Chiral symmetry and pion condensation. I. Model-dependent results

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The successes of current algebra and partial conservation of axial-vector current can be interpreted as indications that the strong interactions are approximately symmetric under chiral SU(2)×SU(2) transformations. In the absence of matter, the explicit chiral-symmetry breaking, which conserves isospin and gives the pion a small mass, defines a unique vacuum state for the theory. The smallness of the chiral-symmetry breaking, however, implies the existence of many other states, approximately degenerate in energy with, but orthogonal to, the true vacuum. In these states, which can be obtained formally from the vacuum by a chiral rotation, in general the expectation value of the π field in the ground state, $\langle\pi\rangle, \neq 0$; thus they contain a "pion condensate." At zero baryon density these states are unphysical, but in a macroscopic system—such as a neutron star—at very high baryon density one of these pion-condensed states might become the true ground state. In this first of a series of papers on the implications of chiral symmetry for pion condensation, we study this phenomenon in the linear σ model. As a consequence of the simplicity of this model (and of a series of justifiable approximations) we are able to calculate analytically the "phase diagram" describing the ground state of infinite nuclear matter as a function of baryon density. We find that above a critical baryon density $\rho_c \simeq O(\rho_{\text{nucl}})$ a phase transition to a pion-condensed ground state occurs. To correct the most obvious phenomenological deficiencies of this simple model we extend the chiral-symmetry approach to include the effects of the $N^*(1236)$ and the ill-understood πN s waves. Further, we indicate briefly additional affects which must be included in any serious quantitative description of real pion condensation. In addition to analyzing the ground state, we examine briefly the spectrum of excited states, an understanding of which is vital to the explanation of the cooling mechanism of neutron stars. In the condensed phase, the meson excitation spectrum contains a "Goldstone boson" associated with the ground state's not being an eigenstate of the conserved operator I_3. However, when electromagnetic interactions are included, this mode disappears via the Higgs phenomenon indicating that the ground state is a superconductor; thus only plasma excitations remain in the meson spectrum. The presence of the pion condensate fosters the β decay of the fermion excitations, and the emitted neutrinos provide the cooling mechanism for neutron stars.

I. INTRODUCTION

Considerable attention in the current literature has focused on the possibility that the ground state of extended regions of nuclear matter at high density ($\rho > \rho_{\text{nucl}}$) is qualitatively different from the "normal" nuclear ground state as inferred from the structure of existing nuclei. In particular, the existence of a "condensed pion phase"—that is, a ground state in which pions, macroscopically occupying a single mode, form a significant constituent—has been proposed and discussed by many authors.[1-5] Most of the impetus for investigations of this new type of ground state has come from its possibly striking astrophysical implications in connection with the properties of neutron stars.[1-5] However, recent studies[6,7] of conceptually similar "abnormal" ground or low-lying excited states of field theories of the strong interactions have served to emphasize the wider interest such possible new phases possess.

To argue the plausibility of a condensed pion phase for the nonspecialist,[5] let us specifically consider the case of neutron-star matter, which is thought to consist primarily of neutrons at very high densities ($\rho > \rho_{\text{nucl}}$) in equilibrium with a smaller number of protons and electrons.[8] At such densities the exclusion principle requires the existence of large Fermi seas and consequently of large chemical potentials,

$$\mu_i \equiv \frac{\partial E}{\partial \rho_i} \quad .$$

Indeed, in neutron-star matter at normal nuclear density ($\rho_{\text{nucl}} = 2.8 \times 10^{14}$ g/cm^3), $\mu_n - \mu_p \approx \frac{3}{4} m_\pi$,[5] and thus the possibility arises that at slightly higher densities, the reaction[9]

$$n \to p + \pi^-$$

which has, crudely speaking, a net energy change $E_i - E_f \simeq \mu_n - \mu_p - m_\pi^{\rm eff}$, could, assuming $m_\pi^{\rm eff} \simeq m_\pi$, reduce the energy of the system if $\Delta E > 0$. If such transitions did occur, since the pions are bosons, they could all occupy that single mode which gave the lowest total energy; the resulting state would therefore be a "pion condensate."

One crucial aspect of this possibility is that it is the effective pion mass, $m_\pi^{\rm eff}$, which enters. Since the pion interacts strongly with the nuclear medium, this can be quite different from m_π. Thus the central issue in any discussion of pion condensation is the evaluation of the interactions of the putative condensed pion mode with the surrounding nuclear matter; in the independent-particle approximation,[4] this reduces to analysis of (off-shell) πN interactions.

That the repulsive s-wave $\pi^- n$ interactions would increase $m_\pi^{\rm eff}$ (see Ref. 10) and therefore inhibit pion condensation has long been recognized.[11] More recently, it was noted that if the pions condensed in a mode with nonzero momentum, the attractive p-wave $\pi^- n$ interactions could overcome the s-wave repulsion, thereby producing a lower total $m_\pi^{\rm eff}$ and favoring pion condensation. On the basis of this observation, many authors,[2-5] using a wide variety of techniques, have sought and found—in simple models including s- and p-wave interactions—a phase transition from the normal nuclear matter ground state to a state with pions condensed in a mode of nonvanishing momentum for densities in the range $\frac{1}{2}\rho_{\rm nucl} < \rho_{\rm crit} < 4\rho_{\rm nucl}$. Attempts to refine the calculations by including other, supposedly less important effects—$\pi\pi$ interactions, more accurate treatment of the $N^*(1236)$, inclusion of NN forces and correlations—have revealed that this phase transition can be quite sensitive to these additional effects. Hence despite its appearance in simple models, the actual existence of pion condensation in neutron stars remains an important open question.

Although it seems clear that the ultimate resolution of this problem can only come from detailed quantitative studies, both the central role of the πN interaction and the possible importance of many smaller hadronic effects suggest that a more unified view of the strong-interaction aspects of pion condensation—a general qualitative framework in which to interpret this phenomenon—would be very valuable. In this paper we shall argue that the approximate $SU(2)\times SU(2)$ chiral symmetry of the strong interactions can provide a framework of significant conceptual and modest calculational utility in confronting the very real intricacies of pion condensation.

To motivate our investigation let us outline briefly the logic that supports it. The successes of current algebra and partial conservation of axial-vector current (PCAC) can be interpreted as indications that the strong interactions are approximately symmetric under chiral $SU(2)\times SU(2)$ transformations and, further, that this symmetry is realized in the "spontaneously broken" or "Goldstone" mode: That is, rather than exhibiting approximately massless or parity-doubled nucleons, the world contains pions which, were nature exactly chiral-symmetric, would be massless Goldstone bosons. The smallness of the pion mass relative to other hadronic parameters is an indication of the degree to which chiral symmetry is respected.

The possible significance of those observations for pion condensation is most easily seen in the context of an explicit realization of chiral symmetry: the σ model of πN interactions.[12] In this model the pion is joined by an isosinglet meson, σ, in a $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\times SU(2)$, and this meson multiplet is coupled in a chirally-invariant manner to the nucleons. In addition, there is an explicit symmetry-breaking term, $\epsilon H_{\rm SB}(\sigma, \vec{\pi})$, which conserves isospin and gives the pion a mass $m_\pi \simeq O(\epsilon)$. As a consequence of this symmetry breaking, the ground state of the theory in the vacuum is uniquely given by $\langle\sigma\rangle = A$, $\langle\pi_i\rangle = 0$; in our present parlance, the normal ground state of the model contains a "σ condensate." Now if $\epsilon = 0$, the state $\langle\sigma\rangle = A$, $\langle\pi_i\rangle = 0$ would be degenerate with all states of the form $\langle\sigma^2 + \vec{\pi}\cdot\vec{\pi}\rangle = A^2$; in other words, states with a "pion condensate," $\langle\pi_i\rangle \neq 0$, would be degenerate with the "normal" states. Thus a consequence of approximate chiral invariance ($m_\pi \ll m_N$) is the existence of possible condensed pion phases which are nearly degenerate in energy with the normal nuclear ground state. When one imposes the constraints appropriate to neutron-star matter—high hadronic density and charge neutrality so that $\mu_n - \mu_p \simeq O(m_\pi)$ = order of chiral-symmetry breaking—one of these condensed pion phases may indeed become the ground state.

But beyond simply suggesting the possibility of pion condensation, chiral invariance has general consequences for the existence and nature of this new phase. Several such consequences—the limiting effect of π-π interactions at large condensate amplitude, the role of the $N^*(1236)$, and the extent to which nuclear correlations and forces can be similar in the pion-condensed and normal phases—will be treated in the course of our discussion.

To present our arguments in detail we begin in Secs. II and III by examining the problem of pion condensation in the σ model, first for simplicity in a version with only the $(\sigma, \vec{\pi})$ mesons and then in the physically interesting case including nucleons. In both cases we establish the existence and discuss briefly the properties of the pion condensed phase. One important result of the discussion of this model is the observation that the phase transition indeed corresponds to a chiral rotation of the ground state [in this case, a specific rotation in the $(\sigma, \vec{\pi})$ space]. This interpretation of the phase transition is in fact a general feature of chiral invariance; the proof of this assertion is, because of its highly technical nature, reserved for a separate article. In Sec. IV we exploit the "chiral rotation" approach to pion condensation to include the effects of the $N^*(1236)$ on the phase transition and to establish a rough "upper bound" in hadronic density below which we can realistically expect pion condensation to occur. In Sec. V we discuss, in a qualitative fashion, the manner in which nuclear correlation effects[2,3,5] can be incorporated into our chiral-symmetry approach. Section VI investigates the nature of the pion excitations in the condensed phase, first in the absence and then in the presence of nucleons. The naive expectation that there exists a massless excitation—the Goldstone boson arising from the spontaneous breakdown of isospin invariance in the condensed phase—is shown to be incorrect when electromagnetic interactions are included. The photon and Goldstone boson combine through the Higgs mechanism to produce a plasmon. At finite temperature, the fermions are unstable against β decay. The pion condensate is shown to enhance dramatically the β-decay rate in the manner originally suggested by Bahcall and Wolf.[1] We conclude the main text with a general discussion of our results and some comments on the phenomenology necessary to improve the calculations of the critical density.

Two appendixes provide supplementary and technical material. Appendix A presents the calculational details relevant to the inclusion of the $N^*(1236)$ in our treatment of pion condensation. Appendix B establishes an important conceptual link between pion condensation and other abnormal states recently found in the σ model.[6]

II. THE σ MODEL WITHOUT NUCLEONS

A. The Lagrangian density[12]

To introduce our ideas in the simplest possible context we begin by considering the standard SU(2)×SU(2) σ model without nucleons.[13] The symmetric part of the Lagrangian density

$$\mathcal{L}_0(x) = \tfrac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi}) + \frac{m_0^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}^2)^2 , \tag{2.1}$$

expressed in terms of the isosinglet field σ and isovector field $\vec{\pi}$, is invariant under the chiral SU(2)×SU(2) transformations generated by linear combinations of the infinitesimal isospin transformations

$$\pi_i \to \pi_i - \epsilon_{ijk} u_j \pi_k , \quad \sigma \to \sigma \tag{2.2}$$

and chiral transformations

$$\pi_i \to \pi_i + v_i \sigma , \quad \sigma \to \sigma - \vec{v}\cdot\vec{\pi} . \tag{2.3}$$

The full Lagrangian density includes a term which, while breaking the invariance under the chiral transformation of (2.3), respects the isospin symmetry generated by the transformations of (2.2). Since the form of the chiral-symmetry breaking will prove important for certain aspects of the possible condensed pion phase, we shall consider two different types of symmetry breaking: The "standard" term[14]

$$\mathcal{L}_{SB}^{(1)} = + c_1 \sigma , \tag{2.4}$$

where c_1 is a positive constant whose value we shall shortly determine, and, for comparative purposes,

$$\mathcal{L}_{SB}^{(2)} = - c_2 \vec{\pi}\cdot\vec{\pi} . \tag{2.5}$$

We shall refer to the symmetry breaking in (2.4) as "$\cos\theta$" symmetry breaking and to that in (2.5) as "$\sin^2\theta$" symmetry breaking; the reason for this nomenclature, which derives from the transformation properties of these terms under chiral rotations, will become apparent later. For brevity and conciseness of notation we define

$$\mathcal{L}^{(i)}(x;\epsilon) \equiv \mathcal{L}_0(x) + \epsilon \mathcal{L}_{SB}^{(i)}(x), \quad i = 1, 2 . \tag{2.6}$$

By choosing ϵ and i appropriately we can discuss the symmetric limit ($\epsilon = 0$) and either of the two broken-symmetry cases ($\epsilon = 1$, $i = 1$ or 2).

We shall treat the Lagrangians $\mathcal{L}^{(i)}$ in the renormalized tree approximation[12] in which the parameters in $\mathcal{L}^{(i)}$ are directly identified with physical observables. For the standard symmetry breaking, using the result that the divergence of the axial-vector current is given by

$$\partial_\mu A_i^\mu \equiv c_1 \pi_i \tag{2.7}$$

leads to the identifications[12]

$$m_\pi^2 = \lambda^2 f_\pi^2 - m_0^2 , \tag{2.8a}$$

$$f_\pi = c_1 / m_\pi^2 , \tag{2.8b}$$

and

$$m_\sigma^2 = 3\lambda^2 f_\pi^2 - m_0^2 . \tag{2.8c}$$

Here m_π and f_π are the physical pion mass and decay constant. If we assume a value of m_σ then m_0, λ, and c_1 are determined by Eqs. (2.8); choosing $m_\sigma \simeq 1$ GeV, for example, gives very roughly $\lambda^2 \simeq 50$, $m_0 \simeq 5m_\pi$. Thus these parameters are "large" in the sense $m_0 \gg m_\pi$ and $\lambda \gg 1$.

In the case of $\mathcal{L}_{SB}^{(2)}$ in lieu of (2.7) one has

$$\partial_\mu A_i^\mu = 2c_2\sigma\pi_i \ , \tag{2.9}$$

so that in the renormalized tree approximation we must choose

$$m_\pi^{\ 2} = 2c_2 \ , \tag{2.10a}$$

$$f_\pi = m_0/\lambda \ , \tag{2.10b}$$

$$m_\sigma^{\ 2} = 2m_0^{\ 2} \ . \tag{2.10c}$$

Again taking the illustrative case $m_\sigma \simeq 1$ GeV, we find λ^2 and m_0 roughly equal to their previous values.[15] Henceforth, in this section, whenever we write m_0 and λ we shall mean the appropriate values determined by Eqs. (2.8) or (2.10).

B. The Hamiltonian density

To study the ground-state energy, it is most convenient to treat the Hamiltonian

$$H^{(i)}(\epsilon) = \int \mathcal{H}^{(i)}(x;\epsilon)\, d^3x \ , \tag{2.11}$$

where the Hamiltonian density is given by

$$\mathcal{H}^{(i)}(x;\epsilon) = \frac{p_{\pi_i}p_{\pi_i} + p_\sigma p_\sigma}{2} + \frac{\nabla\pi_i\cdot\nabla\pi_i + \nabla\sigma\cdot\nabla\sigma}{2} - \frac{m_0^{\ 2}}{2}(\sigma^2 + \vec{\pi}\cdot\vec{\pi}) + \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}\cdot\vec{\pi})^2 - \epsilon\,\mathcal{L}_{SB}^{(i)}(x) \, . \tag{2.12}$$

Here p_{π_i} and p_σ denote the canonical momenta corresponding to the π_i and σ fields, respectively. Working in the tree approximation, which corresponds to considering σ and π_i as classical fields,[16] one can obtain the familiar form[12] of the ground state by minimizing the expectation value of the Hamiltonian with respect to the fields and momenta. Let us recall very briefly the results of this procedure. First, note that the positivity of the $(\nabla\pi_i\cdot\nabla\pi_i)$ and $(\nabla\sigma\cdot\nabla\sigma)$ terms in H guarantees, in the absence of derivative-dependent interactions, that $\langle\pi_i\rangle$ and $\langle\sigma\rangle$, the expectation values of the fields in the ground state, should be spatial constants. Second, observe that our deliberate choice of the "wrong" sign mass term implies that these expectation values are indeed nonzero in the ground state. For $\epsilon = 0$, the minimum occurs on the hypersphere

$$\langle\sigma^2 + \vec{\pi}\cdot\vec{\pi}\rangle = m_0^{\ 2}/\lambda^2 \ . \tag{2.13}$$

Thus the ground state of $H^{(i)}(0)$ is infinitely degenerate.

In the actual physically relevant limit, $\epsilon = 1$, the ground states for $i = 1, 2$ are the nondegenerate states in which, for both $\mathcal{L}_{SB}^{(1)}$ and $\mathcal{L}_{SB}^{(2)}$, $\langle\sigma\rangle = f_\pi$ and $\langle\pi_i\rangle = 0$. Hence in our current terminology, the normal ground state of the σ model already contains a "condensate" of σ mesons. For $\epsilon = 0$, this state is degenerate with many "pion-condensed" states, $\{\langle\sigma\rangle = 0,\ \langle\pi_i\rangle \neq 0\}$, indeed, with all the states in (2.13). All the possible ground states are simply related by SU(2)×SU(2) rotations. Since the chiral-symmetry breaking is small compared to other parameters—$m_\pi \ll M_N$—even for $\epsilon = 1$ we expect that the σ-condensed state—that is, the true ground state of $H^{(i)}(\epsilon)$—and the possible pion-condensed states are nearly degenerate and remain approximately related by a simple chiral rotation. From this point of view, the possibility is clearly indicated that, by a small shift in the physical environment of the $(\sigma, \vec{\pi})$ system, the pion-condensed state could become energetically favored and thus become the ground state in this new environment. In actual neutron-star matter, the constraint imposed by requiring specified hadronic charge and baryon number densities provide precisely the appropriate physical circumstances for this "chiral rotation" of the ground state to occur.

To study this possibility in the present simple model we can consider the problem of minimizing the Hamiltonian in (2.12) subject to the constraint that the charge density

$$\rho_Q \equiv (p_{\pi_1}\pi_2 - p_{\pi_2}\pi_1) = V_3^0 \ , \tag{2.14}$$

which is the time component of the $I_3 = 0$ isospin current, assumes some definite nonzero value, $\langle\rho_Q\rangle$. We have chosen this constraint by analogy to the similar restrictions which one obtains in more realistic models of pion condensation. Introducing a Lagrange multiplier (the charge chemical potential), we consider minimizing

$$H_{\text{eff}}^{(i)} \equiv \int [\mathcal{H}^{(i)}(x;\epsilon) + \mu\rho_Q(x)]\, d^3x$$

with (2.15)

$$\frac{\partial\langle H_{\text{eff}}^{(i)}\rangle}{\partial\mu} \equiv \frac{\partial E_{\text{eff}}^{(i)}}{\partial\mu} = Q = \langle\rho_Q\rangle\, V \ ,$$

where V is the volume of the system. With ρ_Q as given by (2.14), we see that functionally minimizing with respect to the momenta implies[17]

$$p_\sigma = p_{\pi_3} = 0, \quad p_{\pi_1} = -\mu\pi_2, \quad p_{\pi_2} = +\mu\pi_1 \ . \tag{2.16}$$

Substituting these results into (2.15) we are left with the problem of minimizing an effective potential energy,

$$V_{\text{eff}}^{(i)} = \int d^3x\, \mathcal{V}_{\text{eff}}^{(i)}(x) \ ,$$

where

$$\mathcal{V}_{\rm eff}^{(i)} = \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}\cdot\vec{\pi})^2 - \frac{m_0^2}{2}(\sigma^2 + \vec{\pi}\cdot\vec{\pi}) - \frac{\mu^2}{2}(\pi_1^2 + \pi_2^2) - \mathcal{L}_{\rm SB}^{(i)} \; . \tag{2.17}$$

C. $\sin^2\theta$ symmetry breaking

To proceed we must distinguish between the two explicit forms of symmetry breaking. For simplicity, we begin with $\mathcal{L}_{\rm SB}^{(2)}$. In this case the effective potential density is

$$\mathcal{V}_{\rm eff}^{(2)} = \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}\cdot\vec{\pi})^2 - \frac{m_0^2}{2}(\sigma^2 + \vec{\pi}\cdot\vec{\pi}) - \frac{\mu^2}{2}(\pi_1^2 + \pi_2^2) + \frac{m_\pi^2}{2}(\vec{\pi}\cdot\vec{\pi}) \; . \tag{2.18}$$

For $\mu < m_\pi$, it is clear that a state with $\langle \pi_i \rangle \neq 0$, for some i, $\langle\sigma\rangle = 0$ has higher energy than the normal state, $\langle\sigma\rangle = f_\pi = m_0/\lambda$, $\langle\pi_i\rangle = 0$. Thus the ground state remains normal for this range of the chemical potential. For $\mu > m_\pi$, however, explicit minimization establishes that the true ground state of the theory is one of the (infinitely degenerate) states with

$$\langle \pi_1^2 + \pi_2^2 \rangle = \left(\frac{m_0^2 + \mu^2 - m_\pi^2}{\lambda^2}\right)^{1/2} \equiv F(\mu) \; , \tag{2.19}$$

$$\langle\pi_3\rangle = 0 \; ,$$

$$\langle\sigma\rangle = 0 \; .$$

Choosing by convention the ground state to have[18]

$$\langle\pi_1\rangle = F(\mu), \quad \langle\pi_2\rangle = 0 \; , \tag{2.20}$$

we see that in the σ-π_1 plane the nature of the ground state as a function of μ is as shown in Fig. 1(a). At the point $\mu = m_\pi$ the ground state is suddenly rotated from $\langle\sigma\rangle = f_\pi$, $\langle\pi_1\rangle = 0$ to $\langle\sigma\rangle = 0$, $\langle\pi_1\rangle = f_\pi$. Recalling that at[19] $T = 0$ the pressure is given simply by

$$P|_{T=0} = -E_{\rm eff}/V \; , \tag{2.21}$$

where

$$E_{\rm eff} = \int d^3x \langle \mathcal{H}_{\rm eff}(x)\rangle = \int d^3x \langle \mathcal{V}_{\rm eff}(x)\rangle \; ,$$

we find for $\mu < m_\pi$

$$\mathcal{E}_{\rm eff} \equiv E_{\rm eff}/V = \frac{-\lambda^2}{4} f_\pi^4 \; , \tag{2.22}$$

whereas for $\mu > m_\pi$

$$\mathcal{E}_{\rm eff} = -\frac{\lambda^2}{4}\left[f_\pi^2 + \left(\frac{\mu^2 - m_\pi^2}{\lambda^2}\right)\right]^2 \; . \tag{2.23}$$

We see that $P(\mu)$ is as illustrated in Fig. 2(a). The sudden rotation of the ground state at $\mu = m_\pi$ translates into discontinuity in $dP/d\mu|_{\mu=m_\pi}$; hence there is a first-order phase transition at $\mu = m_\pi$.[20]

Two further remarks should be made concerning this phase transition and the nature of the ground state for $\mu > m_\pi$. First, from (2.18) it is apparent that the effective potential—indeed, the full effective Hamiltonian—is a function of $(\pi_1^2 + \pi_2^2)$ only and is therefore invariant under I_3 rotations. The ground state G', however, with $\langle\pi_1\rangle = F(\mu)$, $\langle\pi_2\rangle = 0$, is not invariant, and thus the I_3 symmetry of $H_{\rm eff}$ is spontaneously broken. In this situation one naively expects a "Goldstone boson"[21]—or "soft phonon mode"—to be one of the excitations of the new ground state. Since, however, the boson corresponds to a charged pion, one must consider what effect the inclusion

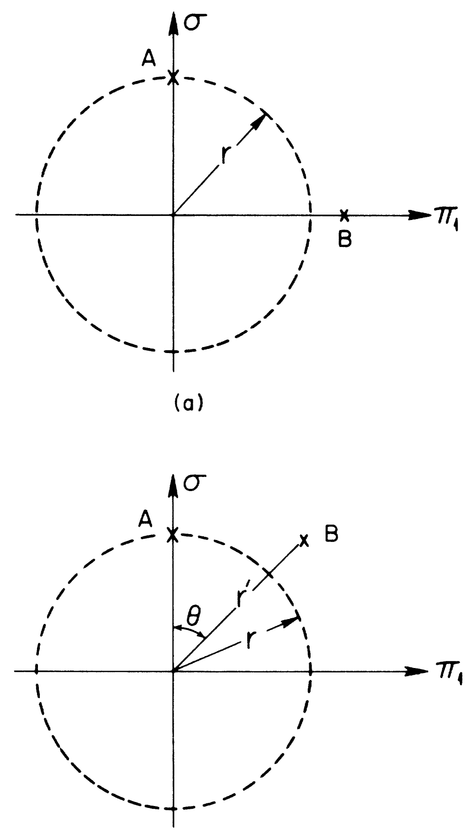


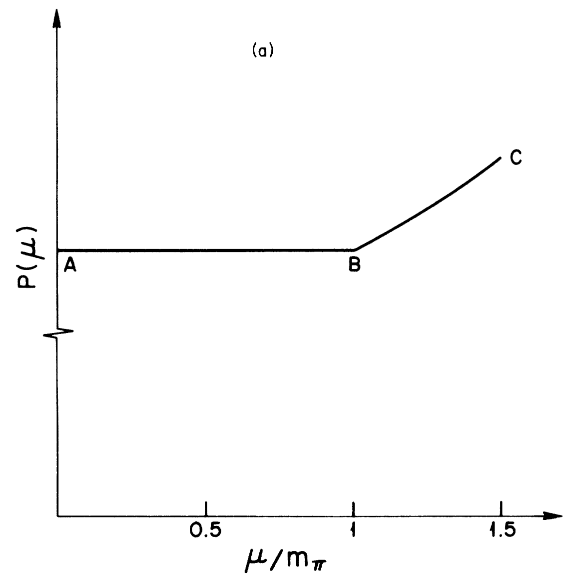
FIG. 1. π-σ diagram. (a) The $(m_\pi^2/2)\pi^2$ symmetry breaking. Point A indicates the ground state for $\mu < m_\pi$. Point B shows the location of the ground state for $\mu > m_\pi$. At B, $\langle\pi_1\rangle = f_\pi^2 + [(\mu^2 - m_\pi^2)/\lambda]^{1/2}$, $\langle\sigma\rangle = 0$. (b) The $f_\pi m_\pi^2\sigma$ symmetry breaking. Point A indicates the ground state for $\mu < m_\pi$. Point B shows the ground state for $\mu = \mu_0 > m_\pi$: $r' = [f_\pi^2 + (\mu_0^2 - m_\pi^2)/\lambda^2]^{1/2}$ and $\cos\theta = (m_\pi^2/\mu_0^2)[1 + (\mu_0^2 - m_\pi^2)/\lambda^2 f_\pi^2]^{-1/2}$.

of electromagnetic interactions will have. In Sec. VI we show in detail that the "Higgs mechanism"—in which the degree of freedom corresponding to the Goldstone boson combines with those of the photon to form a plasma oscillation—will indeed occur. Second, at the point of the phase transition ($\mu = m_\pi$), the two "ground" states, G and G', are exactly related by a chiral rotation. For $\mu \gtrsim m_\pi$, the relation between G and G' is not simply a rotation, but involves a change of "radius" in the (π_1-σ) plane: That is,

$$f_\pi^2 = \langle \pi_1^2 + \sigma^2 \rangle_G \neq \langle \pi_1^2 + \sigma^2 \rangle_{G'}$$

$$= f_\pi^2 + \frac{\mu^2 - m_\pi^2}{\lambda^2} \qquad (2.24)$$

except at $\mu = m_\pi$. However, since $\lambda^2 \gg 1$, for $\mu \simeq O(m_\pi)$, this change in radius is very small. To emphasize the interpretation of this phase transition as a chiral rotation we could have parameterized $\langle\sigma\rangle = A\cos\theta$ and $\langle \pi_1 \rangle = A \sin\theta$ and minimized the effective potential with respect to A and θ. It is clear that this approach yields the correct result: namely, $\theta = 0$, $A = f_\pi$ for $\mu < m_\pi$ and $\theta = \pi/2$, $A = [f_\pi^2 + (\mu^2 - m_\pi^2)/\lambda^2]^{1/2}$, for $\mu > m_\pi$. But further, had we treated the phase transition as purely a chiral rotation—that is, kept A fixed at $A = f_\pi$—we would have made no error at the point of transition and only a slight numerical error[20] for $\mu \gtrsim m_\pi$. These observations will prove useful in the more complicated calculations of Sec. III.



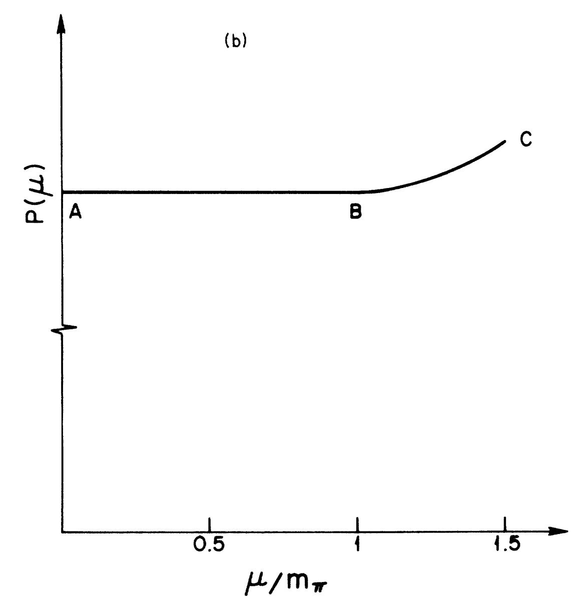


FIG. 2. The pressure vs the chemical potential. (a) In the $(m_\pi^2/2)\pi^2$ symmetry breaking. For AB, $P(\mu) = (\lambda^2/4) f_\pi^4$; the pressure is independent of μ. For BC, $P(\mu) = (\lambda^2/4)[f_\pi^2 + (\mu^2 - m_\pi^2)/\lambda^2]$. Notice the discontinuity in $dP/d\mu$ at $\mu = m_\pi$. (b) In the $f_\pi m_\pi^2 \sigma$ symmetry breaking. For AB, $P(\mu) = (\lambda^2/4) f_\pi^4 + (m_\pi^2 f_\pi^2/2)$. For BC, $P(\mu) = (\lambda^2/4)[f_\pi^2 + (\mu^2 - m_\pi^2)/\lambda^2]^2 + (f_\pi^2 m_\pi^4/2\mu^2)$. At $\mu = m_\pi$, $P(\mu)$ and $dP/d\mu$ are continuous but $d^2P/d\mu^2$ is not.

D. cosθ symmetry breaking

The structure of the ground state in the case of the "standard" symmetry breaking, $\mathcal{L}_{SB}^{(1)}$, is somewhat more complicated than that produced by $\mathcal{L}_{SB}^{(2)}$. Let us therefore treat this case in more detail. Again observing that the effective potential depends on π_1 and π_2 only through $\rho^2 = \pi_1^2 + \pi_2^2$, and that, by inspection $\langle \pi_3 \rangle = 0$ at any minimum, we can reduce the operative part of the effective potential density to

$$\mathcal{V}_{\text{eff}}^{(2)} = \frac{\lambda^2}{4}(\rho^2+\sigma^2)^2 - \frac{m_0^2}{2}(\rho^2+\sigma^2)$$

$$- \frac{\mu^2\rho^2}{2} - f_\pi m_\pi^2 \sigma \ . \qquad (2.25)$$

The conditions $\delta\mathcal{V}_{\text{eff}}^{(2)}/\delta\sigma = \delta\mathcal{V}_{\text{eff}}^{(2)}/\delta\rho = 0$ admit two types of solutions:

$$\langle\rho\rangle = 0, \quad \langle\sigma\rangle = f_\pi \ , \qquad (2.26)$$

and

$$\langle\sigma\rangle = \frac{f_\pi m_\pi^2}{\mu^2} \ ,$$

$$\langle\rho^2\rangle = \frac{m_0^2\mu^2}{\lambda^2} - \frac{f_\pi^2 m_\pi^4}{\mu^4} \equiv r^2(\mu) \ . \qquad (2.27)$$

It is easy to show that for $\mu < m_\pi$, the actual minimum of $V_{\text{eff}}^{(2)}$ is given by the solution in (2.26), and the ground-state energy density is

$$\mathcal{E}^G_{\text{eff}} \equiv E^G_{\text{eff}}/V = -\frac{\lambda^2}{4} f_\pi{}^4 - \frac{m_\pi{}^2 f_\pi{}^2}{2} , \qquad (2.28)$$

independent of μ. As μ passes through m_π, the ground state shifts from the solution in (2.26) to that in (2.27). Following the convention of choosing this new ground state to have $\langle \pi_1 \rangle = r(\mu)$, $\langle \pi_2 \rangle = 0$, we see that the expectation value of π_1 increases smoothly from zero, corresponding to a μ-dependent chiral rotation on the $(\sigma\text{-}\pi_1)$ plane through an angle given by

$$\cos\theta = \frac{m_\pi{}^2}{\mu^2}\left(1 + \frac{\mu^2 - m_\pi{}^2}{f_\pi{}^2\lambda^2}\right)^{1/2} . \qquad (2.29)$$

In this new ground state the energy density is

$$\mathcal{E}^{G'}_{\text{eff}} = -\frac{\lambda^2}{4}\left(f_\pi{}^2 + \frac{\mu^2 - m_\pi{}^2}{\lambda^2}\right)^2 - \frac{f_\pi{}^2 m_\pi{}^4}{2\mu^2} . \qquad (2.30)$$

At $\mu = m_\pi$ one has both $\mathcal{E}^G_{\text{eff}} = \mathcal{E}^{G'}_{\text{eff}}$ and $d\mathcal{E}^G_{\text{eff}}/d\mu = d\mathcal{E}^{G'}_{\text{eff}}/d\mu$; hence the phase transition in this case is second order, as indicated in Fig. 2(b). Once again we note that in the phase above $\mu = m_\pi$ the I_3 symmetry of the effective Hamiltonian is spontaneously broken and a "Goldstone boson" is present.

For both forms of symmetry breaking, then, increasing the charge chemical potential above $\mu = m_\pi$ leads to a new ground state in which the pion field has a nonvanishing expectation value. Beyond illustrating the possibility of pion condensation, these two simple models indicate several general features of this phenomenon. First, they demonstrate the relation of pion condensation to the existence, implied by approximate chiral invariance, of states with $\langle \pi_i \rangle \neq 0$ which are almost degenerate in energy with the normal ground state. Second, they show that the phase transition to the pion-condensed state can be regarded simply as a chiral rotation of the normal ground state. Finally, they establish that the nature of the phase transition depends on the form of chiral-symmetry breaking and thus on the nature of the π-π interaction. In the ensuing sections we shall verify these features in more realistic models of pion condensation.

III. THE σ MODEL WITH NUCLEONS

A. The Lagrangian and Hamiltonian densities

Having clarified the basic concepts of our approach in the simplest conceivable context, we shall next undertake a more quantitative study in the full σ model, which, in addition to permitting us to discuss the realistic case of nuclear matter, has at least some pretensions to being a first approximation to low-energy π-N interactions. We should, however, repeat our basic caveat: Since we are at present interested more in illustrating a new qualitative approach to pion condensation than in presenting detailed quantitative predictions, we shall feel free to make technical as well as physical approximations when these can produce a substantial simplification in our exposition.

The inclusion of nucleons in the σ model leads to a total Lagrangian density whose symmetric part is

$$\mathcal{L}_0(x) = \overline{N}[i\gamma^\mu\partial_\mu - g(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)]N + \mathcal{L}^M_0(x) , \qquad (3.1)$$

where $\mathcal{L}^M_0(x)$, the meson part of the Lagrangian density, is given in Eq. (2.1). Here N represents the nucleon isodoublet

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad (3.2)$$

where p and n are 4-component Dirac spinors. Thus under the infinitesimal isospin and chiral transformations shown in Eqs. (2.2) and (2.3),

$$N \to N' = N + i\frac{\tau_i u_i}{2} N - i\frac{\tau_j v_j}{2}\gamma_5 N . \qquad (3.3)$$

The full Lagrangian is

$$\mathcal{L}^{(i)}(x) = \mathcal{L}^0(x) + \mathcal{L}^{(i)}_{\text{SB}}(x), \quad i = 1, 2 \qquad (3.4)$$

where the symmetry-breaking terms are given by Eqs. (2.4) and (2.5). The Hamiltonian associated with Eq. (3.4) is

$$H^{(i)} = \int d^3x\,\mathcal{H}^{(i)}(x) , \qquad (3.5)$$

with the Hamiltonian density

$$\mathcal{H}^{(i)}(x) = -i\overline{N}\vec{\gamma}\cdot\vec{\nabla}N + g\overline{N}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)N + \mathcal{H}^{(i)}_M(x) , \qquad (3.6)$$

where $\mathcal{H}^{(i)}_M$, $i = 1, 2$ is given by Eq. (2.12) with $\epsilon = 1$, and the γ_i are the standard Dirac matrices.

Assuming that the strong interactions are described by Eq. (3.6), to study the ground state of neutron-star matter we must add two constraints. First, the baryon density must have a prescribed value. Second, when electrons in equilibrium under the reaction $n \to p + e + \nu$ are included, the total charge density must be zero. These constraints are most readily implemented by adding to $\mathcal{H}^{(i)}(x)$ the Lagrange-multiplier terms, corresponding to the charge and baryon-number chemical potentials, and thus studying the effective Hamiltonian

$$\mathcal{H}^{(i)}_{\text{eff}}(x) = \mathcal{H}^{(i)}(x) + \mathcal{H}^{\text{electrons}}_0 + \mu\rho_Q - \nu\rho_B \qquad (3.7)$$

(see Ref. 22) as a function of μ and ν. Here

$$H_0^{\text{electrons}} = \int \mathcal{H}_0^{\text{electrons}}(x)\, d^3x$$

is the energy in the Fermi sea of electrons,

$$\rho_Q = \left(N^\dagger \frac{\tau_3}{2} N + p_{\pi_1}\pi_2 - p_{\pi_2}\pi_1 - \psi_e^\dagger \psi_e \right), \tag{3.8}$$

and

$$\rho_B = N^\dagger N . \tag{3.9}$$

For notational simplicity we shall henceforth suppress the superscript (i) unless we wish to distinguish between the two forms of symmetry breaking.

The expectation value of H_{eff} in the ground state is the free energy of this state. Calling $\langle H_{\text{eff}} \rangle \equiv E_{\text{eff}}$, we see that

$$\left.\frac{\partial E_{\text{eff}}}{\partial \nu}\right|_\mu = -\langle \rho_B \rangle V = -B , \tag{3.10}$$

the total number of baryons, and

$$\frac{\partial E_{\text{eff}}}{\partial \mu} = \langle \rho_Q \rangle V = Q = 0 . \tag{3.11}$$

In actual calculations we shall find it somewhat simpler to study

$$\mathcal{H}'_{\text{eff}}(\mu, \rho_B) \equiv \mathcal{H}_{\text{eff}} + \nu \rho_B \tag{3.12}$$

since then when Eq. (3.11) is satisfied, say for some $\mu = \mu_0$, $E'_{\text{eff}} \equiv \langle H'_{\text{eff}} \rangle$ is just the true ground-state energy at baryon density $\langle \rho_B \rangle$:

$$E'_{\text{eff}}(\mu_0, \langle \rho_B \rangle) = \langle H \rangle \equiv E(\langle \rho_Q \rangle = 0, \langle \rho_B \rangle) . \tag{3.13}$$

Since H_{eff} contains no explicit electron-hadron interactions, it can immediately be split into two parts, and the term involving electrons can readily be evaluated. The contribution from $H_0^{\text{electrons}}$ is simply the energy contained in a Fermi sea of free, relativistic electrons at $T = 0$. Thus

$$\begin{aligned} \mathcal{E}_0^{\text{electrons}} &\equiv E_0^{\text{electrons}} / V \\ &= \frac{1}{4\pi^2} \epsilon_F{}^4 \\ &= \frac{1}{4\pi^2} \mu^4 , \end{aligned}$$

where the final equality follows via $\epsilon_F = \mu_e = \mu$, the chemical potential for (negative) charge. The contribution from the $+\mu\rho_Q$ term is

$$(\mu\rho_Q)_{\text{electrons}} = -\mu\rho_e = -\mu \frac{\mu^3}{3\pi^2} .$$

Thus

$$\mathcal{E}_{\text{eff}}^{\text{electrons}} = -\mu^4 / 12\pi^2 . \tag{3.14}$$

For the hadronic component of H_{eff} the existence of the many-body effects of the nuclear matter precludes application of the simple minimization techniques of Sec. II. However, we can apply a variational technique based on a parametrization of the ground-state expectation values of $(\sigma, \vec{\pi})$ similar to that previously used. Anticipating that the pions will condense in a single mode with nonvanishing momentum, we can parametrize the mesonic part of the pion-condensed ground state, G', by[9]

$$\begin{aligned} \langle \sigma \rangle &= A \cos\theta , \\ \langle \pi_\pm \rangle &= A \sin\theta\, e^{\pm i \vec{k} \cdot \vec{x}} , \\ \langle \pi_0 \rangle &= 0 . \end{aligned} \tag{3.15}$$

By minimizing the total free energy with respect to A,[23] k, and θ, we can determine the nature of G'.

B. The chiral rotation "connecting" normal and condensed pion states

To calculate the free energy we must evaluate

$$E_{\text{eff}} \equiv \langle G' | H_{\text{eff}} | G' \rangle . \tag{3.16}$$

But observing that the state $|G'\rangle$ is just a chiral rotation on the "normal" ground state,[23] $|G^0\rangle$, in which $\langle \sigma \rangle = A$, $\langle \pi_i \rangle = 0$, we see that we can also evaluate E_{eff} by considering

$$E_{\text{eff}} = \langle G^0 | \tilde{H}_{\text{eff}} | G^0 \rangle \equiv \langle G^0 | \int d^3x \tilde{\mathcal{H}}_{\text{eff}} | G^0 \rangle , \tag{3.17}$$

where

$$|G'\rangle = U(k, \theta) |G^0\rangle , \tag{3.18}$$

and the rotated Hamiltonian density, $\tilde{\mathcal{H}}_{\text{eff}}$, is given by

$$\tilde{\mathcal{H}}_{\text{eff}} = U^{-1} \mathcal{H}_{\text{eff}}\, U . \tag{3.19}$$

In terms of the generators of I_3 and A_1 rotations,

$$U = \exp(-i\vec{k} \cdot \int d^3x\, \vec{x}\, V_3^0)\, e^{+i Q_1^5 \theta} \equiv U_1 U_2 . \tag{3.20}$$

Although treating the problem in terms of the rotated Hamiltonian is completely equivalent to using the original Hamiltonian in the rotated states, we shall adopt the former approach because it will better clarify our subsequent general discussion.

After minimizing H_{eff} functionally with respect to the meson momentum, we find that the full hadronic part of the effective Hamiltonian density becomes

$$\mathcal{H}_{\rm eff} = -i\bar{N}\vec{\gamma}\cdot\vec{\nabla}N + g\bar{N}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)N + \frac{\nabla\sigma\cdot\nabla\sigma + \nabla\pi_i\cdot\nabla\pi_i}{2} + \frac{\lambda^2}{4}(\sigma^2+\vec{\pi}^2)^2 - \frac{m_0^2}{2}(\sigma^2+\vec{\pi}^2) - \frac{\mu^2}{2}(\pi_1^2+\pi_2^2) - \epsilon\mathcal{L}_{\rm SB} + \mu\bar{N}\gamma^0\frac{\tau_3}{2}N - \nu'\bar{N}\gamma^0 N, \tag{3.21}$$

where $\nu' = \nu - \frac{1}{2}\mu$. Since all the terms in $H_{\rm eff}^{(i)}$ are invariant under global I_3 rotations, under the local transformation generated by

$$U_1 = \exp(-i\vec{k}\cdot\int d^3x\,\vec{x}\,V_3^0)$$

only terms involving derivatives will be altered. Evaluating

$$\bar{\mathcal{H}}_{\rm eff} = U_1^{-1}\mathcal{H}_{\rm eff}U_1$$

explicitly, we find

$$\bar{\mathcal{H}}_{\rm eff} = \mathcal{H}_{\rm eff} + \frac{\vec{k}\cdot\vec{k}}{2}(\pi_1^2+\pi_2^2) - \bar{N}\vec{\gamma}\cdot\vec{k}\frac{\tau_3}{2}N - \vec{k}\cdot[(\vec{\nabla}\pi_1)\pi_2 - (\vec{\nabla}\pi_2)\pi_1]. \tag{3.22}$$

The global chiral transformation generated by

$$U_2 = e^{+iQ_1^5\theta}$$

changes many of the terms in $\bar{\mathcal{H}}_{\rm eff}$. Restricting our considerations to the ground state, $|G^0\rangle$, however, and replacing the quantum fields $(\sigma, \vec{\pi})$ by the expectation values appropriate to this state, we can write

$$\tilde{\mathcal{H}}_{\rm eff} = U_2^{-1}\bar{\mathcal{H}}_{\rm eff}U_2 = -i\bar{N}\vec{\gamma}\cdot\vec{\nabla}N + g\bar{N}N\langle\sigma\rangle + \frac{\lambda^2}{4}\langle\sigma\rangle^4 - \frac{m_0^2}{2}\langle\sigma\rangle^2 - \nu'\bar{N}\gamma^0N + k_\mu\left(\bar{N}\gamma^\mu\frac{\tau_3}{2}N\cos\theta + \bar{N}\gamma^\mu\gamma_5\frac{\tau_2}{2}N\sin\theta\right) - \frac{k_\alpha k^\alpha}{2}\langle\sigma\rangle^2\sin^2\theta + \delta\mathcal{H}_{\rm eff}\,, \tag{3.23}$$

where $k^\alpha = (\mu, \vec{k})$ is the four-momentum of the condensed pion mode,[5] and the two forms of symmetry breaking are

$$\delta\mathcal{H}_{\rm eff}^{(1)} = -\epsilon f_\pi m_\pi^2\langle\sigma\rangle\cos\theta$$

and

$$\delta\mathcal{H}_{\rm eff}^{(2)} = -\epsilon m_\pi^2\langle\sigma\rangle^2\sin^2\theta\,.$$

We have introduced the four-vector notation for k_α in Eq. (3.23) to highlight the appearance of the (nonmesonic parts of) vector and axial-vector currents, given in the σ model by

$$V_\mu^3 = \bar{N}\gamma_\mu\frac{\tau_3}{2}N \tag{3.24a}$$

and

$$A_\mu^2 = \bar{N}\gamma_\mu\gamma_5\frac{\tau_2}{2}N\,, \tag{3.24b}$$

which arise through the local isospin transformation and subsequent global chiral rotation. From our derivation it is clear that the appearance of these currents in no way depends on specific features of the σ model and indeed stems solely from the relation between G' and G^0. To be more explicit—although at the risk of being somewhat imprecise—we can recall the manner in which Gell-Mann and Lévy[24] calculated the current corresponding to a particular first-kind (global) gauge transformation. By allowing the parameters, Λ_i, of the transformation to be functions of space-time, one can show that $J_i^\mu = \delta\mathcal{L}/\delta(\partial_\mu\Lambda_i)$. Thus since the parameter in the transformation U_1, defined in (3.20), is $\vec{k}\cdot\vec{x}$, the appearance of terms of the form $k_\alpha j^\alpha$ is hardly surprising. This heuristic but essentially correct argument will be made more precise in the following article. Further in Sec. IV, we shall use this observation to include the effects of the $N^*(1236)$ on pion condenstion.

C. Choice of parameters in $\mathcal{H}_{\rm eff}$

In keeping with our intention of ignoring all relativistic closed-loop effects, we must replace the constants in Eq. (3.23) by the appropriate physical values. This leads to three changes. First, we must take $g\langle\sigma\rangle = M$, the physical nucleon mass.[25] Second, in principle we know that to determine $\langle\sigma\rangle \equiv A$ we should minimize $E_{\rm eff}$ with respect to this parameter. However, since the quantities $|\delta H|$, $|k| \simeq O(m_\pi)$ are small compared to m_0 and λ, to high accuracy we can replace A by its value in the normal phase; thus, we take $\langle\sigma\rangle = A = f_\pi$.[23] Third, since the relative normalization of the vector and axial-vector currents between nucleon states is experimentally not 1 but $|G_A/G_V| \simeq 1.24 \equiv g_A$, we must multiply the axial-vector current term by this factor. Since the value of g_A will prove critical in later discussions, let us extend our comments on this point.[26] Although it might at first sight appear that g_A must equal 1 in a theory with a chiral-invariant Lagrangian, this is not the case. In particular, in the "Goldstone mode"—or "spontaneously broken" realization—of the symmetry, g_A is not necessarily 1. Indeed, the only remnant of the chiral invariance of the underlying Lagrangian is the Goldberger-Treiman relation[27]

$$g_A M = f_\pi g_{\pi NN} \tag{3.25}$$

(which is exact in the limit $\epsilon = 0$).[28] Further, if we calculated loop corrections from the σ model Hamiltonian Eq. (3.6), we would see g_A move from 1 to some renormalized value; thus it is completely consistent with the assumed near chiral invariance of the full Hamiltonian to take the experimentally observed value, $g_A \simeq 1.24$, in Eq. (3.23).

Making these three changes in $\tilde{\mathcal{H}}_{\text{eff}}$, we can get the effective Hamiltonian in the form

$$\begin{aligned}\tilde{\mathcal{H}}_{\text{eff}} &= -i\,\overline{N}\vec{\gamma}\cdot\vec{\nabla}N + M\overline{N}N - \nu'\overline{N}\gamma^0 N \\ &\quad + k_\mu\left(\overline{N}\gamma^\mu \frac{\tau_3}{2}N\cos\theta + g_A\overline{N}\gamma^\mu\gamma_5\frac{\tau_2}{2}N\sin\theta\right) \\ &\quad - \frac{k_\mu k^\mu}{2} f_\pi^{\,2}\sin^2\theta + \delta\mathcal{H}_{\text{eff}} \\ &\equiv \mathcal{H}_{\text{eff}} + \Delta\mathcal{H}_{\text{eff}}\,,\end{aligned} \tag{3.26}$$

where

$$\delta\mathcal{H}_{\text{eff}}^{(1)} = -f_\pi^{\,2} m_\pi^{\,2}\cos\theta\,, \tag{3.27a}$$

$$\delta\mathcal{H}_{\text{eff}}^{(2)} = \tfrac{1}{2} f_\pi^{\,2} m_\pi^{\,2}\sin^2\theta\,. \tag{3.27b}$$

To study the possibility of a phase transition from the normal to a pion-condensed state, we must compare the free energies in the two phases, which are given, respectively, by

$$E_{\text{eff}}^{\text{Normal}} \equiv E_{G_0} = \langle G^0 | H_{\text{eff}} | G^0\rangle \tag{3.28}$$

and

$$\begin{aligned}E_{\text{eff}}^{\text{Condensed}} \equiv E_{G'} &= \langle G' | H_{\text{eff}} | G'\rangle \\ &= \langle G^0 | \tilde{H}_{\text{eff}} | G^0\rangle \\ &= \langle G^0 | H_{\text{eff}} + \Delta H_{\text{eff}} | G^0\rangle\,.\end{aligned} \tag{3.29}$$

Here we should emphasize two points. First, when $\theta = \vec{k} = 0$, $\Delta H = 0$, and hence, as in the case of the pure meson problem, the normal state is contained as a limiting case of the condensed-state parameters. This means that we need only consider explicitly the Hamiltonian $\tilde{H}_{\text{eff}}$; as long as we are below the threshold for condensation, minimization of the free energy with respect to θ and k will give the normal state, $\theta = \vec{k} = 0$. The first value of ν at which the minimum shifts from $\theta = 0$ to $\theta = \theta_0 \neq 0$—if such a value of ν exists—determines the baryon density at which the phase transition occurs. Second, since we expect $\mu \sim k \sim O(m_\pi)$, ΔH is a relatively small perturbation—roughly $O(m_\pi/m_N)$—on H_{eff}.

Before explicitly solving for $E_{\text{eff}}(\mu, \nu, k, \theta)$ and minimizing with respect to k and θ, let us digress briefly on the nature of the "normal" ground state. The form of H_{eff} shows that this state corresponds simply to independent neutrons, protons, and electrons in equilibrium under $n \to p + e + \nu$ and with $n_p = n_e$. This is clearly only a crude first approximation to actual nuclear matter, particularly at high densities. If one wished to improve this approximation by including the effects of realistic nucleon-nucleon forces—apart from those induced by the condensed pion mode—additional terms could be added to H_{eff}. To the extent that such forces—for example, the heavy-meson (ω) exchanges which give rise to the nuclear hard core—are approximately chiral-invariant, their contributions to ΔH_{eff} will be small, and thus ΔH_{eff} will remain a small perturbation of H_{eff}.[29] We discuss this point in more detail in Sec. V.

D. Evaluation of the effective energy density

To calculate $\mathcal{E}_{\text{eff}}(\mu, \nu, k, \theta)$ we first observe that the final two terms in $\tilde{\mathcal{H}}_{\text{eff}}$ in Eq. (3.26) have no effect on the nucleon sector. Thus, as in the case of the electrons, we can treat this contribution separately. Turning to the nucleon sector we note that since the nucleons in neutron-star matter are nonrelativistic—$(v_N/c)^2 \lesssim \frac{1}{10}$—we could reduce $\mathcal{H}_{\text{eff}}$ to its nonrelativistic limit and solve the resulting many-body problem.[30] Equivalently, we can derive from $\mathcal{H}_{\text{eff}}$ the nucleon field equation which, reinterpreted as a single-particle equation, assumes the form of a Dirac equation in an external field. Since relativistic effects are small, we can explicitly ignore the sea of antinucleons in the vacuum and, further, we can reduce the Dirac equation to its appropriate nonrelativistic limit. The state of lowest energy is then obtained by filling all the single-nucleon levels which have energy <0 because of the presence of the external field. Combining this nucleon energy with the separate contributions of the pion-condensed mode

$$\mathcal{E}_{\text{eff}}^{M} = \frac{k^2 - \mu^2}{2} f_\pi^{\,2}\sin^2\theta + \delta\mathcal{H}_{\text{eff}} \tag{3.30}$$

and of the electrons

$$\mathcal{E}_{\text{eff}}^{\text{electrons}} = -\mu^4/12\pi^2\,, \tag{3.31}$$

we obtain the full energy density $\mathcal{E}_{\text{eff}}(\mu, \nu, k, \theta)$ for minimization.

The Dirac-type equation which follows from $\tilde{\mathcal{H}}_{\text{eff}}$ in Eq. (3.26) becomes, in momentum space, an eigenvalue equation for single-particle energy levels $E(p)$

$$\mathcal{H}_D N(\vec{p}) \equiv \left\{\vec{\alpha}\cdot\vec{p} + \beta M + (\mu - \vec{\alpha}\cdot\vec{k})\left[\frac{\tau_3}{2}\cos\theta + g_A\frac{\tau_2}{2}\gamma_5\sin\theta\right] - \nu'\right\} N(\vec{p}) = [E(p) + M]\,N(\vec{p})\,. \tag{3.32}$$

We are seeking all $E(p) < 0$. Using the standard approximate Foldy-Wouthuysen transformation approach to remove "odd" operators, which couple large and small components, we find

$$\mathcal{H}_D \simeq \left\{ -\nu' + \frac{[\vec{p} - \vec{k}(\tau_3/2)\cos\theta]^2}{2M} + \mu\frac{\tau_3}{2}\cos\theta - g_A\vec{\sigma}\cdot\vec{k}\sin\theta\,\frac{\tau_2}{2} + \text{smaller terms} \right\} . \quad (3.33)$$

In Eq. (3.33) we observe that the nonrelativistic limit of the isospin current term—$\mu(\tau_3/2)\cos\theta$—gives rise to an s-wave π-N interaction whereas the axial-vector current term—$g_A\vec{k}\cdot\vec{\sigma}\sin\theta\,\tau_2/2$—produces a p-wave term.

Without loss of generality we choose $\vec{k} = k\hat{z}$. Then in a basis in which the nucleon states are written

$$N = \begin{bmatrix} p_\uparrow \\ n_\uparrow \\ p_\downarrow \\ n_\downarrow \end{bmatrix}$$

and when the irrelevant antinucleon states are ignored, H_D assumes the simple matrix form

$$H_D = \begin{bmatrix} a+b & ic & 0 & 0 \\ -ic & a-b & 0 & 0 \\ 0 & 0 & a+b & -ic \\ 0 & 0 & ic & a-b \end{bmatrix} , \quad (3.34)$$

where

$$a = \frac{\vec{p}^{\,2}}{2M} - \nu' + \frac{\vec{k}^2\cos^2\theta}{8M} ,$$

$$b = \frac{\mu}{2}\cos\theta - \frac{\vec{p}\cdot\vec{k}}{2M}\cos\theta ,$$

and

$$c = g_A\frac{k\sin\theta}{2} .$$

The decoupling of spin states leads to just two distinct energy eigenvalues

$$E_\pm(p) = +a \pm (b^2 + c^2)^{1/2} .$$

The eigenfunctions associated with the two eigenvalues $E_\pm$ correspond, in solid-state terminology, to the quasiparticle states which diagonalize the quadratic form equivalent to H_D. We see that we can write

$$u_+(\vec{p},\uparrow) = \cos\varphi\, p(\vec{p},\uparrow) - i\sin\varphi\, n(\vec{p},\uparrow) \quad (3.35a)$$

and

$$u_-(\vec{p},\uparrow) = -i\sin\varphi\, p(\vec{p},\uparrow) + \cos\varphi\, n(\vec{p},\uparrow) , \quad (3.35b)$$

where

$$n(\vec{p},\uparrow) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

in our previous notation so that ($\uparrow$) denotes the direction of the spin and the angle φ is defined by

$$\tan\varphi = \frac{c}{b + (b^2 + c^2)^{1/2}} . \quad (3.35c)$$

Thus in the limit $\varphi \to 0$, that is, as $\theta \to 0$, $u_- \to n$ and $u_+ \to p$. To evaluate the ground-state energy we must fill all states which have energy less than zero. Thus the energy density in the interacting nucleons is just

$$\mathcal{E}_{\text{eff}}^{\text{nucleons}} = \frac{2}{8\pi^3}\int d^3p\,[E_+(p)\,\theta(-E_+) + E_-(p)\,\theta(-E_-)] . \quad (3.36)$$

From Eq. (3.37) one could proceed by numerical integration to calculate the exact $\mathcal{E}_{\text{eff}}$ for various values of μ, ν, θ, k. We choose instead to make two simplifications which, although not without physical justification, are made primarily to enable us to complete the calculation analytically. First, we drop the $\vec{k}\cdot\vec{p}$ and $k^2/2M$ terms in H_D [as given by Eq. (3.33)] and subsequent quantities. This approximation is not totally without validity, since we expect, and shall indeed find, that at the point of the phase transition $p \simeq O((2M\mu_n)^{1/2}) \simeq O((2Mm_\pi)^{1/2})$ whereas $k \simeq O(m_\pi)$ so that $p^2/2M > \vec{k}\cdot\vec{p}/M$. Without the $\vec{k}\cdot\vec{p}$ terms, the quasiparticle Fermi seas become spherically symmetrical and the integrations in Eq. (3.36) can be done trivially. Notice that for $\theta = \pi/2$, the $\vec{k}\cdot\vec{p}$ term in any case vanishes; this fact will prove particularly relevant for the case of $H_{\text{SB}}^{(2)}$ treated below.

Second, insteaa of retaining both the E_+ and E_- Fermi seas, we shall retain only the larger—that is, the E_-—sea.[3,4] This approximation is clearly not exact for $\vec{k} = \theta = 0$, since we know that the normal ground state contains Fermi seas of both neutrons and protons and from Eq. (3.35) we see that

$$u_- \underset{\theta\to 0}{\longrightarrow} n, \quad u_+ \underset{\theta\to 0}{\longrightarrow} p .$$

This same observation, however, establishes that the energy calculated by this approximation differs only slightly from the correct result; indeed, since $\rho_n \gg \rho_p$ in the normal state, one has

$$\mathcal{E}(\rho_n) + \mathcal{E}(\rho_p) \simeq \mathcal{E}(\rho_n + \rho_p) \tag{3.37}$$

to reasonable accuracy. Further, the self-consistency of this approximation at a given θ is readily checked, simply by verifying that $E_+(p)$ is greater than zero at that θ.

A more serious consequence of this approximation follows from its lack of validity for $\theta \simeq 0$. This implies that we cannot at this level study the detailed nature of the threshold for pion condensation except in the case of a first-order phase transition. We stress, however, that for finite θ the one-Fermi-sea approximation is valid: that is, near the point of the phase transition the parameters μ and ν are such that for $\theta > \theta_0$, small but finite, $E_+ > 0$ for all p and thus only the E_- sea is filled. This we shall later demonstrate. Since, as we have argued, the chiral approach to pion condensation is most useful at finite θ, this simplification is quite fortunate.

With these approximations Eq. (3.36) becomes

$$\begin{aligned}\mathcal{E}_{\text{eff}}^{\text{nucleons}} &= \frac{1}{\pi^2}\int_0^{p_F} p^2 dp\, E_-(p) \\ &= \frac{1}{\pi^2}\int p^2 dp \left[\frac{p^2}{2M} - \nu' - \tfrac{1}{2}(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}\right] \\ &= \frac{-2}{15\pi^2}(2M)^{3/2}\left[\nu' + \tfrac{1}{2}(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}\right]^{5/2} .\end{aligned} \tag{3.38}$$

Hence the total free energy as a function of μ, ν, k, and θ becomes

$$\begin{aligned}\mathcal{E}_{\text{eff}} &= \mathcal{E}_{\text{eff}}^{\text{nucleons}} + \mathcal{E}_{\text{eff}}^{\text{pions}} + \mathcal{E}_{\text{eff}}^{\text{electrons}} \\ &= \frac{-2}{15\pi^2}(2M)^{3/2}\left[\nu' + \tfrac{1}{2}(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}\right]^{5/2} + \frac{k^2-\mu^2}{2} f_\pi{}^2\sin^2\theta + \delta\mathcal{H}_{\text{eff}} - \frac{\mu^4}{12\pi^2} ,\end{aligned} \tag{3.39}$$

where the two forms of $\delta\mathcal{H}_{\text{eff}}$ are given in Eq. (3.27).

We are to minimize $\mathcal{E}_{\text{eff}}$ at fixed μ and ν over k and θ with

$$\left.\frac{\partial \mathcal{E}_{\text{eff}}}{\partial \mu}\right|_\nu = \rho_Q = 0$$

and

$$\left.\frac{\partial \mathcal{E}_{\text{eff}}}{\partial \nu}\right|_\mu = -\rho_B .$$

As we remarked above—see Eq. (3.12)—it is somewhat simpler to minimize

$$\mathcal{E}'_{\text{eff}} \equiv \mathcal{E}_{\text{eff}} + \nu\rho_B = \mathcal{E}_{\text{eff}} - \nu(\rho_B)\frac{\partial \mathcal{E}_{\text{eff}}}{\partial \nu} \tag{3.40}$$

at fixed μ and ρ_B. Using Eq. (3.40) and recalling that $\nu' = \nu - \frac{1}{2}\mu$, we obtain

$$\begin{aligned}\mathcal{E}_{\text{eff}}^{(i)} &= \frac{3}{5}\,\frac{(3\pi^2)^{2/3}\rho_B{}^{5/3}}{2M} \\ &\quad + \rho_B\left[\tfrac{1}{2}\mu - \tfrac{1}{2}(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}\right] \\ &\quad + \frac{k^2-\mu^2}{2} f_\pi{}^2\sin^2\theta + \delta\mathcal{H}_{\text{SB}}^{(i)} - \frac{\mu^4}{12\pi^2} .\end{aligned} \tag{3.41}$$

E. $\sin^2\theta$ symmetry breaking

To proceed we must distinguish between the two forms of symmetry breaking. As in Sec. II, we treat the simpler case first. Minimizing $\mathcal{E}_{\text{eff}}^{(1)}$ with respect to k and θ yields, respectively,

$$0 = k\sin^2\theta\left[f_\pi{}^2 - \frac{\rho_B g_A{}^2}{2(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}}\right] \tag{3.42a}$$

and

$$\begin{aligned}0 = \sin\theta\cos\theta &\left[\frac{-\rho_B(g_A{}^2k^2-\mu^2)}{2(g_A{}^2k^2\sin^2\theta + \mu^2\cos^2\theta)^{1/2}}\right. \\ &\left. + (k^2 - \mu^2 + m_\pi{}^2)f_\pi{}^2\right],\end{aligned} \tag{3.42b}$$

which must be satisfied together with the charge neutrality condition $\partial\mathcal{E}'_{\text{eff}}/\partial\mu = 0$, which becomes

$$\begin{aligned}0 = \tfrac{1}{2}\rho_B &\left[1 - \frac{\mu\cos^2\theta}{(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}}\right] \\ &- \mu f_\pi{}^2\sin^2\theta - \frac{\mu^3}{3\pi^2} .\end{aligned} \tag{3.42c}$$

From our discussion of the one-Fermi-sea approximation, it is clear that the charge neutrality condition in (3.42c) can be valid only for θ greater than some small θ_0, since for $\theta < \theta_0$ we must, for consistency, include both Fermi seas. But for finite θ, the small coefficient of the electron term means that—provided that μ is roughly $O(m_\pi)$—it plays little role in either the charge neutrality condition or the effective energy. Thus it is consistent both with our intention to study chiefly finite θ and with our one-Fermi-sea approximation to ignore the contributions of electrons in (3.42). This is then our third approximation.

It is obvious that this approximation, like those of ignoring $\vec{k}\cdot\vec{p}\cos\theta$ terms and of keeping only one Fermi sea, could readily be relaxed at the expense of treating the problem numerically rather than analytically.[5] In addition, after the value of μ is determined in the absence of the electron term, it is a simple matter to determine the angle θ_0 beyond which $\mu^3/3\pi^2$ is negligible compared to the other terms in (3.42c).

To clarify further the physical significance of our approximations and their relation to a more exact treatment, two comments can be made. First, since as $\theta\to 0$, $u_+\to p$ and $u_-\to n$, if we fill only the Fermi sea corresponding to u_-, at $\theta=0$ this amounts to treating pure neutron matter. Thus, assuming our approximations are valid, we see that for $\theta>\theta_0$ the behavior of a pion condensate in actual neutron-star matter is essentially the same as in pure neutron matter. Second, we can view our approach to studying the threshold for pion condensation in the way suggested schematically by Fig. 3: That is, we are studying the approximate threshold density—labeled by $\rho^{(1)}$ in the figure—determined by simple extrapolation from the region $\theta>\theta_0$. Notice that this density must be greater than the actual (or true) critical density—ρ^T—and thus, to the limited extent that the present model accurately reflects $\pi\pi$, πN, and NN interactions, ρ_{chir} provides an "upper bound" beyond which pion condensation should occur.

To study Eqs. (3.42) in the absence of electrons, we can divide the solutions into three classes. The first class, with $\theta=0$, corresponds to the normal ground state, which consists purely of neutrons in our approximation. The effective energy density is then clearly seen to be

$$\mathcal{E}^{(0)}\equiv\mathcal{E}_{\text{eff}}(\theta=0)=\tfrac{3}{5}(3\pi^2)^{2/3}\frac{\rho_B^{5/3}}{2M}\equiv h(\rho_B)\,. \tag{3.43}$$

In the second class of solutions, which exists for $\theta=\theta_0\neq 0$ or $\pi/2$, the minimization conditions lead to nontrivial relations determining θ, μ, and k in terms of ρ_B. Comparing (3.42a) and (3.42b) shows that μ is in fact independent of ρ_B in this phase,

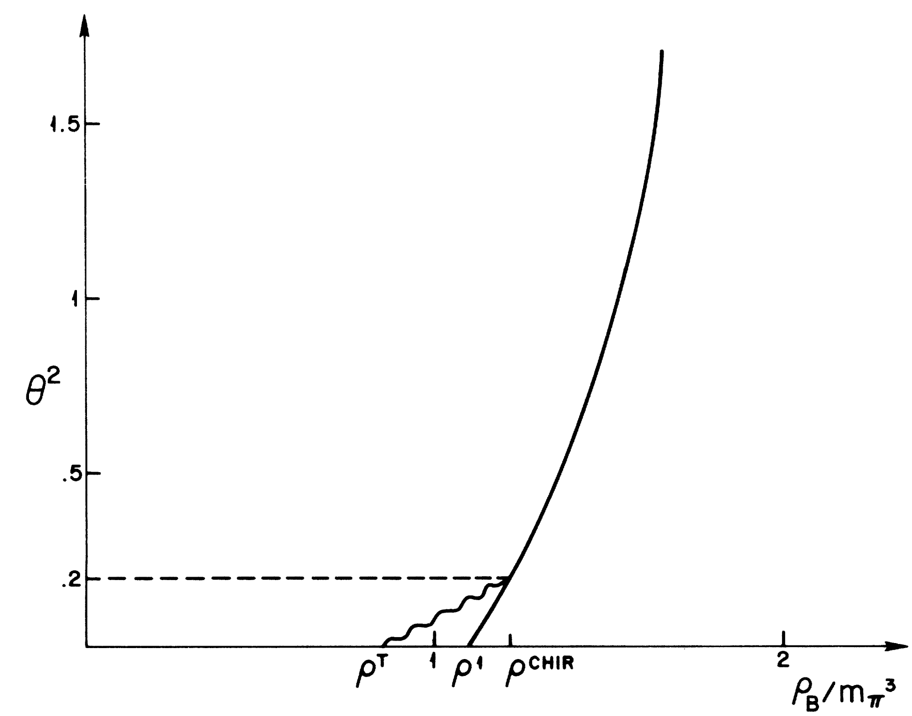


FIG. 3. The chiral angle vs the baryonic density. The angle θ determines the amount of pion condensate. At ρ^T, θ might start deviating from zero through many-body effects. At ρ^{chir}, θ is large enough and the chiral-rotation results are reliable. $\rho^{(1)}$ is the extrapolation to $\theta=0$ of the chiral-rotation results.

$$\mu = \frac{m_\pi g_A}{(g_A - 1)^{1/2}}. \tag{3.44a}$$

Further, we find

$$\sin^2\theta = \frac{1}{g_A^2 - 1}\left(\frac{\rho_B}{2f_\pi^2}\frac{g_A(g_A^2-1)^{1/2}}{m_\pi} - 1\right) \tag{3.44b}$$

and

$$k^2 = \frac{\rho_B}{2f_\pi^2}[g_A(g_A^2-1)^{1/2}]m_\pi + \frac{m_\pi^2 g_A^2}{g_A^2 - 1}. \tag{3.44c}$$

For later reference we note that (3.44b) shows that this second phase can exist only for baryon densities satisfying

$$\frac{2f_\pi^2 m_\pi}{g_A(g_A^2-1)^{1/2}} \leqslant \rho_B \leqslant \frac{g_A 2 f_\pi^2 m_\pi}{(g_A^2-1)^{1/2}}. \tag{3.44d}$$

Using these results to eliminate μ, k, and θ from the expression for $\mathcal{E}_{\text{eff}}$, we find that in this second phase

$$\begin{aligned}\mathcal{E}_{\text{eff}}^{(\theta_0)} &\equiv \mathcal{E}_{\text{eff}}(\theta = \theta_0 \neq 0 \text{ or } \pi/2)\\ &= h(\rho_B) - \frac{\rho_B^2 g_A^2}{8f_\pi^2} + \frac{1}{2}\rho_B\frac{m_\pi g_A}{(g_A^2-1)^{1/2}}\\ &\quad - \frac{1}{2}\frac{f_\pi^2 m_\pi^2}{(g_A^2-1)}.\end{aligned} \tag{3.45}$$

Finally the third solution has $\theta = \pi/2$ and from (3.42a) and (3.42b)

$$k = \frac{\rho_B g_A}{2f_\pi^2} \tag{3.46a}$$

and

$$\mu = \frac{\rho_B}{2f_\pi^2}. \tag{3.46b}$$

Thus in this phase the effective energy density is

$$\begin{aligned}\mathcal{E}_{\text{eff}}^{(\pi/2)} &\equiv \mathcal{E}_{\text{eff}}(\theta = \pi/2)\\ &= h(\rho_B) - \frac{1}{8}\frac{\rho_B^2}{f_\pi^2}(g_A^2 - 1) + \frac{m_\pi^2 f_\pi^2}{2}.\end{aligned} \tag{3.47}$$

The expressions (3.43), (3.45), and (3.47) allow us to determine analytically the ground state as a function of baryon density. Recalling our remark that the second phase cannot exist for

$$\rho_B < \rho_{\text{crit}}^{(1)} = \frac{2f_\pi^2 m_\pi}{g_A(g_A^2-1)^{1/2}}, \tag{3.48}$$

we see that in the region of $0 < \rho_B < \rho_{\text{crit}}^{(1)}$, the normal phase, with an energy density given by $\mathcal{E}_{\text{eff}}^{(0)}$ is the true ground state. At $\rho_{\text{crit}}^{(1)}$ a second-order phase transition occurs,[31] and the pion-condensate amplitude develops smoothly with θ determined by (3.44b). In this region the ground-state energy density is given by $\mathcal{E}_{\text{eff}}^{(\theta_0)}$. When θ reaches $\pi/2$,

$$\rho_B = \rho_{\text{crit}}^{(2)} = \frac{2f_\pi^2 m_\pi}{(g_A^2-1)^{1/2}}g_A, \tag{3.49}$$

beyond which density the phase described by $\mathcal{E}_{\text{eff}}^{(\theta_0)}$ must again become unphysical. A further phase transition, again second order, occurs at $\rho_{\text{crit}}^{(2)}$. Beyond $\rho_{\text{crit}}^{(2)}$, θ remains at $\pi/2$ and the ground state becomes that described by $\mathcal{E}_{\text{eff}}^{(\pi/2)}$. The variation in the nature of the ground state as a function of baryon density is summarized in Fig. 4.

Before we discuss the features of the condensed pion phases, we should verify the self-consistency of the approximations made in our calculations. First, we note that treating the Fermi seas as spherical by dropping the $\vec{k}\cdot\vec{p}\cos\theta$ term is perhaps the weakest approximation; at $\theta = 0$, we see that $p_F \simeq 3m_\pi$, whereas $k \simeq 2m_\pi$. Of course, as θ increases toward $\pi/2$, the neglect of this term becomes more justified. Second, the self-consistency of keeping only one Fermi sea is easily verified. From

$$\Delta E \equiv E_+ - E_- = (\mu^2\cos^2\theta + g_A^2 k^2 \sin^2\theta)^{1/2}, \tag{3.50}$$

it is easy to verify that the Fermi energy $\mathcal{E}_F = p_F^2/2M$ is always smaller than ΔE for all θ in the relevant region of baryon density. Third, the

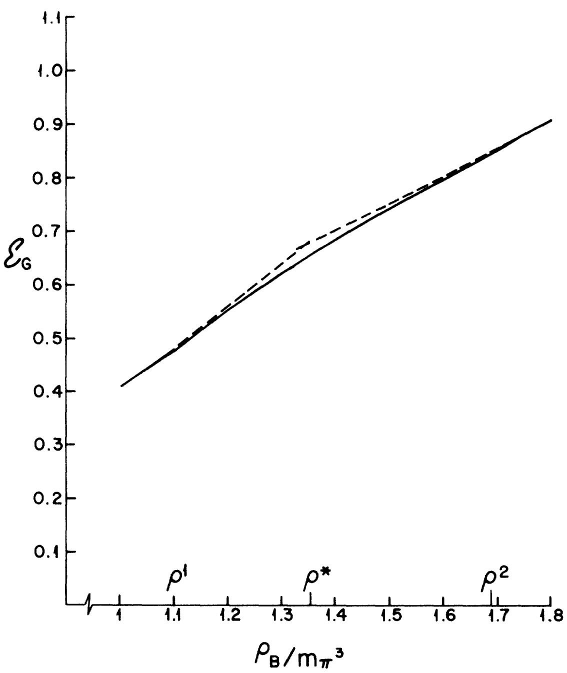


FIG. 4. The actual ground-state energy as a function of baryon density. Below $\rho_{\text{crit}}^{(1)}$, the ground state is "normal." At $\rho_{\text{crit}}^{(1)}$, a pion condensate begins to develop and increases in amplitude until $\theta = \pi/2$ at $\rho_{\text{crit}}^{(2)}$. Between $\rho_{\text{crit}}^{(1)}$ and $\rho_{\text{crit}}^{(2)}$ the ground-state energy density is given by $\mathcal{E}^{(\theta_0)}$ and beyond $\rho_{\text{crit}}^{(2)}$ by $\mathcal{E}^{(\pi/2)}$. The dashed lines show the analytic extensions of $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(\pi/2)}$ into their respective unphysical regions. ρ^* is the point beyond which $\mathcal{E}^{(\pi/2)} < \mathcal{E}^{(0)}$.

value of θ_0 can be estimated by expanding (3.42c) for small θ. We find that

$$0 \simeq \theta^2\left(\frac{\rho_B(g_A{}^2k^2+\mu^2)}{4\mu^2} - \mu f_\pi{}^2\right) - \frac{\mu^3}{3\pi^2}. \qquad (3.51)$$

Our solution amounts to setting the quantity in large parentheses equal to 0; this will be a reasonable approximation for

$$\theta_0{}^2\mu f_\pi{}^2 \gtrsim \frac{\mu^3}{3\pi^2}; \qquad (3.52)$$

with μ as given by (3.44) we find that[32]

$$\theta_0^2 \gtrsim \frac{(2g_A{}^2)}{g_A{}^2-1}\,\frac{1}{3\pi^2} \simeq 0.2\,. \qquad (3.53)$$

Recalling that $\mathcal{E}'_{\text{eff}}(\theta)$ is a function of θ^2 only, we see that higher terms in the expansion of $\mathcal{E}_{\text{eff}}$ will be small and thus the extrapolation from $\theta > \theta_0$ to $\theta = 0$ is a reasonable one.

Despite its simplicity, this model reveals a number of general features of pion condensation in general and of our approach in particular. In the latter category the most important insight comes from considering the condensed phase with $\theta = \pi/2$, which is the actual ground state for $\rho_B > \rho_{\text{crit}}^{(2)}$. In this model the existence of a true ground state with $\theta = \pi/2$ is a happy consequence of the simple angular condition (3.42b). But it serves to point out that the chiral approach allows one to calculate at $\theta = \pi/2$,—where, as one can see immediately from (3.33), considerable simplification obtains—*independent of whether this angle corresponds to a minimum of* $\mathcal{E}'_{\text{eff}}$; if $\mathcal{E}'_{\text{eff}}(\theta=\pi/2) < \mathcal{E}'_{\text{eff}}(\theta=0)$, then $\mathcal{E}'_{\text{eff}}(\theta=0)$ cannot be a minimum and there will exist a pion-condensed ($\theta \neq 0$) phase.[33]

To illustrate these remarks in the present context, we note that from (3.43) and (3.47) the point at which

$$\mathcal{E}'^{(\pi/2)}_{\text{eff}} \equiv \mathcal{E}'_{\text{eff}}(\theta=\pi/2) < \mathcal{E}'_{\text{eff}}(\theta=0) \equiv \mathcal{E}'^{(0)}_{\text{eff}} \qquad (3.54)$$

can be shown to be

$$\rho_B = \rho^* = \frac{m_\pi{}^3}{(g_A{}^2-1)^{1/2}}. \qquad (3.55)$$

Referring to Fig. 4, we see that beyond this density, the inequality in (3.54) holds strictly; of course, since by comparing $\mathcal{E}(\pi/2)$ with $\mathcal{E}(0)$ we have not explicitly minimized with respect to θ, at ρ^* the true ground state is not described by $\mathcal{E}^{(\pi/2)}_{\text{eff}}$; rather, from (3.44b), we see it has, using $2f_\pi{}^2 = m_\pi{}^2$,

$$\sin^2\theta|_{\rho=\rho^*} = \frac{1}{g_A+1}. \qquad (3.56)$$

Nonetheless, the simplicity of this comparison will prove useful in estimating the point of the phase transition in more complicated models of pion condensation.

Another feature of our results which transcends the simple model is the nature of the condensed phase. As in the purely mesonic models of Sec. II, this state is one in which the I_3 invariance of $\mathcal{H}_{\text{eff}}$ is spontaneously broken. The resulting "Goldstone boson" excitation and its fate when electromagnetic interactions are included will be discussed in Sec. VI.

A third feature of this simple model is intriguing more for the questions it raises than those it resolves. From (3.44a) and (3.44b) we observe that the effective mass of the pions in the nuclear medium is, at $\rho = \rho_{\text{crit}}^{(1)}$,

$$(m_\pi^{\text{eff}})|_{\rho_{\text{crit}}^{(1)}} = (\mu^2 - k^2)_{\rho_{\text{crit}}^{(1)}} = -m_\pi{}^2\,. \qquad (3.57)$$

Thus we are confronted with the important question of how best to choose the parameters of the pion-nucleon coupling when the πN amplitude is off-shell. In this simple model the parameter determining the strength of the (p-wave) pion-nucleon interaction is just g_A, since by the Goldberger-Treiman relation

$$g_A M \cong f_\pi g_{\pi NN}\,. \qquad (3.25)$$

Since (3.25) is only approximately valid and since the critical densities are sensitive functions of g_A, it is clear that in the simple model whether one chooses $g_A = g_A^{\text{exp}} \cong 1.24$ or $g_A = f_\pi(g_{\pi NN}^{\text{exp}})/M \cong 1.4$ makes a substantial numerical difference in the predicted threshold densities. Thus, for example, with $g_A = 1.24$, we find the threshold density to be

$$\rho_{\text{crit}}^{(1)} \simeq 2.2\rho_{\text{nucl}}\,, \qquad (3.58a)$$

with the pion-condensate energy and momentum at this point given by

$$\mu|_{\rho_{\text{crit}}^{(1)}} = \frac{g_A m_\pi}{(g_A{}^2-1)^{1/2}} \cong \tfrac{7}{4} m_\pi \qquad (3.58b)$$

and

$$k|_{\rho_{\text{crit}}^{(1)}} = \frac{m_\pi(2g_A{}^2-1)^{1/2}}{(g_A{}^2-1)^{1/2}} \simeq 2m_\pi\,. \qquad (3.58c)$$

With $g_A = 1.4$, these values become[34]

$$\rho_{\text{crit}}^{(1)} \cong 1.4\rho_{\text{nucl}} \qquad (3.59a)$$

$$\mu|_{\rho_{\text{crit}}^{(1)}} \cong \tfrac{3}{2} m_\pi\,, \qquad (3.59b)$$

$$k|_{\rho_{\text{crit}}^{(1)}} \simeq \tfrac{7}{4} m_\pi\,. \qquad (3.59c)$$

It is, however, equally clear that in view of the striking limitations of both the σ model—e.g. lack of hard-core repulsion—and the approximation in which we have treated this model, it is singularly inappropriate to worry about such small variations in the value of g_A. In the following paper, which presents insofar as is possible a model-independent treatment of the implications of chiral invariance for pion condensation, we shall discuss the parametrization of πN interactions in considerably more detail.

Finally we remark that certain aspects of this model are complete artifacts. The most striking example is the instability of the pion condensate. It is easy to show that since the compressibility

$$\kappa \equiv \frac{\partial^2 \mathcal{E}_{\text{eff}}^{(\theta_0)}}{\partial \rho_B{}^2} \tag{3.60}$$

is negative for all $g_A > 1$, the condensed phase seems unstable. This instability results purely from the lack of realistic nucleon-nucleon forces in the model; inclusion of the correct hard-core repulsion, which contributes a term proportional to $\rho_B{}^2$ to $\mathcal{E}'_{\text{eff}}$, will stabilize the system for any reasonable g_A.[5]

F. cosθ symmetry breaking

To study pion condensation in the presence of the "standard" symmetry-breaking term, $\delta\mathcal{H} = -\epsilon\sigma$, we consider the effective energy density

$$\mathcal{E}'_{\text{eff}}(\rho_B, \mu, k, \theta) = \frac{3}{5}\frac{(3\pi^2)^{2/3}}{2M}\rho_B{}^{5/3} + \left(\frac{k^2-\mu^2}{2}\right)f_\pi{}^2\sin^2\theta + \tfrac{1}{2}\rho_B[\mu - (\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}] - m_\pi{}^2 f_\pi{}^2\cos\theta\,, \tag{3.61}$$

where we have, as discussed above, dropped the electron term in obtaining (3.61) from (3.41).

Since this expression is identical to the previous $\mathcal{E}'_{\text{eff}}$ in its dependence on k and μ, Eqs. (3.42a) and (3.42b) remain unchanged. Minimization with respect to θ now requires

$$0 = \sin\theta\left(\frac{1}{2}\rho_B\frac{(\mu^2 - g_A{}^2k^2)\cos\theta}{(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}} + (k^2-\mu^2)\cos\theta + f_\pi{}^2 m_\pi{}^2\right). \tag{3.62}$$

To solve the above equations, consider an expansion of $\mathcal{E}'_{\text{eff}}$ around $\theta = 0$:

$$\mathcal{E}'_{\text{eff}}(\theta) \simeq \mathcal{E}'_{\text{eff}}(0) + \frac{\partial^2\mathcal{E}}{\partial\theta^2}\frac{\theta^2}{2} + \cdots\,, \tag{3.63}$$

where we know there is no linear term from (3.62). When the second-derivative term becomes negative, the minimum can no longer be at $\theta = 0$ and the phase transition will occur. Since this approach not only simplifies our present calculation but also illustrates an important connection between our techniques and others appearing in the literature,[5] we shall adopt it here.

The explicit form of (3.63) for the $\mathcal{E}'_{\text{eff}}$ given in (3.61) becomes

$$\mathcal{E}'_{\text{eff}}(\theta) - \mathcal{E}_{\text{eff}}(0) = -\left[\mu^2 - k^2 - m_\pi{}^2 + \frac{\rho_B(g_A{}^2k^2 - \mu^2)}{2\mu f_\pi{}^2}\right] \times \tfrac{1}{2}(f_\pi\theta)^2 \tag{3.64}$$

Applying the k minimization and charge-neutrality conditions—either directly to (3.64) or by taking the $\theta \to 0$ limit of (3.42a) and (3.42c)—leads to

$$\mu^{\text{crit}} = \rho_B g_A{}^2/2f_\pi{}^2 \tag{3.65a}$$

and

$$k^{\text{crit}} = \rho_B g_A(2g_A{}^2 - 1)^{1/2}/2f_\pi{}^2\,. \tag{3.65b}$$

Substituting these values into (3.64) then determines the critical density to be

$$\rho_B^{\text{crit}} = \frac{2f_\pi{}^2 m_\pi}{g_A(g_A{}^2-1)^{1/2}}, \tag{3.66a}$$

and thus

$$\mu^{\text{crit}} \simeq \frac{g_A m_\pi}{(g_A{}^2-1)^{1/2}} \tag{3.66b}$$

and

$$k^{\text{crit}} \simeq m_\pi(2g_A{}^2-1)^{1/2}/(g_A{}^2-1)^{1/2}\,. \tag{3.66c}$$

Before we discuss more detailed aspects of this phase transition, we should indicate the significance of the expansion in (3.64). It is relatively easy to see that the term in brackets is nothing more than the inverse propagator for pions of four-momentum $k_\alpha = (\mu, \vec{k})$ moving in the pure neutron medium. The free propagator term is obvious, and the appearance of the approximate πN scattering amplitude in the proper self-energy term,

$$\Pi(\mu, k) = \rho_B\frac{(g_A{}^2k^2-\mu^2)}{2\mu} = \rho_B[T_{\pi^- n}^{\text{forward}}(\mu, k)]|_{\text{approx}}\,, \tag{3.67}$$

can be understood by studying Fig. 5. Further, the explicit form the approximate scattering amplitude in (3.64) follows from analyzing the two diagrams in Fig. 6 in a manner consistent with our approximations.

This result—that the coefficient of the second-order term in the expansion of $\mathcal{E}'_{\text{eff}}$ in terms of the pion field amplitude is minus the inverse propagator—is completely general (see Baym and

Flowers[5]) and clarifies at once the connection between calculations which study pion phase transitions in terms of the pion propagator and those which calculate the full effective energy. Further, the higher-order terms in this expansion are the 4-, 6-, 8-, ... point pion amplitudes in the medium. Those readers familiar with the effective action functional, $\Gamma(\Phi)$, will recognize this result immediately, since $\mathcal{E}'_{\text{eff}}$ is in essence $\Gamma(\Phi)$ evaluated at $\Phi = e^{ik_\alpha x}\alpha$. Those unfamiliar with this concept are referred to the sequel, in which these points are treated in some detail.

In most respects—the smooth transition from $\theta = 0$, the off-shell character of the condensed pions, the "spontaneously broken" nature of the ground state—this pion condensate is similar to that found with $\sin^2\theta$ symmetry. In one important respect, however, this phase transition differs from the previous one. From the modified angular condition (3.62) we see that θ varies according to

$$\cos\theta = \frac{m_\pi^{\;2}}{\mu^2}\frac{g_A^{\;2}}{g_A^{\;2}-1} \tag{3.68}$$

for $\mu > m_\pi g_A/(g_A^{\;2}-1)^{1/2}$. Thus θ never quite reaches $\pi/2$. An interesting consequence of this behavior is that this model then provides a nontrivial test of the suggestion made in the previous subsection, that even if $\theta = \pi/2$ is not a minimum one can nevertheless use the density at which

$$\mathcal{E}'_{\text{eff}}\left(\theta = \frac{\pi}{2}\right) < \mathcal{E}'_{\text{eff}}(0) \tag{3.69}$$

as an estimate of the critical density. In this instance, we see directly from (3.61) that

$$\mathcal{E}'_{\text{eff}}\left(\theta = \frac{\pi}{2}\right) = \frac{3}{5}\frac{(3\pi^2)^{2/3}}{2M}\rho_B^{\;5/3} + \frac{\rho_B}{2}(\mu - g_A k) + \frac{(k^2-\mu^2)}{2}f_\pi^{\;2}\,. \tag{3.70}$$

Demanding charge neutrality and minimizing over k leads to $\mu^* = \rho_B/2f_\pi^{\;2}$ and $k^* = g_A\rho_B/2f_\pi^{\;2}$. Thus

$$\mathcal{E}'_{\text{eff}}(\theta = \pi/2) = \frac{\rho_B^{\;2}}{8f_\pi^{\;2}}(1 - g_A^{\;2})\,. \tag{3.71}$$

The inequality in (3.69) then becomes

$$\frac{\rho_B^{\;2}(g_A^{\;2}-1)}{8f_\pi^{\;2}} > m_\pi^{\;2} f_\pi^{\;2} \tag{3.72}$$

or, using $f_\pi \simeq m_\pi/\sqrt{2}$,

$$\rho_B^* = \frac{\sqrt{2}\,m_\pi^{\;3}}{(g_A^{\;2}-1)^{1/2}}\,. \tag{3.73}$$

Comparing this with the actual critical density calculated in (3.66), we find $\rho_B^* = \sqrt{2}\,g_A\,\rho_B^{\text{crit}}$ and hence the estimate obtained by this simple prescription is quite reasonable.

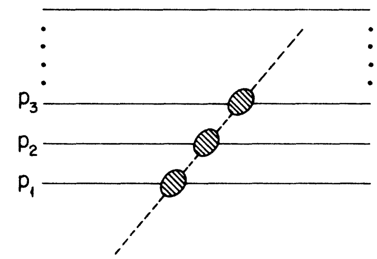


FIG. 5. A schematic illustration of the approximation to the pion self-energy in the nuclear medium.

G. $\delta m\,\overline{N}N$ symmetry breaking

In the previous discussions we have treated only that symmetry breaking which arises in the meson sector of the σ model. This can only be viewed as a first approximation for at least two reasons. First, the mesonic symmetry breaking will in general induce in higher orders of perturbation theory explicit symmetry-breaking terms in the baryonic sector. Second, and we emphasize this point, such baryonic symmetry breaking is important in the phenomenological descriptions of the πN s waves. (We are thinking of the so-called Σ term.)

Perhaps the simplest way to break chiral symmetry in the baryon sector is to endow the nucleon with a small, but nonvanishing, bare mass. Thus let us consider, in conjunction with the standard symmetry breaking in the meson sector, a term in the original Hamiltonian of the form

$$\delta\mathcal{H}_N \equiv \delta m\,\overline{N}N\,, \tag{3.74}$$

with

$$\delta m \simeq O(m_\pi)\,.$$

Under the chiral rotations earlier described, this term transforms to

$$\delta m(\overline{N}N\cos\theta - i\sin\theta\overline{N}\gamma_5\tau_1 N)\,. \tag{3.75}$$

Observing that, in the normal state $|G^0\rangle$, the

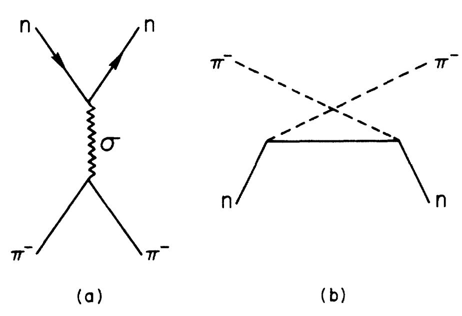


FIG. 6. The $\pi^- n$ scattering amplitude in the σ model. (a) Contributes to the s wave and (b) to both s and p waves.

nucleon pseudoscalar density in the nonrelativistic limit is very small, we see that only the first term in (3.75) contributes. Proceeding as before to calculate $\mathcal{E}'_{\rm eff}(\mu,\rho_B)$, we find

$$\begin{aligned}\mathcal{E}'_{\rm eff}(\mu,\rho_B)&\\ &=\frac{3}{5}\frac{(3\pi^2)^{2/3}}{2M}\rho_B{}^{5/3}\\ &\quad+\rho_B\left[\frac{\mu}{2}+\delta m\cos\theta-\tfrac{1}{2}(\mu^2\cos^2\theta+g_A{}^2k^2\sin^2\theta)^{1/2}\right]\\ &\quad+\frac{(k^2-\mu^2)}{2}f_\pi{}^2\sin^2\theta-m_\pi{}^2f_\pi{}^2\cos\theta\,. \end{aligned}\tag{3.76}$$

From this equation we see that the density at which the energy at $\theta=\pi/2$ becomes less than that at $\theta=0$ is determined by

$$\mathcal{E}'_{\rm eff}(\pi/2)-\mathcal{E}'_{\rm eff}(0)=-\frac{\rho_B{}^2}{8f_\pi{}^2}(g_A{}^2-1)-\delta m\rho_B+m_\pi{}^2f_\pi{}^2\,,\tag{3.77}$$

and thus, with $f_\pi\simeq m_\pi/\sqrt{2}$,

$$\rho_B^*=\frac{\sqrt{2}\,m_\pi{}^3}{(g_A{}^2-1)^{1/2}}\left[\left(1+\frac{2(\delta m)^2}{m_\pi{}^2(g_A{}^2-1)}\right)^{1/2}-\sqrt{2}\,\delta m\right].\tag{3.78}$$

Notice that for $\delta m=0$, this reduces to (3.73). For $\delta m\simeq O(m_\pi)$, however, this effect results in a considerable reduction in the critical density.

Thus the s-wave πN interactions, although considerably weaker than the p waves, can make substantial changes in the pion condensation. In the following paper and in another future publication, we will discuss the phenomenological consequences of the s waves.

IV. INCLUSION OF THE $N^*(1236)$

A. General remarks

In the last section we exhibited the variations in the value of $\rho_B^{\rm crit}$ produced by some of the modifications required to make the naive σ model reflect more accurately low-energy πN interactions. In this section we shall discuss an important additional modification designed to correct two further phenomenological failings of the σ model: first, the underestimation of the p-wave πN interaction near threshold, and second the lack—certainly in our approximation and quite possibly even in a hypothetical all-order calculation—of the $N^*(1236)$ resonance. (We note that other resonances have much higher masses and can be ignored.)

Not surprisingly these two failings are intimately connected. Indeed it seems empirically true that the difference between the actual πN p-wave amplitude and the σ-model Born approximation can be ascribed entirely to the tail of the $N^*(1236)$ resonance. Thus it is clear that one very important step toward constructing a realistic model for analyzing pion condensation is the inclusion of the N^* in our calculation.[35] Our approach will illustrate how this can be accomplished through chiral-invariance arguments.

In Sec. III we established that, solely because of the (x-dependent) chiral rotation relating the pion-condensed state to the normal ground state, the baryonic parts of the vector and axial-vector currents appear in the effective Hamiltonian in the form

$$H_{V-A}\equiv k^\alpha[V_\alpha^{(3)}\cos\theta+A_\alpha^{(2)}\sin\theta]\,,\tag{4.1}$$

which, in the nonrelativistic limit appropriate for nucleons at approximately nuclear densities, is simply

$$H_{V-A}\simeq\mu V_0^{(3)}\cos\theta-\vec{k}\cdot\vec{A}^{(2)}\sin\theta\,.\tag{4.2}$$

Since nothing in this derivation referred to specific fermion fields in the Lagrangian, to incorporate the N^* into our calculation we need only evaluate the matrix elements of these currents between states consisting of both nucleons and N^*.[36] Since there are $4\times4=16$ individual spin and isospin states of the N^*, when these are combined with the $2\times2=4$ nucleon states, the full effective Hamiltonian which generalizes (3.34) becomes a 20×20 matrix. Fortunately, this matrix can be block diagonalized according to spin. Choosing without loss of generality $\vec{k}=k\hat{z}$, we find

$$H_{V-A}=\begin{bmatrix}H_4&&&\\&H_6&&\\&&H_6'&\\&&&H_4'\end{bmatrix},\tag{4.3}$$

where H_4 and H_4' are 4×4 matrices consisting respectively of N^*, $s_z=\frac{3}{2}$ and $s_z=-\frac{3}{2}$ states. H_6 and H_6' are 6×6 matrices consisting of N^* and nucleon $s_z=\frac{1}{2}$ and $s_z=-\frac{1}{2}$ states. The eigenvalues of H_4 and H_4' are the same, as are those of H_6 and H_6'. Consequently at arbitrary θ, there are 10 distinct eigenvalues, each occurring twice. To determine the baryon contribution to the ground state we must, as before, fill all states which, at a given μ and ν, have $E<0$.

The complexity of this system—10 possible distinct Fermi seas—renders a fully analytic treatment (as opposed to numerical analysis) impossible. But an approximate calculation in which only one Fermi sea is filled is tractable. We remind readers that, in the uncondensed state, this one-Fermi-sea approximation amounts to working in pure neutron matter.

To find the appropriate Fermi seas to fill at arbitrary μ, k, and θ, we must find the lowest eigenvalue of the 20×20 matrix. Since the mass splitting between the N and N^*, denoted by Δ ($\Delta \equiv M_{N^*} - M_N$) is about two pion masses ($\Delta \simeq 2m_\pi$), we expect that the lowest-energy eigenvalue will come from the submatrices H_6 and H_6'.

In Appendix A we discuss the explicit forms of the submatrices of H_{V-A} and establish that, for the relevant range of the parameters, the lowest eigenvalue does indeed occur in the 6×6 submatrices. Further, since the diagonalization of these matrices at arbitrary μ, k, and θ is extremely complicated, it is both more illustrative and more in keeping with the qualitative nature of our considerations to use the approximate techniques introduced in Sec. III to study the threshold behavior for θ near zero (to order θ^2, as before) and the simple limiting behavior at $\theta = \pi/2$.

B. The effective energy at $\theta = \pi/2$

The simplicity of the limit $\theta = \pi/2$ suggests that it be discussed first. In this limit, the full effective nonrelativistic Hamiltonian becomes

$$\mathcal{H}_{\text{eff}} = \frac{p^2}{2M} - \nu' - g_A k_z A_z^{(2)} + \underline{\Delta} + \mathcal{E}_M^{(\pi/2)}, \tag{4.4}$$

where $\underline{\Delta}$, the mass-difference operator, is zero for nucleons and equal to Δ for N^*, and where

$$\mathcal{E}_M^{(\pi/2)} = \frac{(k^2 - \mu^2 + \alpha m_\pi^2)}{2} f_\pi^{\,2}, \tag{4.5}$$

with $\alpha = 1$ for $\sin^2\theta$ symmetry breaking and $\alpha = 2$ for $\cos\theta$ symmetry breaking.

In Appendix A we show that the lowest-lying eigenstate in the baryonic part of (4.4) is given by[37]

$$\Lambda_-(p, \nu, \mu, k, \Delta) \simeq \frac{p^2}{2M} - \nu + \tfrac{1}{2}\mu - 3b + \frac{\Delta}{3} + \cdots, \tag{4.6}$$

where $b = g_A' k/2$, and as explained in Appendix A, $g_A' = \frac{3}{5} g_A^{\text{exp}}$. The eigenstate corresponding to this eigenvalue is a mixture of nucleons and N^*'s.

Filling all states with $\Lambda_- < 0$ amounts to filling a spherical Fermi sea up to a Fermi energy of

$$\mathcal{E}_F = \nu - \tfrac{1}{2}\mu + \frac{3g_A' k}{2} - \frac{\Delta}{3}\,. \tag{4.7}$$

The total effective energy is then

$$\mathcal{E}_{\text{eff}}^{(\pi/2)}(\mu, \nu, k, \Delta) = \frac{-2}{15\pi^2} \frac{(2M\mathcal{E}_F)^{5/2}}{2M} + \mathcal{E}_M^{(\pi/2)}; \tag{4.8}$$

again introducing $\mathcal{E}_{\text{eff}}' = \mathcal{E}_{\text{eff}} + \nu\rho_B$, to make the baryon density the independent variable, we obtain

$$\mathcal{E}_{\text{eff}}'^{(\pi/2)}(\mu, \rho_B, k, \Delta) = \tfrac{3}{5}(3\pi^2)^{2/3}\frac{\rho_B^{5/3}}{2M} + \frac{(k^2 - \mu^2 + \alpha m_\pi^{\,2})}{2} f_\pi^{\,2} + \rho_B\left(\tfrac{1}{2}\mu - \frac{3g_A' k}{2} + \frac{\Delta}{3}\right). \tag{4.9}$$

The minimization over k and the condition of charge neutrality imply

$$\frac{\partial \mathcal{E}_{\text{eff}}'^{(\pi/2)}}{\partial k} = 0 = k f_\pi^{\,2} - \rho_B \tfrac{3}{2} g_A' \tag{4.10a}$$

and

$$\frac{\partial \mathcal{E}_{\text{eff}}'^{(\pi/2)}}{\partial \mu} = 0 = -\mu f_\pi^{\,2} + \tfrac{1}{2}\rho_B. \tag{4.10b}$$

Solving these equations for k and μ and substituting into (4.9), we find

$$\mathcal{E}_{\text{eff}}'^{(\pi/2)}(\rho_B) = h(\rho_B) - \frac{\rho_B^{\,2}}{8f_\pi^{\,2}}\left(9g_A'^{\,2} - 1\right) + \frac{\rho_B \Delta}{3} + \alpha m_\pi^{\,2} f_\pi^{\,2}. \tag{4.11}$$

The density beyond which this energy density is lower than that of the normal state is

$$\rho_{B(N,N^*)}^* = \frac{2m_\pi^{\,2}}{9g_A'^{\,2} - 1}\left\{+\frac{\Delta}{3} + \left[\frac{\alpha}{2}\left(9g_A'^{\,2} - 1\right)m_\pi^{\,2} + \frac{\Delta^2}{9}\right]^{1/2}\right\}. \tag{4.12}$$

For $g_A = 1.24$, $g_A' = \frac{3}{5} g_A = 0.75$, $\Delta \simeq 2m_\pi$

$$\rho_{B(N,N^*)}^* \simeq 1.1 m_\pi^{\,3} \sim 2.2\rho_{\text{nucl}} \tag{4.13a}$$

for $\alpha = 1$, i.e., $\sin^2\theta$ symmetry breaking and

$$\rho_{B(N,N^*)}^* \simeq 1.4 m_\pi^{\,3} \simeq 2.8\rho_{\text{nucl}} \tag{4.13b}$$

for $\alpha = 2$, i.e., $\cos\theta$ symmetry breaking.

In both cases we observe that these densities are substantially less than those found in the absence of N^*. We regard these numbers as reasonable indications of the density beyond which pion condensation will occur.

C. The effective energy at $\theta = 0$

To estimate the threshold density for the first appearance of pion condensation, we can expand the effective energy in powers of θ. Using results from Appendix A, where we performed a perturbation calculation in the parameter θ^2, we find that the effective energy at small θ is given by

$$\mathcal{E}_{\text{eff}}'(\theta) = h(\rho_B) - \frac{(f_\pi\theta)^2}{2}\left\{(\mu^2 - k^2 - m_\pi^{\,2}) - \frac{\rho_B}{f_\pi^{\,2}}\left[\frac{\mu}{2} - \frac{4g_A'^{\,2}k^2}{9(\mu+\Delta)} - \frac{25}{18}\frac{g_A'^{\,2}k^2}{\mu} - \frac{4g_A'^{\,2}k^2}{3(\Delta-\mu)}\right]\right\}. \tag{4.14a}$$

This quantity could also be written as

$$\mathcal{E}'_{\rm eff}(\theta) \equiv \mathcal{E}'_{\rm eff}(\theta) - \mathfrak{D}^{-1}\frac{(f_\pi \theta)^2}{2} + \cdots . \tag{4.14b}$$

Again the interpretation of the form of the inverse propagator is clear. The proper self-energy term and forward πN scattering amplitude are related through

$$\Pi(\mu, k) = \rho_B (T_{\pi N})_{\rm forward} . \tag{4.15}$$

In this approximation, the πN amplitude contains in addition to the remnants of the crossed proton Born term and σ exchange, a direct "Born" term from the N^{*-} pole—the term with $1/(\Delta-\mu)$ in (4.14a)—and a crossed Born term from the N^{*+} pole—the term with $1/(\Delta+\mu)$ in (4.14a). Both these terms are shown in Fig. 7. Notice that if $\frac{25}{18}$ is approximated by $\frac{4}{3}$, the proton pole and the N^{*-} pole combine to give the Chew-Low[38] result. Further, observe that in the limit $\Delta \to \infty$, we recover the earlier results involving only neutrons and protons.

The minimization condition on k and the charge-neutrality constraint require, respectively,

$$0 = -k + \rho_B \frac{g_A'^2 k}{f_\pi^2}\left[\frac{4}{9(\mu+\Delta)} + \frac{25}{18\mu} + \frac{4}{3(\Delta-\mu)}\right] \tag{4.16a}$$

and

$$0 = 2\mu - \frac{\rho_B}{f_\pi^2}\left[\tfrac{1}{2} + g_A'^2 k^2 \left(\frac{4}{9(\mu+\Delta)^2} + \frac{25}{18\mu^2} - \frac{4}{3(\Delta-\mu)^2}\right)\right]. \tag{4.16b}$$

Studying these equations in conjunction with the condition $\mathfrak{D}^{-1} = 0$, we find no solutions for the physical values of the parameters g_A ($=1.24$) and Δ ($\cong 2m_\pi$). A more thorough numerical analysis reveals the reason for this situation; for these values of the parameters, there is no minimum with respect to k. Thus when the phase transition occurs—as it does in this model at a density $\rho^{\rm crit} \cong 0.31 m_\pi^3 \cong 0.62\rho_{\rm nucl}$—the lowest energy is obtained by letting $k^2 \to \infty$. It is obvious that for very large k—indeed, for any $k \gg 3m_\pi$—both the physics in this simple model and the approximations in which we have solved it break down entirely: The p waves cannot continue to grow like k^2, higher partial waves must enter $T_{\pi N}$, the Lindhard functions cannot be approximated by their static limits, and the extrapolation of $T_{\pi N}$ to the off-mass-shell point $\mu^2 - k^2 \to -\infty$ is untenable.

To resolve this apparent difficulty we need only recall that for $k < 3m_\pi$ our description of the πN amplitude—including the effects of the N^*—*is* believable. Thus by restricting $k < k_{\rm cutoff} = 3m_\pi$, we can obtain an upper bound on the critical density in the present case. With this cutoff the parameters of the phase transition become

$$k = k_{\rm cutoff} = 3m_\pi , \tag{4.17a}$$

$$\rho^{\rm crit}_{B(N,N^*)} = 0.33 m_\pi^3 \cong 0.66\rho_{\rm nucl} , \tag{4.17b}$$

and

$$\mu = 0.93 m_\pi \tag{4.17c}$$

for $g_A = 1.24$ and $\Delta = 2m_\pi$. One cannot yet, of course, regard this as an accurate estimate of the threshold for pion condensation, for our present calculation explicitly excludes the important inhibiting effects of nuclear correlations[39]; these we shall discuss qualitatively in the next section. Despite this limitation, comparing the critical density in (4.17b) with that in (3.58a) establishes clearly the importance of including correctly the effects of the N^*.

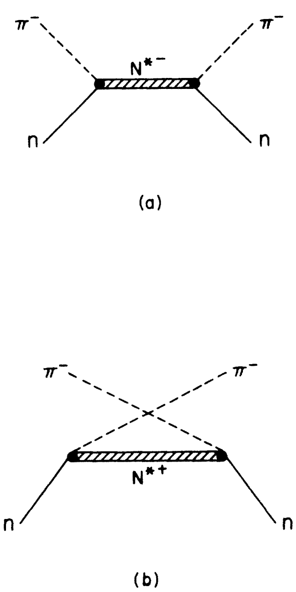


FIG. 7. The additional contribution to the $\pi^- n$ scattering amplitude from N^*. (a) is the direct N^{*-} contribution and (b) is the crossed N^{*+} contribution.

V. CHIRAL SYMMETRY AND NUCLEAR FORCES [2,3,5,29,39]

In the preceding sections we have treated the condensed nuclear matter as if it consisted of otherwise uncorrelated nucleons—or nucleons and N^*'s—moving under the influence of the external condensed pion field. Obviously, any realistic calculation of pion condensation *must* include the significant role that nuclear forces and correlations—for example, the short-range repulsion as realized by ω exchange—play in determining the equation of state. In this section we shall discuss, in a qualitative way, the manner in which these nuclear correlations can be incorporated into the chiral-symmetry approach to pion condensation.

To render the discussion as precise as possible, let us begin by recalling our exact assumptions about the way chiral symmetry enters into the problem of pion condensation. First, we have assumed that the full strong-interaction Hamiltonian has the form

$$H_{\text{nucl}} = H_0 + \delta H, \tag{5.1}$$

where H_0 is a chiral-invariant Hamiltonian density with an energy scale $\simeq 1$ GeV $\simeq O(m_N) \simeq O(m_\omega)$ and δH is an explicit chiral-symmetry-breaking interaction of order m_π. Since nucleons are neither massless nor parity-doubled, we have also assumed that the chiral symmetry in H_0 is realized in the Goldstone manner, so that if δH were zero, m_π would be zero. If these assumptions are not accepted, it becomes difficult to explain the experimental successes of chiral-symmetry predictions.

Second, an assumption akin to but not as specific as the smoothness assumption of PCAC has been implicit in our previous discussion: Namely, that the limit $\delta H \to 0$ is "smooth" in the sense that the particles that exist in the theory for $\delta H \neq 0$ are still present when $\delta H = 0$ and, further that the shift of their masses as $\delta H \to 0$ is $\lesssim O(m_\pi)$, the order of the shift of the pion's mass. A specific example of this occurs in the σ model, with standard symmetry breaking as discussed in Secs. II and III. There, in the tree approximation, the nucleon mass—given by $M = g\langle\sigma\rangle$—shifts, as $\delta H \to 0$, by $(\delta M_N)/(M_N) \simeq 2\%$ for the parameters given in Sec. II.

Third, we have assumed that the condensed pion ground state is related to the normal ground state by a chiral rotation. Since the motivation for this assumption has already been given, we reiterate here only the most obvious caveat; for any abnormal state in which—in the language of the σ model of Secs. II and III—*both* the radius A and the angle θ changed by large amounts, one could not hope to learn anything model-independent from chiral symmetry alone.

Using these three assumptions we have shown that to compare the ground-state energies of the normal and condensed phases we need to study the effective Hamiltonian

$$H_{\text{eff}} = H_0 - \nu'\rho_B + k^\mu[V_\mu^{(3)}\cos\theta + A_\mu^{(2)}\sin\theta] + \delta H^\theta, \tag{5.2}$$

where A^2 is the abstract axial-vector current operator and thus has appropriately renormalized matrix elements, and where δH^θ is the rotated symmetry-breaking term. In what follows we will assume $\cos\theta$ symmetry breaking, and thus δH^θ will effectively be $\delta H\cos\theta$. For any given θ, we can reduce the calculation of the ground-state energy of H_{eff} to a many-body problem as follows. We pull out the c-number terms coming from the condensed mesons and define an approximate nonrelativistic many-body Hamiltonian according to

$$H_{\text{eff}} \to H_{\text{MB}}(\theta) + \frac{k^2-\mu^2}{2} f_\pi^2\sin^2\theta + f_\pi^2 m_\pi^2(1-\cos\theta). \tag{5.3}$$

The θ-dependent many-body Hamiltonian is then taken to be

$$H_{\text{MB}}(\theta) = \sum_i K_i(\theta) + \sum_{i<j} V_{ij}(\theta), \tag{5.4}$$

where the single-particle Hamiltonian

$$K(\theta) = \frac{p^2}{2M} - \nu' + \frac{\mu\tau_3}{2}\cos\theta - g_A\vec{k}\cdot\vec{\sigma}\,\frac{\tau_2}{2}\sin\theta + \delta M\cos\theta \tag{5.5}$$

includes a possible δM symmetry breaking and the V_{ij} are two-particle potentials. These potentials are supposed to come from meson exchange. Our chirally rotated H_{eff} is assumed to produce the same set of mesons as does the usual Hamiltonian at $\theta = 0$. The propagators of these mesons and their couplings to nucleons will be somewhat modified by the θ-dependent terms in (5.2). This will lead to a θ dependence in the nucleon-nucleon potential.

To illustrate the nature of $V(\theta)$ we adopt a simple model where the nucleon-nucleon interaction is generated by π, σ, ρ, and ω exchange, as shown in Figs. 8–9(c). We also include a "Lorentz-Lorentz" term,[40] Fig. 9(d), $V^{\text{L-L}}$, which is supposed to take account of the effect of the repulsive hard core on the attractive one-pion exchange. This effect acts to reduce the net attraction in the channel with pion quantum numbers. When θ is nonzero the mesons interact with the condensed pion field. This changes their propagators, as shown in Fig. 10. For heavy-meson exchange we can model this interaction by imagining that the meson mass squared is a θ-dependent quantity $m^2 + \delta m^2(\theta)$, where $\delta m^2(\theta) \sim m_\pi^2$. We then estimate the θ dependence of a heavy-meson exchange potential to be

$$\left|\frac{V(\theta)-V(0)}{V(0)}\right| \sim \frac{\delta m^2(\theta)}{m^2} \sim \frac{m_\pi^2}{m^2}. \tag{5.6}$$

For ω, ρ, and σ exchange as well as for $V^{\text{L-L}}$, this is a few percent. The θ-dependent pieces of V^ω, V^ρ, V^σ, and $V^{\text{L-L}}$ will then contribute to the ground-state energy density a term of order $\rho_B V\delta m^2(\theta)/m^2 \sim \rho_B V m_\pi^2/m^2$, where V is some typical potential energy per nucleon. Taking $V \sim m_\pi$ this is negligi-

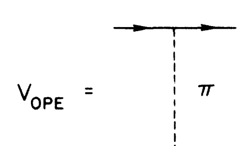


FIG. 8. The one-pion-exchange potential.

ble compared to the θ-dependent energy density of order $\rho_B m_\pi$ coming from the single-particle terms in $H_{\rm MB}$. Therefore, we can safely set

$$\begin{aligned} V^\omega(\theta) &\approx V^\omega(0) \equiv V^\omega, \\ V^\rho(\theta) &\approx V^\rho(0) \equiv V^\rho, \\ V^\sigma(\theta) &\approx V^\sigma(0) \equiv V^\sigma, \end{aligned} \tag{5.7}$$

and

$$V^{\rm L\text{-}L}(\theta) \approx V^{\rm L\text{-}L}(0) \equiv V^{\rm L\text{-}L}.$$

For one-pion exchange, on the other hand, it is easy to see that

$$\left|\frac{V^{\rm OPE}(\theta) - V^{\rm OPE}(0)}{V^{\rm OPE}(0)}\right| \sim 1, \tag{5.8}$$

and that we cannot neglect the θ dependence of $V^{\rm OPE}$.[41]

With this model and the approximation in Eq. (5.7), the many-body Hamiltonian is

$$H_{\rm MB} = \sum K_i(\theta) + \sum_{i<j} \left[V_{ij}^\omega + V_{ij}^\rho + V_{ij}^{\rm L\text{-}L} + V_{ij}^\sigma + V_{ij}^{\rm OPE}(\theta)\right]. \tag{5.9}$$

We can simplify $H_{\rm MB}$ by making a spin-isospin transformation which diagonalizes $K(\theta)$. Let

$$U^{-1}(\theta)(\mu\tau_3\cos\theta - g_A \vec{k}\cdot\vec{\sigma}\tau_2\sin\theta)U(\theta) = \tau_3(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2}; \tag{5.10}$$

then $H_{\rm MB}$ can be transformed to

$$\begin{aligned} U^{-1}(\theta)H_{\rm MB}U(\theta) &= H'_{\rm MB} \\ &= \sum_i \frac{p_i{}^2}{2M} - \nu' + \frac{\tau_3}{2}(\mu^2\cos^2\theta + g_A{}^2k^2\sin^2\theta)^{1/2} \\ &\quad + \delta m^2\cos\theta + \sum_{i<j}(V_{ij}^\omega + V_{ij}^\sigma) \\ &\quad + \sum_{i<j} U^{-1}(\theta)\left[V_{ij}^\rho + V_{ij}^{\rm L\text{-}L} + V_{ij}^{\rm OPE}(\theta)\right]U(\theta), \end{aligned} \tag{5.11}$$

where we have taken a nonrelativistic limit so that V^ω and V^σ are independent of spin as well as isospin. In $H'_{\rm MB}$ the θ dependence of the potential energy comes both from the chiral-symmetry breaking in $V^{\rm OPE}(\theta)$ and from the fact that V^ρ and $V^{\rm L\text{-}L}$ do not commute with U. In this connection, it should be understood that *U is not a chiral transformation*. It is a spin-isospin transformation as opposed to a chiral rotation which is a γ_5-isospin transformation. The way in which the potential transforms under U has nothing to do with chiral symmetry.

From the form of $H'_{\rm MB}$ one sees that the repulsive hard-core V^ω and medium-range attraction V^σ are the same in the condensed and normal states. They can therefore be ignored. The other potentials V^ρ,

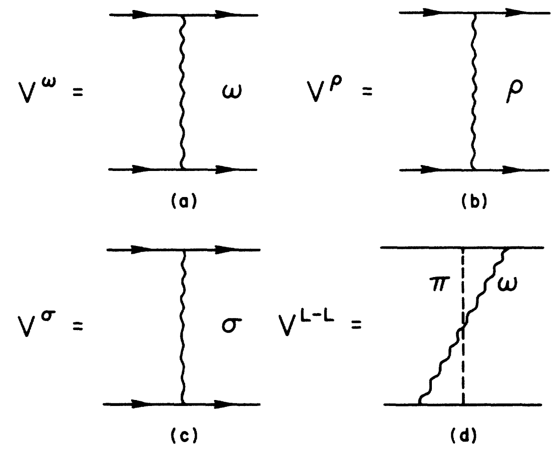


FIG. 9. The "heavy-meson" exchange potentials: (a) the ω, spin- and isospin-independent; (b) the ρ, a spin- and isospin-dependent force; (c) the σ, an intermediate-range attraction; (d) the effect of short-range repulsions on one-pion exchange.

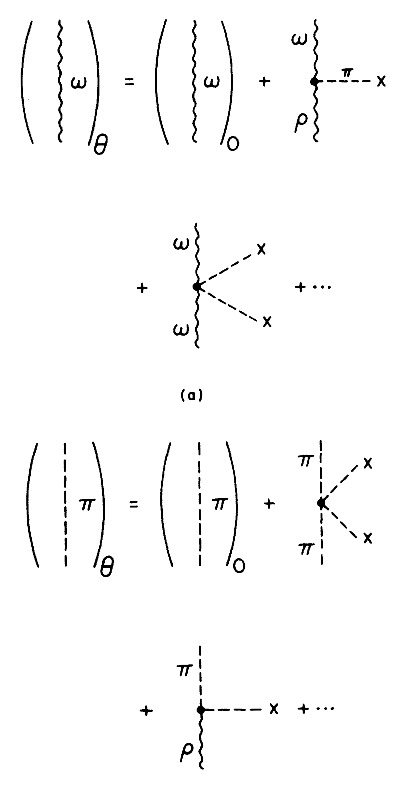


FIG. 10. (a) The modification to the ω-exchange potential—the nonrelativistic limit of the ω propagator—in the presence of the condensed field. (b) The modification to the one-pion-exchange potential—the nonrelativistic limit of the pion propagator—in the presence of the condensed field.

$V^{\text{L-L}}$, and V^{OPE} cannot be disregarded and their omission is a major defect of our model. Actually, chiral symmetry has little to say about V^{ρ} or $V^{\text{L-L}}$. Within the context of this paper they would be viewed as phenomenological additions to the σ model as was the N^*. On the other hand, $V^{\text{OPE}}(\theta)$ is determined by chiral symmetry and will be correctly given by the σ model. In lowest order the contribution of V^{OPE} to the ground-state energy is given by the diagram in Fig. 11. The θ dependence of this diagram, which we have not computed, is complicated by the fact that the pion as well as nucleon propagators depend strongly on θ.

We hope to return to some of these nuclear form effects in a future publication.[39] With regard to these effects, the one advantage that our formalism has is that they can be estimated by computing at $\theta = \pi/2$, where considerable simplification occurs.

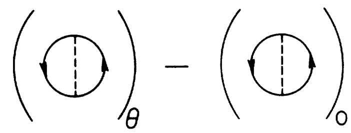


FIG. 11. The contribution to the energy shift caused on the modification of the one-pion-exchange potential; actually, all propagators and couplings should be evaluated in the presence of the external condensed field.

VI. EXCITATIONS AND β DECAY

In this section we study the low-lying excitations in the condensed state. For a neutron star low lying means of order kT or a few keV.[19] This is an extremely small energy on a nuclear or pionic scale.

Our model clearly suggests a fermionic spectrum of excitations composed of particles above and holes beneath the Fermi energy. It also suggests a spectrum of excitations associated with the meson fields. However, it will be argued below and shown in more detail in the following paper that when the electromagnetic field is taken into account the only low-lying meson mode becomes a plasmon. There might be bosonic modes associated with collective oscillations of the Fermi surface. Low-lying modes of this type are usually associated with a broken symmetry. In the following paper it is pointed out that there is no reason to expect such modes in the present context. Thus, we expect to end up with only a fermion excitation spectrum and a plasmon.

At finite temperatures neutron stars can cool by neutrino emission. This may be viewed as the β decay of quasiparticles. At the end of this section we will see how to treat this process. We hope to give more detailed calculations in a later publication.

A. Meson excitations

Here we will treat only the problem of mesons alone, as was done in Sec. II. We will see that there are no low-lying excitations. When nucleons are present the situation is considerably more complicated. This will be touched on in the following paper.

We choose the standard symmetry breaking $c_1\sigma = m_\pi^2 \mathfrak{A}\sigma$, but our results are actually independent of the type of symmetry breaking. From Sec. II we know that for $\mu > m_\pi$, the expectation values of the field are

$$\begin{aligned}
&\sigma = \mathfrak{A}\cos\theta,\\
&\pi_1 = \mathfrak{A}\sin\theta,\\
&\cos\theta = \frac{m_\pi}{\mu},\\
&\mathfrak{A} = A + \frac{\mu^2}{8\lambda A} + O\left(\frac{1}{\lambda^2}\right).
\end{aligned} \tag{6.1}$$

As in Sec. III it is convenient to make a chiral rotation and work with fields whose expectations values are $\sigma = \mathfrak{A}$ and $\pi = 0$. The rotated Lagrangian is

$$\begin{aligned}
\mathcal{L} = {}& \tfrac{1}{2}[(\partial_\mu\sigma + k_\mu\pi_2\sin\theta)^2 + (\partial_\mu\pi_1 + k_\mu\pi_2\cos\theta)^2 + (\partial_\mu\pi_2 - k_\mu\pi_1\cos\theta - k_\mu\sigma\sin\theta)^2 + (\partial_\mu\pi_3)^2]\\
&- \lambda(\sigma^2 + \vec{\pi}^2 - A^2)^2 + c_1(\cos\theta\,\sigma - \sin\theta\,\pi_1),
\end{aligned}$$

where we have introduced the four-vector notation

$$k^\mu = (\mu, 0, 0, 0). \tag{6.3}$$

To find the excitations we set $\sigma = \mathfrak{A} + \sigma'$ and expand in powers of the fields σ' and π, keeping only the quadratic terms. There are no linear terms since the fields in (5.1) minimize the energy. The result is

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\sigma' + k_\mu\pi_2\sin\theta)^2 + (\partial_\mu\pi_1 + k_\mu\pi_2\cos\theta)^2 + (\partial_\mu\pi_3)^2 + (\partial_\mu\pi_2 - k_\mu\pi_1\cos\theta - k_\mu\sigma\sin\theta)^2 - k^2(\pi_1^2 + \pi_2^2 + \pi_3^2) - m^2\sigma'^2], \tag{6.4}$$

where terms of order λ^{-1} have been dropped, and m^2 is large.

From (6.4) we see immediately that π_3 decouples from the other fields and propagates like a particle with mass $(k^2)^{1/2} = \mu$. This is far too heavy to be a low-lying object on the scale described above. The large mass—$(m^2 + \mu^2\sin^2\theta)$—of the σ' effectively removes it from low-lying excitation. For this reason, we drop σ' from $\mathfrak{L}$. We are then left with the simpler Lagrangian involving only π_1 and π_2,

$$\mathfrak{L} = \tfrac{1}{2}[(\partial_\mu \pi_1)^2 + (\partial_\mu \pi_2)^2 + 2k^\mu \cos\theta(\pi_2 \partial_\mu \pi_1 - \pi_1 \partial_\mu \pi_2) - k^2 \sin^2\theta\, \pi_1{}^2]\,. \tag{6.5}$$

If one looks for solutions to Lagrange's equations of the form $\pi_i = c_i \exp(-i\omega t + i\vec{q}\cdot\vec{x})$, $i = 1, 2$, one finds that there is one branch with the property that $\omega(\vec{q}) \to 0$ as $|\vec{q}| \to 0$. This mode is the Goldstone boson which arises from the fact that the ground state is not an eigenstate of the conserved operator I_3. It is a possible low-lying excitation. We shall demonstrate that it disappears when the photon field is included. The other branch to the spectrum of (6.5) has an energy of order k^2 as $|q| \to 0$ and like the π_3 mode is not of interest.

The electromagnetic field A^ν enters the Lagrangian through its kinetic energy $-\frac{1}{4}(F^{\mu\nu})^2$ and through its coupling to charged mesons. The easiest way to see how A^μ couples to the chirally rotated mesons in (6.3) is to note that k^μ and A^μ can only appear in the combination $k^\mu + eA^\mu$. Thus we add $-\frac{1}{4}(F^{\mu\nu})^2$ and replace k^μ by $k^\mu + eA^\mu$ in (6.3). Again we expand in powers of A^μ, σ', and π keeping only quadratic terms. As before, π_3 decouples and can be dropped. Taking also the formal limit $m_\sigma \to \infty$ and dropping σ' yields, after some algebra,

$$\mathfrak{L} = -\tfrac{1}{4}(F^{\mu\nu})^2 + \tfrac{1}{2}[(\partial_\mu \pi_2 - e\mathfrak{Q}\sin\theta A_\mu)^2 - 4\pi_1 k^\mu \cos\theta(\partial_\mu \pi_2 - e\mathfrak{Q}\sin\theta A_\mu) + (\partial_\mu \pi_1)^2 - k^2\sin^2\theta\, \pi_1{}^2]\,, \tag{6.6}$$

where a total divergence proportional to $k^\mu \partial_\mu(\pi_1\pi_2)$ has been dropped. Let us now define a new field

$$B^\mu = A^\mu - \frac{1}{e\mathfrak{Q}\sin\theta}\partial^\mu \pi_2\,, \tag{6.7}$$

which is just a gauge transformation on A^μ. Since $F^{\mu\nu}$ is gauge-invariant, $\mathfrak{L}$ then becomes

$$\mathfrak{L} = -\tfrac{1}{4}(F^{\mu\nu})^2 + \frac{m_\gamma{}^2}{2}(B_\mu)^2 + 2e\mathfrak{Q}\sin\theta\cos\theta\, \pi_1 k^\mu B_\mu + \tfrac{1}{2}(\partial_\mu \pi_1)^2 - \tfrac{1}{2}k^2\sin^2\theta\, \pi_1{}^2\,, \tag{6.8}$$

where $F^{\mu\nu}$ is now defined as

$$F^{\mu\nu} = \frac{\partial}{\partial x_\nu}B^\mu - \frac{\partial}{\partial x_\mu}B^\nu\,, \tag{6.9}$$

and

$$m_\gamma = e\mathfrak{Q}\sin\theta \tag{6.10}$$

appears as a photon mass. One can easily convince oneself that (6.8) has no modes for which $\omega(q) \to 0$ as $|q| \to 0$. There are no longer any low-lying excitations.

The fact that the Goldstone boson disappeared at the expense of the photon's getting a mass is no surprise. This is a consequence of the obvious fact that condensed charged pions make a superconductor. In the language of superconductivity, we recognize a magnetic penetration depth (or London depth[42]) $l = m_\gamma{}^{-1}$ of a few fermis.[4]

B. Fermi excitations

In our models there is only one Fermi sea available for the creation of low-lying excitations. With our approximations the (effective) energies of the fermion states are just

$$\epsilon(p) = E_+(p)\,. \tag{6.11}$$

In a more realistic calculation M would be replaced by some effective mass and (6.11) would be expected to hold only near the Fermi surface.

C. β decay

The Hamiltonian responsible for β decay[43] is

$$\mathcal{H}_W = \frac{G}{\sqrt{2}}(V_1^\mu + iV_2^\mu + A_1^\mu + iA_2^\mu)\bar{\nu}(\gamma^\mu(1+\gamma_5))e + (\text{H.c.}), \tag{6.12}$$

where V and A are the vector and axial-vector currents defined above. To discuss β decay in the condensed pion phase, it is convenient to apply our chiral rotation to $\mathcal{H}_W$. We will then be working with the ground state where $\langle\pi\rangle = 0$. The result of applying the rotation to $\mathcal{H}_W$ is to make the replacement

$$V_1^\mu + iV_2^\mu + A_1^\mu + iA_2^\mu \to e^{i\vec{k}\cdot\vec{x}}\{V_1^\mu + A_1^\mu + i[\cos\theta(V_2^\mu + A_2^\mu) - \sin\theta(V_3^\mu + A_3^\mu)]\}\,. \tag{6.13}$$

The candidate process for neutrino emission is fermion of momentum $\vec{p} \to$ fermion of momentum $\vec{p}'$ + electron + neutrino. Let us consider the energetics of this decay. Because of the factor $e^{i\vec{k}\cdot\vec{x}}$ in (6.13), the momentum balance will be

$$\vec{p} = \vec{p}' - \vec{k} + \vec{p}_e + \vec{p}_\nu\,, \tag{6.14}$$

where $\vec{p}_e$ and $\vec{p}_\nu$ are the electron and neutrino energies. To get energy conservation straight we have to remember that the hadronic weak current changes I_3 for the hadrons by one unit. Therefore, if we use effective energies for the hadronic component but not the leptons, the energy balance is

$$\epsilon_{\text{eff}}(p) = \epsilon_{\text{eff}}(p') - \mu + (p_e{}^2 + m_e{}^2)^{1/2} + |\vec{p}_\nu|\,. \tag{6.15}$$

Referring to Eq. (6.11) we see that this reduces to

$$\frac{p^2}{2m} = \frac{p'^2}{2m} - \mu + (p_e{}^2 + m_e{}^2)^{1/2} + |\vec{p}_\nu|\,. \tag{6.16}$$

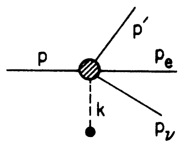


FIG. 12. The induced β decay in the presence of the pion condensate.

Equations (6.14) and (6.16) can be interpreted as a fermion picking up four-momentum $(\mu, \vec{k})$ from a condensed pion and then decaying into another fermion and the lepton pair (see Fig. 12). This is just the process suggested years ago by Bahcall and Wolf.[1,4] One can easily convince oneself that for $p_F \gtrsim k/2$ it is kinematically allowed.

The fermion excitations are nonrelativistic linear combinations of n and p or more generally nucleons and N^*'s. In either case it is easy to evaluate the required matrix elements of the rotated weak current in (6.13). For a given θ one should therefore be able to do rather well in calculating the cooling rate due to neutrino emission. As mentioned above, we hope to return to this in a future publication.

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APPENDIX A: THE MATRIX ELEMENTS $\mathcal{H}_{\rm eff}$ WHEN N^*(1236) IS INCLUDED

In the text, we established that to include the N^* in the σ model calculation of pion condensation we needed to evaluate the matrix element of the operator

$$H \equiv \mu V_0^{(3)} \cos\theta - g_A k_z A_z^{(2)} \sin\theta + \underline{\Delta} \tag{A1}$$

between nucleon and N^*(1236) states. Here $V_0^{(3)}$ and $A_z^{(2)}$ are the usual vector and axial-vector currents and $\underline{\Delta}$, the mass-difference operator is diagonal with elements equal to zero for the nucleons and $M_{N^*} - M_N \simeq 2m_\pi$ for the N^*.

The full (20×20) matrix representing H in the $N \oplus N^*$ subspace can be block diagonalized, each block having the same value for the spin projection. The blocks H_4 and H_4' have $s_z = \frac{3}{2}$ and $-\frac{3}{2}$, and the blocks H_6 and H_6' have respectively $s_z = \frac{1}{2}$ and $-\frac{1}{2}$. To calculate the explicit matrix elements of the submatrices H_4 and H_6 we will use the

$$(\mathrm{SU}(2))_{\rm spin} \otimes (\mathrm{SU}(2))_{\rm isospin} \subset \mathrm{SU}(4)$$

quark-model wave functions available in the literature.[44] Actually, since that model predicts for nulceons

$$\left|\frac{g_A}{g_V}\right| = \frac{5}{3}, \tag{A2}$$

while experimentally $|g_A| = 1.24|g_V|$, we take the numerical predictions only for the ratios of the couplings for nucleons to those for N^*.[45] Thus we modify the axial-vector term in (A1) to

$$g_A k_z A_z^{(2)} \sin\theta \rightarrow (g_A^{\rm exp}\tfrac{3}{5})(A_z^{(2)})_{\rm quark\ model} \sin\theta \equiv g_A' A_z^{(2)} \sin\theta. \tag{A3}$$

Recalling that in the SU(4) quark model

$$A_z^{(2)} = \tfrac{1}{2} \sum_\alpha \sigma_{z(\alpha)} \tau_{(\alpha)}^{(2)}, \tag{A4}$$

where the sum over α is over the distinct quarks in the baryon, we see that the nonvanishing axial-vector matrix elements in H_4 are

$$\langle ++ | A_z^{(2)} | + \rangle = -\frac{\sqrt{3}}{2} i, \tag{A5a}$$

$$\langle + | A_z^{(2)} | 0 \rangle = -i, \tag{A5b}$$

and

$$\langle 0 | A_z^{(2)} | - \rangle = -\frac{\sqrt{3}}{2} i, \tag{A5c}$$

where $|q\rangle$ denotes an N^* state of charge q. Hence the full submatrix H_4 assumes the form

$$H_4 = \begin{bmatrix} 3a+\Delta & \sqrt{3}ib & 0 & 0 \\ -\sqrt{3}ib & a+\Delta & 2ib & 0 \\ 0 & -2ib & -a+\Delta & \sqrt{3}ib \\ 0 & 0 & -\sqrt{3}ib & -3a+\Delta \end{bmatrix}, \tag{A6a}$$

where we have used the basis

$$\psi = \begin{bmatrix} N^{*++} \\ N^{*+} \\ N^{*0} \\ N^{*-} \end{bmatrix}, \tag{A6b}$$

and where we have introduced

$$a = \frac{\mu}{2}\cos\theta \quad\text{and}\quad b = \frac{g_A k}{2}\sin\theta. \tag{A6c}$$

The four eigenvalues at arbitrary θ are clearly given by

$$\lambda_\pm^{(1)} = \Delta \pm 3(a^2+b^2)^{1/2} \tag{A7a}$$

and

$$\lambda_\pm^{(2)} = \Delta \pm (a^2+b^2)^{1/2}. \tag{A7b}$$

The lowest eigenvalue is clearly $\lambda_-^{(1)}$. For $\theta \simeq 0$,

$$\lambda_-^{(1)} \cong \Delta - \tfrac{3}{2}\mu + \cdots \tag{A8a}$$

and for $\theta \simeq \pi/2$,

$$\lambda_-^{(1)} \cong \Delta - \tfrac{3}{2} g_A' k\,. \tag{A8b}$$

In the (6×6) submatrix with $s_z = \frac{1}{2}$, the nonvanishing matrix elements of the axial-vector current are

$$\langle ++|A_z^{(2)}|+\rangle = \frac{-i}{2\sqrt{3}}, \tag{A9a}$$

$$\langle ++|A_z^{(2)}|p\rangle = \frac{i2}{\sqrt{6}}, \tag{A9b}$$

$$\langle p|A_z^{(2)}|n\rangle = \frac{-i}{2}\,\frac{5}{3}, \tag{A9c}$$

$$\langle p|A_z^{(2)}|0\rangle = \frac{-i2}{\sqrt{18}}, \tag{A9d}$$

$$\langle +|A_z^{(2)}|n\rangle = \frac{i2}{\sqrt{18}}, \tag{A9e}$$

$$\langle +|A_z^{(2)}|0\rangle = \frac{-i}{3}, \tag{A9f}$$

$$\langle 0|A_z^{(2)}|-\rangle = \frac{-i}{2\sqrt{3}}, \tag{A9g}$$

and

$$\langle n|A_z^{(2)}|-\rangle = \frac{-2i}{\sqrt{6}}. \tag{A9h}$$

Thus in the basis

$$\psi = \begin{bmatrix} N^{*++} \\ N^{*+} \\ p \\ n \\ N^{*0} \\ N^{*-} \end{bmatrix}, \tag{A10}$$

H_6 is given by

$$\begin{bmatrix} 3a+\Delta & \frac{ib}{\sqrt{3}} & \frac{-4ib}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{-ib}{\sqrt{3}} & a+\Delta & 0 & \frac{-i4b}{\sqrt{18}} & \frac{2ib}{3} & 0 \\ \frac{4ib}{\sqrt{6}} & 0 & a & \frac{5}{3}ib & \frac{4ib}{\sqrt{18}} & 0 \\ 0 & \frac{4ib}{\sqrt{18}} & -\frac{5}{3}ib & -a & 0 & \frac{4ib}{\sqrt{6}} \\ 0 & \frac{-2ib}{3} & \frac{-4ib}{\sqrt{18}} & 0 & -a+\Delta & \frac{ib}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{-4ib}{\sqrt{6}} & \frac{-ib}{\sqrt{3}} & -3a+\Delta \end{bmatrix}, \tag{A11}$$

with $a = \mu\cos\theta/2$ and $b = g_A' k \sin\theta/2$. We will restrict ourselves to the form of the eigenvalues for $\theta \simeq 0$ and $\theta = \pi/2$, since we do not know the general analytic expression for any θ.

(a) $\theta \simeq 0$. We will solve the eigenvalue problem here through a perturbation-theory expansion in θ.

The lowest eigenvalue in the region $\mu \simeq m_\pi$, where we expect a phase transition, is to zeroth order the neutron state, i.e.,

$$\lambda_-|_{\theta=0} = -\frac{\mu}{2}. \tag{A12}$$

To second order in perturbation theory

$$\lambda_- = \lambda_-|_{\theta=0} + \frac{\theta^2}{2}\left[\frac{\mu}{2} - \frac{4g_A'^2 k^2}{9(\mu+\Delta)} - \frac{25 g_A'^2 k^2}{18\mu} - \frac{4 g_A'^2 k^2}{3(\Delta-\mu)}\right]. \tag{A13}$$

The contribution of this state to $\mathcal{E}'_{\text{eff}}$ is therefore

$$(\mathcal{E}'_{\text{eff}})_{\text{baryons}} = h(\rho_B) + \rho_B\left(\frac{\mu}{2} + \lambda_-\right). \tag{A14}$$

(b) $\theta = \pi/2$. At $\theta = \pi/2$, the eigenvalues of H_6 can be calculated by brute force. It is simpler, however, to note that they will be the same as those of the operator $-g_A k A_z^{(3)} + \Delta$, which in the basis (A10) has the block-diagonal form

$$\begin{bmatrix} -b+\Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{3}+\Delta & \frac{-8}{\sqrt{18}}b & 0 & 0 & 0 \\ 0 & \frac{-8}{\sqrt{18}}b & -\frac{5}{3}b & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{3}+\Delta & \frac{-8b}{\sqrt{18}} & 0 \\ 0 & 0 & 0 & \frac{-8b}{\sqrt{18}} & \frac{5}{3}b & 0 \\ 0 & 0 & 0 & 0 & 0 & b+\Delta \end{bmatrix}, \tag{A15}$$

with $b = g_A k/2$. Again we expect $b \simeq m_\pi$ so that the lowest eigenvalue in (A15) is given by

$$\lambda_-^{(p,+)} = -b + \frac{\Delta}{2} - 2b\left(1 + \frac{\Delta}{6b} + \frac{\Delta^2}{16b^2}\right)^{1/2}. \tag{A16}$$

This eigenvalue can be approximated by

$$\lambda_-^{(p,+)} \simeq -3b + \frac{\Delta}{3} - \frac{\Delta^2}{18b} + \text{smaller terms} \tag{A17}$$

in the region of $b \sim m_\pi$ and $\Delta \sim 2m_\pi$.

APPENDIX B: OTHER ABNORMAL STATES IN THE σ MODEL

In the Introduction we alluded to the investigation of abnormal states in finite nuclei recently conducted by Lee and Wick. Since a part of their study included a discussion of the σ model, it seems appropriate to clarify the relation of their work to ours. This is perhaps most directly accomplished by reproducing, in our notation and with our approach, their specific results in the σ model.

The abnormal state considered by Lee and Wick is one in which the normal σ expectation value, $\langle\sigma\rangle \equiv A = f_\pi$ is significantly reduced, $A \simeq 0$, so that the nucleon mass, which in the tree approximation to the σ model is given by

$$M = gA\,,$$

approaches zero. We shall term this an "abnormal-σ" state. Although the pion degrees of freedom are not involved in this phase transition, both our parametrization [Eq. (3.15)] of the meson fields and our variational approach can be applied directly; we need simply take $k=\theta=0$ in (3.15) and vary the expression for $\mathcal{E}'_{\rm eff}$ with respect to A.

One aspect of the previous calculations which can *not* be immediately utilized is the explicit form of $\mathcal{E}'_{\rm eff}$. It is clear from both (3.41) and (3.45) that if we replace the nucleon mass by $g\langle\sigma\rangle = gA$, we can never find a phase transition with $A \gtrsim 0$, since the nonrelativistic nucleon kinetic energy term,

$$(\mathcal{E}'_{\rm eff})|_{\rm kinetic} = \frac{3}{5}\,\frac{(3\pi^2)^{2/3}}{2M}\,\rho_B^{\,5/3}\,, \tag{B1}$$

becomes infinite as $A \to 0$. Thus we must calculate the correct relativistic energy for the nucleons if we wish to study the region $A \simeq 0$.

We start from the effective Hamiltonian appropriate to infinite neutron matter and ignore the charge chemical potential since the σ's are also neutral. Thus

$$\mathcal{H}_{\rm eff} = -i\bar{N}\vec{\gamma}\cdot\vec{\nabla}N + gA\bar{N}N - \nu\bar{N}\gamma^0 N + \mathcal{E}_M\,, \tag{B2}$$

where the meson field energy density is given by

$$\mathcal{E}_M = \frac{\lambda^2}{4}A^4 - \frac{m_0^{\,2}}{2}A^2 - f_\pi m_\pi^{\,2}A\,. \tag{B3}$$

Note that for the purposes of comparison we have chosen the standard symmetry-breaking term in (B3). Further, recall that since we are working in infinite matter the gradient and momentum terms corresponding to the neutral σ will vanish. In this respect, our calculation is simpler than that in Ref. 6.

Writing the Hamiltonian in (B2) in momentum space and considering it as a Dirac Hamiltonian, we obtain in essence the free Dirac equation for a particle of mass gA in a constant external potential

$$\mathcal{H}_D N_i(\vec{p}) \equiv (\vec{\alpha}\cdot\vec{p} + \beta gA - \nu)N_i(\vec{p}) = E_i N_i(\vec{p})\,. \tag{B4}$$

Hence the energies of the "particle" states are

$$E_+ = [\vec{p}^{\,2} + (gA)^2]^{1/2} - \nu\,. \tag{B5}$$

Since all states with $E_+ < 0$ are filled in the ground state, we find the simple relation between ν and the baryon density

$$\rho_B = \frac{1}{\pi^2}\int_0^{p_F} p^2 dp = \frac{1}{3\pi^2}[\nu^2 - (gA)^2]^{3/2}\,, \tag{B6}$$

where $p_F = [\nu^2 - (gA)^2]^{1/2}$. The total energy in the Fermi sea is then

$$\mathcal{E}_{\rm eff}(\nu, A) = \frac{1}{\pi^2}\int_0^{p_F} p^2 dp\, E_+(p) = \frac{\nu}{12\pi^2}[\nu^2 - (gA)^2]^{1/2}[-\nu^2 + \tfrac{5}{2}(gA)^2] - \frac{(gA)^4}{8\pi^2}\ln\left\{\frac{\nu + [\nu^2 - (gA)^2]^{1/2}}{gA}\right\}. \tag{B7}$$

As in the text we find it simpler to work with

$$\mathcal{E}'_{\rm eff}(\rho_B, A) \equiv \mathcal{E}_{\rm eff}(\nu, A) + \nu\rho_B = \mathcal{E}_0(\rho_B, A)\,. \tag{B8}$$

In terms of $p_F = (3\pi^2\rho_B)^{1/3}$, $\mathcal{E}_0(\rho_B)$ takes on the familiar form[46]

$$\mathcal{E}_0(\rho_B, A)|_{\text{Fermi sea}} = \frac{1}{8\pi^2}p_F[p_F^{\,2} + (gA)^2]^{1/2}[2p_F^{\,2} + (gA)^2] - \frac{(gA)^4}{8\pi^2}\ln\left\{\frac{p_F + [p_F^{\,2} + (gA)^2]^{1/2}}{gA}\right\}. \tag{B9}$$

Hence the total "energy" density is given by

$$\mathcal{E}_0(\rho_B, A) = \mathcal{E}_0(\rho_B, A)|_{\text{Fermi sea}} + \frac{\lambda^2}{4}A^4 - \frac{m_0^{\,2}}{2}A^2 - f_\pi m_\pi^{\,2}A\,, \tag{B10}$$

and the actual ground state—at a given ρ_B—is determined by minimizing (B10) with respect to A. Clearly at $\rho_B = 0$ (B10) reduces to the normal equation determining A in the tree approximation,

$$\frac{\lambda^2}{4}A^4-\frac{m_0^2}{2}A^2-f_\pi m_\pi^2 A\,, \tag{B11}$$

with the solution

$$A=A_0=f_\pi=\left(\frac{m_0^2+m_\pi^2}{\lambda^2}\right)^{1/2}.$$

In inverse fermis, $A_0=0.5\ \mathrm{F}^{-1}$.

Since the form of (B9) renders a completely analytic treatment difficult, we have calculated $\mathcal{E}_0(\rho_B, A)$ numerically and plotted the results versus A at different baryon densities in Fig. 13. For purposes of computing these figures we have taken the parameters to be exactly those specified in the text: That is, $\lambda^2=50$, $m_0=4m_\pi$, $f_\pi=m_\pi/\sqrt{2}$, $g=M/A_0\simeq 10$.

By comparing Fig. 13(c) and 13(d) we observe that a sharp transition from the "normal" nuclear phase ($A\simeq A_0$) to the "abnormal-σ" state ($A\simeq 0$) occurs at a density, ρ_B^{crit}, bounded by $1.59\rho_B^{\mathrm{nucl}} < \rho_B^{\mathrm{crit}} < 1.84\rho_B^{\mathrm{nucl}}$.

This density is substantially less than the critical densities for the pion phase transitions found in pure neutron matter for the simple σ model. Why, then, have we ignored this abnormal phase

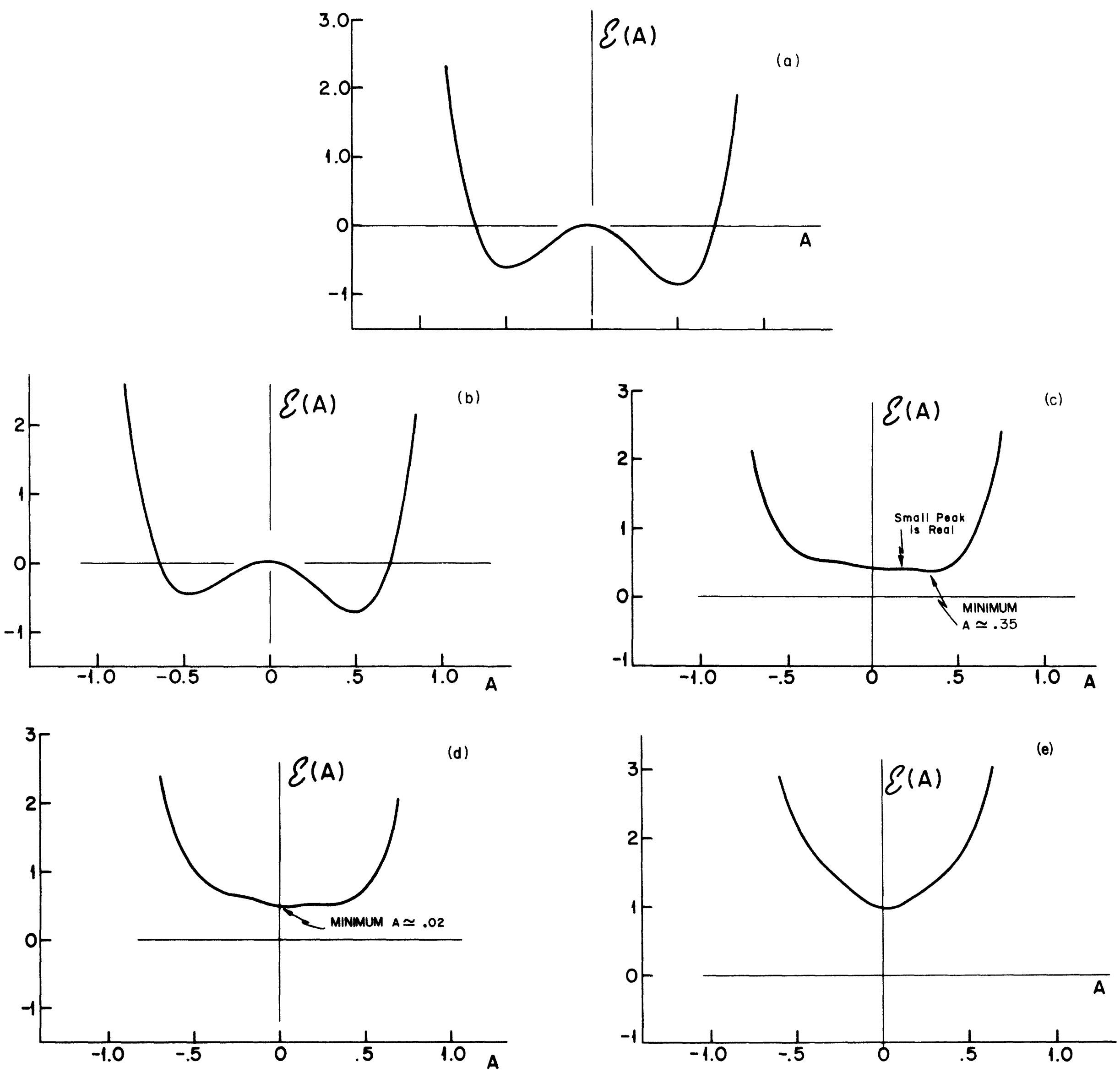

FIG. 13. The ground-state energy-density function $\mathcal{E}_0(\rho_B, A)$ plotted vs A. (a) $p_F=0$, $\rho_B=0$, (b) $p_F=1$, $\rho_B=0.2\rho_B^{\mathrm{nucl}}$, (c) $p_F=2$, $\rho_B=1.59\rho_B^{\mathrm{nucl}}$, (d) $p_F=2.1$, $\rho_B=1.84\rho_B^{\mathrm{nucl}}$, (e) $p_F=2.5$, $\rho_B=3.10\rho_B^{\mathrm{nucl}}$.

in our discussions and assumed that $A \simeq A_0$ for densities beyond $1.8\rho_B^{\text{nucl}}$? Should we not have allowed all the parameters—A, k, and θ—to vary in our search for the abnormal ground state?

Since the responses to these questions provide yet another illustration of the central theme of our study, let us present them in some detail. As we have consistently emphasized, it is the approximate chiral symmetry of the hadronic interactions which in our view motivates the study of the pion condensation. The existence of "pion-condensed" states very nearly degenerate with the "normal" vacuum is a consequence of chiral invariance. So, too, is the appearance of the vector and axial-vector currents in the rotated Hamiltonian (3.23).

To the extent that the σ model is a realization of chiral invariance—and to the extent that we study only quantities that depend solely or primarily on chiral invariance—its predictions reflect the greater generality of that model-independent symmetry. For this reason, we felt it justified to use the σ model as a means of introducing our approach to pion condensation. Furthermore, since the "abnormal-σ" state is not related to the normal state by an approximate chiral transformation—indeed, the invariant $(\sigma^2+\vec{\pi}^2)$ differs dramatically in the two states—it is difficult to know how generally the specific results of the σ model for this state can be believed. Thus, for example, how will the short-range nuclear forces—which presumably differ substantially in the two phases—affect the "abnormal-σ" phase transition? And how should one interpret the sensitivity of this transition to the basically arbitrary parameter, m_σ.[47] Finally it is worth noting that the critical density associated with the inclusion of N^* and obtained from chiral considerations is appreciably lower than the "abnormal-σ" critical density.

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[2]A. B. Migdal, Zh. Eksp. Teor. Fiz. 61, 2210 (1971) [Sov. Phys.—JETP 34, 1184 (1972)]; Phys. Rev. Lett. 31, 247 (1973); Phys. Lett. 45B, 448 (1973); Nucl. Phys. A210, 421 (1973); S. Barshay and G. Vagradov, Phys. Lett. 41B, 130 (1972); S. Barshay, V. Rostokin, and G. Vagradov, *ibid.* 43B, 271 (1973); S. Barshay, G. Vagradov, and G. E. Brown, *ibid.* 43B, 359 (1973). These authors approach the pion-condensation question by studying instabilities in the pion field as indicated by the pion self-energy in the medium.

[3]R. F. Sawyer, Phys. Rev. Lett. 29, 382 (1972); D. J. Scalapino, *ibid.* 29, 386 (1972); R. F. Saywer and D. J. Scalapino, Phys. Rev. D 7, 953 (1973); 8, 1260(E) (1973); R. F. Sawyer and A. C. Yao, *ibid.* 7, 1579 (1973). These authors study pion condensation by comparing explicitly the "ground states" of condensed and normal phases and determining the conditions for the condensed state to be favored energetically.

[4]J. Kogut and J. T. Manassah, Phys. Lett. 41A, 129 (1972); J. T. Manassah, *ibid.* 45A, 375 (1973).

[5]G. Baym, Phys. Rev. Lett. 30, 1340 (1973); G. Baym and E. Flowers, Nucl. Phys. A222, 29 (1974); C. K. Au and G. Baym, Nucl. Phys. A236, 500 (1974).

[6]T.-D. Lee and G. C. Wick, Phys. Rev. D 9, 2291 (1974).

[7]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974).

[8]We shall consistently ignore muons, which are also present in very small quantities. The extent to which one can think of matter at these high densities in terms of an independent-particle model is discussed in Ref. 1. [See also R. G. Palmer and P. W. Anderson, Phys. Rev. D 9, 3281 (1974).] Further, we shall consider explicitly only the case of infinite, approximately uniform, nuclear matter; complications arising from the actual structure of neutron stars as determined by gravitational forces will be ignored.

[9]For definiteness we shall concentrate on the possibility of π^- condensation. The possibility of π^0 condensation is treated in several of the articles listed in Refs. 2, 3, and 5.

[10]In this heuristic discussion we are being somewhat cavalier about the "net energy change" and "effective pion mass." In particular, we are using m_π^{eff} as an abbreviation for the total effect of pion interactions with the medium on the pion's energy.

[11]W. D. Langer, L. C. Rosen, J. M. Cohen, and A. G. W. Cameron, Astrophys. Space Sci. 5, 259 (1969).

[12]For a detailed pedagogical discussion of the σ model and references to the original papers on this subject, we refer readers to B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972).

[13]Many of the results in this section were earlier reported in R. F. Dashen and J. T. Manassah, Phys. Lett. 47A, 453 (1974); 50B, 460 (1974). These articles also contain a general model-independent discussion of pion condensation in the absence of nucleons.

[14]This form of symmetry breaking leads to canonical PCAC, $\partial_\mu A_i^\mu \propto \pi_i$ and to the Weinberg scattering-length predictions for $\pi\pi$ scattering.

[15]Although λ^2 and m_0^2 must differ slightly in the two cases—one has, for example, $f_\pi = [(m_0^2+m_\pi^2)/\lambda^2]^{1/2}$ for $i=1$, and $f_\pi = m_0/\lambda$ for $i=2$—the important point for future reference is that both λ and m_0 are "large", i.e., $\lambda \gg 1$, $m_0 \gg m_\pi$.

[16]D. G. Boulware and L. S. Brown, Phys. Rev. 172, 1628

(1968); see also S. Coleman, in *Secret Symmetry*, proceedings of the 1973 International Summer School of Physics "Ettore Majorana" (to be published).

[17]G. Baym (Ref. 5) has shown in general that if there exists a nonzero pion-condensed amplitude, it must vary in time according to the appropriate charge chemical potential: Thus $\langle\pi_{\mp}(x,t)\rangle = e^{\pm i\mu t}\langle\pi_{\mp}(x)\rangle$. This result follows directly from Hamilton's equation $\delta H/\delta p_{\pi_i} = \dot{\pi}_i$ and the minimization condition [Eq. (2.16)] of $H_{\rm eff}$ with respect to p_{π_i}.

[18]Since π_1 is a pseudoscalar (2.16) implies that normal parity is spontaneously broken in this new ground state. We shall treat this point in more detail in a later paper.

[19]To determine the structure of the energy levels it is of course sufficient to use the zero-temperature limit. But in addition, neutron-star matter is indeed very "cold," since on the scales of the masses, energies, and chemical potentials involved ($E \gtrsim 10$–100 MeV), the thermal energy scale is very small ($kT_0 \simeq 75$ keV) even though $T_0 \simeq 10^8\,^\circ$K is, by normal standards, very large.

[20]Of course the form of A as a function of μ can be important for determining certain properties of the condensed phase. But we see that the nature of the phase transition—whether, for example, it is first or second order—can be seen from the behavior of $\mathcal{E}$ in θ.

[21]J. Goldstone, Nuovo Cimento 19, 155 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962); P. W. Higgs, Phys. Lett. 12, 132 (1964); Phys. Rev. Lett. 13, 508 (1964); Phys. Rev. 145, 1156 (1966).

[22]The choice of signs for the chemical potential is pure convention and is made solely for later convenience.

[23]Strictly speaking, U is not a unitary operator acting in a Hilbert space: That is, in the infinite-volume limit we are considering, U acting on $|G^0\rangle$ produces a state which is orthogonal to all states in the Hilbert space containing $|G^0\rangle$. This is the reason that $\tilde{H}_{\rm eff}$ can have a different ground-state energy from $H_{\rm eff}$, although it would appear that they are related by a unitary transformation. For any *local* operator, $O(x)$, however, $\tilde{O}(x) \equiv U^{-1}O(x)U$ *is* well defined. In practice, we shall transform only local operators, not states, and thus no "infinite-volume" problems will arise. The references in the text to the transformation properties of states should be regarded as picturesque rather than precise.

[24]M. Gell-Mann and M. Lêvy, Nuovo Cimento 16, 53 (1960).

[25]This nucleon mass is the actual observed value to the extent that all nuclear forces are included in the Hamiltonian.

[26]Here we discuss only the theoretical aspects of the value of g_A. Later we treat the phenomenological side of this problem.

[27]M. L. Goldberger and S. B. Treiman, Phys. Rev. 110, 1178 (1958). For a modern exposition, see S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[28]That there must exist such a relation in the limit $\epsilon = 0$ can be seen by observing that three parameters in the Lagrangian—m_0, λ, and g—determine, when PCAC is used, four physical parameters—M_N, g_A, f_π, and $g_{\pi NN}$ (recall $m_\pi = 0$ when $\epsilon = 0$ in the Goldstone mode). Hence there must exist one relation among these physical parameters which expresses the underlying chiral invariance of the Lagrangian; this is Eq. (3.25).

[29]This remark does not imply that the inclusion of additional nuclear forces will not alter certain aspects of the phase transition but rather only that the same approach and interpretation apply in more realistic cases. It is, for example, physically clear that the critical density can be changed by including strong NN forces. More formally, one can see this in perturbation theory, where the states and energy levels of $H_{\rm eff}$ clearly influence the effect of the perturbation $\Delta H_{\rm eff}$ on $E'_{\rm eff}$. See Sec. V, R. Rajaraman, Phys. Lett. 48B, 179 (1974); W. Weise and G. Brown, *ibid.* 48B, 297 (1974).

[30]This is basically the approach used in Refs. 2–5. Thus we choose the equivalent alternative approach.

[31]Thus there is a discontinuity in the second derivative of $\mathcal{E}_{\rm eff}$ as a function of ρ_B at the critical density.

[32]This numerical result assumes $g_A = g_A^{\rm exp} = 1.24$.

[33]This assumes no pathology at $\theta = \pi$.

[34]We have deliberately avoided expressing ρ, μ, and k more accurately in order to emphasize the unreliability of estimates based on such a crude model.

[35]Many authors have emphasized this result and calculated, in a variety of models, the effects of this resonance.

[36]The arguments in favor of treating the N^* resonance as an independent particle under neutron-star conditions appear in R. F. Dashen and R. Rajaraman, Phys. Rev. D 10, 694 (1974); 10, 708 (1974); and earlier references quoted there.

[37]The higher-order correction terms to this approximate result—the first of which is shown in (A17)—can easily be seen to be small in the region of the phase transition.

[38]G. Chew, Phys. Rev. Lett. 9, 233 (1963); F. Low, *ibid.* 9, 279 (1963).

[39]Recent detailed investigations of the effects of nucleon-nucleon correlations are reported in G. Bertsch and M. Johnson, Phys. Rev. D (to be published), and S. O. Backmann and W. Weise, Phys. Lett. 55B, 1 (1975). An analytic calculation of these effects in the simple models discussed here is presented in G. Baym, D. Campbell, R. Dashen, and J. Manassah (unpublished).

[40]M. O. Ericson and T. E. O. Ericson, Ann. Phys. (N.Y.) 36, 323 (1966). See also the discussions of Refs. 2, 3, 5, 29, and 39 and G. Baym and G. E. Brown (unpublished).

[41]Actually, we should probably also separate off the long-range tail of two-pion exchange as well. For simplicity, we ignore this refinement.

[42]See, for example, A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw-Hill, New York, 1971).

[43]M. Gell-Mann and R. Feynman, Phys. Rev. 109, 193 (1958); E. C. G. Sudarshan and R. E. Marshak, *ibid.* 109, 1860 (1958).

[44]W. Thirring, in *Quantum Electrodynamics*, proceedings of the IV Schladming conference on nuclear physics, edited by P. Urban (Springer, Berlin, 1965) [Acta Phys. Austriaca, Suppl. 2 (1965)], pp. 205–211. This paper is perhaps more easily accessible in the form reprinted in J. J. Kokkedde, *The Quark Model* (Benjamin, New York, 1969), pp. 183–189. Readers unfamiliar with the manipulations that follow are strongly urged to consult this reference.

[45]This is the prescription given by the Melosh transformation between "current quarks and constituent quarks" [H. J. Melosh, Phys. Rev. D 9, 1095 (1974)]. This scheme has been very successful phenomenologically and it would be surprising if its prediction for the N^*N^* matrix element of the axial-vector current were seriously in error.

[46]This expression is consistent with that in Ref. 6 when one recalls that whereas we are considering pure neutron matter—for purposes of comparison with our earlier results—Lee and Wick treat symmetric nuclear matter. The difference is a factor of 2 in the statistical weight of a given momentum state.

[47]In Ref. 6, Lee and Wick argue that for $m_\sigma \simeq M$, $\rho_B^{\text{crit}} \propto \rho_B^{\text{nucl}}(m_\sigma/M)^2$. With $m_\sigma = 2$ GeV, our calculations show that $\rho_B^{\text{crit}} = 8.5\rho_B^{\text{nucl}}$.