PHYSICAL REVIEW D VOLUME 12, NUMBER 4 15 AUGUST 1975

Gravitational energy-momentum on nonmaximal surfaces*

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(Received 18 February 1975)

Gravitational Cauchy data defined iteratively from an input set which is of pure linearized gauge form representing a nonflat hypersurface appear to possess tachyonic energy-momentum P^μ. We show that such systems do behave correctly: P^μ is positive timelike, and vanishing of the energy implies flat space. Although the P^μ begin at quartic order in the inputs, it is possible to exhibit explicitly the helicity-2 character of the gravitational excitations on which both the energy and the flatness conditions depend.

The extreme nonlinearity of the gravitational constraints complicates explicit evaluation of the physical content of a given initial data set, particularly on an arbitrary spacelike surface. We shall deal here with a Cauchy problem which apparently has tachyonic character: vanishing energy, but not momentum. The data are defined iteratively in terms of an input which is "pure gauge" at the linear level, representing a nonflat hypersurface. Owing to the non-Abelian gauge invariance, however, this system does not describe flat space. [A Yang-Mills analog is $A_i^a = \lambda \partial_i \Lambda^a$, where λ is an expansion parameter; the magnetic field does not vanish to $O(\lambda^2)$.] It will be shown that the undesirable momentum actually does vanish, and that the lowest-order energy is positive, vanishing only for flat space. It is also gratifying that the energy, as well as the conditions for flatness, are defined entirely in terms of a purely transverse-traceless (helicity-2) field excitation. These solutions are thus in full agreement with the physically expected properties of gravitational systems.

The gravitational initial data (g_{ij}, π^{ij}) satisfy the four constraints[1]

$$gR = \pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2, \qquad \nabla_j \pi^{ij} = 0, \tag{1}$$

where all operations are with respect to the positive 3-metric g_{ij} on the surface, and the π^{ij} are the conjugate densities describing extrinsic curvature. The energy-momentum (M, P^i) are defined in the usual way in terms of the linear parts of the constraints. We shall consider for concreteness a particular class of initial data here, but the results apply to more general ones which share the property of having pure gauge parts to lowest order in some expansion parameter. The data are defined by[2]

$$g_{ij} = \phi^4 \delta_{ij}, \qquad \pi^{ij} = \phi^2[(LW)_{ij} + \tfrac{2}{3}\delta_{ij}\nabla^2\sigma], \tag{2}$$

where ∇^2 is the flat Laplacian and $(LW)_{ij} \equiv W_{i,j} + W_{j,i} - \frac{2}{3}\delta_{ij} W_{l,l}$ (we have written an explicit ∇^2 with σ for future convenience, but there is no loss of generality involved). The boundary conditions required for P^μ to be well defined are[1] that $\phi \sim 1 + O(r^{-1})$ and $\pi^{ij} \sim O(r^{-2})$ in Cartesian frames at infinity, and ϕ and W_i will be defined as functional power series in σ by the constraints, Eq. (1), which read

$$-8\nabla^2\phi = \phi^5[(LW)^2 - \tfrac{2}{3}(\nabla^2\sigma)^2] \equiv \rho, \tag{3}$$

$$-2\partial_j[\phi^6 (LW)_{ij}] = \tfrac{4}{3}\phi^6 \partial_i \nabla^2\sigma \equiv j_i, \tag{4}$$

while M and P^i are just proportional, respectively, to $\int d^3r\, \rho$ and $\int d^3r\, j^i$. To order 1, we find $\phi_1 = 0$, $\pi_1^{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)\sigma$, which corresponds to a pure gauge excitation of the field, since the Gauss-Codazzi flat-space imbedding conditions

$$W_{ij} \equiv R_{ij} - g^{-1}(\pi_{im}\pi^m{}_j - \tfrac{1}{2}\pi\pi_{ij}) = 0, \qquad \nabla_m(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi) - (j \leftrightarrow m) = 0 \tag{5}$$

are satisfied, and of course $P_1^\mu \equiv 0$ since j_1^i is a gradient. Physically, we have a bumpy $(\pi \neq 0)$ hypersurface in flat space-time to this order. To order 2, however, space is no longer flat, for while $\pi_2^{ij} \equiv 0$, we find $(\sigma_i \equiv \sigma_{,i},\ \sigma_{ij} \equiv \sigma_{,ij})$

$$-8\phi_2 = \nabla^{-2}\partial_i\partial_j S_{ij}, \qquad S_{ij} \equiv (\delta_{ij}\sigma_k{}^2 - \sigma_i\sigma_j) \tag{6}$$

so that W_{ij} does not in general vanish to this order. Because ϕ_2 is a divergence, M_2 still vanishes. One would expect from the fact that nontrivial excitations start at $O(2)$ that P^μ, being at least bilinear in field excitations, would start at $O(4)$. However, if one computes π_3^{ij}, it does not vanish, unlike ϕ_3, and this leads to an apparently tachyonic situation since $M_3 \equiv 0$, while

$$P_3^i \sim \int d^3r\, \phi_2 \partial_i \nabla^2\sigma \tag{7}$$

does not manifestly vanish. Fortunately, closer analysis of this nonlocal functional shows that j^i

is in fact itself a divergence:

$$8\int d^3r\,\phi_2\partial_i\nabla^2\sigma = 8\int \sigma_i\nabla^2\phi_2$$

$$= -\int \sigma_i S_{jk,jk}$$

$$= -\int [\sigma_{ij}\partial_k(\sigma_j\sigma_k) + \sigma_j{}^2\nabla^2\sigma_i]$$

$$= -\int [(\sigma_j{}^2)_{,i}\nabla^2\sigma + \sigma_j{}^2\nabla^2\sigma_i]$$

$$= 0. \quad (8)$$

The angular momentum also vanishes because of this property of j_i that it is of the form $\partial_j V^{ij}$, V symmetric.

We now come to quartic order, where the P^μ should come into play; $P_4^i = 0$ since $\pi_4^{ij} = 0$, but there is a ϕ_4 which is no longer a total divergence,[3] leading to

$$M_4 \sim \int d^3r\{5\phi_2[(LW_1)^2 - \tfrac{2}{3}(\nabla^2\sigma)^2] + 2(LW_1)(LW_3)\}, \quad (9)$$

which can be reduced to

$$M_4 \sim \int d^3r[S_{ij,ij}(-\nabla^{-2})S_{lm,lm} + 4\sigma_l{}^2 S_{ij,ij}], \quad (10)$$

where the first term is positive but nonlocal in the σ's and the second is local but nonpositive. However, a lengthy calculation shows that M_4 has the remarkably simple form

$$M_4 \sim \tfrac{1}{2}\int d^3r[\nabla S_{ij}^{\mathrm{TT}}]^2 \geq 0, \quad (11)$$

where S_{ij}^{TT} is the transverse-traceless[1] projection of S_{ij}, which is the hallmark of helicity 2. Thus the quartic mass has appropriate dependence on a TT excitation (itself quadratic in σ).

Does $M_4 = 0$ imply that space is flat to the appropriate (quadratic) order? To see this, consider the relevant part, W_2^{ij} of Eq. (5), and define $\overline{W}_{ij} \equiv W_{ij} - \frac{1}{2}\delta_{ij}W_{kk}$. This tensor is clearly traceless, since the trace of W_{ij} is the definition of the scalar constraint in Eq. (1). It is also transverse, since the combination $G_{ij} \equiv R_{ij} - \frac{1}{2}\delta_{ij}R$ is identically transverse, and the terms quadratic in π_1^{ij} can be shown to share this property by direct differentiation. Therefore, $\overline{W}_{ij}$ is a pure TT tensor, and may be shown explicitly to be proportional to the only available TT tensor:

$$\overline{W}_{ij} = \overline{W}_{ij}^{\mathrm{TT}} \sim \nabla^2 S_{ij}^{\mathrm{TT}}. \quad (12)$$

The flatness conditions are just the requirement that the two quantities S^{TT} vanish. But $M_4 = 0$ is equivalent to $S^{\mathrm{TT}} = 0$, which completes the proof.[4] Note that $S^{\mathrm{TT}} = 0$ does not require $\sigma = 0$, since one can have intrinsically curved surfaces ($\phi_2 \neq 0$) appropriately imbedded ($\pi_1^{ij} \neq 0$) in flat space-time.

The above results are gratifying since proofs of positive energy[5] have always been complicated by the nondefinite character of the combination $(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2)$ when $g_{ij} \neq \delta_{ij}$, and this was the source of our problems too, since for $\pi = 0$ the "kinetic energy" contribution is manifestly positive. Of course, one can always (in principle) make a gauge transformation for any asymptotically flat system to a surface $\pi = 0$; in our case this would correspond to a first-order transformation to a nearby maximal hypersurface with $\pi_1^{ij} = 0$. With this regauging, both g_{ij} and π^{ij} would start at $O(2)$ in σ on the new surface. But then one could apply the known results of the linearized approximation to these transformed data, that for arbitrary weak fields (g_{ij}, π^{ij}) satisfying the linearized constraints the total P^μ reads

$$M_L \sim \int d^3r[(\tfrac{1}{2}\nabla g_{ij}^{\mathrm{TT}})^2 + (\pi_{ij}^{\mathrm{TT}})^2] \geq 0,$$
$$\vec{\mathrm{P}}_L \sim \int d^3r\,\pi^{\mathrm{TT}}\vec{\nabla}g^{\mathrm{TT}}. \quad (13)$$

When $M_L = 0$ the flatness conditions are satisfied to linear order, and $M_L^2 - \vec{\mathrm{P}}_L^2 > 0$ for any asymptotically flat excitation.[6] But the only quantity available to construct g^{TT} or π^{TT} for the transformed data in our case is of course S^{TT}, in agreement with our explicit results.

We conclude that initial-value systems with $\pi \neq 0$ do not clash with the required physical properties of gravitational systems, particularly of P^μ and the flat-state conditions, although a general explicit proof will not be easy to construct.

It is a pleasure to acknowledge particularly valuable discussions with T. Regge, C. Teitelboim, and R. Wald, whose insistence that the apparent paradoxes be solved without leaving the usual framework was salutary. The troublesome and therefore interesting initial data which caused this investigation were first posed by N. Ó Murchadha.

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*Work supported in part by the National Science Foundation.

[1]For the initial-value framework, definitions of P^μ, and boundary conditions see R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. 117, 1595 (1960); 122, 997 (1961).

[2]N. Ó Murchadha (private communication).

[3]One may think of ρ_4 as involving Coulomb interactions

among ρ_2's; since only the integrals of $T^{0\mu}$, but not the $T^{\mu\nu}$ themselves, are (linear) gauge invariants in the linearized theory, it is not surprising that M_4, unlike M_2, is nonzero.

[4]We suspect that this connection between M and W_{ij} is very general; if $M=0$ is to imply flatness, there must always exist a way of expressing M as a positive functional of W_{ij}. In our example, $M_4 \sim \int W_{ij}(-\nabla^{-2})W_{ij}d^3r$.

[5]D. Brill and S. Deser, Ann. Phys. (N.Y.) 50, 542 (1968); D. Brill, S. Deser, and L. D. Faddeev, Phys. Lett. 26A, 538 (1968). Advances in some important cases have recently been made by P. S. Jang and R. Geroch (unpublished).

[6]S. Deser, J. Trubatch, and S. Trubatch, Nuovo Cimento 39, 1159 (1965). In our particular example, the π^{ij} happen to vanish at $O(2)$ on the new surface.