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Einstein-Cartan cosmologies with a magnetic field

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This note points out that the natural generalization of the Maxwell equations in spaces having torsion leads to a breakdown of the gauge invariance and charge conservation principle. The equations are, however, taken in a form so as to preserve gauge invariance; and the possibility of bounce is investigated in the presence of a magnetic field for Bianchi type-I universes with $p = 0$ and $p = \rho$.

I. INTRODUCTION

In recent years, attempts have been made by Trautman[1] and others (very readable reviews have been given by Kuchowicz[2] and Hehl[3,4]) to link up spin with geometry by considering the affinities to be nonsymmetric. The basic geometry is thus non-Riemannian and the field equations are obtained from a variational principle where the metric tensor components and the affinities are varied independently. The antisymmetric part of the affinity is coupled with the intrinsic spin density of material particles and, in particular, vanishes in the absence of spin.

In the study of cosmological models with spinning dust particles, it has been claimed by Kopczynski,[5] Trautman,[6] Isham *et al.*,[7] and Tafel[8] that the new formalism, commonly called the Einstein-Cartan theory, leads to a spin-spin interaction which opposes the usual gravitational attraction of matter and thus may arrest the collapse to singularities in cosmological models. However, early workers like Trautman and Isham *et al.* have held that the spin-spin interaction is effective only if the spins of the particles are aligned. It has been suggested that alignment might be brought about by a high-enough magnetic field in the early phase of the universe. It then seems imperative to study the role of magnetic fields in this formalism—this, as far as the present author is aware, has not been done so far. (It has recently been argued by Hehl *et al.*[9] that even in the absence of alignment, the spin interactions may be present as there are some squared terms involving the spin in the Einstein-Cartan field equations. Even then it would be of interest to investigate the early conditions of universes with a magnetic field.) Indeed, a magnetic field not only contributes to the energy density but, as will appear in the following sections, is also associated with shear. Both of these would augment the gravitational effect as is apparent from the expansion equations deduced by the present author[10] and by Stewart and Hájíček.[11] The present note shows that even in the presence of a magnetic field, there exist solutions where the collapse is halted by the action of spin.

II. FIELD EQUATIONS AND THEIR INTEGRATION

As has been already pointed out by Hehl,[4] the Maxwell field does not couple to torsion. It is easy to see that if it did couple to torsion, one would have to sacrifice the charge conservation principle. If one tries to obtain the electromagnetic field equations from the variational principle with the usual Lagrangian,

$$\delta \int (\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^{\mu}A_{\mu})(-g)^{1/2}dx^0dx^1dx^2dx^3 = 0 ,$$

with

$$F_{\mu\nu} \equiv A_{\mu}|_{\nu} - A_{\nu}|_{\mu} \neq A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}$$

(where a vertical line indicates covariant differentiation with the nonsymmetric affinity, a semicolon denotes the covariant derivative with the Christoffel indices, and a comma denotes the ordinary partial derivative), one finds that the gauge invariance is lost. The variational principle gives

$$F^{\mu\nu}|_{\nu} = J^{\mu}$$

or, if $Q_{\alpha}{}^{\sigma}{}_{\beta}$ be the antisymmetric part of the affinity,

$$F^{\mu\nu}{}_{;\nu} + Q_{\alpha}{}^{\mu}{}_{\nu}F^{\alpha\nu} = J^{\mu}$$

so that

$$\frac{1}{(-g)^{1/2}}[J^{\mu}(-g)^{1/2}]_{,\mu} = \frac{1}{(-g)^{1/2}}[Q_{\alpha}{}^{\mu}{}_{\nu}F^{\alpha\nu}(-g)^{1/2}]_{,\mu} ,$$

or

$$[(J^{\mu} - Q_{\alpha}{}^{\mu}{}_{\nu}F^{\alpha\nu})(-g)^{1/2}]_{,\mu} = 0 .$$

Thus if J^{μ} is identified as the charge current vector. then in general the charge conservation principle is not valid. It therefore seems more appropriate to adopt the prescription that the electromagnetic field tensor is to be defined by

$$F_{\mu\nu} \equiv A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}$$

when we have the usual Maxwell's field equations,

$$F^{\mu\nu}{}_{;\nu} = J^{\mu} ,$$

$$[J^{\mu}(-g)^{1/2}]_{,\mu} = 0 .$$

To have an idea of the influence of the magnetic field we consider the Bianchi type-I line element

$$ds^2 = c^2dt^2 - e^{2\phi}dx^2 - e^{2\theta}dy^2 - e^{2\psi}dz^2 , \tag{1}$$

where ϕ, θ, and ψ are functions of t alone. The energy-momentum tensor for the fluid plus electromagnetic field is

$$T^{\mu}{}_{\nu} = (p + \rho c^2)v^{\mu} v_{\nu} - p\delta^{\mu}_{\nu} + \frac{1}{4\pi}\left(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^{\mu}_{\nu} - F^{\mu\alpha}F_{\nu\alpha}\right) ,$$

where $v^{\mu} = (1/c)\delta^{\mu}_0$. In general, there will be an interaction between the spins and the magnetic field tending to bring about the alignment of spins. However, for simplicity, we shall restrict ourselves to the condition where the alignment has already been brought about and so no torque is exerted on the elementary magnetic dipoles. We shall take the spin as well as the magnetic field to be in the x direction, so that only $\mathcal{S}_{23}$ $(= -\mathcal{S}_{32})$ and F_{23} $(= -F_{32})$ of the spin tensor and the electromagnetic field tensor are nonvanishing.

The condition $T^{23} = 0$ gives $\dot{\psi} = \dot{\theta}$ and the other field equations are

$$\frac{G}{c^2}(8\pi p - B^2) = G^1_1 + 64\pi^2\mathcal{S}^2G^2 , \tag{2}$$

$$\frac{G}{c^2}(8\pi p + B^2) = G^2_2 + 64\pi^2\mathcal{S}^2G^2 , \tag{3}$$

$$\frac{G}{c^2}(8\pi\rho c^2 + B^2) = G^0_0 - 64\pi^2\mathcal{S}^2G^2 , \tag{4}$$

where B is the intensity of the magnetic field and $\mathcal{S}$ is the spin density given by $\mathcal{S}^2 = \frac{1}{2}\mathcal{S}_{ik}\mathcal{S}^{ik}$. We have the conservation relations

$$64\pi^2\mathcal{S}^2 = S^2R^{-6} , \tag{5}$$

$$B^2 = b^2e^{-4\psi} , \tag{6}$$

$$d(\rho c^2R^3) + 3pR^2dR = 0 , \tag{7}$$

where S and b are constants and $R^3 = \exp(\phi + 2\psi)$. G^{α}_{β} are the Einstein tensor components $R^{\alpha}_{\beta} - \frac{1}{2}R\delta^{\alpha}_{\beta}$ for the line element (1).

From Eqs. (2) and (3) we get

$$\frac{d}{dt}[R^3(\dot{\psi} - \dot{\phi})] = 2Gc^2B^2R^3 \tag{8}$$

so that with $B \neq 0$, the shear is nonvanishing. To proceed further, we require a relation between the pressure and the energy density. We consider first the case of dust $p = 0$. From Eqs. (3) and (4) we get

$$\dot{\phi} + \dot{\psi} = AtR^{-3} , \tag{9}$$

where the constant A is given by

$$A = 8\pi G\rho R^3 \tag{10}$$

and an arbitrary constant of integration has been absorbed in the time variable. In view of Eq. (9) and the definition of R, we get

$$\dot{\psi} = \frac{3\dot{R}}{R} - \frac{At}{R^3} , \tag{11}$$

$$\dot{\phi} = \frac{2At}{R^3} - \frac{3\dot{R}}{R} . \tag{12}$$

Substituting from the above in Eq. (8) and using relation (6) we get

$$\frac{d^2}{dt^2}(R^3) = \tfrac{3}{2}A + Gc^2b^2e^{\phi - 2\psi} . \tag{13}$$

Adding Eq. (2) and (3) and using relations (11) and (12) and also remembering that $p = 0$, we get after a little reduction

$$24At\dot{R}R^{-4} - 30R^{-2}\dot{R}^2 - 6R^{-1}\ddot{R} + R^{-6}(2G^2S^26A^2t^2) + AR^{-3} = 0 . \tag{14}$$

Equation (14) is essentially the expansion relation deduced by Stewart and Hájíček[11] and a little later by Tafel[8]:

$$\theta_{,\alpha}v^{\alpha} + \tfrac{1}{3}\theta^2 - \dot{v}^{\alpha}{}_{;\alpha} + 2\sigma^2 - (\omega_{ik} + \mathcal{S}_{ik})(\omega^{ik} + \mathcal{S}^{ik}) + 4\pi(\rho + 3p) = 0 ,$$

with the difference that in our case there is a magnetic field but vanishing rotation $(\omega_{ik} = 0)$. Thus the Stewart-Hájíček equation becomes in this case

$$\theta_{,\alpha}v^{\alpha} + \tfrac{1}{3}\theta^2 - \dot{v}^{\alpha}{}_{;\alpha} + 2\sigma^2 - 2\mathcal{S}^2 + 4\pi\left(\rho + 3p + \frac{B^2}{4\pi}\right) = 0 . \tag{15}$$

A straightforward calculation of σ^2 and B^2 from Eqs. (8), (11), and (12) shows that Eqs. (14) and (15) are identical. [In Eq. (15), $G = c = 1$.]

We have not been able to integrate Eq. (14) in complete generality. However, we note first two cases where the solutions are already known.

(i) $B = 0$, the Trautman[6]-Kopczynski[5] case. In this case, Eq. (13) is readily integrated to give

$$R^3 = \tfrac{3}{4}At^2 + \alpha t + \frac{S^2G^2 - \alpha^2}{A} = \tfrac{3}{4}A\left(t + \frac{2\alpha}{3A}\right)^2 + \frac{S^2G^2 - \frac{4}{3}\alpha^2}{A} , \tag{16}$$

where α is an arbitrary constant of integration related to the shear, and another constant of integration has been determined by using Eq. (14). In this case the occurrence of a singularity is avoided if $G^2S^2 > \frac{4}{3}\alpha^2$. (Compare the discussion of Stewart and Hájíček.[11])

(ii) $S=0$. The solution in this case has been given by Thorne[12] in the form

$$t = a_0(e^\psi + 2\beta)(e^\psi - \beta)^{1/2} , \tag{17}$$

$$R^3 = e^{3\psi} + 4\beta e^{2\psi} - 8\beta^2 e^\psi , \tag{18}$$

where

$$a_0{}^2 = \frac{4}{3A} = \frac{1}{6\pi G\rho R^3} , \tag{19}$$

$$\beta = \frac{3b^2}{A} . \tag{20}$$

Here of course there is always a singularity of vanishing R^3.

Finally, we present a particular solution in the neighborhood of $t=0$ to exhibit the existence of singularity-free solutions even in the presence of magnetic fields. We take

$$e^\psi = e^{-\phi} = R^3 , \tag{21}$$

$$R^6 = \frac{G^2S^2t^2}{c^4} + \mu^2 . \tag{22}$$

Equations (13) and (14) are approximately satisfied if

$$A\mu \ll \frac{2S^2G^2}{c^4} , \tag{23}$$

$$b = \frac{G^{1/2}}{c}\mu S . \tag{24}$$

Thus the situation represented by Eqs. (21) and (22) is realizable near $R = R_{\min} = \mu^{1/3}$, if there is a magnetic field present satisfying the inequality (23). At the minimum of R, both $\dot\psi$ and $\dot\phi$ vanish, but while e^ψ is a minimum e^ϕ is a maximum and the universe has a cigarlike form, the long axis lying in the common direction of the spin and the magnetic field. As time goes on, the term $G^2S^2R^{-6}$ becomes small and the situation approaches the case of Thorne as given in case (ii) above. The expansion is still anisotropic although all the directions are expanding. Still later when $e^\psi \gg \beta$, the solution approaches the isotropic Friedmann universe.

For the solution given by Eqs. (21) and (22), we have

$$R_{\min}{}^3 = \mu = \frac{cb}{G^{1/2}S} < \frac{2S^2G^2}{c^4A} = \frac{2S^2G}{8\pi c^4\rho R^3} \sim 1\ \text{cm}^3 .$$

Thus the minimum occurs at a volume which is smaller than even that in the corresponding Trautman case[6] of vanishing magnetic field. With $R_{\min} < 1$ cm (the Trautman value), Eq. (24) gives

$$\frac{GN^2\hbar^2}{4B_{\max}{}^2c^2} < 1 ,$$

with (25)

$N \sim 10^{79}$ (N being the number of baryons),

$B_{\max} > 10^{39}$ gauss. [The inequality (23) and Eq. (24) set an upper bound to b vis-à-vis the spin S and the density ρ. However, for a given S, b varies directly as μ ($=R_{\min}{}^3$). So as $R_{\min}$ decreases, b decreases; but according to Eqs. (6) and (21), the decrease of b is accompanied by an increase of B. Thus the upper bound of b corresponds to a lower bound of B.] The lower bound of B far exceeds the critical value ~10^{13} gauss at which one would expect the field to show quantum effects (cf. Euler and Kockel[13]). However, our approximate solution is a very special one in the sense that it has been obtained under condition (23). It seems likely that other solutions exist where the magnetic field does not attain so enormous a value.

An explicit integration of the field equations in terms of elementary functions is easily obtained in case $p=\rho$. Such an equation of state may appear unphysical. However, we present the solution because of its simplicity and because it gives us some idea of the general state of affairs when the pressure is nonvanishing. Equations (2) and (3) now give

$$\dot\phi + \dot\psi = \alpha R^{-3} \tag{26}$$

so that

$$\dot\psi = \frac{3\dot R}{R} - \frac{\alpha}{R^3} , \tag{27}$$

$$\dot\phi = \frac{2\alpha}{R^3} - \frac{3\dot R}{R} , \tag{28}$$

where α is an arbitrary constant of integration. Equation (14) is now replaced by

$$24\alpha\dot R R^{-4} - 30R^{-2}\dot R^2 - 6R^{-1}\ddot R + (2G^2S^2 - 6\alpha^2 - 2Q)^{-6}R = 0 , \tag{29}$$

where the constant Q is defined by

$$Q = 8\pi G\rho R^6 . \tag{30}$$

Equation (29) has the first integral,

$$\frac{d}{dt}(R^3) = 4\alpha + \beta t R^{-3} , \tag{31}$$

with

$$\beta \equiv S^2 - 3\alpha^2 - Q \ . \tag{32}$$

Thus R^3 has a minimum if $\beta > 0$, which would correspond to the spin dominating over the combined gravitational influence and the shear. The integral of Eq. (31) is, in this case,

$$|R^6 - 4\alpha R^3 t - \beta t^2| = A^2 \left[\frac{(\mu - 2\alpha)t + R^3}{(\mu + 2\alpha)t - R^3}\right]^{2\alpha/\mu} ,$$

with

$$\mu^2 \equiv \beta + 4\alpha^2$$

and A is an arbitrary constant of integration which determines the value of R at $t = 0$. The minimum of R occurs at

$$t = -4\alpha R_{\min}{}^3 \beta^{-1} ,$$

$$R_{\min}{}^3 = A\left(\frac{\mu - 2\alpha}{\mu + 2\alpha}\right)^{2\alpha/\mu} .$$

In the case $\beta \leqslant 0$, the integrals of Eq. (31) are

$$|R^6 - 4\alpha R^3 t - \beta t^2| = A^2 \exp\left(\frac{4\alpha t}{R^3 - 2\alpha t}\right) \text{ for } \beta + 4\alpha^2 = 0$$

$$= A^2 \exp\left[-\frac{4\alpha}{\mu}\tan^{-1}\left(\frac{R^3 - 2\alpha t}{\mu t}\right)\right]$$

$$\text{for } -\mu^2 \equiv \beta + 4\alpha^2 < 0 \ .$$

Substituting from Eq. (31) in Eq. (8) we get

$$B^2 = \beta R^{-12}(R^6 - 4\alpha R^3 t - \beta t^2) ,$$

so that if B be nonvanishing we have

$$e^{\psi} = R^3 |R^6 - 4\alpha R^3 t - \beta t^2|^{-1/4} ,$$

$$e^{\phi} = R^{-3} |R^6 - 4\alpha R^3 t - \beta t^2|^{1/2} ,$$

while if $B = 0$, one has if $\beta \neq 0$ (in the case $\beta = 0$, $R_{\min}$ coincides with the singular state $R = 0$)

$$R^6 - 4\alpha R^3 t - \beta t^2 = 0$$

or

$$R^3 = (2\alpha \pm \mu)t ,$$

with

$$\mu^2 = 4\alpha^2 + \beta \geqslant 0 .$$

The asymptotic nature of these solutions near $R = 0$ in the case of vanishing spin has been studied by Thorne.[12]

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