General-relativistic quantum field theory: An exactly soluble model

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The massive scalar and Dirac fields quantized on a de Sitter background geometry prove to be exactly soluble models in general-relativistic field theory. The Feynman Green's function is computed for both the scalar and Dirac fields. A dimensional regularization procedure applied in coordinate space facilitates the calculation of their respective effective Lagrangians, which describe the vacuum corrections due to closed matter loops. The model is found to be renormalizable. There is no creation of real particle pairs.

I. INTRODUCTION

It is well known that the quantum corrections to a field theory may be summarized by adding to the classical action functional S an effective action W.[1,2] The functional $\Gamma = S + W$ contains all the information to be extracted from the theory. In general Γ may only be calculated in perturbation theory; however, in certain cases of high symmetry Γ may be calculated in closed form.

In this paper we show how this approach can be used to solve the problem of quantizing matter fields in a given curved space-time. We present exact solutions for Γ for both Klein-Gordon and Dirac fields coupled to a gravitational field of constant curvature (de Sitter space).[3]

Inevitably divergences arise which must be removed from the final answer in a coordinate-invariant manner. This is accomplished by applying the dimensional regularization procedure[4] directly in coordinate space.

In Sec. II it is shown how the effective Lagrangian and hence the effective action may be calculated from a knowledge of the Feynman Green's function. The effective Lagrangian is calculated in closed form in Sec. III and the result is compared with a perturbative treatment in Secs. IV and V. The following results were found: (i) The quadratic and quartic infinities from the perturbative treatment arise also in the exact solution and are absorbed by a renormalization of the gravitational and cosmological constants. (ii) The logarithmic infinity which cannot be absorbed by renormalization is ambiguous in the exact theory. A closer examination of the perturbation theory reveals that it too is really ambiguous; the logarithmic infinity occurs with a definite coefficient only if we require the effective Lagrangian to possess an expansion in integral powers of the curvature. In de Sitter space this ambiguous term is of no consequence since it does not contribute to the energy-momentum tensor. Hence this particular model is renormalizable. (iii) The perturbation series is an asymptotic series which is valid for small curvature (large mass), but does not converge. It therefore provides no information about the large-curvature behavior. (iv) Independently of the magnitude of the curvature, there is no particle production in the de Sitter model. (v) Dimensional regularization can be applied without recourse to Fourier analysis.

Similar results are found for the Dirac equation in Sec. VI. In Sec. VII we consider a special case of the problem of quantizing in a coordinate system covering only part of the space. This particular case is of additional interest in that it allows a comparison of x-space and p-space dimensional regularization.

II. THE GENERAL THEORY

For a scalar field ϕ with an action functional[5]

$$S[\phi] = -\tfrac{1}{2}\int dx\, g^{1/2}(\phi_{,\alpha}\phi^{,\alpha} + m^2\phi^2),$$

the vacuum-to-vacuum transition amplitude is given by the functional integral

$$\langle 0\,\text{out}|\,0\,\text{in}\rangle = N^{-1}\int d[\phi]\, e^{iS[\phi]},$$

where N is a normalization constant independent of the metric. The effective Lagrangian $\mathcal{L}_1(x)$ is defined by

$$\exp\left(i\int \mathcal{L}_1 dx\right) = \langle 0\,\text{out}|\,0\,\text{in}\rangle .$$

These last two expressions are properly defined only if there exist asymptotic regions.

By considering

$$\int d[\phi]\,\frac{\delta}{\delta\phi(x)}[\phi(x')e^{iS[\phi]}] = 0,$$

we see that the Feynman Green's function

$$G(x,x')=\frac{i\int d[\varphi]\phi(x)\phi(x')e^{iS[\phi]}}{\int d[\phi]e^{iS[\phi]}}$$

satisfies the equation

$$g^{1/2}(\Box-m^2)G(x,x')=-\delta(x,x'). \quad (1)$$

From the definition of $\mathfrak{L}_1$ we see that

$$\int dx\frac{\partial\mathfrak{L}_1}{\partial m^2}=\frac{i}{2}\int dx\, g^{1/2}G(x,x)+\frac{i}{N}\frac{\partial N}{\partial m^2}.$$

The second term on the right-hand side of this equation is independent of the metric, so let us discard it; then removing the x integration yields the relation

$$\frac{\partial\mathfrak{L}_1}{\partial m^2}=\frac{i}{2}g^{1/2}G(x,x). \quad (2)$$

(2) is meaningful even in a space that is not asymptotically flat. We shall assume that (2) holds in general, and that $\mathrm{Im}\mathfrak{L}_1$ may be interpreted as the rate of particle production from the vacuum. It should be noted that these assumptions are implicit in Schwinger's calculation.[1]

$G(x,x)$ is a highly singular object in four dimensions, but we may regularize it by working in n dimensions (n complex) and analytically continuing $G(x,x)$ from a region of the n plane in which it is regular. The infinities of the theory then appear as poles at $n=4$. Equation (1) does not define $G(x,x)$ uniquely since there remains the freedom to add to G any solution of the homogeneous equation. We shall complete the specification of G by appealing to generalizations of the analytic properties that G enjoys in flat space, namely that G is analytic in the lower half m^2 plane and in the upper half $(x-x')^2$ plane.

In a general field theory (2) would give $\mathfrak{L}_1$ correctly to $O(\hbar)$ in an expansion in powers of Planck's constant. In a linear field theory there are no further terms and (2) is exact.

III. THE GREEN'S FUNCTION IN THE DE SITTER MODEL

de Sitter space[6] is a maximally symmetric space of constant (positive) curvature. It may be realized in n dimensions as the hyperboloid of revolution

$$\frac{1}{K}=-\xi_0^{\,2}+\xi_1^{\,2}+\cdots+\xi_n^{\,2}$$

in a Minkowski space of $(n+1)$ dimensions with metric

$$ds^2=-d\xi_0^{\,2}+d\xi_1^{\,2}+\cdots+d\xi_n^{\,2}.$$

The Riemann tensor in de Sitter space can be simply expressed in terms of the metric as

$$R_{\alpha\beta\gamma\delta}=K(g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\delta}).$$

Using ξ^μ ($\mu=0,1,\ldots,n-1$) as coordinates on the space, the metric takes the form

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{K\xi_\mu\xi_\nu}{1-K\xi^2}\text{ for }K\xi^2<1,$$

where $\eta_{\mu\nu}=\mathrm{diag}(-1,1,\ldots,1)$, $\xi_\mu=\eta_{\mu\nu}\xi^\nu$, and $\xi^2=\eta_{\mu\nu}\xi^\mu\xi^\nu$. We note in passing, however, that these coordinates cover only half the space; we shall return to this point in connection with the Fulling phenomenon in Sec. VII. An alternative coordinate system may be introduced in which the metric takes the form

$$ds^2=\frac{1}{K}\left(-d\theta^2+\cosh^2\theta\{d\chi_1^{\,2}+\sin^2\chi_1[d\chi_2^{\,2}+\sin^2\chi_2(d\chi_3^{\,2}+\cdots)]\}\right). \quad (3)$$

The complete hyperboloid is covered by the coordinate range

$$-\infty<\theta<\infty,$$

$$0\leq\chi_1,\ldots,\chi_{n-1}\leq\pi,\quad 0\leq\chi_n\leq 2\pi.$$

Let us denote points of the hyperboloid by x, x', etc. and the vectors corresponding to them in the embedding space by ξ, ξ', etc. Thus, we may define

$$\sigma(x,x')=\tfrac{1}{2}(\xi-\xi')^2.$$

Since de Sitter space is maximally symmetric, $G(x,x')$ depends on x and x' only through σ. The biscalar σ is defined as above rather than as the world function, say, since there are points in the space that cannot be joined by any geodesic. It is easily verified that the derivatives of σ satisfy the identities

$$\sigma^{;\alpha}\sigma_{;\alpha}=\sigma(2-K\sigma)$$

and (4)

$$\sigma_{;\alpha\beta}=g_{\alpha\beta}(1-K\sigma).$$

In view of these identities we have

$$(\Box-m^2)F(\sigma)=-\left[\sigma(K\sigma-2)\frac{d^2}{d\sigma^2}+n(K\sigma-1)\frac{d}{d\sigma}+m^2\right]F(\sigma) \tag{5}$$

for an arbitrary function F.

In order to find a Green's function, we need a representation of $\delta(x,x')$ in n dimensions on the de Sitter hyperboloid. In flat space we have

$$\delta(x,x')=\int\frac{d^np}{(2\pi)^n}e^{ip(x-x')-i\epsilon p^2},$$

with the limit $\epsilon\to 0$ understood. Performing the integration yields

$$\delta(x,x')=\frac{ie^{-in\pi/4}}{(4\pi\epsilon)^{n/2}}e^{i(x-x')^2/4\epsilon}.$$

Now a representation of $\delta(x,x')$ in the tangent space to the hyperboloid is also a representation in the hyperboloid. A suitable representation is therefore

$$\delta(x,x')=ie^{-in\pi/4}\frac{e^{i\sigma/2\epsilon}}{(4\pi\epsilon)^{n/2}}g^{1/2}. \tag{6}$$

In fact (6) has been chosen with a certain prescience. In order to complete the specification of the Feynman Green's function we shall require G to be analytic in the upper half σ plane and in the lower half m^2 plane. In view of the structure of (6) and the above remarks let us seek a solution to (1) of the form

$$G(x,x')=-\frac{e^{-in\pi/4}}{(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{n/2}}f(s)e^{i\sigma/2s}, \tag{7}$$

where $f(s)$ is to be chosen. This incorporates the required analyticity property with respect to σ. From (5) it is seen that this will be a solution provided

$$\frac{e^{i\sigma/2\epsilon}}{\epsilon^{n/2}}=i\int_0^\infty\frac{ds}{s^{n/2}}f(s)\mathfrak{M}e^{i\sigma/2s}, \tag{8}$$

where $\mathfrak{M}$ is the operator defined by

$$\mathfrak{M}=Ks^2\frac{d^2}{ds^2}-[(n-2)Ks-i]\frac{d}{ds}+\left(m^2-\frac{in}{2s}\right).$$

The fact that $\mathfrak{M}$ is a second-order operator is a distinctive feature of gravitation; in flat-space field theories the analogous operator is usually of first order.[7]

Integrating (8) by parts yields

$$\frac{e^{i\sigma/2\epsilon}}{\epsilon^{n/2}}=i\int_0^\infty ds\,e^{i\sigma/2s}\mathfrak{M}^\dagger(f(s)s^{-n/2})+i\Delta,$$

where $\mathfrak{M}^\dagger$ is the adjoint of $\mathfrak{M}$ and Δ is given by

$$\Delta=\left\{\frac{-iK\sigma}{2}fs^{-n/2}e^{i\sigma/2s}-Ke^{i\sigma/2s}\frac{d}{ds}(fs^{2-n/2})-[(n-2)Ks-i]fs^{-n/2}e^{i\sigma/2s}\right\}_0^\infty.$$

f is required to satisfy

$$\mathfrak{M}^\dagger(fs^{-n/2})=0,$$

i.e.,

$$Ks^2f''+(2Ks-i)f'+\left[m^2-\frac{n}{2}\left(\frac{n}{2}-1\right)K\right]f=0, \tag{9}$$

subject to the boundary condition $f(0)=1$. The point $s=0$ is an irregular singular point of (9), so the single boundary condition determines the solution uniquely as a hypergeometric function[8]:

$$f(s)={}_2F_0(\tfrac{1}{2}+i\alpha,\tfrac{1}{2}-i\alpha;-iKs)$$
$$=(iKs)^{-1/2-i\alpha}\Psi\left(\tfrac{1}{2}+i\alpha,1+2i\alpha;\frac{1}{iKs}\right), \tag{10}$$

with

$$\alpha^2=\frac{m^2}{K}-\frac{(n-1)^2}{4}.$$

In the first instance let us take n real. Then since the asymptotic solutions to (9) are $s^{-1/2\pm i\alpha}$, we see that for $n>1$ the integral in (7) converges at infinity and

$$\Delta=\frac{-ie^{i\sigma/2\epsilon}}{(4\pi\epsilon)^{n/2}}.$$

Substituting (10) into (7) and performing the integration yields (see Appendix A)

$$G(x,x')=\frac{iK^{n/2-1}}{(4\pi)^{n/2}}\frac{\Gamma(\frac{1}{2}(n-1)+i\alpha)\Gamma(\frac{1}{2}(n-1)-i\alpha)}{\Gamma(\frac{1}{2}n)}\times{}_2F_1\left(\frac{n-1}{2}+i\alpha,\frac{n-1}{2}-i\alpha;\frac{n}{2};1-\frac{K}{2}(\sigma+i\epsilon)\right), \tag{11}$$

where the $i\epsilon$ in the argument of ${}_2F_1$ denotes the side of the cut on which the hypergeometric function is to be evaluated for $\sigma < 0$. There can exist only two linearly independent solutions of (1) that depend on x and x' only through $\sigma(x, x')$. These may be taken to be (11) and its complex conjugate. From this observation it follows that (11) is the unique maximally symmetric solution to (1) that is analytic in the upper half σ plane. [The properties of $G(x, x')$ are examined further in Appendix B.] For $n < 2$ the limit $\sigma \to 0$ in (11) is finite, and we obtain

$$G(x, x) = \frac{iK^{n/2-1}}{(4\pi)^{n/2}} \frac{\Gamma(\frac{1}{2}(n-1)+i\alpha)\Gamma(\frac{1}{2}(n-1)-i\alpha)}{\Gamma(\frac{1}{2}+i\alpha)\Gamma(\frac{1}{2}-i\alpha)} \times \Gamma(1-\tfrac{1}{2}n). \tag{12}$$

$G(x, x)$ is analytic in n apart from simple poles. It may, therefore, be extended throughout the complex n plane. Expanding (12) about $n = 4$ yields

$$G(x, x) = \frac{i(m^2 - 2K)}{8\pi^2(n-4)} - \frac{3iK}{16\pi^2} + \frac{i(m^2-2K)}{16\pi^2}\left[\ln\frac{K}{4\pi} + \psi(\tfrac{3}{2}+i\alpha) + \psi(\tfrac{3}{2}-i\alpha) + \gamma - 1\right] + O(n-4), \tag{13}$$

where $\psi(z) = (d/dz)\ln\Gamma(z)$ and $\gamma = -\psi(1)$ is Euler's constant.

IV. THE PERTURBATION EXPANSION

To evaluate $G(x, x')$ perturbatively we seek a solution of (9) having the asymptotic form

$$f(s) \sim e^{-im^2 s} \sum_{r=0}^{\infty} \left(\frac{K}{m^2}\right)^r f_r(im^2 s), \tag{14}$$

with $f_0(0) = 1$ and $f_r(0) = 0$, for $r > 0$. This series can be only an asymptotic series since by (10) $f(s)$ has a branch point at $Ks = 0$.

Substituting (14) in (7) we obtain

$$G(x, x') = \frac{-e^{-in\pi/4}}{(4\pi)^{n/2}} \times \sum_{r=0}^{\infty}\left(\frac{K}{m^2}\right)^r \int_0^{\infty} \frac{ds}{s^{n/2}} e^{-im^2 s} e^{i\sigma/2s} f_r(im^2 s).$$

In order to avoid problems of convergence as $s \to \infty$ we take m^2 to have an infinitesimal imaginary part. This expansion is essentially a special case of the perturbation theory obtained by DeWitt for a general background metric.[2]

It is convenient to rotate the contour of integration to the negative imaginary axis. Defining $u = -im^2 s$ and $\omega = \frac{1}{2}(n-4)$ we have

$$G(x, x) \sim \frac{i}{(4\pi)^{2+\omega}} \sum_{r=0}^{\infty} K^r (m^2)^{1-r+\omega} I_r(\omega), \tag{15}$$

where

$$I_r(\omega) = \int_0^{\infty} \frac{du}{u^{2+\omega}} e^{-u} f_r(u). \tag{16}$$

From (9) we see that the $f_r(u)$ satisfy the recurrence relation

$$\frac{d}{du} f_r(u) = -u^2 \frac{d^2}{du^2} f_{r-1}(u) + (2u^2 - 2u)\frac{d}{du} f_{r-1}(u) + [(2+\omega)(1+\omega) + 2u - u^2] f_{r-1}(u). \tag{17}$$

With the stated initial conditions (14), the first three functions f_r are

$$f_0(u) = 1,$$
$$f_1(u) = -\tfrac{1}{3}u^3 + u^2 + (2+\omega)(1+\omega)u,$$
$$f_2(u) = \tfrac{1}{18}u^6 - \tfrac{11}{15}u^5 + [\tfrac{5}{2} + \tfrac{1}{3}(2+\omega)(1+\omega)]u^4 + [-2 + \tfrac{5}{3}(2+\omega)(1+\omega)]u^3 + \tfrac{1}{2}(3+\omega)(2+\omega)(1+\omega)\omega u^2.$$

Equation (16) defines $I_r(\omega)$ for $\mathrm{Re}\,\omega < -1$. For ω outside this region, $I_r(\omega)$ is defined by analytic continuation. $I_0(\omega)$ and $I_1(\omega)$ are found to have poles at $\omega = 0$ ($n = 4$), but $I_2(\omega)$ is regular as $\omega \to 0$. From the form of (17) it follows that $I_r(0)$ is finite for $r \geq 2$. The Laurent expansions of I_0, I_1, and I_2 about $\omega = 0$ are

$$I_0(\omega) = \frac{1}{\omega} - (1-\gamma) + O(\omega),$$
$$I_1(\omega) = -\frac{2}{\omega} - (\tfrac{7}{3} + 2\gamma) + O(\omega),$$
$$I_2(\omega) = \tfrac{29}{15} + (\tfrac{401}{90} + \tfrac{29}{15}\gamma)\omega + O(\omega^2).$$

Substituting in (15) and comparing the result with the exact answer (13), we see that the perturbation expansion yields the pole terms exactly.

V. RENORMALIZATION

Having calculated the Green's function, we find the effective Lagrangian from (2). Consider first the asymptotic form of $\mathcal{L}_1$ for small K,

$$\mathcal{L}_1(x) \sim -\tfrac{1}{2} g^{1/2} \frac{1}{(4\pi)^{2+\omega}} \sum_{r=0}^{\infty} \frac{K^r (m^2)^{2-r+\omega}}{2-r+\omega} I_r(\omega). \tag{18}$$

The first two terms of this series are singular as $\omega \to 0$ since $I_0(\omega)$ and $I_1(\omega)$ have poles at $\omega = 0$. The third term is also singular since the denominator vanishes for $\omega = 0$. For the moment let us restrict our attention to the first two terms, which correspond precisely to pole terms in the exact theory.

The Einstein Lagrangian is

$$\mathcal{L}_0 = -\frac{1}{16\pi G} g^{1/2} R - \lambda g^{1/2},$$

where R, the Ricci scalar, is related to the curvature K of de Sitter space by

$$R = -n(n-1)K.$$

G is the Newtonian constant and λ is the cosmological constant. The Lagrangian corrected for the effects of the quantum matter field is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1.$$

The quartic and quadratic divergences may be absorbed by renormalization of λ and G. To accomplish this, define renormalized constants by

$$\begin{aligned} \lambda_R &= -(g^{-1/2}\mathcal{L})|_{K=0} \\ &= \lambda + \frac{1}{2}\frac{(m^2)^{2+\omega}}{(4\pi)^{2+\omega}}\frac{I_0(\omega)}{2+\omega}, \\ \frac{n(n-1)}{16\pi G_R} &= \frac{\partial}{\partial K}(g^{-1/2}\mathcal{L})\Big|_{K=0} \\ &= \frac{n(n-1)}{16\pi G} - \frac{1}{2}\frac{(m^2)^{1+\omega}}{(4\pi)^{2+\omega}}\frac{I_1(\omega)}{1+\omega}. \end{aligned} \tag{19}$$

The third term in (18) appears to be nonrenormalizable. However, if we turn to the exact theory, using (12) and (2), we find that $\mathcal{L}_1$ has a pole at $\omega = 0$ with residue

$$-\frac{1}{2(4\pi)^2}\int dm^2 (m^2 - 2K).$$

Here, therefore, the K^2 pole arises as an arbitrary "constant" of integration.

The discrepancy is resolved if we do not insist that the perturbation series be an expansion in integer powers of K. In that case, on performing the m^2 integration we are free to add a "constant" of integration which, on dimensional grounds, must have the form

$$c(\omega)K^{2+\omega}g^{1/2}, \tag{20}$$

with $c(\omega)$ an arbitrary function of ω only. In terms of the pseudospherical coordinates (3) can be written

$$g_{\mu\nu} = \frac{1}{K}\gamma_{\mu\nu},$$

with $\gamma_{\mu\nu}$ being the metric tensor on the unit hyperboloid. Consequently the contribution of the arbitrary term (20) to the effective action is

$$c(\omega)\int \gamma^{1/2} d\theta\, d\chi_1 \cdots d\chi_n,$$

which is a pure number independent of K and m, and so may be discarded. To illustrate the point we may calculate the effective energy-momentum tensor $T^1_{\mu\nu}$. Since $T^1_{\mu\nu}$ must be maximally symmetric, $T^1_{\mu\nu} = (1/n)g_{\mu\nu}T^1$.

Employing the metric (3) we have

$$\frac{\partial g^{\mu\nu}}{\partial K} = \frac{1}{K}g^{\mu\nu}$$

and therefore

$$\begin{aligned} \frac{\partial \mathcal{L}_1}{\partial K} &= \frac{\partial \mathcal{L}_1}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial K} \\ &= \frac{1}{2K}g^{1/2}T^1_{\mu\nu}g^{\mu\nu} \\ &= \frac{1}{2K}g^{1/2}T^1. \end{aligned}$$

This enables us to calculate T^1 from the explicit form of $\mathcal{L}_1$. It is clear that a term of the form (20) will make no contribution.

On the other hand, $c(\omega)$ can be chosen to cancel the K^2 pole term in (18). If $c(\omega) = -1/\omega$, we have

$$\frac{1}{\omega}K^2(m^2)^\omega - \frac{1}{\omega}K^{2+\omega} \to K^2 \ln\frac{m^2}{K} \quad \text{as } \omega \to 0.$$

In the exact theory, from (12) and (2) we have

$$\begin{aligned} g^{-1/2}\mathcal{L}_1 &= \frac{-K^{1+\omega}}{2(4\pi)^{2+\omega}}\Gamma(-1-\omega) \\ &\quad \times \int dm^2 \frac{\Gamma(\frac{3}{2}+i\alpha+\omega)\Gamma(\frac{3}{2}-i\alpha+\omega)}{\Gamma(\frac{1}{2}+i\alpha)\Gamma(\frac{1}{2}-i\alpha)}. \end{aligned}$$

Using (19), we obtain

$$\begin{aligned} g^{-1/2}\mathcal{L} &= \frac{n(n-1)K}{16\pi G_R} - \lambda_R \\ &\quad - \frac{1}{2(4\pi)^{2+\omega}}\left[\Gamma(-1-\omega)K^{1+\omega}\int dm^2 \frac{\Gamma(\frac{3}{2}+i\alpha+\omega)\Gamma(\frac{3}{2}-i\alpha+\omega)}{\Gamma(\frac{1}{2}+i\alpha)\Gamma(\frac{1}{2}-i\alpha)} - \frac{(m^2)^{2+\omega}}{2+\omega}I_0(\omega) - \frac{K(m^2)^{1+\omega}}{1+\omega}I_1(\omega)\right]. \end{aligned}$$

Choosing the constant of integration to eliminate the pole at $\omega=0$ in the term in brackets, and performing the limit $\omega \to 0$, we find

$$g^{-1/2}\mathfrak{L} = \frac{3K}{4\pi G_R} - \lambda_R$$
$$-\frac{1}{32\pi^2}\left\{(\tfrac{1}{2}m^4 - 2Km^2)\ln\frac{K}{m^2} + \tfrac{1}{4}m^4 - \tfrac{2}{3}Km^2 + \int dm^2(m^2-2K)[\psi(\tfrac{3}{2}+i\alpha)+\psi(\tfrac{3}{2}-i\alpha)]\right\}. \quad (21)$$

In this case therefore, the theory is renormalizable.

VI. THE DIRAC EQUATION

The action functional for a real (neutral) spinor field may be taken to be

$$S[\psi] = \frac{i}{2}\int g^{1/2}\bar{\psi}(\gamma^\mu\psi_{;\mu} + m\psi)dx.$$

Defining the Feynman Green's function for the spinor field by

$$G(x,x') = \frac{i\int d[\psi]\psi(x)\psi^{\sim}(x')e^{iS[\psi]}}{\int d[\psi]e^{iS[\psi]}},$$

we find that the equations analogous to (1) and (2) are

$$ig^{1/2}\gamma[\gamma^\mu G_{;\mu}(x,x') + mG(x,x')] = -\delta(x,x') \quad (22)$$

and

$$\frac{\partial\mathfrak{L}_1}{\partial m} = -\tfrac{1}{2}g^{1/2}\,\mathrm{tr}[\gamma G(x,x)]. \quad (23)$$

The symbol tr refers to a trace taken over spinor indices. $G(x,x')$ is a bispinor, that is, it transforms like the product $\psi(x)\psi^{\sim}(x')$. In view of this, let us seek to solve (22) by the ansatz

$$G(x,x') = H(x,x')\Phi(x,x')\gamma^{-1}(x'),$$

with $H(x,x')$ transforming like the product $\psi(x)\psi^{\sim}(x)$ and $\Phi(x,x')$ being a bispinor subject to the condition

$$\Phi(x,x) = \text{unit matrix}.$$

In view of the identities (4) satisfied by the derivatives of σ, H can only depend on x and x' through σ and $\sigma_{;\mu}$. Therefore it must have the structure

$$H = A(\sigma) + B(\sigma)\sigma_{;\alpha}\gamma^\alpha.$$

In order to solve (22) we shall also require a knowledge of the derivatives of Φ, so let us define $Z_\mu(x,x')$ by

$$\Phi_{;\mu}(xx') = Z_\mu(x,x')\Phi(x,x').$$

Z_μ is constrained by the integrability condition

$$\Phi_{;\mu\nu} - \Phi_{;\nu\mu} = \tfrac{1}{4}R_{\mu\nu\alpha\beta}\gamma^\alpha\gamma^\beta\Phi = -\frac{K}{4}[\gamma_\mu,\gamma_\nu]\Phi$$

to satisfy

$$Z_{\mu;\nu} - Z_{\nu;\mu} + [Z_\mu, Z_\nu] = -\frac{K}{4}[\gamma_\mu,\gamma_\nu]. \quad (24)$$

The simplest solution of (24) is

$$Z_\mu = \frac{i\sqrt{K}}{2}\gamma_\mu;$$

with this choice (22) becomes an equation for A and B which may be cast in the form

$$\sigma_{;\alpha}\gamma^\alpha\left[A' + \left(m - \frac{(n-2)}{2}i\sqrt{K}\right)B\right] + \sigma_{;\alpha}\sigma^{;\alpha}B' + \sigma_{;\alpha}{}^{\alpha}B + \left(m + \frac{in\sqrt{K}}{2}\right)A = ig^{-1/2}\delta(x,x').$$

Using (6) this equation is seen to be equivalent to the pair

$$K\frac{dA}{dz} + [2m - i(n-2)\sqrt{K}]B = 0,$$

$$\left[z(z-1)\frac{d^2}{dz^2} + n(z-\tfrac{1}{2})\frac{d}{dz} + \left(\frac{n}{2} - \frac{im}{\sqrt{K}}\right)\left(\frac{n-2}{2} + \frac{im}{\sqrt{K}}\right)\right]A = -\left(m - i\frac{(n-2)}{2}\sqrt{K}\right)e^{-in\pi/4}\frac{e^{iz/K\epsilon}}{(4\pi\epsilon)^{n/2}} \quad (25)$$

(where we have set $z = K\sigma/2$). This equation is similar to the equation satisfied by the Green's function of Sec. III. A calculation precisely analogous to that of Sec. III establishes a more general result that also covers (25); namely, that the solution of

$$z(z-1)\frac{d^2w}{dz^2}+[(a+b+1)z+(c-a-b-1)]\frac{dw}{dz}+abw=e^{(i\pi/2)(c-a-b-1)}\,\epsilon^{c-a-b-1}e^{iz/\epsilon}, \tag{26}$$

for $\mathrm{Re}a, \mathrm{Re}b, \mathrm{Re}(c-a-b-1)>0$, that is analytic in the region $\mathrm{Im}z \geq 0$ except, perhaps, for $z=0$ is

$$w=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;1-(z+i\epsilon)). \tag{27}$$

Comparing (25) with (26) and (27) we see that

$$A(\sigma)=-\left[m-i\frac{(n-2)}{2}\sqrt{K}\right]\frac{K^{n/2-1}}{(4\pi)^{n/2}}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\times{}_2F_1\left(a,b;c;1-\frac{K\sigma}{2}\right), \tag{28}$$

with $a=\frac{1}{2}(n-2)+im/\sqrt{K}$, $b=\frac{1}{2}n-im/\sqrt{K}$, and $c=\frac{1}{2}n$.

Now since $\mathrm{tr}\,\gamma^\alpha=0$, we have

$$\mathrm{tr}[\gamma G(x,x)]=nA(0),$$

which is finite for $n<2$. Thus by (23) and (28)

$$g^{-1/2}\frac{\partial\mathcal{L}_1}{\partial m}=(2+\omega)\frac{mK^{1+\omega}}{(4\pi)^{2+\omega}}\frac{\Gamma(2+\omega+i\beta)\Gamma(2+\omega-i\beta)}{\Gamma(1+i\beta)\Gamma(1-i\beta)}\times\Gamma(-1-\omega), \tag{29}$$

with $\beta^2=m^2/K$. Expanding (29) asymptotically about $K=0$ and integrating we find

$$g^{-1/2}\mathcal{L}_1=\frac{\Gamma(-1-\omega)(m^2)^{2+\omega}}{2(4\pi)^{2+\omega}}+K\frac{\Gamma(-1-\omega)(m^2)^{1+\omega}}{(4\pi)^{2+\omega}}\frac{(2+\omega)^2(3+2\omega)}{12}+O(K^2).$$

As in Sec. V we take

$$\mathcal{L}=g^{1/2}\frac{n(n-1)}{16\pi G}K-g^{1/2}\lambda+\mathcal{L}_1$$

and define renormalized constants by

$$\lambda_R=-(g^{1/2}\mathcal{L})|_{K=0}=\lambda-\frac{\Gamma(-1-\omega)(m^2)^{2+\omega}}{2(4\pi)^{2+\omega}},$$

$$\frac{n(n-1)}{16\pi G_R}=\frac{\partial}{\partial K}(g^{-1/2}\mathcal{L})\bigg|_{K=0}=\frac{n(n-1)}{16\pi G}+\frac{\Gamma(-1-\omega)(m^2)^{1+\omega}}{(4\pi)^{2+\omega}}\frac{(2+\omega)^2(3+2\omega)}{12}.$$

In terms of the renormalized constants we obtain a finite Lagrangian as $\omega\rightarrow 0$,

$$g^{-1/2}\mathcal{L}=\frac{3K}{16\pi G_R}-\lambda_R+\frac{1}{(4\pi)^2}\left(\frac{m^4}{4}-\frac{7}{6}Km^2\right)+\frac{1}{(4\pi)^2}\left(\frac{m^4}{2}+Km^2\right)\ln\frac{K}{m^2}+\frac{1}{(4\pi)^2}\int dm^2(m^2+K)\left[\psi\left(2+\frac{im}{\sqrt{K}}\right)+\psi\left(2-\frac{im}{\sqrt{K}}\right)\right]. \tag{30}$$

Apart from numerical factors this is of the same form as (21).

VII. THE FULLING PHENOMENON

Given an arbitrary space-time manifold it is not possible, in general, to find a coordinate system that will cover it entirely. With this in mind, we note that a slight relaxation of the analyticity conditions imposed on G may result in the occurrence of the "Fulling phenomenon."[9] That is, a solution of (1) obtained in a coordinate system that does not cover the whole space differs from the solution that would be obtained in a coordinate system that does. To see how this happens let us seek to solve (1) in the coordinates of Sec. III.

Without loss of generality we may take $\xi'_\mu=0$

($\mu = 0, 1, \ldots, n-1$); then, in an obvious notation, (1) becomes

$$(\partial^2 - K\xi^\alpha \xi^\beta \partial_{\alpha\beta} - nK\xi^\alpha \partial_\alpha - m^2)G(\xi) = -\delta(\xi). \quad (31)$$

Maximal symmetry requires $G(\xi)$ to be a function of ξ^2 (which is a function of σ). The purpose of the analyticity condition is to define the way in which analytic continuation is to be effected around the singularity $\sigma = 0$. Since $2\sigma \sim \xi^2$ for σ small, we might (mistakenly) assume that analyticity in the upper half σ plane is equivalent to analyticity in the upper half ξ^2 plane. Writing (31) in terms of the variable $z = K\xi^2$ and using a representation for $\delta(\xi)$ analogous to (6), we find

$$\left\{ z(z-1)\frac{d^2}{dz^2} + \left[\frac{(n+1)}{2} z - \frac{n}{2}\right]\frac{d}{dz} + \frac{m^2}{4K}\right\} G(z) = iK^{n/2-1}\frac{e^{-in\pi/4}}{(4\pi\epsilon)^{n/2}} e^{iz/\epsilon}.$$

Comparison with Eq. (26) reveals the solution as

$$G(\xi) = \frac{iK^{n/2-1}}{(4\pi)^{n/2}} \frac{\Gamma(a)\Gamma(b)}{\Gamma(\frac{1}{2})} {}_2F_1(a, b; \tfrac{1}{2}; 1 - K\xi^2), \quad (32)$$

with $a = \frac{1}{4}(n-1) + \frac{1}{2}i\alpha$, $b = \frac{1}{4}(n-1) - \frac{1}{2}i\alpha$.

By virtue of the easily established relation

$$\xi^2 = \sigma(2 - K\sigma) \quad (33)$$

and the formulas (valid for general a and b)

$$\Gamma(a)\Gamma(a+\tfrac{1}{2}) = \sqrt{\pi}\, 2^{1-2a}\Gamma(2a)$$

and

$${}_2F_1(a, b; \tfrac{1}{2}; \zeta^2) = \frac{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{2\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})} \times \left[{}_2F_1(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}(1+\zeta)) + {}_2F_1(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}(1-\zeta))\right],$$

we may recast (32) in the form

$$G(\xi) = \frac{iK^{n/2-1}}{(4\pi)^{n/2}} \frac{\Gamma(\frac{1}{2}(n-1)+i\alpha)\Gamma(\frac{1}{2}(n-1)-i\alpha)}{\Gamma(\frac{1}{2}n)} \times \left[{}_2F_1\left(\frac{n-1}{2}+i\alpha, \frac{n-1}{2}-i\alpha; \frac{n}{2}; 1-\frac{K\sigma}{2}\right) + {}_2F_1\left(\frac{n-1}{2}+i\alpha, \frac{n-1}{2}-i\alpha; \frac{n}{2}; \frac{K\sigma}{2}\right)\right].$$

The first term is identical with (11) and so represents the response of the field to a "charge" at the point $\xi = 0$. The second term corresponds to a "charge" located at the point antipodal to $\xi = 0$. In view of (33) we see that the occurrence of this parasitic term is due to the fact that while ξ^2 is only zero in the light cone of the origin, its "covariantization" $\sigma(2 - K\sigma)$ has another zero outside the coordinate patch when $\sigma = 2/K$; that is, on the light cone of the point antipodal to the origin. Thus the δ convergent sequence that we have employed represents in reality two "charges."

We remark finally that (31) may also be solved by Fourier transformation, and the integrals evaluated using conventional p-space dimensional regularization. The solution obtained is identical to the one presented above in which the regularization has been performed directly in coordinate space.

VIII. DISCUSSION

Since $\mathcal{L}$ in (21) and (30) is purely real, there is no particle creation in de Sitter space. This is surprising if we recall that pair creation in de Sitter space is the gravitational analog of Schwinger's calculation[1] of charged particle production in a constant electric field. One might expect pair creation to be important when $K \sim m^2$, both by analogy with Schwinger's results, and from the fact that this is the condition for the Compton wavelength of the particle to be comparable with the radius of curvature. That this is the case despite the nonexistence of a global timelike Killing vector is surprising. We may compare the plane monochromatic electromagnetic wave where the stationary nature of the external field seems to preclude the otherwise expected pair creation. From another point of view, pair production does not occur because the scalar field is coupled to zero four-momentum gravitons.

It is, unfortunately, not clear whether the renormalizability of the exact theory is a result of the high symmetry of the model or another indication of the failure of perturbation theory. One explanation of how it comes about in the de Sitter model is that the four-divergence which can be neglected in the effective Lagrangian of the general theory[2] is here $24K^2$. That is, it is of the same form as the terms that are usually retained. This suggests that renormalizability is an idiosyncrasy of the model.

It is an interesting feature of the model that it has been possible to use dimensional regularization in x space rather than p space. The high symmetry allows a natural extension of the de Sitter manifold to n dimensions (for integer n) and an appropriate choice of a δ-convergent sequence as a representation of the δ function. Again, it is not clear to what extent this can be generalized to other background geometries.

Finally, one might note that all the calculations could be carried through without the explicit use of particular coordinates. We have introduced coordinates only for pedagogical reasons. Even allowing for the special symmetry, this is a nontrivial observation.

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APPENDIX A

Consider Goldstein's integral[10]

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;1-z)=\int_0^\infty dt\, t^{a+b-c-1}e^{-zt}\,{}_2F_0(c-a,c-b;-1/t), \tag{A1}$$

with Rea, Reb, Re$z>0$. If also Im$z>0$, then the path of integration may be rotated to the negative imaginary axis. With the substitution $t=(iKs)^{-1}$, (A1) becomes

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;1-z)=e^{(i\pi/2)(c-a-b)}K^{c-a-b}\int_0^\infty ds\, s^{c-a-b-1}e^{iz/Ks}\,{}_2F_0(c-a,c-b;-iKs). \tag{A2}$$

The right-hand side of this equation is an analytic function of z in the upper half z plane. It follows that (A2) is valid throughout the region Im$z>0$. This establishes (11).

APPENDIX B

In order to examine the properties of the Green's function we make use of the relation[8]

$${}_2F_1(a,b;c;1-z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c;z)+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,z^{c-a-b}\,{}_2F_1(c-a,c-b;c-a-b+1;z). \tag{B1}$$

Using the formula

$$\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$$

(B1) may be cast in the form

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;1-z)=&\frac{1}{\Gamma(c-a)\Gamma(c-b)}\frac{\pi}{\sin\pi(c-a-b)}\\ &\times\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-c+1)}\,{}_2F_1(a,b;a+b-c+1;z)\right.\\ &\left.-z^{c-a-b}\,\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a-b+1)}\,{}_2F_1(c-a,c-b;c-a-b+1;z)\right].\end{aligned} \tag{B2}$$

Letting $c\to a+b-1$, we obtain

$$\begin{aligned}&\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-1)}\,{}_2F_1(a,b;a+b-1;1-z)\\ &=\frac{1}{z}+\frac{1}{\Gamma(a-1)\Gamma(b-1)}\sum_{r=0}^\infty\frac{\Gamma(a+r)\Gamma(b+r)}{r!\,(r+1)!}z^r\left[\ln z+\psi(a+r)+\psi(b+r)-\psi(1+r)-\psi(2+r)\right].\end{aligned} \tag{B3}$$

Comparing this with (11), and using the formulas

$$\frac{1}{z+i\epsilon}=\frac{1}{z}-i\pi\delta(z),\quad \ln(z+i\epsilon)=\ln|z|+i\pi\theta(-z)$$

valid for z real, we find that in four dimensions

$$\mathrm{Re}G(x,x')=\frac{K}{16\pi}\delta(z)-\theta(-z)\frac{1}{16\pi}(m^2-2K)\,{}_2F_1(\tfrac{3}{2}+i\alpha,\tfrac{3}{2}-i\alpha;\ 2;z),$$

$$\mathrm{Im}G(x,x')=\frac{K}{16\pi^2}\left\{\frac{1}{z}+\frac{1}{\Gamma(\frac{1}{2}+i\alpha)\Gamma(\frac{1}{2}-i\alpha)}\right.$$

$$\left.\times\sum_{r=0}^{\infty}\frac{\Gamma(\frac{3}{2}+i\alpha+r)\Gamma(\frac{3}{2}-i\alpha+r)}{r!\,(r+1)!}z^r\left[\ln|z|+\psi(\tfrac{3}{2}+i\alpha+r)+\psi(\tfrac{3}{2}-i\alpha+r)-\psi(1+r)-\psi(2+r)\right]\right\},$$

where we have set $z=K\sigma/2$.

In order to obtain the limiting form of $G(x,x')$ as $K\to 0$ we make use of the series expansions of the hypergeometric functions and the formula

$$\Gamma(z+\gamma)/\Gamma(z)\sim z^\gamma \quad \text{as } z\to\infty.$$

From (B2) we find

$$\lim_{K\to 0}G(x,x')=\frac{-i}{(4\pi)^{2+\omega}}\frac{\pi}{\sin\pi(1+\omega)}\left[\sum_{r=0}^{\infty}\frac{1}{r!\,\Gamma(2+\omega+r)}(m^2)^{1+\omega+r}\left(\frac{\sigma}{2}\right)^r-\sum_{r=0}^{\infty}\frac{1}{r!\,\Gamma(-\omega+r)}(m^2)^r\left(\frac{\sigma}{2}\right)^{-1-\omega+r}\right].$$

Noting that the sums in this expression are the power-series expansions of Bessel functions, we obtain finally

$$\lim_{K\to 0}G(x,x')=\frac{-i}{(4\pi)^{2+\omega}}\left(-\frac{2m^2}{\sigma}\right)^{(1+\omega)/2}\frac{\pi}{\sin\pi(1+\omega)}\left[J_{1+\omega}((-2m^2\sigma)^{1/2})-e^{-(1+\omega)\pi i}J_{-1-\omega}((-2m^2\sigma)^{1/2})\right]$$

$$=\frac{-\pi e^{-\omega\pi i}}{(4\pi)^{2+\omega}}\left(-\frac{2m^2}{\sigma}\right)^{(1+\omega)/2}H^{(2)}_{1+\omega}((-2m^2\sigma)^{1/2}),$$

which is precisely the Feynman function in Minkowski space-time as computed from

$$G_{\text{Minkowski}}(x,x')=\frac{1}{(2\pi)^n}\int\frac{d^np}{p^2+m^2}e^{ip(x-x')}.$$

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