approaches the point $\lambda=0$ through complex points of the λ plane.

Our calculations in this section show how sensitively the character of the fixed singularity at $l=-\frac{1}{2}$ depends upon the rate of decrease of $g^2(r)$ for $r\to 0$. These results may also be of importance for field theory in connection with sums of ladder graphs where the effective parameter is $g^4(\xi)$ while $g^2(\xi)$ vanishes logarithmically as the scale parameter ξ tends to infinity.

V. GENERAL ANALYTIC PROPERTIES

We can consider the singular structures of the scattering amplitudes obtained for our potential models from the more general point of view of the analytic properties of these amplitudes in the (l,k^2) manifold. We have found that, breaking the scale covariance of the straight g^2r^{-2} potential, the fixed cut disappears: As $g^2(r)\to 0$ for $r\to 0$, it contracts, so to speak, to the point $l=-\frac{1}{2}$. Some kind of singularity, if any, remains there. We may ask: What kind of singular structures can we have at this fixed point in the l plane?

If there are no branch-point surfaces in the (l,k^2) manifold which can act as shielding cuts and no physical zero-mass particles, then the elastic unitarity condition is very restrictive in selecting the possible fixed singularities in the l plane.[3,7,12] We can have either soft branch points, where the amplitude remains bounded, or certain types of branch points of infinite order and essential singularities where the approaches to the singular point, along which the amplitude becomes unbounded, have the appropriate dependence upon the two-particle threshold in order to comply with the unitarity condition.[13] Typically, in our examples these approaches correspond to accumulations of Regge poles which become complex above the threshold.

Other fixed hard singularities would require shielding cuts. But these are moving branch points and they cannot be present in the l plane as long as multiparticle t-channel ($t \sim k^2$ for potential scattering) thresholds are included.[8] The reason is that, without such thresholds, the branch-point surfaces cannot be removed from the t-plane for large values of Rel, where the amplitude is known to be regular in the cut t-plane. These moving branch points cannot disappear from the physical sheet through the elastic cut, nor through left-hand branch lines. Hence we conclude that, in potential scattering, the singular structures remaining at $l = -\frac{1}{2}$ for $g(r) \to 0$ are either soft branch points or essential singularities and special branch points of the type described above. This is just what we find in our examples. In particular, the interesting essential singularity is connected with an accumulation of Regge poles at $l = -\frac{1}{2}$. Every Regge pole satisfies the elastic unitarity condition and so does the essential singularity.

On the basis of these general considerations of possible singularities, we can now discuss the implication of asymptotically free field theories of the results obtained in potential scattering. As long as field-theoretical calculations do not take into account multiparticle channels to the extent that moving branch points for shielding can be generated, and as long as no zero-mass particles are present, we can have only certain special singularities at the position in the l plane where a fixed cut collapses for $g \to g_\infty = 0$. These are the types of singularities described above which are able to comply by themselves with the unitarity condition. The situation described above can be found as a result of renormalized ladder calculations, where scale-breaking vertices and propagator corrections have been inserted. Although ladder graphs contain not only two-particle, but also multiparticle intermediate states, these are generally not sufficient to generate the required branch-point trajectories.

Of course, even with multiparticle states included so that shielding cuts can in principle be generated, the field theory may well choose the type of fixed singularity we have seen in our potential models. These singularities comply by themselves with two-particle t-channel unitarity. Our point is only that with shielding cuts, other types of fixed singularities become possible which are not allowed in potential scattering. Also certain field theories simply may not contain branch-point surfaces with the right properties for the shielding mechanism.

In a field where the coupling $g^2(\xi)$ vanishes logarithmically as the scale parameter $\xi \to \infty$, we may have an analogy to our potential (3.3) giving rise to an accumulation of Regge poles. An explicit example is the $(\phi^3)_6$ theory (ϕ^3-coupling theory in six dimensions), which has been discussed by Lovelace.[2] Here the coupling g^2 vanishes logarithmically in the high-momentum limit, and the renormalized ladder approximation leads to an amplitude which has an accumulation of Regge poles at $l = -1$. In this theory the point $l = -1$ is the one to which the highest fixed cut obtained with the bare ladder sum contracts for $g^2 \to 0$; it is an analog of the point $l = -\frac{1}{2}$ in our potential model. On the other hand, $(\phi^4)_4$-coupling theory, which is renormalizable but not asymptotically free, has fixed branch points in the l plane if one calculates the amplitude points in the renormalized ladder approximation. The highest fixed branch cut is centered around the point $l = 0$, and the right-hand branch point depends only upon the finite fixed point $g_\infty > 0$ of the theory. As we have pointed out in Sec. II, this theory should be associated with the potential (1.2) for $g(0) > 0$.

VI. REMARKS AND SPECULATIONS

For quantum electrodynamics, it is well known that, for example, the electron-electron scattering amplitude has a fixed branch point in the j plane above $j = 1$, as long as the calculation is restricted to a summation of ladder diagrams.[14] The branch point results from ladders involving photon-photon scattering diagrams, and its position depends upon α^2, corresponding to g^4 in our notation. It is expected to be modified by the inclusion of a sufficiently complete set of multiparticle t-channel states which allows for the generation of moving branch points. In any case, the branch point above $j = 1$ has to disappear from the amplitude because of s-channel unitarity. In a more intuitive fashion, it can be argued[14] that absorptive corrections will transform it into a complex branch-point trajectory of the form $\alpha(t) = 1 + \text{const} \times \sqrt{t}$ near $t = 0$ which saturates the Froissart bound. In fact, any bare amplitude which requires complete absorption for the restoration of s-channel unitarity gives rise to a trajectory of this type.[15]

While for ordinary renormalizable theories like quantum electrodynamics a modification of the fixed cuts in the j plane may be expected, and certainly is required in the case of branch points above $j = 1$, it is an open question whether in asymptotically free theories the fixed singularities obtained on the level of two-particle t-channel unitarity are altered by the inclusion of multiparticle states. As we have pointed out in the last section, the kind of fixed singularities one obtains are typical for compliance with the elastic unitarity condition, while the inclusion of multiparticle states can make shielding cuts possible and hence other types of fixed singularities can occur. Nevertheless, this does not mean that the singularities we have obtained must be changed; the situation may be different for different theories. If an essential singularity appears at $j=1$, it must also comply with s-channel unitarity (Froissart bound). Together with the reality properties of the amplitude, this is a strong restriction. It may imply some modification of the singularity by multiparticle t-channel thresholds.

In asymptotically free gauge theories,[16] even lowest-order calculations of the anomalous dimensions require more than two-particle states. It must be left to future work to find the exact properties of fixed singularities in these theories. In particular, also the influence of spin remains to be explored in connection with the existence of accumulation points.[11]

Finally, we must mention the problem of the actual calculation of the high-energy limit of hadron scattering amplitudes in terms of an underlying field theory. We may ask what remains of these fixed singular structures we have discussed if we consider now the j plane of the hadron-hadron scattering amplitude. If we try to approach the problem from first principles via the elementary fields, the confinement of quarks and gluons could become relevant. On the other hand, it may be that the essential features of our simple potential calculations retain some meaning if applied directly to the hadron scattering amplitude. Models of hadrons constructed on the basis of asymptotically free non-Abelian gauge theories are not expected to have *physical* zero-mass particles. The vector mesons either acquire masses through symmetry-breaking mechanisms, or they are assumed to be permanently confined as, for instance, in certain color gauge theories. Therefore, the physical scattering amplitudes of hadrons should satisfy the two-particle unitarity requirements we have used in our considerations. Still, there remain the question of massive multiparticle states and many other problems. We hope to come back to these problems elsewhere.

Note added in proof. After completion of this paper, we found two references which consider potentials with a short-distance behavior like $r^{-2}(\ln r)^{-n}$ for purposes other than ours. Only wave functions and their limits for $r\to 0$ are discussed. The references are: J. M. Charap and N. Dombey, Phys. Lett. 9, 210 (1964); H. Cornille and E. Predazzi, J. Math. Phys. 6, 1730 (1965).

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[1]Brief reports about the results of this paper have been presented by one of us (R.O.) in invited talks given at the International Conference on High Energy Physics, Palermo, Italy, 1975 (unpublished), and the 1975 Black Forest Discussion Meeting on Hadron-Hadron Scattering, Todtnauberg, Germany, 1975 (unpublished).
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