Scalar binding of quarks*

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(Received 19 March 1975)

A scalar potential which is linear in the radius is used to confine relativistic individual quark states. Calculation results are compared with those of the "bag" model.

A new model for the structure of hadrons has appeared[1] in which the constituent fields are confined in a finite region called a "bag." It is the purpose of this comment to describe a simple model in which relativistic quark states are confined by a central potential and to compare some results with those of the "bag" model.

Quarks are assumed to obey Dirac's equation, and if we replace the rest mass in his equation for a free particle by $\kappa^2 r$ where κ is a characteristic wave number and r is the radius from a fixed point we have ($\hbar = c = 1$)

$$E\psi = (\alpha, p)\psi + \beta\kappa^2 r\psi , \tag{1}$$

where ψ is the four-component wave function for a quark. Literally Eq. (1) describes the motion of a particle having no rest mass in a scalar potential.[2] It can be separated in spherical coordinates by the standard method. The two functions of radius, ψ_a and ψ_b, called the large and small components, respectively, obey[3]

$$\begin{aligned}(E-\kappa^2 r)\psi_a + \frac{d\psi_b}{dr} + \frac{j+1}{r}\psi_b = 0,\\ (E+\kappa^2 r)\psi_b - \frac{d\psi_a}{dr} + \frac{j-1}{r}\psi_a = 0,\end{aligned} \tag{2}$$

where, if l be the orbital angular momentum in ψ_a, $j = l+1$ when the total angular momentum is $l+\frac{1}{2}$ and $j = -l$ when the total is $l-\frac{1}{2}$. Owing to the appearance of $\kappa^2 r$ in these equations, they have regular singular points in addition to the customary one at $r = 0$ (and the irregular one at $r = \infty$). The second-order differential equations which one derives for ψ_a or ψ_b or any linear combination of them are then of the same general type (i.e., up to elementary algebraic substitutions) as those for spheroidal harmonics.[4] In brief[5] let us define

$$s \equiv |j|,$$

$$P \equiv r^{1-s} e^{(1/2)\kappa^2 r^2}(\psi_a + \psi_b) = \sum_0^\infty p_k r^k . \tag{3}$$

Then, through Eq. (2) the p_k obey a recursion relation

$$(k+1)(k+s)(k+2s+1)p_{k+1} + Ejp_k + (k+s+1)[E^2 - 2\kappa^2(k+s)]p_{k-1} = 0 . \tag{4}$$

Now, the p_k must be bounded so as $k \to \infty$, Eq. (4) shows that

$$kp_{k+1} = 2\kappa^2 p_{k-1}, \text{ as } k \to \infty .$$

Therefore, the only acceptable series for P (i.e., for square integrable ψ_a and ψ_b) must converge at large n like the series

$$\sum^\infty (n!)^{-1/2}[-2^{1/2}\kappa r]^n .$$

The ratio of successive p_k is then negative and

$$\frac{p_{k+1}}{p_k} = \frac{p_k}{p_{k-1}}\left[1 + O\left(\frac{1}{k}\right)\right] .$$

This condition plus the requirement that if $i < 0$, $p_i = 0$ is sufficient to determine the eigenvalues for E as roots of an infinite continued fraction.[4] It is pertinent to compare eigenvalues obtained in this way with those of the nonrelativistic isotropic harmonic oscillator of frequency ω. They are given by

$$E' = (n + \tfrac{3}{2})\omega , \tag{5}$$

where $n = 0, 1, 2, \ldots$ and as n increases they become increasingly degenerate. The eigenvalues E of Eq. (1) are such that E^2 obeys very nearly the same equation as $2E'$, with ω replaced by κ^2. There is a small spin-orbit splitting. The six lowest eigenvalues for E^2/κ^2 are shown in Fig. 1. The corresponding numerical values are given in Table I, where the quantum number n refers to the nearest value that appears in the nonrelativistic case, Eq. (5). From the symmetry of Eq. (4) one sees that the negative eigenvalues for E have the same absolute values as in Table I with reversed signs of j. The eigenstates in this model are thus nearly evenly spaced in E^2.

Approximate eigenfunctions for the lowest state of positive energy may be obtained by applying the variational principle to the expectation value of E^2. Let

$x \equiv \kappa r$;

then, to second degree polynomials, one finds

$$\psi_a = (1 + 0.042\,87x + 0.004\,57x^2)e^{-x^2/2},$$
$$\psi_b = -x(0.463\,30 - 0.083\,67x)e^{-x^2/2}. \quad (6)$$

The expectation values for E and β with these functions are 1.6193κ and 0.7308, respectively, in excellent agreement with exact values.

We now assume the quarks to belong to SU(6) and to have "color," so we may approximate states of the nucleons by putting three quarks in the ground state, $j = 1$, $n = 0$. One can then compute[6] expectation values of magnetic moments and of g_A/g_V in the beta decay of nucleons. The value of κ is chosen so that three times the lowest eigenvalue is the mass of the proton, M. This gives

$$\kappa = 0.2058M.$$

The complete (unnormalized) wave function is

$$\Psi = \begin{pmatrix} \psi_a \\ -i\dfrac{(\sigma, r)}{r}\psi_b \end{pmatrix}, \quad (7)$$

with ψ_a and ψ_b as given in Eq. (6). Taking the average value of the magnetic moment for a proton,

$$\frac{e}{2}\langle\Psi|(r\times\alpha)_z|\Psi\rangle = \frac{0.2811}{\kappa}e$$
$$= 2.732\left(\frac{e}{2M}\right),$$

in good agreement with the observed 2.79. From SU(6) the corresponding value for the neutron is $-\frac{2}{3}$ of this, i.e., -1.821 instead of -1.91 nuclear magnetons.

The static value for g_A/g_V from SU(6) is $\frac{5}{3}$, and this is to be multiplied by $\langle\sigma_z\rangle$, which when calculated either using Eq. (7) or from the exact value of $\langle\beta\rangle$ is 0.8206. This gives

$$g_A/g_V = 1.368,$$

to be compared with the observed value 1.26. It seems remarkable that independent quark states which are quantized about a fixed point should lead to quite good agreement among numerical values for energy, magnetic moments, and g_A/g_V.[7] The agreement is not quite so good for the root-mean-square radius of charge distribution in the proton. At a given instant the mean-square radius of a quark, $\langle r_1^2\rangle$ about the center of "mass" of all three, $\vec{\rho} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)$, is

$$\langle r_1^2\rangle = \langle\Psi|(\vec{r} - \vec{\rho}^2)|\Psi\rangle$$
$$= \tfrac{2}{3}\langle\Psi|r_1^2|\Psi\rangle$$
$$\cong 1.16\kappa^{-1},$$
$$\langle r_1^2\rangle^{1/2} = 1.07 \text{ fermi},$$

in not very good agreement with the observed value of about 0.92 fermi.[8,9]

Corresponding numerical results are similar in the "bag" model.[10] The difference is that in that model a fraction of the nucleon mass is represented in the volume energy of the "bag." Consequently the contribution from the quark states is less to the mass and therefore higher to the magnetic moment of the nucleon. This, however, is compensated by the quark states being more relativistic than in the model presented in this note. This is reflected in the fact that g_A/g_V (which is the average value of $\frac{5}{3}\sigma_z$) is much lower in the "bag" model than in the present one.

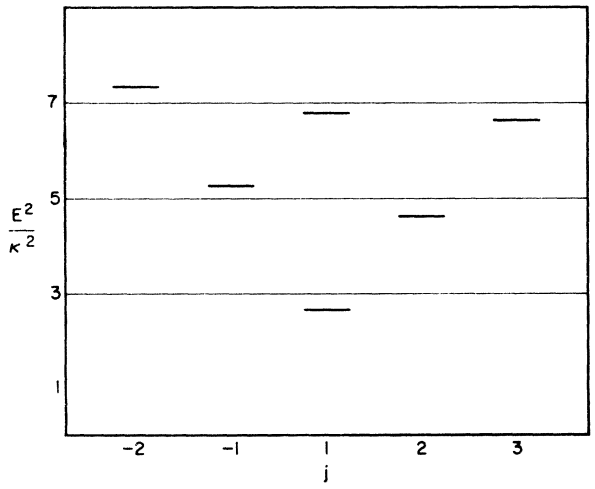

FIG. 1. Spectrum of E^2/κ^2 as function of j. The horizontal lines at odd integers represent the nonrelativistic values.

I wish especially to express my appreciation to Professor Low and Professor Feynman for their helpful discussions on this subject.

TABLE I. Eigenvalues of lowest states.

n	j	E^2/κ^2	$2E'/\omega$	E/κ
0	1	2.6228	3	1.6195
1	−1	5.2629	5	2.2941
	2	4.6079	5	2.1466
2	−2	7.3132	7	2.7043
	1	6.7740	7	2.6027
	3	6.6018	7	2.5694

*Work performed under the auspices of the U. S. Energy Research and Development Administration.

[1]A. Chodos, R. L. Jaffee, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[2]A linear vector potential leads also to interesting but quite different results. Y. M. Cho, Nuovo Cimento 23A, 550 (1974).

[3]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1958), 4th edition, p. 269, *et seq.*

[4]R. L. McLaurin, Trans. Camb. Philos. Soc. 17, 33 (1898); M. F. Manning, Phys. Rev. 48, 161 (1935); J. A. Stratton, P. M. Morse, L. J. Chu, and R. C. Hutner, *Elliptic Cylinder and Spheroidal Wave Functions* (Wiley, New York, 1941).

[5]Details appear in C. L. Critchfield, Los Alamos Report No. LA-UR-74-1813 (unpublished).

[6]R. Van Royen and V. F. Weisskopf, Nuovo Cimento 50A, 617 (1967).

[7]Professor Feynman (private communication) has pointed out that any purely scalar confining potential is apt to give good relativistic corrections to the static values in the independent quark model.

[8]B. Borkowski, P. Peuser, G. G. Simon, V. H. Walther, and R. D. Wendling, Nucl. Phys. A222, 269 (1974).

[9]The total magnetic moment is the same in such instantaneous states as in the independent model because the c.m. motion contributes to the former.

[10]F. Low, private communication.