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Entropy extremum of relativistic self-bound systems: A geometric approach

J. Katz and Y. Manor

The Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem, Israel

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Conditions for entropy extremum of self-bound relativistic systems of particles, in stationary axisymmetric motion, are obtained under the following constraints: (i) The system is kept isolated in a geometrical sense, implying that the total mass-energy and total angular momentum, defined by the asymptotic behavior of the metric, are kept constant, (ii) the total number of particles is kept constant, and (iii) Einstein's constraint equations are imposed on a spacelike hypersurface. It is shown that if the system is in mechanical equilibrium, the total entropy is an extremum for all trial nonequilibrium configurations that satisfy the constraints and respect the symmetry, if and only if (i) Einstein's dynamical equations are satisfied, (ii) the temperature and the chemical free energy, as seen from infinity, are constant, and (iii) the system is rigidly rotating. The proof does not depend on a particular functional expression for the mass or for the angular momentum; consequently, no related Lagrange multipliers have been used.

I. INTRODUCTION

With relativistic systems of particles of one type, having an axisymmetric configuration and rotating in a stationary way, we mainly have in mind relativistic rotating stars. One may also think about rotating stellar clusters to which our results are applicable as well.[1] The system is supposed to be in local thermodynamic equilibrium. Such systems are characterized, as a whole, by four quantities: the total entropy S, the total number of particles N, the total mass (or mass-energy) M, and the total angular momentum J.

Contrary to what happens in classical thermodynamics of homogeneous systems, S and N are not on the same footing as M and J. Entropy and particle number are related to locally defined thermodynamic functions and are well-defined functionals. Mass and angular momentum are defined far away from the system, in fact at infinity, and are related to the asymptotic form of the metric. Thus, entropy and number of particles have a proper thermodynamic definition while mass and angular momentum are geometrically well defined.

Evidently, integrals may be constructed in terms of thermodynamic functions and of the metric components whose value is equal to M or J. This may be done in many different ways. However, if thermodynamic functions are present in the integrands, the integrals cannot be equal to M or J unless some of Einstein's equations are satisfied.

In classical thermodynamics of uniform systems, entropy and energy are complementary with respect to equilibrium and stability conditions of the system: To a maximum of entropy ($\delta S = 0$, $\delta^2 S < 0$) at fixed total energy, angular momentum, and particle number correspond a minimum of energy at fixed total entropy, angular momentum, and particle number. In relativistic rotating stars such a complementarity also exists, provided energy and angular momentum are properly defined as integrals of matter and field functions. Thus, the elegant energy extremum theorems of Hartle and Sharp[2] and the generalized (and improved) form due to Bardeen[3] may easily be cast into entropy extremum theorems with corresponding constraints.

However, since S, N, M, and J are all defined *a priori* one may as well analyze the conditions for extremum of entropy without resort to any particular functional expressions whose integrals are equal to M and J. The way to keep the system isolated consists of fixing the total number of particles and restricting all trial nonequilibrium configurations of matter and fields to be such that the metric has an invariant asymptotic form[4] insuring that M and J are kept constant. This approach is in no way complementary to any energy extremum theorem. It is in essence the approach of Tolman[5] in his analysis of thermodynamic equilibrium conditions of general relativistic fluids in static spherically symmetric states. It has also something in common with the method used by Cocke,[6] who is explicitly concerned with applying a variational principle to the entropy of a spherically symmetric and static fluid. As far as we know, this approach has never been used to analyze rotating systems. This is the object of the present work.

We have analyzed the conditions for extremum of entropy of relativistic rotating self-bound systems by applying a variational principle to S, keeping the system isolated in the sense we just described. The main result may be stated as follows: If the system is kept isolated and satisfies Einstein's constraint equations on some spacelike hypersurface, the total entropy of the system is an extremum *with matter in mechanical equilibrium*[7] if, and only if, Einstein's dynamical equations are satisfied, the temperature and the chemical free energy, both as measured from infinity, are uniform, and the system is rigidly rotating.

Compared with the energy extremum theorems of Hartle and Sharp[2] and of Bardeen[3] this is a weaker result, because mechanical equilibrium has to be imposed. It seems, however, that this is all one can obtain without defining the energy and the angular momentum twice: once at spatial infinity and once as functionals whose values are equal to the values defined at infinity.

In Sec. II we shall describe the physical and mathematical background, while in Sec. III we shall analyze various constraints we impose on the system. The consequences of constrained variational principle applied to the entropy are developed in Sec. IV and the results as well as comparison with previous work are discussed in Sec. V.

II. PHYSICAL AND MATHEMATICAL BACKGROUND

A. Geometrical background

Consider physical pseudo-Riemannian manifolds[8] which admit two Killing fields: a timelike one,[9] ξ^λ $(\lambda, \mu, \nu, \rho, \sigma = 0, 1, 2, 3)$, and a spacelike one, η^λ. Local "cylindrical coordinates" exist, $(x^\lambda) \equiv (x^0 \equiv t,\ x^1 \equiv \phi,\ x^2 \equiv \rho,\ x^3 \equiv z)$, in which $\xi^\lambda = \delta^\lambda_0$ and $\eta^\lambda = \delta^\lambda_1$. The metric is, in addition, invariant under a $(t, \phi) \to (-t, -\phi)$ reversal, because it turns out that these are the physically relevant metrics. Spacetime is supposed to be asymptotically flat, and at large asymptotic spatial distances, one can find coordinates $(x^\lambda) \equiv (x^0 \equiv t,\ x^1 \equiv x,\ x^2 \equiv y,\ x^3 \equiv z)$ in which $\xi^\lambda = \delta^\lambda_0$, $\eta^\lambda = (0, -y, x, 0)$ and in which at fixed t and for $r \equiv (x^2 + y^2 + z^2)^{1/2} \to \infty$, the metric $g_{\mu\nu}$ tends to the form[10]

$$\hat{g}^{00} = 1 + M/(2\pi r) + O(r^{-2}), \tag{1}$$

$$\hat{g}^{k0} = \eta^k J/(4\pi r^3) + O(r^{-3}), \tag{2}$$

$$\hat{g}^{kl} = -\delta^{kl} + O(r^{-2}) \tag{3}$$

$(k, l, m, n = 1, 2, 3)$, where M and J are two numbers. In appropriate units[11] they represent the mass and angular momentum, as sensed from infinity, of some bounded axisymmetric stationary system of particles.

B. Kinematical background

The space contains such a system[12] of particles, all of one type with mass $m = 1$. They are rotating around the axis of symmetry and their motion is described by a unit timelike field u^λ (with $u^\lambda u_\lambda = 1$) which is everywhere in the local ξ-η plane. Thus there exist two scalar functions ζ and Ω, whose Lie derivatives relative to ξ and η are zero (a symmetry condition) and such that

$$u^\lambda \equiv v^\lambda/\zeta \equiv (\xi^\lambda + \Omega\eta^\lambda)/\zeta\,. \tag{4}$$

In cylindrical coordinates, $\zeta^{-1} = u^0$ while Ω appears as the angular velocity; both are (ρ, z)-dependent only. With u^λ one may define a proper volume element $\mathcal{V}$ and a proper volume V by integrating over any spacelike hypersurface Σ extending to spatial infinity:

$$V = \int_\Sigma \mathcal{V} \equiv \int_\Sigma \hat{u}^\lambda d\Sigma_\lambda\,. \tag{5}$$

We shall restrict the hypersurfaces Σ to be invariant (intransitive) varieties[13] of the one-parameter group generated by η^λ; in other words, we shall impose the condition

$$\eta^\lambda d\Sigma_\lambda = 0\,. \tag{6}$$

Physical interpretations are most easy on hypersurfaces of "simultaneity" ($t = \text{const}$). Condition (6) implies that the property of simultaneity will not be affected by a rotation around the axis of symmetry.

C. Local thermodynamics

We shall assume that local thermodynamic equilibrium holds. Thus, a local entropy density function $s(x)$ exists which may be expressed as a functional of two thermodynamic functions describing the system: the density of particles (or density of stars if we are dealing with a cluster) $n(x)$ and the "internal energy" density $\sigma(x)$. Both have zero Lie derivatives in the ξ and η directions. Local temperature $T(x)$ and local gravitochemical potential[14] $\mu(x)$ are defined by the differential form of $s(x)$, namely,

$$ds = (1/T)d\sigma - (\mu/T)dn\,, \tag{7}$$

while local pressure $p(x)$ is given by the following standard expression:

$$p = \mu n + Ts - \sigma\,. \tag{8}$$

Local thermodynamic functions are associated with global quantities; for instance, the total entropy S is given by

$$S = \int_\Sigma s\mathcal{V}\,. \tag{9}$$

Up to this point, we have introduced 13 functions: $g_{\mu\nu}$, Ω, σ, and n.

D. Functional variation of the total entropy

Consider a slightly different geometry,[15] $g_{\mu\nu} + \delta g_{\mu\nu}$, a different thermodynamic state of the system of particles (or stars) $\sigma + \delta\sigma$ and $n + \delta n$, and a different state of rotation $\Omega + \delta\Omega$. We shall restrict the infinitesimal changes to those that preserve the symmetry; this may be done by taking $\delta\xi^\lambda = \delta\eta^\lambda = 0$.

The change in total entropy, δS, is calculated with the help of (7) and (8), which may be written in the following equivalent form:

$$\delta(s\hat{u}^\lambda) = (1/T)\delta(\sigma\hat{u}^\lambda) + (p/T)\delta\hat{u}^\lambda - (\mu/T)\delta(n\hat{u}^\lambda). \tag{10}$$

$$T^{\mu\nu} \equiv (\sigma + p)u^\mu u^\nu - pg^{\mu\nu} \tag{11}$$

is the energy tensor of the system. We shall be interested in writing δS in terms of three scalars β, γ, α, and one tensor $E^{\mu\nu}$ which are defined in terms of $\zeta, \Omega, T, \sigma, p$, the Ricci tensor of the metric $R^{\mu\nu}$, and the scalar curvature R:

$$\beta \equiv (T\zeta)^{-1}, \quad \gamma \equiv \Omega\beta, \quad \alpha \equiv \mu/T \tag{12}$$

and

$$E^{\mu\nu} \equiv (R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R) - T^{\mu\nu} \equiv G^{\mu\nu} - T^{\mu\nu}. \tag{13}$$

The factor β^{-1} is the local temperature as seen from infinity, (α/β) represents the local gravitochemical potential as seen from infinity, while $E^{\mu\nu} = 0$ are, of course, Einstein's equations. They are not imposed.

With these definitions, S may be written in the following form,[16] the proof of which is not completely straightforward and is given in the Appendix:

$$\delta S = \int_\Sigma [\beta(\delta\mathfrak{M} - \mathfrak{D}) - \gamma\delta\mathfrak{J} - \alpha\delta\mathfrak{N} - \beta(v^\nu\delta\hat{E}^\lambda_\nu + \tfrac{1}{2}v^\lambda\hat{E}_{\mu\nu}\delta g^{\mu\nu})d\Sigma_\lambda]. \tag{14}$$

In this identity,[17]

$$\mathfrak{M} \equiv 2\partial_\nu(D^{[\lambda}\hat{\xi}^{\nu]})d\Sigma_\lambda = 2\xi^\nu\hat{R}^\lambda_\nu d\Sigma_\lambda, \tag{15}$$

$$\mathfrak{J} \equiv -\partial_\nu(D^{[\lambda}\hat{\eta}^{\nu]})d\Sigma_\lambda = -\eta^\nu\hat{R}^\lambda_\nu d\Sigma_\lambda, \tag{16}$$

and

$$\mathfrak{N} \equiv n\hat{u}^\lambda d\Sigma_\lambda, \tag{17}$$

while $\mathfrak{D}$ is defined as

$$\mathfrak{D} \equiv \tfrac{1}{2}(\delta\mathfrak{M} + v^\lambda\hat{g}^{\mu\nu}\delta R_{\mu\nu}d\Sigma_\lambda). \tag{18}$$

The differential forms $\mathfrak{M}$ and $\mathfrak{J}$ are Komar's[18] conserved forms for energy and angular momentum. With the asymptotic conditions (1)–(3), it is easily shown that

$$\int_\Sigma \mathfrak{M} = M \text{ and } \int_\Sigma \mathfrak{J} = J. \tag{19}$$

The total number of particles is given by

$$N = \int_\Sigma \mathfrak{N}. \tag{20}$$

With (6), $\mathfrak{D}$ becomes divergencelike, say of the form $\partial_\nu\mathfrak{D}^{[\mu\nu]}$ where

$$\mathfrak{D}^{[\mu\nu]} \equiv \delta(D^{[\mu}\hat{\xi}^{\nu]}) + \{\hat{g}^{\rho\sigma}(\delta\Gamma^{[\mu}_{\rho\sigma}) - (\delta\Gamma^\sigma_{\rho\sigma})\hat{g}^{\rho[\mu}\}v^{\nu]} \tag{21}$$

and in which $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols. This $\mathfrak{D}^{[\mu\nu]}$, according to (1), (2), and (3), goes to zero at spatial infinity like r^{-4}. Thus, as a consequence of Stokes's theorem, the Σ integral of $\mathfrak{D}$ is identically equal to zero.

Since $(\delta\mathfrak{M} - \mathfrak{D})$ and $(\delta\mathfrak{J})$ are both divergencelike, we shall write them, respectively, in the form $\partial_\nu\hat{\mathfrak{M}}^{[\mu\nu]}d\Sigma_\mu$ and $\partial_\nu\hat{\mathfrak{J}}^{[\mu\nu]}d\Sigma_\mu$. Both antisymmetric tensor densities $\hat{\mathfrak{M}}^{[\mu\nu]}$ and $\hat{\mathfrak{J}}^{[\mu\nu]}$ are linear and homogeneous in $(\delta g^{\rho\sigma})$ and $[D_\lambda(\delta g^{\rho\sigma})]$. We may thus write them as follows, dropping the brackets around the indices:

$$\mathfrak{M}^{\mu\nu} \equiv (\delta g^{\rho\sigma})\mathfrak{M}^{\mu\nu}_{\rho\sigma} + (D_\lambda\delta g^{\rho\sigma})\mathfrak{M}^{\mu\nu\lambda}_{\rho\sigma}, \tag{22}$$

$$\mathfrak{J}^{\mu\nu} \equiv (\delta g^{\rho\sigma})\mathfrak{J}^{\mu\nu}_{\rho\sigma} + (D_\lambda\delta g^{\rho\sigma})\mathfrak{J}^{\mu\nu\lambda}_{\rho\sigma}. \tag{23}$$

The second term on the right-hand side is completely antisymmetric in μ, ν, and λ. With (22) and (23) a new identity for δS may be written, which will be used below; it contains integrals on the 2-dimensional boundary B of the system of particles (i.e., the intersection of the world tube of the matter with Σ) in addition to Σ integrals:

$$\delta S = \int_B (\beta\hat{\mathfrak{M}}^{\mu\nu} - \gamma\hat{\mathfrak{J}}^{\mu\nu})d\Sigma_{\mu\nu} + \int_B (\delta g^{\rho\sigma})(\hat{\mathfrak{M}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda\beta - \hat{\mathfrak{J}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda\gamma)d\Sigma_{\mu\nu} - \int_\Sigma [\alpha\delta\mathfrak{N} + (\beta v^\nu\delta\hat{E}^\lambda_\nu + \hat{F}^\lambda_{\rho\sigma}\delta g^{\rho\sigma})d\Sigma_\lambda]. \tag{24}$$

In this expression

$$\begin{aligned} F^\lambda_{\rho\sigma} &\equiv \tfrac{1}{2}v^\lambda E_{\rho\sigma} + [\mathfrak{M}^{\lambda\nu}_{\rho\sigma}\partial_\nu\beta + D_\mu(\mathfrak{M}^{\mu\nu\lambda}_{\rho\sigma}\partial_\nu\beta)] - [\mathfrak{J}^{\lambda\nu}_{\rho\sigma}\partial_\nu\gamma + D_\mu(\mathfrak{J}^{\mu\nu\lambda}_{\rho\sigma}\partial_\nu\gamma)] \\ &\equiv \tfrac{1}{2}v^\lambda E_{\rho\sigma} + \mathfrak{M}^\lambda_{\rho\sigma} - \mathfrak{J}^\lambda_{\rho\sigma}. \end{aligned} \tag{25}$$

E. Coordinate invariance and related identities

For an arbitrary infinitesimal one-parameter transformation of coordinates $\delta x^\lambda = \theta^\lambda(x)\delta\tau$, the free scalar S is invariant. With (8) taken into account, a straightforward calculation shows that what remains of δS is as follows:

$$0=(\delta S/\delta\tau)$$

$$=\int_\Sigma \hat{\theta}^\lambda[\zeta\beta D_\nu E^\nu_\lambda+(\sigma+p)\{(\xi^\nu u_\nu)\partial_\lambda\beta+(\eta^\nu u_\nu)\partial_\lambda\gamma\}-\eta\partial_\lambda\alpha]\,u^\rho d\Sigma_\rho\,. \tag{26}$$

The boundary contributions are identically equal to zero because of (7) and no other term appears besides those with $u^\rho d\Sigma_\rho$. This identity holds for any change of coordinates θ^λ, if the term in brackets is identically equal to zero. This result may also be obtained from the Gibbs-Duhem equality derived from (7) and (8):

$$n\partial_\lambda\alpha=(\sigma+p)\,\partial_\lambda(1/T)+(1/T)\,\partial_\lambda p\,. \tag{27}$$

If β, γ, and α are uniform, which is the case in thermodynamic equilibrium, (26) reduces to

$$D_\nu E^\nu_\lambda=-D_\nu T^\nu_\lambda=0\,. \tag{28}$$

Equations (28) are those of mechanical equilibrium of the system of particles. The fact that mechanical equilibrium is implied by thermodynamic equilibrium is a known property in Newtonian theory and has also been exploited for relativistic self-bound systems.[3] Identity (26) is closely analogous to the generalized Bianchi identities deduced from the action integral from which Einstein's equations are derived. It is well known that the condition for this action to be invariant under arbitrary coordinate transformations leads precisely to the condition (28) of mechanical equilibrium. The existence of the Bianchi-like identities (26) will have important consequences regarding the results of applying a variational principle to S, as we shall see.

III. CONSTRAINTS ON THE VARIATIONS OF TOTAL ENTROPY

A. Einstein's dynamical constraints

We shall now limit all trial configurations for $g_{\mu\nu}$, σ, n, and Ω to satisfy Einstein's constraint equations (or initial value equations). Initial value equations depend on the hypersurface Σ. If n^λ is the local unit normal vector in the direction of $d\Sigma^\lambda$, the constraint equations are defined by

$$H_\mu\equiv E^\lambda_\mu n_\lambda=0\,. \tag{29}$$

We thus impose (29) and

$$\delta H_\mu=0\,. \tag{30}$$

When this is done, the terms in (14) that contain $E_{\mu\nu}$ and $\delta E_{\mu\nu}$ may be reduced as follows:

$$(v^\nu\delta\hat{E}^\lambda_\nu+\tfrac{1}{2}\hat{E}_{\mu\nu}v^\lambda\delta g^{\mu\nu})n_\lambda$$
$$=v^\lambda(\hat{E}_{\lambda(\mu}n_{\nu)}+\tfrac{1}{2}\hat{E}_{\mu\nu}n_\lambda)\,\delta(g^{\mu\nu}-n^\mu n^\nu)$$
$$\equiv\tfrac{1}{2}\hat{H}_{\mu\nu}\delta e^{\mu\nu}\,. \tag{31}$$

So, one is left with a linear combination of only six linearly independent terms since $e^{\mu\nu}n_\nu=0$. With (29) and (30), δS becomes a linear homogeneous functional of δn and $\delta g^{\mu\nu}$ only. The constraint equations have effectively been used to eliminate two functions, σ and Ω, in terms of the remaining eleven ones, n and $g_{\mu\nu}$.

B. Conditions for isolation

We shall be interested in extremizing the entropy with fixed mass, fixed angular momentum, and fixed particle number. Consider first the constraints related to the asymptotic behavior of $g_{\mu\nu}$. According to (1) and (2), $\delta M=\delta J=0$ implies only that, at spatial infinity, $\delta\hat{g}^{00}$ is $O(r^{-2})$ and $\delta\hat{g}^{k0}$ is $O(r^{-3})$. Such conditions of isolation are, however, too weak and not very useful because S is defined by an integral whose integrand is identically zero outside the boundary B. We shall impose the slightly stronger condition

$$\int_{\mathcal{B}}\hat{\mathfrak{M}}^{\mu\nu}d\Sigma_{\mu\nu}=\int_{\mathcal{B}}\hat{\mathfrak{J}}^{\mu\nu}d\Sigma_{\mu\nu}=0 \tag{32}$$

on any closed 2-surface $\mathcal{B}$ that does not cut the boundary B but may be infinitesimally close to B. If (32) holds, then according to (19)

$$\delta M=\delta\int_\Sigma\mathfrak{M}=\int_\Sigma\delta\mathfrak{M}=\int_\Sigma(\delta\mathfrak{M}-\mathfrak{D})$$
$$=\int_\Sigma\partial_\nu\hat{\mathfrak{M}}^{\mu\nu}d\Sigma_\mu$$
$$=\int_{\mathcal{B}\to\infty}\hat{\mathfrak{M}}^{\mu\nu}d\Sigma_{\mu\nu}$$
$$=0\,, \tag{33}$$

and similarly $\delta J=0$. Conditions (32) are thus sufficient in order to insure that M and J are kept constant.

We shall see below that (32) is also necessary for S to be an extremum in the following sense. As a result of applying a constrained variational principle to S (see Sec. IV) we shall find that β, γ, and α are constants. Consequently, for arbitrary variations of $g_{\mu\nu}$ the variation of the entropy will reduce to a boundary integral, the first one on the right-hand side of (24). So S will be extremal only if (32) is satisfied. Conditions (32) are thus also necessary conditions as far as M and J are concerned.

In addition to fixed M and J we shall impose the condition that the total number of particles is fixed; that is, according to (20),

$$\delta N = \int_\Sigma \delta \mathfrak{N} = 0 . \tag{34}$$

IV. CONSTRAINED VARIATIONAL PRINCIPLE AND EXTREMUM OF ENTROPY

A. The variational principle

Consider all trial configurations that satisfy Einstein's constraint equations (29) and (30). If in addition the system is isolated [conditions (32) and (34)], the total constrained variation of S reduces to the following form obtained from (24) and (25):

$$\delta S = \int_B (\delta g^{\rho\sigma})(\hat{\mathfrak{M}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda \beta - \hat{\mathfrak{J}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda \gamma) d\Sigma_{\mu\nu} - \int_\Sigma (\alpha - \alpha_0)\, \delta\mathfrak{N} - \int_\Sigma (\hat{K}_\lambda \delta n^\lambda + \hat{K}_{\rho\sigma}\delta e^{\rho\sigma}) d\Sigma , \tag{35}$$

where α_0 is a constant Lagrange multiplier and $d\Sigma_\lambda \equiv n_\lambda d\Sigma$, while

$$K_\lambda \equiv n_\mu n^\nu (\mathfrak{M}^\mu_{\lambda\nu} - \mathfrak{J}^\mu_{\lambda\nu}) , \tag{36}$$

$$K_{\rho\sigma} \equiv \tfrac{1}{2}\beta H_{\rho\sigma} + n_\lambda (\mathfrak{M}^\lambda_{\rho\sigma} - \mathfrak{J}^\lambda_{\rho\sigma}) . \tag{37}$$

Let us then apply to S a variational principle. For fixed $g_{\mu\nu}$ and arbitrary variations of n, the entropy is extremal, $\delta S = 0$, if

$$\alpha = \alpha_0 = \alpha_B , \tag{38}$$

where an index B indicates the value on the boundary. Next consider arbitrary variations of $g_{\mu\nu}$ such that

$$(\delta g^{\rho\sigma})_B = (D_\lambda \delta g^{\rho\sigma})_B = 0 . \tag{39}$$

Then $\delta S = 0$, if

$$K_\lambda = 0 \quad \text{and} \quad K_{\rho\sigma} = 0 . \tag{40}$$

These ten equations are not independent since S is invariant under arbitrary coordinate transformations. They are related by a set of Bianchi-like identities which may easily be written in terms of K_λ and $K_{\rho\sigma}$. A simple and useful form of these identities has already been obtained in (26); if we take account of (38), they become the following equations:

$$\zeta\beta D_\nu T^\nu_\lambda = (\sigma + p)[(\xi^\nu u_\nu)\partial_\lambda \beta + (\eta^\nu u_\nu)\partial_\lambda \gamma] . \tag{41}$$

In terms of numbers of variables and number of equations, the situation is now as follows: for a given field n^λ (i.e., a given Σ + a coordinate system) one has the ten equations (40) of which only six are independent. These six independent equations contain eight variables: $e_{\rho\sigma}$, β, and γ. Two equations are thus missing, say for β and γ. One way to obtain a complete set of equations is to impose mechanical equilibrium [condition (28)]. Then, according to (41) and with $(\sigma + p) \neq 0$ everywhere except perhaps on the boundary or on the axis of symmetry (see footnote 12), one obtains

$$q_\lambda \equiv (\xi^\nu u_\nu)\partial_\lambda \beta + (\eta^\nu u_\nu)\partial_\lambda \gamma = 0 . \tag{42}$$

This represents only two independent equations since β and γ have zero Lie derivatives in the ξ and η directions.

Equations (42) have some interesting implications. It turns out,[19] in the presence of stationary heat flows, that the entropy production is positive if the heat-flow vector is proportional to q_λ. It follows that chemical equilibrium (38) and mechanical equilibrium (28) imply the absence of heat flows in a rotating system. If the system does not rotate ($\gamma = 0$) the absence of heat flow immediately implies that the system is also in thermodynamic equilibrium: β = const. However, in a rotating system, mechanical equilibrium and chemical equilibrium are not enough to insure thermodynamic equilibrium or uniform β and γ.

B. Entropy extremum theorem

In considering extremum properties we shall drop condition (39) and adopt *arbitrary* variations of $g_{\mu\nu}$. The value of δS reduces now to a boundary contribution; in fact, with (32), (35), (38), and (40), one obtains

$$\delta S = \int_B (\delta g^{\rho\sigma})(\hat{\mathfrak{M}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda \beta - \hat{\mathfrak{J}}^{\mu\nu\lambda}_{\rho\sigma}\partial_\lambda \gamma) d\Sigma_{\mu\nu} . \tag{43}$$

We shall see below that the only solutions of Eq. (40) and Eq. (41) for β and γ are

$$\beta = \beta_B = \text{const} , \tag{44}$$

$$\gamma = \gamma_B = \text{const} . \tag{45}$$

This will imply [see (37) together with (25)] that (40) reduces to Einstein's dynamical equations

$$E_{kl} = 0 \quad (\text{with } E^0_\mu = 0) . \tag{46}$$

It also implies that the entropy is extremal, $\delta S = 0$, for arbitrary constrained variations of the variables since nothing is left of the second member of (43). Thus, if β and γ are uniform, the following theorem is established: For every trial configuration of matter and fields that keeps the system axisymmetric, satisfies Einstein's constraint equations, (29) and (30), and for which the system remains isolated in the sense of (32) and (34), the entropy is an extremum for the matter in mechanical equilibrium, Eq. (28), if, and only if, Einstein's dynamical equations (46) are satisfied, temperature as seen from infinity is uniform, Eq. (44), chemical free energy as seen from infinity is uniform, Eq. (36), and the system is rigidly rotating, Eq. (45).

It remains to be shown that β and γ can only be uniform.

C. Thermodynamic equilibrium

Away from the axis of rotation the metric may be written as follows[20]:

$$ds^2 = (e^U + \omega^2 g_{11})dt^2 - 2\omega g_{11} dt\, d\phi + g_{11} d\phi^2 + g_{ab}\, dx^a\, dx^b \qquad a, b = 2, 3\,, \quad (47)$$

in which U, g_{11} (<0), ω, and g_{ab} depend only on $(x^a) = (x^2 \equiv \rho, x^3 \equiv z)$. The existence of this form of the metric depends on the form of the velocity field as given in (4) (nonconvective motion) and on very reasonable and general topological properties of spacetime. Let us put together the equations for β and γ, namely (42) and $K_\lambda = 0$ from (40), and write them in terms of the metric (47). A tedious but straightforward calculation leads to the following equations: From (42), and since $u_0 \neq 0$ everywhere,

$$\partial_a \beta + (u_1/u_0)\, \partial_a \gamma = 0\,; \qquad (48)$$

there are two nontrivially satisfied equations in the set $K_\lambda = 0$. One may be written as follows:

$$g^{ab}\{\omega \times \partial_a[\omega(-g_{11})^{1/2}] \times \partial_b \beta - (-g_{11})^{1/2} \times \partial_a \omega \times \partial_b \gamma\} = 0\,. \qquad (49)$$

The second one, with the help of (48), reduces to

$$\partial_a(f g^{ab} \partial_b \gamma) = 0\,, \qquad (50)$$

in which f is the following positive-definite function:

$$f \equiv -g_{11} e^U (u^0/u_0)(-\det g_{kl})^{1/2}\,. \qquad (51)$$

Equation (50) is an elliptic equation of the form

$$\Delta\gamma + h^a(x^b)\, \partial_a \gamma = 0\,. \qquad (52)$$

If γ has either a maximum or a minimum at some interior point of our system, then according to Hopf's theorem, i.e., the maximum principle,[21] γ is a constant. Thus the only nonuniform solutions of (52), if they exist, have a maximum and a minimum on B. Consider a contour[22] C, the intersection of B with a hypersurface $\phi = \text{const}$. In order to analyze the behavior of γ on B, we shall take near C, in the hypersurface $\phi = \text{const}$, coordinates x^a in such a way that the equation of C is[23] given by $x^3 = \text{const}$. In these coordinates, γ on the contour C depends only on x^2. Thus, according to (48),

$$(f g^{22} \partial_2 \gamma)_C = \text{const}. \qquad (53)$$

Since γ assumes its maximum and its minimum on B, and since fg^{22} is everywhere negative, it follows from (53) that $\partial_a \gamma$ can only be zero on B and thus γ is uniform in the system and on the boundary of the system. According to (48), β is then also uniform. This completes the proof of the extremum theorem of the entropy.

Equation (49) has not been used in the proof. It is a redundant one. It may, however, be interesting to note that with (49) one does not need Hopf's theorem if one supposes analyticity of γ. Also one does not need to go to Fermi coordinates in order to prove that β and γ are uniform.[24]

V. DISCUSSION

The entropy extremum theorem developed above is distinctly different from the energy extremum theorems and is not complementary to any of them. We did not use any functional definition of energy or angular momentum and relied only on the geometrical definition of these quantities. For this reason, the theorem is without equivalent in the Newtonian limit where energy and angular momentum are defined in terms of functionals and where (just the opposite of what happens in Einstein's theory of gravitation) the number of particles *is* defined by the asymptotic value of the gravitational field.

The striking feature in proving that the extremum of entropy leads to uniform temperature as seen from infinity and to uniform angular velocity, without using Lagrange multipliers, is the fact that no boundary conditions have been introduced for β and γ or their derivatives. Nor had we to fix the values of constants of integration on the basis of a regular behavior near the axis of rotation. Analyticity of β and γ is not even needed.

In this respect, Tolman's[5] proof of the uniformity of β in static spherically symmetric systems is a weak one, the more so that the full set of Einstein's equations was taken into account. Tolman obtained a differential equation for $\beta(r)$ whose solution is $\partial_r \beta(r) = \text{const}$. In order to obtain uniform β this derivative has to be equated to zero. This amounts, as we saw in Sec. IV, to impose what we are really trying to establish, that there is no heat flow. As we noted previously, uniform temperature as seen from infinity in spherically symmetric systems is a direct consequence of chemical equilibrium ($\alpha = \alpha_B$) and mechanical equilibrium [Eq. (28)].

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APPENDIX

Here we prove identity (14). From (13) we have

$$\hat{R}^\lambda_\nu = \hat{T}^\lambda_\nu + \tfrac{1}{2}\delta^\lambda_\nu \hat{R} + \hat{E}^\lambda_\nu = (\hat{T}^\lambda_\nu - \tfrac{1}{2}\delta^\lambda_\nu \hat{T}^\rho_\rho) + (\hat{E}^\lambda_\nu - \tfrac{1}{2}\delta^\lambda_\nu \hat{E}) \tag{54}$$

in which

$$E \equiv g^{\mu\nu}E_{\mu\nu} \tag{55}$$

and

$$T^\rho_\rho = g^{\mu\nu}T_{\mu\nu} = \sigma - 3p\,. \tag{56}$$

Contracting (54) with v^ν and successively using (4), (11), and (56) we obtain after some rearrangements

$$v^\lambda(\hat{\sigma} + 3\hat{p}) = \hat{m}^\lambda - 2\Omega\hat{j}^\lambda + v^\lambda\hat{E} - 2v^\nu\hat{E}^\lambda_\nu\,, \tag{57}$$

where

$$\hat{m}^\lambda = 2\hat{R}^\lambda_\nu\xi^\nu, \quad \hat{j}^\lambda = -\hat{R}^\lambda_\nu\eta^\nu\,. \tag{58}$$

The conditions of the variations are such that with (4) one has

$$\delta v^\lambda = \eta^\lambda\delta\Omega\,. \tag{59}$$

Again, using (11), (13), and (59) one obtains

$$u^\lambda(\hat{\sigma}+\hat{p})u_\nu\delta v^\nu = (\hat{T}^\lambda_\nu + \delta^\lambda_\nu\hat{p})\delta v^\nu = \hat{R}^\lambda_\nu\delta v^\nu - \tfrac{1}{2}\delta^\lambda_\nu\hat{R}\delta v^\nu - \hat{E}^\lambda_\nu\delta v^\nu + \delta^\lambda_\nu\hat{p}\delta v^\nu\,. \tag{60}$$

Or, after some rearrangements

$$u^\lambda(\hat{\sigma}+\hat{p})u_\nu\delta v^\nu = \tfrac{1}{2}(\hat{\sigma}+3\hat{p})\delta v^\lambda - 2\hat{p}\delta v^\lambda - \hat{j}^\lambda\delta\Omega + \tfrac{1}{2}\hat{E}\delta v^\lambda - \hat{E}^\lambda_\nu\delta v^\nu\,. \tag{61}$$

Now, (4) implies that

$$u_\lambda v^\lambda = \zeta\,, \tag{62}$$

hence

$$\delta\zeta = \tfrac{1}{2}u^\mu u^\nu\zeta\delta g_{\mu\nu} + u_\lambda\delta v^\lambda\,. \tag{63}$$

Multiplying (11) by $\frac{1}{2}\delta g_{\mu\nu}$ and remembering that $\delta(-g)^{1/2} = \frac{1}{2}\hat{g}^{\mu\nu}\delta g_{\mu\nu}$ one gets

$$\tfrac{1}{2}(\hat{\sigma}+\hat{p})u^\mu u^\nu\delta g_{\mu\nu} = \tfrac{1}{2}\hat{T}^{\mu\nu}\delta g_{\mu\nu} + p\delta(-g)^{1/2}\,. \tag{64}$$

Now multiply (63) by $(\hat{\sigma}+\hat{p})$, use (64) and move $p\delta(-g)^{1/2}$ to the left-hand side; the result is

$$(\hat{\sigma}+\hat{p})\delta\zeta - p\zeta\delta(-g)^{1/2} = \tfrac{1}{2}\zeta\hat{T}^{\mu\nu}\delta g_{\mu\nu} + (\hat{\sigma}+\hat{p})u_\lambda\delta v^\lambda\,; \tag{65}$$

with

$$R^{\mu\nu}\delta g_{\mu\nu} = -R_{\mu\nu}\delta g^{\mu\nu} \tag{66}$$

and (13) one obtains

$$\begin{aligned}\tfrac{1}{2}\hat{T}^{\mu\nu}\delta g_{\mu\nu} &= \tfrac{1}{2}[\hat{R}^{\mu\nu}\delta g_{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\hat{R}\delta g_{\mu\nu} - \hat{E}^{\mu\nu}\delta g_{\mu\nu}]\\ &= \tfrac{1}{2}[-\hat{R}_{\mu\nu}\delta g^{\mu\nu} - R\delta(-g)^{1/2} - \delta\hat{E} + g_{\mu\nu}\delta\hat{E}^{\mu\nu}] = \tfrac{1}{2}[-\delta\hat{R} + \hat{g}^{\mu\nu}\delta R_{\mu\nu} - \delta\hat{E} + g_{\mu\nu}\delta\hat{E}^{\mu\nu}]\\ &= \tfrac{1}{2}[\hat{g}^{\mu\nu}\delta R_{\mu\nu} + g_{\mu\nu}\delta\hat{E}^{\mu\nu} + \delta(\hat{\sigma} - 3\hat{p})]\,.\end{aligned} \tag{67}$$

Inserting (67) into (65) and multiplying by u^λ one has

$$u^\lambda[(\hat{\sigma}+\hat{p})\delta\zeta - p\zeta\delta(-g)^{1/2} + 3\zeta\delta\hat{p}] = u^\lambda(\hat{\sigma}+\hat{p})u_\nu\delta v^\nu + \tfrac{1}{2}[g_{\mu\nu}\delta\hat{E}^{\mu\nu} + \hat{g}^{\mu\nu}\delta R_{\mu\nu}]v^\lambda + \tfrac{1}{2}v^\lambda\delta(\hat{\sigma}+3\hat{p})\,. \tag{68}$$

From (57)

$$\tfrac{1}{2}(\hat{\sigma}+3\hat{p})\delta v^\lambda = -\tfrac{1}{2}v^\lambda\delta(\hat{\sigma}+3\hat{p}) + \tfrac{1}{2}\delta\hat{m}^\lambda - \delta(\Omega\hat{j}^\lambda) - \delta(v^\nu\hat{E}^\lambda_\nu) + \tfrac{1}{2}\delta(v^\lambda\hat{E})\,. \tag{69}$$

Combine (69) with (61) and obtain

$$u^\lambda(\hat{\sigma}+\hat{p})u_\nu\delta v^\nu + \tfrac{1}{2}v^\lambda\delta(\hat{\sigma}+3\hat{p}) = \tfrac{1}{2}\delta\hat{m}^\lambda - \delta(2\Omega\hat{j}^\lambda) - 2\delta(v^\nu\hat{E}^\lambda_\nu) + \delta(v^\lambda\hat{E}) - 2\hat{p}\delta v^\lambda + \Omega\delta\hat{j}^\lambda + v^\nu\delta\hat{E}^\lambda_\nu - \tfrac{1}{2}v^\lambda\delta\hat{E}\,. \tag{70}$$

Defining

$$\hat{\mathfrak{D}}^\lambda \equiv \tfrac{1}{2}(\delta\hat{m}^\lambda + v^\lambda\hat{g}^{\mu\nu}\delta R_{\mu\nu}) \tag{71}$$

[cf. (18)] and substituting (70) in the right-hand side of (68) one has

$$u^\lambda[(\hat{\sigma}+\hat{p})\delta\zeta - p\zeta\delta(-g)^{1/2} + 3\zeta\delta\hat{p}] + 2\hat{p}\delta v^\lambda = \hat{\mathfrak{D}}^\lambda - 2\delta(\Omega\hat{j}^\lambda) + \Omega\delta\hat{j}^\lambda - 2\delta(v^\nu\hat{E}^\lambda_\nu) + \delta(v^\lambda\hat{E}) + v^\nu\delta\hat{E}^\lambda_\nu - \tfrac{1}{2}v^\lambda\hat{E}^{\mu\nu}\delta g_{\mu\nu}\,. \tag{72}$$

Because the following identity holds

$$\begin{aligned}\delta[(\hat{\sigma}+3\hat{p})v^\lambda]-2\hat{p}\delta v^\lambda-u^\lambda[(\hat{\sigma}+\hat{p})\delta\zeta-p\zeta\delta(-g)^{1/2}+3\zeta\delta\hat{p}]\\ &=\zeta\delta(\sigma\hat{u}^\lambda)+\sigma\hat{u}^\lambda\delta\zeta+3v^\lambda\delta\hat{p}+3\hat{p}\delta v^\lambda-2\hat{p}\delta v^\lambda-u^\lambda\hat{\sigma}\delta\zeta-u^\lambda\hat{p}\delta\zeta+v^\lambda p\delta(-g)^{1/2}-3v^\lambda\delta\hat{p}\\ &=\zeta\delta(\sigma\hat{u}^\lambda)+\hat{p}\delta v^\lambda-u^\lambda\hat{p}\delta\zeta+v^\lambda p\delta(-g)^{1/2}=\zeta\delta(\sigma\hat{u}^\lambda)+\zeta\hat{p}\delta u^\lambda+\zeta p u^\lambda\delta(-g)^{1/2}\\ &=\zeta[\delta(\sigma\hat{u}^\lambda)+p\delta\hat{u}^\lambda]\end{aligned} \tag{73}$$

or equivalently

$$u^\lambda[(\hat{\sigma}+\hat{p})\delta\zeta-p\zeta\delta(-g)^{1/2}+3\zeta\delta\hat{p}]+2\hat{p}\delta v^\lambda=\delta[(\hat{\sigma}+3\hat{p})v^\lambda]-\zeta[\delta(\sigma\hat{u}^\lambda)+p\delta\hat{u}^\lambda], \tag{74}$$

one obtains, on substituting (74) into (72),

$$\zeta[\delta(\sigma\hat{u}^\lambda)+p\delta\hat{u}^\lambda]=\delta[(\hat{\sigma}+3\hat{p})v^\lambda]+\delta(2\Omega\hat{j}^\lambda)+2\delta(v^\nu\hat{E}^\lambda_\nu)-\delta(v^\lambda\hat{E})-\hat{\mathfrak{D}}^\lambda-\Omega\delta\hat{j}^\lambda-v^\nu\delta\hat{E}^\lambda_\nu+\tfrac{1}{2}v^\lambda\hat{E}^{\mu\nu}\delta g_{\mu\nu}. \tag{75}$$

Put the variation of (57) in place of the first member of the right-hand side of (75) and rearrange to establish

$$\zeta[\delta(\sigma\hat{u}^\lambda)+p\delta\hat{u}^\lambda]=\delta\hat{m}^\lambda-\Omega\delta\hat{j}^\lambda-\hat{\mathfrak{D}}^\lambda-v^\nu\delta\hat{E}^\lambda_\nu+\tfrac{1}{2}v^\lambda\hat{E}^{\mu\nu}\delta g_{\mu\nu}. \tag{76}$$

Now make use of (10), divide by ζ, and rewrite (76), by using (12), in the form

$$\delta(s\hat{u}^\lambda)=\beta(\delta\hat{m}^\lambda-\hat{\mathfrak{D}}^\lambda)-\gamma\delta\hat{j}^\lambda-\alpha\delta(n\hat{u}^\lambda)-\beta(v^\nu\delta\hat{E}^\lambda_\nu+\tfrac{1}{2}v^\lambda\hat{E}^{\mu\nu}\delta g_{\mu\nu})d\Sigma_\lambda, \tag{77}$$

which upon multiplication by $d\Sigma_\lambda$, and with the definitions (15), (16), (17), and (18) for $\mathfrak{M}$, $\mathcal{J}$, $\mathfrak{N}$, and $\mathfrak{D}$, respectively, yields

$$\delta(s\hat{u}^\lambda d\Sigma_\lambda)=\beta(\delta\mathfrak{M}-\mathfrak{D})-\gamma\delta\mathcal{J}-\alpha\delta\mathfrak{N}-\beta(v^\nu\delta\hat{E}^\lambda_\nu+\tfrac{1}{2}v^\lambda\hat{E}^{\mu\nu}\delta g_{\mu\nu})d\Sigma_\lambda. \tag{78}$$

Finally, by integrating (78) over Σ and using (5) and (9), one obtains expression (14), which is an identity equivalent to (10).

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[1]This is true, not because of formal analogies in the equations of equilibrium, but because relativistic clusters have a maximum of entropy for equilibrium configurations. This may be proved by statistical-mechanics arguments. See G. Horwitz and J. Katz, unpublished.

[2]J. B. Hartle and D. H. Sharp, Astrophys. J. 147, 317 (1967).

[3]J. M. Bardeen, Astrophys. J. 162, 71 (1970).

[4]In fact, as we shall see, weaker conditions may be imposed.

[5]R. C. Tolman and P. Ehrenfest, Phys. Rev. 36, 1791 (1930).

[6]W. J. Cocke, Ann. Inst. Henri Poincaré 2, 283 (1965).

[7]In the process of varying S, trial configurations are not equilibrium configurations.

[8]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, New York, 1973).

[9]The metric $g_{\mu\nu}$ has signature -2; we shall denote $\det(g_{\mu\nu})$ by g (<0) and $A(-g)^{1/2}$ by $\hat{A}$ for every quantity A.

[10]A. Papapetrou, Proc. R. Irish. Acad. 52, 11 (1948).

[11]Units: $c=\hbar=8\pi G=k=1$ in standard notations.

[12]For simplicity we shall consider simply connected configurations; no empty blobs in the matter.

[13]L. Eisenhart, *Riemannian Geometry* (Princeton Univ. Press, Princeton, 1949).

[14]Also called chemical free energy, see C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[15]As will appear clearly in Sec. IV C [see (47)], the 10 components of the metric are not independent. They will, however, be treated as such in our variation calculation. For a justification of this procedure, see Bardeen (Ref. 3), p. 77.

[16]This identity is useful in studying not only entropy extremum of rotating relativistic systems [G. Horwitz and J. Katz, Ann. Phys. (N.Y.) 76, 301 (1973)], but also their stability [J. Katz and G. Horwitz, Astrophys. J. 194, 439 (1974)]. In addition, the identity is useful in establishing the connection between various energy or entropy extremum theorems [J. Katz, J. Phys. A 5, 781 (1972).

[17]Square brackets around indices means antisymmetrization; ∂_λ represents an ordinary derivative relative to x^λ, while D_λ is a covariant differentiation.

[18]A. Komar, Phys. Rev. 113, 934 (1959).

[19]J. Katz and G. Horwitz, in *Modern Developments in Thermodynamics*, edited by B. Gal-Or (Wiley, New York, 1973).

[20]See B. Carter, in *Black Holes*, 1972 Les Houches Lectures, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), pp. 159–166.

[21]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Vol. II.

[22]To be precise, the word contour means, in this work, interior contour. Insisting on mathematical rigor demands working with a curve infinitesimally displaced from the contour C towards the interior of the system.

[23]Such coordinates always exist if the differentiability class of C is at least the same as that of the manifold

(C^3). The coordinates may be further specialized to Fermi coordinates. Strictly speaking, C has to be univalent but one may always remove a point or use two coordinate patches [see N. J. Hicks, *Notes on Differential Geometry* (Van Nostrand, Princeton, 1965)].

[24]Had we known something about the behavior of the coefficients of $\partial_a \gamma$ or $\partial_a \beta$ in (49), we could have used this information instead of one equation of the pair (28).