Scattering and absorption of electromagnetic waves by a Schwarzschild black hole*

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The scattering and absorption of electromagnetic waves by a spherically symmetric nonrotating black hole is studied in the Schwarzschild background, by means of the known expansion of the modified Debye potentials in partial waves. The power reflection coefficients and the phase shifts of the partial waves are evaluated at both high and low frequencies. Then the scattering and absorption cross sections of the black hole are determined. It is shown that the black hole is almost unable to absorb electromagnetic waves when the wavelength of the radiation is greater than the Schwarzschild radius.

I. INTRODUCTION

Several papers have been recently published on electromagnetic (em) wave propagation in a strong gravitational field. One of the most interesting problems of this kind is the problem of em wave propagation in the presence of a black hole. This problem has been investigated in a spherically symmetric background[1-4] and in a Kerr background.[5,6] In both cases it is possible to write decoupled and separable equations not only for electromagnetic[7-9,1,5] but also for gravitational[5,10,11] test fields (namely for fields not affecting the background).

For em fields in sourceless regions and in spherical static gravitational fields, separation was first obtained by Wheeler, by expanding the quadripotential in terms of four-dimensional vector harmonics.[7] This result has been generalized to the case where sources are present by Ruffini, Tiomno, and Vishveshwara.[1]

By introducing the modified Debye potentials, Mo and Papas have obtained more general equations holding for a class of nonstatic spherical gravitational fields and also for radially inhomogeneous media (in sourceless regions).[9] Decoupled equations may be obtained also in the presence of both sources and radially inhomogeneous media.[12]

In this paper we treat the scattering and absorption of em waves by a Schwarzschild black hole. We use the expansion of the modified Debye potentials in partial waves and calculate the reflection coefficients of the partial waves. Then we calculate the absorption cross section of the black hole both at high frequencies and at low frequencies.

At high frequencies our results confirm and extend the results already obtained by Mashhoon.[13] As to the scattering cross section for small angles, we prove that the expression given by this author for high frequencies is valid also at low frequencies. As to low-frequency reflection coefficients, Matzner[14] considered scalar waves. Price[8,15] dealt with scalar, electromagnetic, and gravitational radiation emitted by a collapsing object, whereas the present paper deals with waves impinging onto the black hole.

In the sequel we use the geometrized mks system, in which the light velocity $c=1$, and in which the mass is measured in meters, and the electric and the magnetic permeability are pure numbers.

II. THE WAVE EQUATION FOR THE RADIAL FUNCTION

Let us consider the Schwarzschild frame of reference $\{t, r, \theta, \phi\}$ with the metric

$$ds^2 = (1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ .$$

Here M is the mass of the black hole and $2M$ is the Schwarzschild radius.

In sourceless regions one can introduce two scalar potentials U, V known as modified Debye potentials,[9] in terms of which the physical fields $\vec{E}$, $\vec{B}$ are given by

$$\vec{E} = \frac{1}{\epsilon}\nabla\times[\nabla\times(V\vec{r})] - \left(1-\frac{2M}{r}\right)^{-1/2}\frac{\partial}{\partial t}[\nabla\times(U\vec{r})]\ ,$$

$$\vec{B} = \nabla\times[\nabla\times(U\vec{r})] + \mu\left(1-\frac{2M}{r}\right)^{-1/2}\frac{\partial}{\partial t}[\nabla\times(V\vec{r})]\ .$$

Here $\vec{E}$ and $\vec{B}$ are locally measurable quantities[16] for observers fixed at $r, \theta, \phi = \text{const}$, ϵ and μ denote the electric and magnetic permeability, respectively, and ∇ is the conventional del operator in the spatial coordinates $\{r, \theta, \phi\}$ with length interval

$$d\sigma^2 = (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ .$$

The modified Debye potentials satisfy the equations

$$\frac{\partial}{\partial r}\left[\frac{1}{\mu}\left(1-\frac{2M}{r}\right)\frac{\partial rU}{\partial r}\right]+\frac{1}{\mu r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 U}{\partial\phi^2}\right]-\frac{r}{1-2M/r}\frac{\partial}{\partial t}\left(\epsilon\frac{\partial U}{\partial t}\right)=0$$

and

$$\frac{\partial}{\partial r}\left[\frac{1}{\epsilon}\left(1-\frac{2M}{r}\right)\frac{\partial rV}{\partial r}\right]+\frac{1}{\epsilon r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 V}{\partial\phi^2}\right]-\frac{r}{1-2M/r}\frac{\partial}{\partial t}\left(\mu\frac{\partial V}{\partial t}\right)=0,$$

respectively.

Let us restrict ourselves to propagation in a vacuum, $\epsilon=\mu=1$. By writing

$$\left.\begin{matrix}U\\V\end{matrix}\right\}=\frac{1}{r}\Psi_l(r)e^{i\omega t}Y_{lm}(\theta,\varphi),$$

where $Y_{lm}(\theta,\phi)$ denotes the (l,m)th ordinary spherical harmonic, we obtain the following equation for the radial function Ψ_l in both cases:

$$\frac{d^2\Psi_l}{dr^{*2}}+\left[\omega^2-\left(1-\frac{2M}{r}\right)\frac{l(l+1)}{r^2}\right]\Psi_l=0, \tag{1}$$

where

$$r^*=r+2M\ln\left(\frac{r}{2M}-1\right). \tag{2}$$

Equation (1) has the same form as the Schrödinger equation in one dimension for a particle of energy ω^2 in the potential

$$U_l(r^*)=\left(1-\frac{2M}{r}\right)\frac{l(l+1)}{r^2}. \tag{3}$$

The quantity

$$K_l(r^*)=\left[\omega^2-\left(1-\frac{2M}{r}\right)\frac{l(l+1)}{r^2}\right]^{1/2} \tag{4}$$

can also be interpreted as a wave number which depends on the radial coordinate. It is real with a minimum at $r=3M$, if the potential peak is smaller than ω^2. Otherwise there is a finite region where $K_l(r^*)$ is an imaginary quantity. In this case two values of r exist, say r_1 and r_2, for which $K_l(r^*)$ vanishes (a third zero of K_l is negative, and therefore of no interest). They are usually called turning points. One finds

$$r_1=\frac{2}{\omega}\left[\frac{l(l+1)}{3}\right]^{1/2}\cos\frac{\eta}{3},$$
$$r_2=\frac{2}{\omega}\left[\frac{l(l+1)}{3}\right]^{1/2}\cos\frac{\eta-2\pi}{3}, \tag{5}$$

where

$$\eta=\arccos\{-3\omega M(3/[l(l+1)])^{1/2}\}$$

and arccos denotes the principal value of the inverse trigonometric function, so that $2M\le r_1\le r_2$.

When the (angular) frequency ω is smaller than the critical frequency ω_c given by

$$\omega_c=\left(\frac{2}{27}\right)^{1/2}\frac{1}{M}, \tag{6}$$

the turning points exist for all partial waves, that is, for all values of l. When $\omega>\omega_c$, the turning points exist only for high-l waves; more precisely, they exist if l is greater than the critical parameter l_c given by

$$l_c(l_c+1)=27\omega^2M^2. \tag{7}$$

The properties of the solutions of Eq. (1) depend strongly on the existence or absence of the turning points and on their distance.

III. THE POWER REFLECTION COEFFICIENTS AT HIGH FREQUENCIES

With reference to Eq. (3), it can be noted that $U_l(r^*)$ vanishes as $1/r^{*2}$ for $r^*\to+\infty$ $(r\to\infty)$ and as $\exp(r^*)$ for $r^*\to-\infty$ $(r\to 2M+0)$. At the same time, the wave number $K_l(r^*)$ tends to the constant value ω. Accordingly, for large values of r^*, the solutions of Eq. (1) can be written as the superposition of waves of the type $\exp(i\omega r^*)$ and $\exp(-i\omega r^*)$. However, the usual boundary condition requires[2-4,14,17] that the wave be purely ingoing for $r^*\to-\infty$. Thus the solution has the asymptotic behaviors

$$\Psi_l\sim A_l\exp(i\omega r^*)\quad(r^*\to-\infty), \tag{8}$$

$$\Psi_l\sim\exp(i\omega r^*)+R_l\exp(-i\omega r^*)\quad(r^*\to+\infty), \tag{9}$$

where R_l is the amplitude reflection coefficient of the potential barrier $U_l(r^*)$ for the lth wave. Obviously $|R_l|^2$ is the power reflection coefficient.

At high frequencies $(\omega\gg\omega_c)$ R_l can be determined by solving Eq. (1) by means of the WKB approximation, which in its recently developed form[18] may be applied even in the case where real turning points do not exist $(l<l_c)$. In Appendix A we prove that the WKB method is certainly reliable when the wavelength of radiation is much less than the Schwarzschild radius $R_S=2M$, and l is either greater or slightly smaller than l_c.

Let us consider the case where the turning points exist. The power reflection coefficient $|R_l|^2$ is given by[18]

$$|R_l|^2 = \frac{\exp(2\theta_l)}{1+\exp(2\theta_l)} , \tag{10}$$

where

$$\theta_l = \int_{r_1^*}^{r_2^*} [U_l(r^*) - \omega^2]^{1/2} dr^* . \tag{11}$$

In Eq. (11) r_1^* and r_2^* denote the turning points for the variable r^*. The integral may be expressed in terms of elliptic integrals,

$$\theta_l = 2\omega[r_2(2r_1+r_2)]^{-1/2}\left\{ \left[\frac{l(l+1)}{\omega^2} + \tfrac{1}{2}r_1 r_2\right]K(k) - \frac{r_2(r_2+2r_1)}{2}E(k) - 2Mr_1\pi(\alpha^2, k) + \frac{4M^2 r_1}{2M-r_1}\pi\left(\frac{2M\alpha^2}{2M-r_1}, k\right)\right\}, \tag{12}$$

where

$$\alpha^2 = 1 - \frac{r_1}{r_2}, \quad k^2 = \frac{r_2^2 - r_1^2}{r_2(r_2+2r_1)} ,$$

and $K(k)$, $E(k)$, and $\pi(\alpha^2, k)$ denote the complete elliptic integrals of the first, second, and third kinds, respectively.[19]

For $l \gg l_c$ one obtains

$$\theta_l = [l(l+1)]^{1/2}\left\{\ln \frac{2[l(l+1)]^{1/2}}{\omega M} + C_l\right\} , \tag{13}$$

where C_l is at most of the order of unity.

When l is slightly greater than l_c we may set

$$U_l(r^*) = \frac{l(l+1)}{9M^2}\left[\frac{1}{3} - \left(\frac{r}{3M} - 1\right)^2\right] . \tag{14}$$

The turning points are approximately given by

$$r_{1,2} = 3M\left\{1 \mp \frac{1}{\sqrt{3}}\left[1 - \frac{27\omega^2 M^2}{l(l+1)}\right]^{1/2}\right\} ,$$

and Eq. (11) yields

$$\theta_l = \frac{\pi}{2}[l(l+1)]^{1/2}\left[1 - \frac{27\omega^2 M^2}{l(l+1)}\right] . \tag{15}$$

In the second case, $l \lesssim l_c$, the power reflection coefficient is still given by Eq. (10), but now θ_l has the form

$$\theta_l = i\int_{\tilde{r}_1^*}^{\tilde{r}_2^*} [\omega^2 - \tilde{U}_l(r^*)]^{1/2} dr^* . \tag{16}$$

In Eq. (16) $\tilde{r}_1^*$ and $\tilde{r}_2^*$ denote complex-conjugated zeros of the wave number ($\mathrm{Im}\,\tilde{r}_1^* < 0$, $\mathrm{Im}\,\tilde{r}_2^* > 0$), and the function $\tilde{U}_l(r^*)$ is the analytic continuation of $U_l(r^*)$ in the complex plane. It is to be noted that $\tilde{r}_1^*$ and $\tilde{r}_2^*$ are branch points of the wave number $\tilde{K}_l(r^*)$. Consequently, it is a many-valued function. We choose its phase so that on the real axis $\tilde{K}_l(r^*) = K_l(r^*)$. Moreover, in Eq. (16) the path of integration must not turn around the branch points. As a result, θ_l turns out to be (real and) negative. Since l is slightly smaller than l_c, we may use Eq. (14). The zeros of the wave number are now given by

$$\tilde{r}_{1,2} = 3M\left\{1 \mp \frac{i}{\sqrt{3}}\left[1 - \frac{l(l+1)}{27\omega^2 M^2}\right]^{1/2}\right\} ,$$

and Eq. (16) yields

$$\theta_l = \frac{\pi}{2}[l(l+1)]^{1/2}\left[1 - \frac{27^2\omega^2 M^2}{l(l+1)}\right] . \tag{17}$$

Equation (17) coincides with Eq. (15). From Eqs. (10), (15), and (17) it is clear that for $l \to l_c = 3\sqrt{3}\,\omega M$ the reflection coefficient tends to $\frac{1}{2}$. It turns out, however, that at high frequencies the partial waves are almost completely reflected or transmitted according to whether $l \gtrsim l_c$ or $l \lesssim l_c$ (see Fig.1).

For example, at $\omega = 10^2\omega_c$ one finds $1 - |R_l|^2 \approx 10^{-5}$ and $|R_l|^2 \approx 10^{-5}$ for $(l - l_c)/l_c = 10^{-2}$ and $(l_c - l)/l_c = 10^{-2}$, respectively.

For $l \ll l_c$ the WKB approximation is not applicable and we must use another technique for the calculation of R_l.

Equation (1) is equivalent to the integral equation[20]

$$\Psi_l(r^*) = \frac{i}{2\omega}\int_{-\infty}^{+\infty} e^{-i\omega|r^* - \xi|}U_l(\xi)\Psi_l(\xi)d\xi + e^{i\omega r^*} .$$

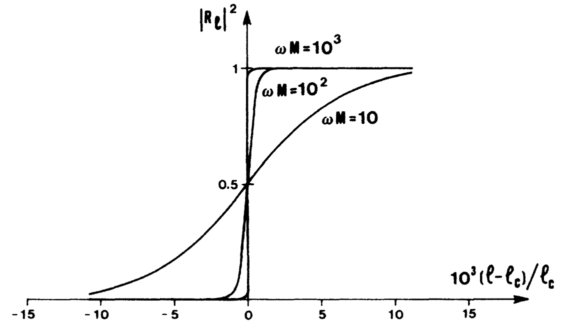

FIG. 1. The power reflection coefficient $|R_l|^2$ vs $10^3(l - l_c)/l_c$.

For $r^* \to +\infty$ we obtain

$$\Psi_l(r^*) = \frac{i}{2\omega} e^{-i\omega r^*} \int_{-\infty}^{+\infty} e^{i\omega\xi} U_l(\xi)\Psi_l(\xi) d\xi + e^{i\omega r^*} + O\left(\frac{1}{r^*}\right) .$$

Consequently,

$$R_l = \frac{i}{2\omega} \int_{-\infty}^{+\infty} e^{i\omega\xi} U_l(\xi)\Psi_l(\xi) d\xi . \quad (18)$$

It appears from Eq. (18) that the reflection coefficient R_l is proportional to the Fourier transform of the infinitely differentiable function $U_l\Psi_l$. Accordingly, it tends to zero more rapidly than any negative power of ω when $\omega \to \infty$.

For $l \ll l_c$ the potential barrier may be treated as a little perturbation, and the Born approximation may be used.

By setting

$$\Psi_l(\xi) = e^{i\omega\xi} \quad (19)$$

and recalling expression (3) for U_l, Eq. (18) yields

$$|R_l|^2 = \left[\frac{l(l+1)}{4\omega M}\right]^2 |I(\omega M)|^2 , \quad (20)$$

where

$$I(\omega M) = 2M \int_{-\infty}^{+\infty} e^{2i\omega r^*} \left(1 - \frac{2M}{r}\right) \frac{dr^*}{r^2} = \int_{-\infty}^{+\infty} \exp\{4i(\omega M)[x + \ln(x-1)]\} \frac{dx}{x^2} .$$

Equation (20) makes evident the dependence of R_l on l. It has already been noted that R_l tends to zero very rapidly when $\omega \to \infty$.

IV. THE POWER REFLECTION COEFFICIENTS AT LOW FREQUENCIES

As already noted, for $\omega \ll \omega_c$, where ω_c is given by Eq. (6), real turning points exist for all partial waves. From Eq. (5) we obtain

$$r_1 = 2M\left[1 + \frac{4\omega^2 M^2}{l(l+1)} + O\left(\frac{\omega^4 M^4}{l^4}\right)\right], \quad (21)$$

$$r_2 = \frac{[l(l+1)]^{1/2}}{\omega} - M\left[1 + O\left(\frac{\omega M}{l}\right)\right], \quad (22)$$

where $O(x)$ denotes a quantity of order x.

In order to find an expression for R_l at low frequency, we have treated Eq. (1) by means of the technique already used by Ipser and Fackerell in investigations on gravitational waves.[21,22]

The method consists of finding approximate solutions of Eq. (1) separately in three regions, namely on the left-hand side of the barrier $r \lesssim r_1$, in the region where the potential peak dominates $r_1 \lesssim r \ll r_2$, and on the right-hand side of the peak $r_1 \ll r \leq \infty$, and then matching them by imposing the usual continuity conditions.

In the region $r < r_1$ the function $U_l(r^*)$ falls off as $\exp(r^*)$ for $r^* \to -\infty$ and reduces to a half over a distance $\Delta r^* \approx M$, much smaller than the wavelength $\lambda = 2\pi/\omega$, since $\omega \ll \omega_c$. Consequently, the function (8)

$$\Psi_l = e^{i\omega r^*}$$

represents a good approximation to the solution of Eq. (1) not only for $r^* \to -\infty$, that is, $r \to 2M$, but in the whole region $2M < r \lesssim r_1$.

From Eqs. (2) and (21) it is clear that $r_1^* \approx 2M \ln\{4\omega^2 M^2/[l(l+1)]\}$, so that for $r \simeq r_1$ we have

$$\Psi_l \simeq 1 + i\omega r^* . \quad (23)$$

For $r_1 < r \ll r_2$ where the potential peak dominates, we can neglect the term ω^2 in the expression of $K_l{}^2$. With such an approximation, Eq. (1) takes the form, in terms of the variable r,

$$r(r-2M)\frac{d^2\Psi_l}{dr^2} + 2M\frac{d\Psi_l}{dr} - l(l+1)\Psi_l = 0 . \quad (24)$$

This is a hypergeometric equation. Two independent solutions of it are (see Appendix B)

$$\Psi_l^{(1)}(r) = \frac{r}{2M} P_l\left(\frac{r}{M} - 1\right) - \frac{1}{2(2l+1)}\left[P_{l+1}\left(\frac{r}{M} - 1\right) - P_{l-1}\left(\frac{r}{M} - 1\right)\right],$$
$$\Psi_l^{(2)}(r) = \frac{r}{2M} Q_l\left(\frac{r}{M} - 1\right) - \frac{1}{2(2l+1)}\left[Q_{l+1}\left(\frac{r}{M} - 1\right) - Q_{l-1}\left(\frac{r}{M} - 1\right)\right], \quad (25)$$

respectively.

where P_l is the Legendre polynomial and Q_l is the Legendre function of the second kind.

For $r \to 2M$, $\Psi_l^{(1)}(r)$ and $\Psi_l^{(2)}(r)$ behave like

$$\Psi_l^{(1)} \sim 1 + l(l+1)\left(\frac{r}{2M} - 1\right)$$
$$\Psi_l^{(2)} \sim -\tfrac{1}{2}\ln\left(\frac{r}{2M} - 1\right) \quad (26)$$

From Eqs. (23) and (26) we infer that at $r \approx r_1$, the function $\exp(i\omega r^*)$ fits smoothly onto

$$\Psi_l = (1 + 2i\omega M)\Psi_l^{(1)} - 4i\omega M \Psi_l^{(2)} ,$$

or also, by neglecting terms of the order of ωM, onto the function $\Psi_l^{(1)}$. For $r \gg r_1$ Eq. (1) may be replaced by

$$\frac{d^2\Psi_l}{dr^{*2}} + \left[\omega^2 - \frac{l(l+1)}{r^{*2}}\right]\Psi_l = 0 . \quad (27)$$

It is well known that this equation has the solutions

$$\Psi_l^{(A)} = C_A\left[\frac{\pi\omega r^*}{2}\right]^{1/2} J_{l+1/2}(\omega r^*) ,$$

$$\Psi_l^{(B)} = C_B\left[\frac{\pi\omega r^*}{2}\right]^{1/2} (-1)^{l+1} J_{-l-1/2}(\omega r^*) ,$$

where $J_{l+1/2}$ and $J_{-l-1/2}$ denote Bessel cylindrical functions of the first kind, and C_A, C_B denote two constant coefficients. For $\omega r^* \ll l$ we may retain the first term of the expansions and write

$$\Psi_l^{(A)} \simeq C_A \frac{(\omega r^*)^{l+1}}{(2l+1)!!} ,$$
$$\Psi_l^{(B)} \simeq C_B (2l-1)!!(\omega r^*)^{-l} . \quad (28)$$

In order to match the function (25) with the functions (28), we note that, as appears from the derivation, Eq. (24) may be used up to such values of r or r^* so that

$$\omega r \simeq \omega r^* \ll [l(l+1)]^{1/2} \simeq l .$$

Accordingly, for

$$l\omega M \ll \omega r \simeq \omega r^* \ll l \quad (29)$$

(which is satisfied since we have assumed $\omega M \ll 1$) we can use the asymptotic expression of the functions (25), namely

$$\Psi_l^{(1)}(r) \simeq \frac{(2l)!}{(l-1)!(l+1)!}\left(\frac{r}{2M}\right)^{l+1} ,$$
$$\Psi_l^{(2)}(r) \simeq \frac{(l+1)!(l-1)!}{2(2l+1)!}\left(\frac{2M}{r}\right)^{l} \quad (30)$$

It appears that $\Psi_l^{(1)}$ and $\Psi_l^{(2)}$ are monotonically increasing and monotonically decreasing functions of r, respectively. Hence, $\Psi_l^{(2)}$ becomes less and less important when r increases, and may be neglected. By matching the solutions in the region (29) we find

$$C_A = \frac{(2l)!(2l+1)!!}{(l+1)!(l-1)!}(2\omega M)^{-l-1} . \quad (31)$$

Thus the suitable solution up to $r^* \to \infty$ is

$$\Psi_l = C_A\left[\frac{\pi\omega r^*}{2}\right]^{1/2} J_{l+1/2}(\omega r^*) . \quad (32)$$

As for $r^* \to \infty$ we have $\Psi_l^{(A)} \to C_A \sin(\omega r^* - l\pi/2)$, and we find that the power transmission coefficient $T_l = 1 - |R_l|^2$ is given by

$$T_l \simeq 4\left[\frac{(l+1)!(l-1)!}{(2l)!(2l+1)!!}\right]^2 (2\omega M)^{2l+2} . \quad (33)$$

Of course, these calculations give only the leading terms in the expansions of the coefficients. By a method described in Appendix C one finds that the approximations introduced yield an error proportional to ωM in the expression (33) of T_l.

It may be interesting to note that one arrives at the same expression of the power transmission coefficient T_l of the barrier in the case where the radiation is emitted by a collapsing object, so that the partial waves are purely outgoing radial waves at $r^* \to \infty$.[15,8,12]

V. THE ABSORPTION CROSS SECTION

The knowledge of the power reflection and transmission coefficient allows us to evaluate the absorption cross section σ_{abs}:

$$\sigma_{\text{abs}} = \frac{\pi}{2\omega^2}\sum_{l=1}^{\infty}(2l+1)(2 - |R_l^U|^2 - |R_l^V|^2)$$
$$= \frac{\pi}{2\omega^2}\sum_{l=1}^{\infty}(2l+1)(T_l^U + T_l^V) .$$

Here $|R_l^U|^2$ and $|R_l^V|^2$ are the reflection coefficients of the U waves and V waves. In our problem $R_l^U = R_l^V = R_l$, $T_l^V = T_l^V = T_l$, and

$$\sigma_{\text{abs}} = \frac{\pi}{\omega^2}\sum_{l=1}^{\infty}(2l+1)T_l . \quad (34)$$

In the high-frequency limit, we can set

$$T_l = 0 \text{ for } l > l_c ,$$
$$T_l = 1 \text{ for } l < l_c$$

in Eq. (34), and find[13]

$$\sigma_{\text{abs}}^{(\infty)} = 27\pi M^2 . \quad (35)$$

This is a quantity of the same order as the geometrical cross section. This result may also be obtained by pointing out that light rays do or do not go down the black hole according to whether the impact parameter is less than or greater than the critical parameter $p = 3\sqrt{3}M$.

Consider now low frequencies, such that $\omega M \ll 1$. Since T_l tends to zero as ω^{2l+2}, the main contribution to σ_{abs} comes from the first partial wave for which

$$T_1 = \tfrac{4}{9}(2\omega M)^4 . \quad (36)$$

One finds

$$\sigma_{\text{abs}} = \tfrac{4}{3}\pi(2M)^4\omega^2 . \quad (37)$$

It appears that the absorption cross section vanishes for $\omega \to 0$ as the squared frequency. The black hole is almost unable to absorb em radiation at $\omega \ll \omega_c$, or at wavelengths λ much greater than the Schwarzschild radius $R_S = 2M$. By recalling Eq. (35) we can write

$$\sigma_{\text{abs}} = \tfrac{64}{81}\pi^2\left(\frac{R_S}{\lambda}\right)^2\sigma_{\text{abs}}^{(\infty)} .$$

For example, if $\lambda = 10^2 R_S$, we have

$$\sigma_{abs} \simeq 5\times 10^{-4}\sigma_{abs}^{(\infty)} .$$

As far as we know, Eq. (37) constitutes a result that has not been previously published. Let us now examine the case $\lambda \approx M$.

The transmission coefficients at only a few values of l are expected to be close to unity when ω is a little greater than ω_c. If we set, for a rough estimate, $T_1 = 1$, $T_l = 0$, $(l \neq 1)$, and $\omega = \omega_c$ we obtain

$$\sigma_{abs} \approx \sigma_{abs}^{(\infty)} , \quad \omega \gtrsim \omega_c .$$

On the other hand, all the transmission coefficients are much less than unity when ω is less (but not much less) than ω_c, thus

$$\sigma_{abs} \ll \sigma_{abs}^{(\infty)} , \quad \omega \lesssim \omega_c .$$

Hence ω_c or something not very different from ω_c is the cutoff frequency for the absorption of em waves by a Schwarzschild black hole.

VI. THE PHASE SHIFTS

The power reflection coefficients do not allow one to compute the scattering cross section. It is necessary to evaluate the complex-amplitude reflection coefficients, mainly their phases in dependence on l.

Let us define[14] the phase shifts δ_l by means of the asymptotic expression of Ψ_l valid for $r^* \to +\infty$,

$$\Psi_l \simeq \sin\left(\omega r^* - l\frac{\pi}{2} + \delta_l\right). \tag{38}$$

For the evaluation of δ_l the analysis of the preceding sections is not sufficient. For this purpose, however, the WKB method may be used at all frequencies, provided that $l \gg 1$ (see Appendix A). We restrict ourselves to the case where the turning points exist, and, moreover, the imaginary part of δ_l may be neglected (so that $|R_l|^2 \simeq 1$ and the scattering is elastic). The WKB standard formula[18] yields

$$\delta_l = (l + \tfrac{1}{2})\frac{\pi}{2} + \lim_{r^*\to\infty}\left[\int_{r_2^*}^{r^*} K_l(\xi)d\xi - \omega r^*\right]. \tag{39}$$

In this expression the neglected terms are at most of the order of l^{-1} (see Appendix A). The integral $\int K_l(\xi)d\xi$ may be expressed in terms of incomplete elliptic integrals of the first, second, and third kinds,[19] but the limit appearing in Eq. (39) cannot be expressed in a closed form.[12] Here we give only an asymptotic formula valid for high-l waves,

$$\delta_l \simeq -2\omega M \ln(l/4\omega M) - \omega M . \tag{40}$$

We can note that, for $l \to \infty$, the phase shifts differ from the "Coulomb" ones only by a constant.[14,13] This may be shown by starting from the form of the radial equation given by Mo and Papas.[9] The function

$$f_l = (1 - 2M/r)^{-1/2}\Psi_l(r)$$

satisfies the equation

$$f_l''(r) + \left[\frac{\omega^2}{(1-2M/r)^2} - \frac{l(l+1)}{r(r-2M)} + \frac{M}{r^2}\,\frac{2r-3M}{(r-2M)^2}\right] f_l = 0 ,$$

or also, by neglecting the terms in r^{-3} and r^{-4} and provided that $l \gg \omega M$, the equation

$$f_l''(r) + \left[\omega^2 + \frac{4M\omega^2}{r} - \frac{l(l+1)}{r^2}\right] f_l = 0 .$$

From this Coulomb approximation we obtain[23,12]

$$\delta_l = -\arg\Gamma(l + 1 + 2i\omega M) + 2\omega M \ln(4\omega M) ,$$

where Γ is the Euler gamma function.

For $l \to \infty$

$$\delta_l = -2\omega M \ln\left(\frac{l}{4\omega M}\right) . \tag{41}$$

This expression differs from (40) by the l-independent term ωM. This term does not change the interference pattern of high-l waves, but may be important if one considers even low values of l, for which (40) does not hold.

VII. THE SCATTERING CROSS SECTION

The simplest expression for the scattering cross section of a black hole can be deduced from the equation given by Mo and Papas,[9]

$$\sigma(\theta,\phi) = \frac{\cos^2\phi}{4\omega^2}\left|\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\left[e^{-2i\delta_l}T_l(\theta) + e^{-2i\eta_l}\pi_l(\theta)\right]\right|^2 + \frac{\sin^2\phi}{4\omega^2}\left|\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\left[e^{-2i\delta_l}\pi_l(\theta) + e^{-2i\eta_l}T_l(\theta)\right]\right|^2 ,$$

where δ_l and η_l are the phase shifts of the U waves and the V waves, respectively, and

$$\pi_l(\theta) = \frac{P_l^1(\cos\theta)}{\sin\theta}, \quad T_l(\theta) = \frac{d}{d\theta}P_l^1(\cos\theta) .$$

Here P_l^1 are associated Legendre functions.

In our case $\delta_l = \eta_l$, accordingly, the scattering cross section turns out to be independent of ϕ:

$$\sigma(\theta) = \frac{1}{4\omega^2}\left|\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\,e^{-2i\delta_l}\left[\pi_l(\theta) + T_l(\theta)\right]\right|^2 . \tag{42}$$

When one deals with long-range potentials, at small scattering angles the main contribution to $\sigma(\theta)$ comes from high-l waves. For such values of l, we can use Eq. (40), and, at small θ, we can write

$$\frac{\pi_l(\theta)+T_l(\theta)}{l(l+1)} \simeq P_l(\cos\theta) .$$

In this way Eq. (42) takes the form of the usual scattering cross section for scalar waves. In particular, the Rutherford law holds, according to which

$$\sigma(\theta) \simeq 16M^2\theta^{-4} \tag{43}$$

for small θ. Mashhoon presents it in his wave treatment of the high-frequency scattering.[13] The present derivation, however, is valid at every frequency. In other words, we conclude that Eq. (43) is acceptable even for $\omega M \ll 1$.

VIII. CONCLUDING REMARKS

It has been shown in Sec. V that, as far as the em energy absorption is concerned, a black hole behaves in substantially different ways, depending upon whether the wavelength λ is larger or smaller than the radius $R_S = 2M$ of the black hole. For $\lambda \lesssim 2M$ the usual picture of the black hole as a trap for radiation appears to be rather inadequate, as already stated by Matzner.[14]

The case of higher-spin fields, e.g., the gravitational radiation, is analogous to the case of electromagnetic waves, but only the partial waves exist with $l \geq 2$. Fackerell found[22] that the transmission coefficient for $l \geq 2$ vanishes for $\omega \to 0$ as ω^{2l+2}. Note the close analogy with our results.

The cutoff frequency for em energy absorption is expected to be a very low frequency. In the case of black holes formed by the collapse of a neutron star, their mass M is expected to be greater than a critical mass[24] of the order of the solar mass $M_0 \simeq 1.5\times10^3$ m. Accordingly, the cutoff wavelength $\lambda \approx 2M$ corresponds to a frequency smaller than 100 kHz, which is a very low frequency indeed, as far as radio astronomy is concerned. We can pose the question of whether the presence of black holes in the universe alters the black-body thermal radiation and produces an excess of low frequencies.

To this end, let us consider a volume L^3 where N black holes exist.

Let $E_0 = \rho L^3$ be the energy contained in L^3 and E be the electromagnetic energy absorbed by these black holes during the time t, at some wavelength less than the average radius of the black holes. For a rough evaluation we set $E \simeq N\rho\sigma_{\text{abs}} t$. Here ρ is the energy density at the above-mentioned wavelength and σ_{abs} is the average absorption cross section. Since the mass comprised in the black holes is at most some percent of the whole mass of the universe,[25] and the mass density is about 10^{-30} g/cm$^3 \approx 10^{-54}$ mks geometrized units, we obtain

$$E/E_0 < 10^{-54} M t .$$

An upper bound to t is the average age of the black holes, which is at most of the order of the age of the universe, 10^{10} years $\approx 10^{26}$ m. Even if M were much greater than the critical mass of the neutron stars, E/E_0 would turn out to be enormously less than unity. We conclude that the alteration of the black-body spectrum due to the low-frequency small absorption of the black holes is practically unappreciable.

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APPENDIX A

The present analysis has the purpose of establishing an upper bound to the error we introduce in the calculation of $|R_l|^2$ and T_l by using the WKB approximation.

Let us consider the quantity

$$\mu = \int_c \left| K_l^{-1/2} \frac{d^2}{dr^{*2}} K_l^{-1/2} \right| |dr^*| , \tag{A1}$$

where the integration path c (Figs. 2 or 3) goes from $-\infty$ to $+\infty$ along the real axis. The semicircles are included in it in order to avoid the singularities of the function $K_l^{-1/2} d^2 K_l^{-1/2}/dr^{*2}$, that is, the zeros of K_l.

It may be proved that[18]

$$\frac{\Delta T_l}{T_l} \lesssim \mu \text{ if } l > l_c ,$$

$$\frac{\Delta |R_l|^2}{|R_l|^2} \lesssim \mu e^{|\theta_l|} \text{ if } l < l_c .$$

For a rough evaluation of μ in the case of Fig. 2

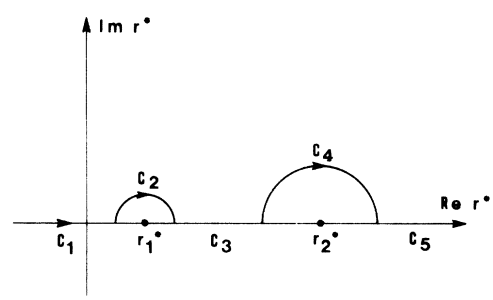

FIG. 2. The integration path of the μ integral in the case where $l \gg l_c$.

we set

$$K_l^2 \simeq \omega^2 \quad \text{on } C_1 \text{ and } C_5, \tag{A2}$$

$$K_l^2 \simeq -U_l(r^*) \quad \text{on } C_3 \,, \tag{A3}$$

$$K_l^2 \simeq \left(\frac{\partial k_l^2}{\partial r^*}\right)_{r^*=r^*_{1,2}} (r^* - r^*_{1,2}) \quad \text{on } C_{2,4} \,. \tag{A4}$$

Expressions (A4) require the radii R_2 and R_4 of C_1 and C_4 to be small enough:

$$R_2 \lesssim M, \quad R_4 \lesssim l/\omega \,.$$

In the case of Fig. 3 we set

$$K_l^2 \approx \frac{1}{2}\left(\frac{\partial^2 k_l^2}{\partial r^{*2}}\right)_{r=3M} (r^* - r_0^*)^2$$

on the semicircle, with the condition $|r^* - r_0^*| \lesssim M$, and $k_l^2 \approx \omega^2$ on the rest of the integration path. In both cases one finds

$$\mu \approx (\omega M)^{-1} \,.$$

Hence,

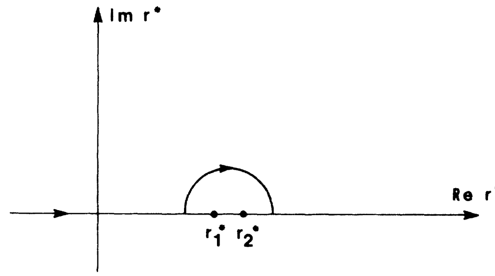

FIG. 3. The integration path of the μ integral in the case where $l \simeq l_c$.

$$\frac{\Delta T_l}{T_l} \lesssim \frac{\lambda}{M} \qquad \text{for } l > l_c \,,$$

$$\frac{\Delta |R_l|^2}{|R_l|^2} \lesssim \frac{\lambda}{M} e^{|\theta_l|} \quad \text{for } l < l_c \,.$$

A similar technique can be used for the phase shifts. We note that the WKB approximation is a good approximation in the region $2M \ll r \ll l/\omega$ when $l \gg 1$, because there the wave number changes slowly. Accordingly, the WKB approximation holds on both sides of r_2^*, and we may consider the connection formula[18]

$$\Psi_l(r_0^*) = |K_l(r_0^*)|^{-1/2}\exp\left[\int_{r_2^*}^{r_0^*} |K_l(\xi)|d\xi\right] + A\exp\left[-\int_{r_2^*}^{r_0^*} |K_l(\xi)|d\xi\right] - \frac{2(1+\sigma_1)}{K_l^{1/2}(r^*)}\cos\left[\int_{r_2^*}^{r^*} K_l(\xi)d\xi - \frac{\pi}{4} + \sigma_2\right] .$$

Here r_0^* is a point on the left of r_2^* and r^* is a point on the right of r_2^*; moreover,

$$\sigma_1, \sigma_2 \lesssim \mu(r_0^*, r^*) \,,$$

where

$$\mu(r_0^*, r^*) = \int_{r_0^*}^{r^*} \left| K_l^{-1/2}(\xi) \frac{d^2}{dr^{*2}} K_l^{-1/2}(\xi)\right| |d\xi| \,.$$

The integration path must not turn around the singular point r_2^*. If $r_0^* = \alpha r_2^*$ with $\alpha < 1$ but not too small, one finds

$$\mu(r_0^*, r^*) \approx \frac{1}{l}$$

for all r^* and all frequencies. Thus the WKB standard equation (40) introduces an error at most of the order $1/l$ in the calculation of δ_l.

APPENDIX B

Consider the functions

$$\Psi_l^{(1)} = \frac{r}{2M} P_l\left(\frac{r}{M} - 1\right) - \frac{1}{2(2l+1)}\left[P_{l+1}\left(\frac{r}{M} - 1\right) - P_{l-1}\left(\frac{r}{M} - 1\right)\right] , \tag{B1}$$

where P_l are the Legendre polynomials. Let

$$\xi = \frac{r}{M} - 1 \,.$$

By differentiating $\Psi_l^{(1)}$ and recalling the relation

$$\frac{dP_{l+1}}{d\xi} - \frac{dP_{l-1}}{d\xi} = (2l+1)P_l(\xi) \,, \tag{B2}$$

we obtain

$$\frac{d\Psi_l^{(1)}}{dr} = \frac{\xi+1}{2M}\frac{dP_l}{d\xi} \,, \tag{B3}$$

$$\frac{d^2\Psi_l^{(1)}}{dr^2} = \frac{1}{2M^2}\left[\frac{dP_l}{d\xi} + 2(\xi+1)\frac{d^2P_l}{d\xi^2}\right] . \tag{B4}$$

On the other hand, we have

$$\begin{aligned} &(l+1)P_{l+1}(\xi) - (2l+1)\xi P_l(\xi) + lP_{l-1}(\xi) = 0 \,, \\ &lP_{l-1}(\xi) + (\xi^2 - 1)\frac{dP_l}{d\xi} - l\xi P_l(\xi) = 0 \,, \end{aligned} \tag{B5}$$

which allow us to write Eq. (B1) in the form

$$\Psi_l(r) = \frac{\xi+1}{2} P_l(\xi) + \frac{\xi^2 - 1}{2l(l+1)}\frac{dP_l}{d\xi} \,. \tag{B6}$$

By introducing (B3), (B4), and (B6), Eq. (24) becomes

$$(1-\xi^2)\frac{d^2P_l}{d\xi^2}+2\xi\frac{dP_l}{d\xi}-l(l+1)P_l(\xi)=0 .$$

This is just the Legendre equation. Thus $\Psi_l^{(1)}(r)$ actually obeys Eq. (24). Similar remarks apply to

$$\Psi_l^{(2)}=\frac{r}{2M}Q_l\left(\frac{r}{M}-1\right)-\frac{1}{2(2l+1)}\left[Q_{l+1}\left(\frac{r}{M}-1\right)-Q_{l-1}\left(\frac{r}{M}-1\right)\right],$$

since Q_l also obey the relations (B2) and (B5).

APPENDIX C

We restrict ourselves to the case where $l=1$. The equation

$$\frac{d^2\Psi_1}{dr^{*2}}=\left[\left(1-\frac{2M}{r}\right)\frac{2}{r^2}-\omega^2\right]\Psi_1 \qquad (C1)$$

can be solved by an iterative method. Let us consider the actual solution of (C1) which satisfies the boundary conditions

$$\Psi_1(r_0^*)=\frac{r_0^2}{4M^2},\quad \left(\frac{d\Psi_1}{dr^*}\right)_{r^*=r_0^*}=\frac{r_0}{2M^2}\left(1-\frac{2M}{r_0}\right),$$

where r_0 is a point internal to the potential barrier, e.g., $r_0=3M$, and $r_0^*=r_0+2M\ln(r_0/2M-1)$. As a first approximation, we take the static solution

$$\Psi_1^{(1)}=\frac{r}{2M}P_1\left(\frac{r}{M}-1\right)-\frac{1}{6}\left[P_2\left(\frac{r}{M}-1\right)-1\right]=\left(\frac{r}{2M}\right)^2,$$

and introduce this function on the right-hand side of Eq. (C1). Integrating up to $r_1^*\approx 2M\ln(2\omega^2M^2)$ yields

$$\left(\frac{d\Psi_1}{dr^*}\right)_{r^*=r_1^*}=\frac{r}{2M^2}\left(1-\frac{2M}{r}\right)-\left(\frac{\omega}{2M}\right)^2\int_{r_0}^{r_1}\frac{r^2dr}{1-2M/r},$$

and then

$$\Psi_1(r_1^*)=\left(\frac{r_1}{2M}\right)^2-\left(\frac{\omega}{2M}\right)^2\int_{r_0}^{r_1}\int_{r_0}^{r}\frac{r^2dr}{1-2M/r}. \qquad (C2)$$

The last term on the right-hand side of (C2) constitutes a first correction to Ψ_1. We find

$$\Delta\Psi=\left(\frac{\omega}{2M}\right)^2\int_{r_0}^{r_1}\int_{r_0}^{r}\frac{r^2dr}{1-2M/r}\simeq(\omega M\ln\omega M)^2 .$$

Thus the static solution $\Psi_1^{(1)}$ can be used up to r_1^* with an error of the order of $(\omega M\ln\omega M)^2$. In the same way one could show that $\Psi=\Psi_1^{(2)}$ and $\Psi=e^{i\omega r^*}$ can be used up to r_1^* with errors of order $(\omega M)^2$. Accordingly, we can write the connection formula

$$e^{i\omega r^*}\to(1+2i\omega M)\Psi_1^{(1)}-4i\omega M\Psi_1^{(2)}$$

by neglecting terms of the order of $(\omega M)^2$.

On the contrary, $\Psi_1^{(1)}$ cannot be used up to $r_2^*\simeq\sqrt{2}/\omega$. The correction

$$\Delta\Psi=\left(\frac{\omega}{2M}\right)^2\int_{r_0}^{r_2}\int_{r_0}^{r}\frac{r^2dr}{1-2M/r}=-\frac{\omega^2r_2^3}{6M}\left[1+O\left(\frac{M}{r}\right)\right]$$

is of the same order $(\omega M)^{-2}$ as $\Psi_1^{(1)}(r_2)$. But we can use the iterative method again by introducing

$$\Psi_1=\left(\frac{r}{2M}\right)^2+\sum_{k=3}^{\infty}\alpha_k\left(\frac{r}{2M}\right)^k$$

on the right-hand side of (C1).

After lengthy but straightforward calculations, we obtain the even-power series

$$\Psi_1(r)=\sum_{k=1}^{\infty}\alpha_k'\left(\frac{r}{2M}\right)^{2k}, \qquad (C3)$$

where

$$\alpha_k'=\frac{(-1)^{k+1}(2\omega M)^{2k-2}}{2k(2k-1)(2k-2)\cdots4\times3}\sum_{m=0}^{\infty}\sum_{\alpha+\beta+\cdots+\gamma=m}\frac{2^m}{[2k(2k-1)]^\alpha[(2k-2)(2k-3)]^\beta\cdots[4\times3]^\gamma},$$

and terms of the order of $(\omega M)^{-1}$ for $r\simeq r_2$ have been neglected.

This approximation is equivalent to replacing Eq. (C1) with the equation

$$\frac{d^2\Psi}{dr^2}=\left[\frac{2}{r^2}-\omega^2\right]\Psi,$$

whose solutions are

$$\Psi_1^{(A)}=\left(\frac{\pi\omega r}{2}\right)^{1/2}J_{3/2}(\omega r),$$

$$\Psi_1^{(B)}=\left(\frac{\pi\omega r}{2}\right)^{1/2}J_{-3/2}(\omega r). \qquad (C4)$$

Owing to the parity of the series (C3), we can write

$$\Psi_1(r)=C\Psi_1^{(A)}(r).$$

The constant C is easily found to be given by C

$=3/(2\omega M)^2$.

From what proceeds it follows that we introduced an error of the order of ωM in the calculation of the transmission coefficient.

For $l>1$, $\Psi_l^{(1)}$ is a polynomial. However, only the highest power term is important for $r\simeq r_2$. By iterations this term generates a series proportional to $\Psi_l^{(A)}$, with an error proportional to ωM.

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