Study of (3, 3*) ⊕ (3*,3) ⊕ (8,8) symmetry breaking in a linear σ model*

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Employing several linear SU(3) σ models developed by Schechter, Ueda, and collaborators, we study the effects of various possible chiral-symmetry–breaking terms in the Lagrangian belonging to the (3, 3*) ⊕ (3*, 3) ⊕ (8, 8) representations of SU(3) × SU(3). In this approach the pseudoscalar mesons (π, K, η, η') and scalar mesons (ϵ, κ, σ, σ') are assigned to (3, 3*) ⊕ (3*, 3) and the model is used to describe the scalar and pseudoscalar mass spectra, scalar meson decays, $\eta' \to \eta\pi\pi$, and the $\pi\pi$ and πK scattering lengths. Inclusion in the Lagrangian of isospin-violating terms and an effective nonleptonic weak interaction allows treatment of electromagnetic and weak effects as well. We find that a form of chiral symmetry breaking suggested by Okubo is at least as successful as the Gell-Mann–Oakes–Renner (3, 3*) ⊕ (3*, 3) form. The scheme of Sirlin and Weinstein is also satisfactory. Pure (8, 8) symmetry breaking is unacceptable within the context of the present model.

I. INTRODUCTION

The possible role of chiral SU(3)×SU(3) symmetry[1] in elementary particle physics has been the focus of considerable attention. Since it is obviously not an exact symmetry of nature (e.g., the axial-vector currents are not conserved), some of the most interesting questions, as in the case of SU(3), deal with the way in which chiral symmetry is broken. An attractive and popular idea[2,3] is that the underlying strong Lagrangian is approximately SU(3)×SU(3)-symmetric, but that, in the chiral-symmetric limit, the vacuum is not invariant under chiral transformations, i.e., the symmetry is spontaneously broken. We are then not committed to the existence of parity doublets in the particle spectrum as would be the case for a chiral-invariant vacuum. The massless bosons associated with spontaneous symmetry breakdown[4-6] are, in this case, the octet of pseudoscalar mesons, which acquire mass through the explicit chiral-symmetry-breaking terms in the Lagrangian. It is with these latter terms that the present investigation is concerned.

With regard to the transformation properties of the symmetry-breaking term, the simplest choice[1-3] is to assign it to the (3, 3*)⊕(3*, 3) representation of SU(3)×SU(3). Then, the symmetry-breaking Hamiltonian density $\mathcal{H}_{SB}$ takes the form

$$\mathcal{H}_{SB} = -c_0 u_0 - c_8 u_8 , \tag{1.1}$$

where the u_i ($i = 0, 1, \ldots, 8$) are nine scalar density operators belonging to the (3, 3*)⊕(3*, 3) representation. While it was originally thought[1] that c_8/c_0 would be small, ensuring that SU(3) would be a better symmetry than SU(3)×SU(3), arguments were given by Gell-Mann, Oakes, and Renner[2] (GMOR) for the very appealing alternative that $c_8/c_0 \approx -\sqrt{2}$, implying that SU(2)×SU(2) symmetry is better than SU(3). This would explain the small mass of the pion compared to all other hadrons.[7]

Though it has been employed in many investigations, there have been no definitive tests of the GMOR model. The study by Kim and von Hippel[8] of σ terms in meson-baryon scattering seemed to provide a positive test, but their analysis has recently been called into question.[9] In addition, the large value of the πN σ term found by Cheng and Dashen[10,11] may not be compatible with the GMOR model if chiral symmetry breaking is assumed to be small, and if scale-invariance[12,13] considerations are ignored.

For these reasons and others, alternative schemes of SU(3)×SU(3) breaking have been considered. The most popular takes $\mathcal{H}_{SB}$ as belonging to the (8, 8) representation.[14] However, if one adopts the point of view that the smallness of the pion mass directly reflects the approximate SU(2)×SU(2) invariance of the strong Hamiltonian, then this assignment must be rejected in favor of one in which the dominant part of $\mathcal{H}_{SB}$ has the form of Eq. (1.1) with[15] $c \approx -\sqrt{2}$. In fact, an interesting possibility is that the (3, 3*)⊕(3*, 3) part of $\mathcal{H}_{SB}$ is SU(2)×SU(2)-invariant and that small admixtures of other representations break SU(2)×SU(2). We would then have

$$\mathcal{H}_{SB} = -c_0(u_0 - \sqrt{2}\, u_8) + \mathcal{H}'_{SB} \tag{1.2}$$

where $\mathcal{H}'_{SB}$ is a small SU(2)×SU(2)-breaking part of $\mathcal{H}_{SB}$, its SU(3)×SU(3)-breaking properties being *a priori* unspecified.

A model of this type was first discussed by Okubo,[16] who argued that if the only SU(3)-breaking part of $\mathcal{H}_{SB}$ belongs to (3, 3*)⊕(3*, 3), then $\mathcal{H}_{SB}$ must have the structure given in Eq. (1.2), where

$\mathcal{H}'_{SB}$ is now restricted to be SU(3)-invariant.[17] A Hamiltonian of the form in Eq. (1.2) has also been considered by Sirlin and Weinstein,[18] who chose $\mathcal{H}'_{SB}$ to belong to (8, 8), but allowed it to break SU(3). It was shown that the model could accommodate a large value of the πN σ term as well as a large value of the $I=0$ S-wave $\pi\pi$ scattering length a_0, for which there may be some evidence.[19]

The object of the present investigation is to study the proposal of Sirlin and Weinstein and compare it with the GMOR model. For this purpose we employ a linear SU(3) σ model[20] which has been developed and used extensively by Schechter and Ueda and their collaborators[21-26] in a number of applications. This model is based on a Lagrangian which is constructed out of SU(3) nonets of pseudoscalar (π, K, η, η') and scalar $(\epsilon, \kappa, \sigma, \sigma')$ fields assigned to the $(3, 3^*) \oplus (3^*, 3)$ representation of SU(3)×SU(3). In their calculations, Schechter *et al.* worked to lowest order in the interaction terms of the Lagrangian; higher-order corrections (loops) were ignored. Their Lagrangian contains terms which are SU(3)×SU(3)-invariant and others which break this symmetry and SU(3). Schechter *et al.* assumed $(3, 3^*)\oplus(3^*, 3)$ symmetry breaking[27] in their work; we will extend this to include (8, 8) symmetry breaking as well.

With regard to the SU(3)×SU(3)-invariant terms in the Lagrangian there are two cases of interest. The first, in which the chiral-invariant part V_0 is allowed to be an arbitrary, nonderivative function of the basic fields, is the case assumed in most of the work of Schechter *et al.* However, the interesting special case in which V_0 is restricted by renormalizability considerations was also studied.[26,28] We will refer to these two cases as the general model and the renormalizable model, respectively.

Spontaneous breakdown is accommodated in the σ models by treating the Lagrangians semiclassically to find the equilibrium point of the system (i.e., by locating the extremum of the "potential" terms in the Lagrangian). The physical fields are defined as the displacement of the original fields from their equilibrium values.

Although, in the general model, the form of V_0 is left unspecified, it turns out that many physical quantities can be determined solely from a knowledge of the symmetry-breaking term V_{SB}. However, a complete description is impossible without the imposition of further restrictions. Such restrictions were found[23] to be available from scale-invariance[13] arguments.

To get more information out of the general model, Schechter *et al.* postulated that V_0 is scale-invariant. The possibility that the terms in the Lagrangian which break chiral SU(3)×SU(3) are the only operator terms which break scale invariance has been seriously considered by several authors.[29] The alternative to this[30] calls for the presence in the Lagrangian of operator terms which are chiral-invariant but which break scale invariance. We will find, in fact, that the renormalizable model conforms to this latter possibility.

The symmetry-breaking term V_{SB} was taken by Schechter *et al.* to have its simplest, nontrivial form, namely, a linear[31] combination of the scalar fields of the model. If a combination of the σ and σ' fields is chosen, this is equivalent to the model of Eq. (1.1). With this choice of V_{SB} the model was used[21,23] to describe the strong mass spectrum, scalar-meson decays ($\sigma\to 2\pi$, $\kappa\to K\pi$, etc.) and the decay $\eta'\to\eta\pi\pi$.

If the $I=1$ ϵ field is included in V_{SB} [this is equivalent to adding a u_3 term to Eq. (1.1)], isospin violation is introduced. Terms of this type may arise from higher-order electromagnetic tadpole contributions. The Lagrangian, thus extended, was employed[22,24] to discuss electromagnetic mass shifts and the $\eta\to 3\pi$ decay.

Finally, with the addition of an effective nonleptonic weak-interaction term in the Lagrangian, the decays $K\to 2\pi$ and $K\to 3\pi$ were studied.[25]

It was found from these investigations that the GMOR scheme, with $c_8/c_0\approx -\sqrt{2}$ in Eq. (1.1), was favored. Our program will be to repeat the calculations of Schechter *et al.* using

$$\mathcal{H}_{SB} = -c_0u_0 - c_8u_8 - d_0z_0 - d_8z_8 \tag{1.3}$$

for the isospin-conserving part of the symmetry-breaking interaction; z_0 and z_8 are the SU(3) singlet and $I=0$ octet members of the (8, 8) representation. To deal with isospin-violating effects, we will add a combination of u_3 and z_3 to Eq. (1.3). The effective nonleptonic interaction of Goswami *et al.*[25] will be used to describe the $K\to 2\pi$ and $K\to 3\pi$ decays.

As mentioned above, special attention will be paid to the Sirlin-Weinstein hypothesis,[18] which has

$$\mathcal{H}_{SB} = -c_0(u_0 - \sqrt{2}\,u_8) - d_0z_0 - d_8z_8\,. \tag{1.4}$$

We will be especially interested in the Okubo case,[16] which is taken here to mean

$$\mathcal{H}_{SB} = -c_0(u_0 - \sqrt{2}\,u_8) - d_0z_0 \tag{1.5}$$

(although the GMOR Hamiltonian belongs to the Okubo class). The models represented by Eqs. (1.3)–(1.5) will be investigated for both the general and renormalizable forms of V_0. In addition, the results corresponding to several other special cases of Eq. (1.3) will be described.

In performing calculations with the model, we will include only the lowest-order nontrivial contributions to a particular quantity or process (i.e., we will work in the tree approximation). In the renormalizable model, at least, there is apparently some justification[32] for this procedure.

The numerical predictions presented below will, in general, be the result of a best fit to certain of the experimentally better-determined quantities. For example, our inputs generally include the pion, kaon, and η masses as well as the pion and kaon decay constants, f_π and f_K. The symmetry-breaking parameters can be determined from the value of pseudoscalar masses. In addition, the σ mass, and sometimes the σ' mass and σ-σ' mixing angle, must be used as input to calculate certain decay parameters. When isospin-violating effects are being considered, additional inputs (such as the π^+-π^0 mass difference and the K^+ and K^0 masses) are required to determine the additional parameters.

We will defer the detailed discussion of our results until later. Several general conclusions should be mentioned at this point, however. Of the possible symmetry-breaking schemes contained in Eq. (1.3) (such as $c_0u_0+d_8z_8$, d_8z_8, etc.) we can rule out, in the context of the present model, all but those of GMOR [Eq. (1.1)], Sirlin and Weinstein [Eq. (1.4)], and Okubo [Eq. (1.5)]. In particular, pure (8, 8) symmetry breaking leads to unacceptable predictions.

While it is difficult to choose between the schemes of Sirlin and Weinstein and of Okubo, since the former has an additional parameter, it appears that, in the present σ-model calculation, the Okubo form of symmetry breaking works at least as well as the GMOR form does.

Since the basic scalar and pseudoscalar fields are assigned to $(3,3^*)\oplus(3^*,3)$ and the symmetry-breaking terms have components in (8, 8), it is impossible to have PCAC (partial conservation of axial-vector current) [or PCVC (partial conservation of vector current)] satisfied as an operator identity. This is in contrast with most previous work with σ models. We could have built operator PCAC into the present model by assigning the basic fields to appropriate combinations of $(3,3^*)\oplus(3^*,3)$ and (8, 8). However, this would have committed us to the existence of many other, unobserved mesons.

We do not feel that the present assignments of mesons and symmetry-breaking terms necessarily affect any of the good predictions of the soft-meson current-algebra method. In fact, we find that most of the symmetry-breaking choices considered here favor a rather small admixture of (8, 8). Thus, if indeed maximum smoothness in off-mass-shell extrapolations requires operator PCAC to be satisfied, the present models should have reasonably smooth off-mass-shell extrapolation.

The paper is organized as follows: In Sec. II we present a review and summary of the main features of the linear SU(3)×SU(3) σ model in both its general and renormalizable forms; the structure of the symmetry-breaking terms is analyzed in Sec. III with special emphasis on the (8, 8) contributions; in Secs. IV–VI the respective predictions of the model for strong, electromagnetic, and weak quantities or processes are derived; the details of the numerical analysis are given in Sec. VII, followed by our conclusions in Sec. VIII.

II. A SUMMARY OF THE THEORY

In this section we will give a brief description of the basic concepts involved in both the general linear SU(3) σ model[21-25] and its renormalizable version.[26] This will include a summary of the major formulas that will be employed throughout this paper. The general formalism involves eighteen basic fields, $M_{\bar{a}}^{b}$ and $M_a^{\bar{b}}$ $(a,b=1,2,3)$, which transform according to the $(3,3^*)$ and $(3^*,3)$ representations of SU(3)×SU(3), respectively. The upper (lower) indices denote the 3 (3*) representation of SU(3) and the barred (unbarred) indices denote the right- (left-) hand space of chiral SU(3) ×SU(3).

We may define the Hermiticity relation

$$(M_{\bar{a}}^{b})^\dagger = M_b^{\bar{a}} \tag{2.1}$$

and the transformation of the fields under parity as

$$PM_{\bar{a}}^{b}(\vec{x},t)P^{-1}=M_a^{\bar{b}}(-\vec{x},t)\,. \tag{2.2}$$

These relations allow us to decompose the fields as

$$\begin{aligned} M_{\bar{a}}^{b} &= S_a^b + i\phi_a^b\,,\\ M_a^{\bar{b}} &= S_a^b - i\phi_a^b\,, \end{aligned} \tag{2.3}$$

where the Hermitian matrices S_a^b and ϕ_a^b represent nonets of scalar $(\epsilon,\kappa,\sigma,\sigma')$ and pseudoscalar (π,K,η,η') fields, respectively. For matrix notation we identify

$$\begin{aligned} M_{\bar{a}}^{b} &\to (M)_{ab}\,,\\ M_a^{\bar{b}} &\to (M^\dagger)_{ab}\,. \end{aligned} \tag{2.4}$$

We consider a Lagrangian density of the form

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M\partial^\mu M^\dagger)-V_0-V_{\mathrm{SB}}-V_{\mathrm{wk}}\,, \tag{2.5}$$

where V_0 is a nonderivative, chiral SU(3)×SU(3)-invariant function built from M and $M^\dagger$. V_{SB} is an explicit symmetry-breaking form which can be chosen to belong to any representation of chiral SU(3)×SU(3). V_{wk} is the weak-interaction Lagrangian. The structure of V_{SB} and V_{wk} will be specified in Sec. III.

The stable ground state of the system must now be determined.[4-6] To do this we treat the Lagrangian semiclassically[5,23] and determine the equilibrium point by imposing the extremum conditions on the "potential" $V = V_0 + V_{SB}$:

$$\left\langle\frac{\partial V_0}{\partial \phi}\right\rangle_0 + \left\langle\frac{\partial V_{SB}}{\partial \phi}\right\rangle_0 = 0\,,$$
$$\left\langle\frac{\partial V_0}{\partial S}\right\rangle_0 + \left\langle\frac{\partial V_{SB}}{\partial S}\right\rangle_0 = 0\,, \tag{2.6}$$

where $\langle\rangle_0$ indicates that the enclosed expression is to be evaluated at the equilibrium point. The physical fields are then taken to be

$$\tilde{\phi} = \phi - \langle\phi\rangle_0\,,$$
$$\tilde{S} = S - \langle S\rangle_0\,. \tag{2.7}$$

Parity conservation requires that

$$\langle\phi\rangle_0 = \left\langle\frac{\partial V_0}{\partial \phi}\right\rangle_0 = \left\langle\frac{\partial V_{SB}}{\partial \phi}\right\rangle_0 = 0\,. \tag{2.8}$$

We can also choose

$$\langle S_a^b\rangle_0 = \delta_a^b \alpha_a \quad \text{(no sum)}\,. \tag{2.9}$$

Isotopic-spin invariance of the ground state requires

$$\alpha_1 = \alpha_2 = \alpha\,. \tag{2.10}$$

We also define the quantity

$$\omega = \alpha_3/\alpha\,, \tag{2.11}$$

which is a measure of the SU(3) noninvariance of the ground state.

The Lagrangian density is now expanded about the ground state in terms of the physical fields to give

$$\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}(\partial_\mu \tilde{\phi}\partial^\mu \tilde{\phi}) + \frac{1}{2}\,\mathrm{Tr}(\partial_\mu \tilde{S}\partial^\mu \tilde{S}) - \frac{1}{2}\sum_{a,b,c,d}\left(\left\langle\frac{\partial^2 V}{\partial\phi_a^b\partial\phi_c^d}\right\rangle_0 \tilde{\phi}_a^b\tilde{\phi}_c^d + \left\langle\frac{\partial^2 V}{\partial S_a^b\partial S_c^d}\right\rangle_0 \tilde{S}_a^b\tilde{S}_c^d\right)$$
$$- \frac{1}{2}\sum_{a,b,c,d,e,f}\left\langle\frac{\partial^3 V}{\partial S_a^b\partial\phi_c^d\partial\phi_e^f}\right\rangle_0 \tilde{S}_a^b\tilde{\phi}_c^d\tilde{\phi}_e^f - \frac{1}{4!}\sum_{a,b,c,d,e,f,g,h}\left\langle\frac{\partial^4 V}{\partial\phi_a^b\partial\phi_c^d\partial\phi_e^f\partial\phi_g^h}\right\rangle_0 \tilde{\phi}_a^b\tilde{\phi}_c^d\tilde{\phi}_e^f\tilde{\phi}_g^h + \cdots\,. \tag{2.12}$$

$\langle\partial^2 V/\partial\phi\partial\phi\rangle_0$ may be identified with the matrix of pseudoscalar-meson masses squared and $\langle\partial^3 V/\partial S\partial\phi\partial\phi\rangle_0$ with the $S\phi\phi$ coupling constants. Similar identifications can be made with the other terms appearing in the expansion.

So far we have not specified any properties of V_0 other than that it is SU(3)×SU(3)-invariant. We will now describe the two models for V_0 used in our analysis.

A. General model

It can be shown[21] that any nonderivative SU(3)×SU(3)-invariant form for V_0 can be expressed as a function of the chiral invariants

$$I_1 \equiv \mathrm{Tr}(MM^\dagger)\,,$$
$$I_2 \equiv \mathrm{Tr}(MM^\dagger MM^\dagger)\,,$$
$$I_3 \equiv \mathrm{Tr}(MM^\dagger MM^\dagger MM^\dagger)\,, \tag{2.13}$$
$$I_4 \equiv 6(\det M + \det M^\dagger)\,.$$

In the general model V_0 is considered to be an arbitrary function of $I_1, \ldots, I_4$. We introduce the following definitions for future use:

$$V_i = \left\langle\frac{\partial V_0}{\partial I_i}\right\rangle_0\,, \tag{2.14}$$

$$V_{ij} = \left\langle\frac{\partial^2 V_0}{\partial I_i \partial I_j}\right\rangle_0\,. \tag{2.15}$$

From Eq. (2.12) we see that in order to calculate masses, coupling constants, etc. we need to know second and higher derivatives of V_0 with respect to the relevant fields. Since, in the general model, V_0 is arbitrary, these derivatives cannot be calculated directly. However, they can be determined[23] from derivatives of V_{SB}, which is not arbitrary in our approach.

The requirement that V_0 be a chiral invariant and obey the equilibrium conditions allows us to relate the derivatives of V_0 to those of V_{SB}, when evaluated at the equilibrium point. We are interested in the following relations[23]:

$$(\alpha_a-\alpha_b)\left\langle\frac{\partial^2 V_0}{\partial S_b^a \partial S_f^e}\right\rangle_0=\delta_a^f\delta_e^b\left(\left\langle\frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0-\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0\right), \tag{2.16}$$

$$(\alpha_a+\alpha_b)\left\langle\frac{\partial^2 V_0}{\partial \phi_b^a \partial \phi_f^e}\right\rangle_0=-\delta_a^f\delta_e^b\left(\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0+\left\langle\frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0\right)-24V_4\delta_a^b\delta_e^f\alpha_1\alpha_2\alpha_3/\alpha_e\,, \tag{2.17}$$

where V_4 is arbitrary, and

$$(\alpha_a+\alpha_b)\left\langle\frac{\partial^3 V_0}{\partial S_h^g \partial \phi_f^e \partial \phi_b^a}\right\rangle_0=\delta_e^b\left\langle\frac{\partial^2 V_0}{\partial S_h^g \partial S_f^a}\right\rangle_0+\delta_a^f\left\langle\frac{\partial^2 V_0}{\partial S_h^g \partial S_b^e}\right\rangle_0-\delta_g^b\left\langle\frac{\partial^2 V_0}{\partial \phi_f^e \partial \phi_h^a}\right\rangle_0$$
$$-\delta_a^h\left\langle\frac{\partial^2 V_0}{\partial \phi_f^e \partial \phi_b^g}\right\rangle_0+12i\delta_a^b\left\langle\frac{\partial^2}{\partial S_h^g \partial \phi_f^e}\left[\frac{\partial V_0}{\partial I_4}(\det M-\det M^\dagger)\right]\right\rangle_0. \tag{2.18}$$

The last term contributes only when all fields are isoscalar, and we will not use this equation in that case. We will also need[23]

$$(\alpha_a+\alpha_b)\left\langle\frac{\partial^4 V_0}{\partial \phi_n^m \partial \phi_h^g \partial \phi_f^e \partial \phi_b^a}\right\rangle_0=\delta_e^b\left\langle\frac{\partial^3 V_0}{\partial \phi_n^m \partial \phi_h^g \partial S_f^a}\right\rangle_0+\delta_g^b\left\langle\frac{\partial^3 V_0}{\partial \phi_n^m \partial \phi_f^e \partial S_h^a}\right\rangle_0+\delta_m^b\left\langle\frac{\partial^3 V_0}{\partial \phi_h^g \partial \phi_f^e \partial S_n^a}\right\rangle_0$$
$$+\delta_a^f\left\langle\frac{\partial^3 V_0}{\partial \phi_n^m \partial \phi_h^g \partial S_b^e}\right\rangle_0+\delta_a^h\left\langle\frac{\partial^3 V_0}{\partial \phi_n^m \partial \phi_f^e \partial S_b^g}\right\rangle_0+\delta_a^n\left\langle\frac{\partial^3 V_0}{\partial \phi_h^g \partial \phi_f^e \partial S_b^m}\right\rangle_0$$
$$+12i\delta_b^a\left\langle\frac{\partial^3}{\partial \phi_n^m \partial \phi_h^g \partial \phi_f^e}\left[\frac{\partial V_0}{\partial I_4}(\det M-\det M^\dagger)\right]\right\rangle_0. \tag{2.19}$$

Again the last term contributes only when all fields are isoscalar, a case we will not consider.

In the above equations based on chiral invariance we do not obtain any useful information if all fields involved are isoscalar (except for a relation between the η and η' masses which can be obtained by eliminating V_4).

We can impose more constraints on these quantities if we require that V_0 also be invariant under scale transformations.[13,23] Thus, V_{SB} breaks scale invariance as well as SU(3)×SU(3). This requirement gives us the basic relation

$$\mathrm{Tr}\left(\phi\frac{\partial V_0}{\partial \phi}+S\frac{\partial V_0}{\partial S}\right)=4V_0\,. \tag{2.20}$$

This equation can then be differentiated and evaluated at the equilibrium point to give the requirements

$$\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^2 V_0}{\partial S_a^a \partial S_b^b}\right\rangle_0=-3\left\langle\frac{\partial V_{SB}}{\partial S_b^b}\right\rangle_0 \quad (b=1,2,3) \tag{2.21}$$

and

$$\sum_{a=1}^{3}\alpha_a\left\langle\frac{\partial^3 V_0}{\partial S_a^a \partial \phi_e^f \partial \phi_g^h}\right\rangle_0=2\left\langle\frac{\partial^2 V_0}{\partial \phi_e^f \partial \phi_g^h}\right\rangle_0. \tag{2.22}$$

At first glance these constraints and others involving higher-order derivatives of V_0 seem to imply restrictions on many classes of coupling constants. Closer examination shows that they are useful only to relate the σ and σ' masses and the coupling constants involving only isoscalar fields. For coupling constants involving other fields no additional information other than that implied by the relation between the σ and σ' masses is obtained.

B. Renormalizable model

We now discuss the special case in which the model is taken to be renormalizable.[26] This implies that no term in V_0 (or in V_{SB}) can be higher than fourth degree in the fields ϕ and S. Specifically, in terms of the invariants defined in Eq. (2.13), the expression for V_0 is restricted to have the form

$$V_0=aI_1+bI_1^2+cI_2+dI_4\,. \tag{2.23}$$

Noting the definitions given in Eqs. (2.14) and (2.15) we find (in the isospin-symmetric limit)

$$V_1 = a + 2b(2\alpha^2 + \alpha_3{}^2),$$
$$V_2 = c,$$
$$V_3 = 0, \qquad (2.24)$$
$$V_4 = d,$$

and

$$V_{11} = 2b.$$

Therefore

$$V_0 = [V_1 - V_{11}\alpha^2(2+\omega^2)]\, I_1 + \frac{V_{11}}{2} I_1{}^2 + V_2 I_2 + V_4 I_4. \qquad (2.25)$$

Naturally all the relations developed in the general model are applicable in this case.

III. FORM OF THE SYMMETRY-BREAKING AND WEAK-INTERACTION TERMS

We will assume that the (nonweak) symmetry-breaking part of the Lagrangian has the decomposition

$$V_{\rm SB} = V_{\rm SB}^{I=0} + V_{\rm SB}^{I=1} + V_{\rm SB}^{\gamma}, \qquad (3.1)$$

where $V_{\rm SB}^{I=0}$ and $V_{\rm SB}^{I=1}$ are chosen to belong to the combination of SU(3) × SU(3) representations $(3, 3^*) \oplus (3^*, 3) \oplus (8, 8)$ and where $I = 0$ and $I = 1$ denote the isospin content of the particular terms. The isospin-conserving term $V_{\rm SB}^{I=0}$ is, of course, necessary for a realistic description of strong phenomena. Its structure, in terms of the basic fields ϕ and S, will be given below in Sec. III A. The isospin-violating terms $V_{\rm SB}^{I=1}$ and $V_{\rm SB}^{\gamma}$ are required for a treatment of electromagnetic phenomena and will be discussed in Sec. III B. The weak-interaction term $V_{\rm wk}$ will be described in Sec. III C.

A. The isospin-conserving part

We assume that the isospin- (and parity-) conserving part of $V_{\rm SB}$ has the form given in Eq. (1.3), which, to reiterate, is

$$V_{\rm SB}^{I=0} = -c_0 u_0 - c_8 u_8 - d_0 z_0 - d_8 z_8, \qquad (3.2)$$

where u_0 and u_8 are the scalar SU(3) singlet and $I = 0$ octet members of the $(3, 3^*) \oplus (3^*, 3)$ representation of SU(3)×SU(3), and z_0 and z_8 are the scalar members of (8, 8) with the same respective SU(3) assignments.

First we consider the $(3, 3^*) \oplus (3^*, 3)$ part. Following Schechter and Ueda[21,23] we identify appropriate combinations of the scalar fields, S^a_b, with u_0 and u_8. Then (with repeated Greek indices summed from 1 to 3)

$$u_0 = \frac{1}{\sqrt{3}} S^\alpha_\alpha \qquad (3.3)$$

and

$$u_8 = \frac{1}{\sqrt{6}} (S^1_1 + S^2_2 - 2S^3_3). \qquad (3.4)$$

Thus,

$$-c_0 u_0 - c_8 u_8 = -c_0\left(\frac{1}{\sqrt{3}} S^\alpha_\alpha\right) - c_8\left[\frac{1}{\sqrt{6}}(S^1_1 + S^2_2 - 2S^3_3)\right] \qquad (3.5)$$

$$\equiv -g_0 S^\alpha_\alpha - g_3 S^3_3, \qquad (3.6)$$

where

$$g_0 = \frac{1}{\sqrt{6}}(\sqrt{2}\, c_0 + c_8)$$

and (3.7)

$$g_3 = -(\tfrac{3}{2})^{1/2} c_8.$$

Next, we analyze the (8, 8) contribution in more detail. The simplest expression transforming as (8, 8) is a bilinear combination of the M's. Taking the direct product of $(3, 3^*)$ and $(3^*, 3)$ one has

$$(3, 3^*) \otimes (3^*, 3) = (8, 8) \oplus (8, 1) \oplus (1, 8) \oplus (1, 1). \qquad (3.8)$$

Thus, for a form that transforms as (8, 8) we must remove the singlet contributions in the right-hand (barred) space and the left-hand (unbarred) space. Writing this product in terms of M's gives

$$\begin{aligned} M^a_{\bar b} M^{\bar c}_d &= (M^a_{\bar b} M^{\bar c}_d - \tfrac{1}{3}\delta^c_b M^a_{\bar\alpha} M^{\bar\alpha}_d - \tfrac{1}{3}\delta^a_d M^\alpha_{\bar b} M^{\bar c}_\alpha + \tfrac{1}{9}\delta^a_d \delta^c_b M^\alpha_{\bar\beta} M^{\bar\beta}_\alpha) \Rightarrow (8, 8) \\ &+ \tfrac{1}{3}\delta^c_b (M^a_{\bar\alpha} M^{\bar\alpha}_d - \tfrac{1}{3}\delta^a_d M^\alpha_{\bar\beta} M^{\bar\beta}_\alpha) \Rightarrow (8, 1) \\ &+ \tfrac{1}{3}\delta^a_d (M^\alpha_{\bar b} M^{\bar c}_\alpha - \tfrac{1}{3}\delta^c_b M^\alpha_{\bar\beta} M^{\bar\beta}_\alpha) \Rightarrow (1, 8) \\ &+ \tfrac{1}{9}\delta^a_d \delta^c_b M^\alpha_{\bar\beta} M^{\bar\beta}_\alpha, \Rightarrow (1, 1), \end{aligned} \qquad (3.9)$$

where the representations generated by each term are indicated. Hence, for the (8, 8) part, we must consider the tensor

$$T^{a\bar{c}}_{\bar{b}d} = M^a_{\bar{b}} M^{\bar{c}}_d - \tfrac{1}{3}\delta^c_b M^a_{\bar{\alpha}} M^{\bar{\alpha}}_d - \tfrac{1}{3}\delta^a_d M^{\alpha}_{\bar{b}} M^{\bar{c}}_{\alpha} + \tfrac{1}{9}\delta^a_d \delta^c_b M^{\alpha}_{\bar{\beta}} M^{\bar{\beta}}_{\alpha}, \tag{3.10}$$

where

$$T^{\alpha\bar{c}}_{\bar{b}\alpha} = T^{a\bar{\alpha}}_{\bar{\alpha}d} = 0 \tag{3.11}$$

by construction. We are interested in the even-parity nonet in the usual decomposition:

$$8 \otimes 8 = 27 + 10 + \overline{10} + 8_S + 8_A + 1. \tag{3.12}$$

This nonet has the form

$$E^t_s = -T^{t\bar{\alpha}}_{\bar{s}\alpha} - T^{\alpha\bar{t}}_{\bar{\alpha}s} + \delta^t_s T^{\alpha\bar{\beta}}_{\bar{\alpha}\beta}. \tag{3.13}$$

We can now identify

$$z_0 = \frac{1}{\sqrt{3}} E^{\alpha}_{\alpha} \tag{3.14}$$

and

$$z_8 = \frac{1}{\sqrt{6}}(E^1_1 + E^2_2 - 2E^3_3). \tag{3.15}$$

Paralleling the (3, 3*)⊕(3*, 3) case we write

$$d_0 z_0 + d_8 z_8 \equiv h_0 E^{\alpha}_{\alpha} + h_3 E^3_3, \tag{3.16}$$

where

$$h_0 = \frac{1}{\sqrt{6}}(\sqrt{2}\,d_0 + d_8)$$

and (3.17)

$$h_3 = -(\tfrac{3}{2})^{1/2} d_8.$$

From Eq. (3.13)

$$E^{\alpha}_{\alpha} = T^{\alpha\bar{\beta}}_{\bar{\alpha}\beta}$$

and

$$E^3_3 = -T^{3\bar{\alpha}}_{\bar{3}\alpha} - T^{\alpha\bar{3}}_{\bar{\alpha}3} + T^{\alpha\bar{\beta}}_{\bar{\alpha}\beta}. \tag{3.18}$$

Using Eqs. (2.3) and (3.10) we can translate these expressions into S's and φ's and obtain

$$E^{\alpha}_{\alpha} = (S^{\alpha}_{\alpha} S^{\beta}_{\beta} + \phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}) - \tfrac{1}{3}(S^{\alpha}_{\beta} S^{\beta}_{\alpha} + \phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha}) \tag{3.19}$$

and

$$E^3_3 = \tfrac{4}{3}(S^3_{\alpha} S^{\alpha}_3 + \phi^3_{\alpha}\varphi^{\alpha}_3) - 2(S^3_3 S^{\alpha}_{\alpha} + \phi^3_3 \phi^{\alpha}_{\alpha}) + (S^{\alpha}_{\alpha} S^{\beta}_{\beta} + \varphi^{\alpha}_{\alpha}\varphi^{\beta}_{\beta}) - \tfrac{5}{9}(S^{\alpha}_{\beta} S^{\beta}_{\alpha} + \phi^{\alpha}_{\beta}\phi^{\beta}_{\alpha}). \tag{3.20}$$

Consequently, the complete form of $V_{SB}^{I=0}$ is

$$V_{SB}^{I=0} = -g_0 S^{\alpha}_{\alpha} - g_3 S^3_3 - h_0[(S^{\alpha}_{\alpha} S^{\beta}_{\beta} + \phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}) - \tfrac{1}{3}(S^{\alpha}_{\beta} S^{\beta}_{\alpha} + \phi^{\alpha}_{\beta}\varphi^{\beta}_{\alpha})] - h_3[\tfrac{4}{3}(S^3_{\alpha} S^{\alpha}_3 + \varphi^3_{\alpha}\phi^{\alpha}_3) - 2(S^3_3 S^{\alpha}_{\alpha} + \phi^3_3 \phi^{\alpha}_{\alpha}) + (S^{\alpha}_{\alpha} S^{\beta}_{\beta} + \phi^{\alpha}_{\alpha}\phi^{\beta}_{\beta}) - \tfrac{5}{9}(S^{\alpha}_{\beta} S^{\beta}_{\alpha} + \phi^{\alpha}_{\beta}\varphi^{\beta}_{\alpha})], \tag{3.21}$$

where the g's and h's are defined by Eqs. (3.7) and (3.17), respectively.

B. Isospin-violating part

The isospin-violating (parity-conserving) part of the Lagrangian[22,24] will be used to describe electromagnetic mass shifts, the decay $\eta \to 3\pi$, and corrections to the $K \to 2\pi$ decays (including an attempt to account for the decay $K^+ \to \pi^+\pi^0$). The term $V_{SB}^{I=1}$ is assumed to have the form

$$V_{SB}^{I=1} = -c_3 u_3 - d_3 z_3, \tag{3.22}$$

where u_3 and z_3 are the $I = 1$, $I_3 = 0$, even-parity members of the (3, 3*)⊕(3*, 3) and (8, 8) representations of SU(3)×SU(3), respectively. This part of the Lagrangian may be considered to arise from an electromagnetic tadpole mechanism, schematically represented in Fig. 1. From an analysis analogous to that in Sec. III A above we can express $V_{SB}^{I=1}$ in terms of the basic fields as

$$V_{SB}^{I=1} = -g_1(S^1_1 - S^2_2) + \tfrac{2}{3}h_1[(S^1_1 S^1_1 - S^2_2 S^2_2) + (\varphi^1_1\phi^1_1 - \phi^2_2\varphi^2_2) - 2(S^1_3 S^3_1 - S^2_3 S^3_2) - 2(\phi^1_3\varphi^3_1 - \phi^2_3\varphi^3_2) + 3(S^1_1 - S^2_2)S^3_3 + 3(\phi^1_1 - \phi^2_2)\phi^3_3], \tag{3.23}$$

where

$$g_1 \equiv c_3/\sqrt{2}$$

and (3.24)

$$h_1 \equiv d_3/\sqrt{2}.$$

The term V^{γ}_{SB} is introduced to account for the effect of ordinary single-photon exchange, as pictured in Fig. 2, on the pion and kaon masses.

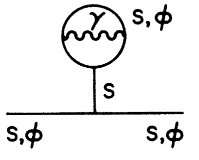

FIG. 1. Electromagnetic tadpole diagram.

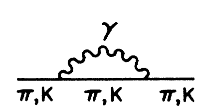

FIG. 2. Single-photon-exchange contribution to π and K electromagnetic self-masses.

It has the form

$$V^{\gamma}_{SB} = d_\pi \phi_1^2\phi_2^1 + d_K \phi_1^3\phi_3^1. \tag{3.25}$$

C. The weak interaction

In order to describe the weak nonleptonic decays of K mesons into two and three pions, $K\to 2\pi$ and $K\to 3\pi$, respectively, we include in the Lagrangian a weak interaction of the current-current type. The nonleptonic part of this weak interaction is assumed to be the $|\Delta S| = 1$ member of an SU(3) octet and is taken to be [25]

$$V_{wk} = -\frac{G}{\sqrt{2}} X\,\mathrm{Tr}(J_\mu J^\mu U), \tag{3.26}$$

where $G = 1.026\times 10^{-5} m_p^{-2}$ is the universal Fermi constant, X is a normalization parameter, and U is the matrix

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.27}$$

J_μ is the weak hadronic current constructed from the Noether currents of the Lagrangian without V_{wk}

$$J_\mu = V_\mu + A_\mu - \tfrac{1}{3}\mathrm{Tr}[A_\mu], \tag{3.28}$$

$$V_\mu = -i\,\{\tilde S \overleftrightarrow{\partial}_\mu \tilde S + \phi \overleftrightarrow{\partial}_\mu \phi + [\underline{\alpha}, \partial_\mu \tilde S]\},$$

and (3.29)

$$A_\mu = \phi \overleftrightarrow{\partial}_\mu \tilde S - \tilde S \overleftrightarrow{\partial}_\mu \phi - \{\underline{\alpha}, \partial_\mu \phi\},$$

where

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

If we define the pion and kaon decay constants f_π and f_K in the usual way,

$$[(2\pi)^3 2\omega_\pi]^{1/2}\langle 0|\partial_\mu A_1^{\mu 2}|\pi^+\rangle = f_\pi m_\pi^2$$

and (3.30)

$$[(2\pi)^3 2\omega_K]^{1/2}\langle 0|\partial_\mu A_1^{\mu 3}|K^+\rangle = f_K m_K^2,$$

then, to lowest order, the model gives

$$f_\pi = 2\alpha, \tag{3.31}$$

$$f_K = \alpha + \alpha_3. \tag{3.32}$$

To sum up, the full Lagrangian density to be employed in the following can be written as

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi) + \tfrac{1}{2}\mathrm{Tr}(\partial_\mu S\partial^\mu S) - V_0 - V_{SB}^{I=0} - V_{SB}^{I=1} - d_\pi\phi_1^2\phi_2^1 - d_K\phi_1^3\phi_3^1 - V_{wk}\,. \tag{3.33}$$

IV. STRONG-INTERACTION CALCULATIONS

In this section we apply the model to the description of purely strong effects including the meson mass spectrum, strong decays, and $\pi\pi$ and πK scattering lengths. We will thus begin by neglecting all symmetry-breaking terms in the Lagrangian except for $V_{SB}^{I=0}$, which is given in Eq. (3.21). We will take $\alpha_1 = \alpha_2 = \alpha$ [corresponding to an SU(2)-invariant ground state].

Calculations based on the general model of the chiral-symmetric part V_0 are given in Sec. IV A. The additional results obtainable from the renormalizable model are presented in Sec. IV B with the details of these calculations given in Appendix B.

A. The general model

1. Mass spectrum

From the expansion of the Lagrangian in Eq. (2.12) one sees that the scalar and pseudoscalar meson masses are given, to lowest order, by second derivatives of the Lagrangian with respect to the scalar and pseudoscalar fields. We will begin by considering the pseudoscalar mass spectrum. Using Eq. (2.17) and Table I one finds for the pion (here and in the following the particle symbol will also stand for its mass)

$$\pi^2 = \left\langle \frac{\partial^2 V}{\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 \tag{4.1}$$

$$= \left\langle \frac{\partial^2 V_0}{\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 \tag{4.2}$$

$$= \left(\frac{g_0}{\alpha} + \tfrac{2}{3}h_0(5+3\omega) + \tfrac{26}{9}h_3\right) + \left(\tfrac{2}{3}h_0 + \tfrac{10}{9}h_3\right). \tag{4.3}$$

Thus

$$\pi^2 = \frac{g_0}{\alpha} + 2h_0(2+\omega) + 4h_3. \tag{4.4}$$

Also

$$K^2 = \left\langle \frac{\partial^2 V}{\partial\phi_1^3\partial\phi_3^1}\right\rangle_0$$

$$= \frac{(2g_0 + g_3)/\alpha + 4h_0(2+\omega) + \tfrac{2}{3}h_3(4-\omega)}{1+\omega}. \tag{4.5}$$

Turning now to the η-η' system, the η-η' mixing angle θ_P is defined by

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix}. \tag{4.6}$$

We can identify

TABLE I. Expressions for the derivatives of V_{SB} with respect to the scalar and pseudoscalar fields, evaluated at the equilibrium point. We write the expressions as $\langle\rangle_0 = a_1 g_0 + a_2 g_3 + a_3 h_0 + a_4 h_3 + a_5 g_1 + a_6 h_1$. All derivatives not listed are zero at the equilibrium point.

Equilibrium point derivative	a_1	a_2	a_3	a_4	a_5	a_6
$\left\langle \frac{\partial V_{SB}}{\partial S_1^1} \right\rangle_0$	-1	0	$-\frac{2}{3}(2\alpha_1+3\alpha_2+3\alpha_3)$	$-\frac{2}{9}(4\alpha_1+9\alpha_2)$	-1	$\frac{2}{3}(2\alpha_1+3\alpha_3)$
$\left\langle \frac{\partial V_{SB}}{\partial S_2^2} \right\rangle_0$	-1	0	$-\frac{2}{3}(3\alpha_1+2\alpha_2+3\alpha_3)$	$-\frac{2}{9}(9\alpha_1+4\alpha_2)$	$+1$	$-\frac{2}{3}(2\alpha_2+3\alpha_3)$
$\left\langle \frac{\partial V_{SB}}{\partial S_3^3} \right\rangle_0$	-1	-1	$-\frac{2}{3}(3\alpha_1+3\alpha_2+2\alpha_3)$	$\frac{4}{9}\alpha_3$	0	$2(\alpha_1-\alpha_2)$
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_1^1} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0$	0	0	$-\frac{4}{3}$	$-\frac{8}{9}$	0	$\frac{4}{3}$
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_2^2} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0$	0	0	-2	-2	0	0
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_3^3} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0$	0	0	-2	0	0	2
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_2^2 \partial S_2^2} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0$	0	0	$-\frac{4}{3}$	$-\frac{8}{9}$	0	$-\frac{4}{3}$
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_2^2 \partial S_3^3} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0$	0	0	-2	0	0	-2
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_3^3 \partial S_3^3} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0$	0	0	$-\frac{4}{3}$	$\frac{4}{9}$	0	0
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^2 \partial S_2^1} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 - d_\pi$	0	0	$\frac{2}{3}$	$\frac{10}{9}$	0	0
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^3 \partial S_3^1} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^3 \partial \phi_3^1} \right\rangle_0 - d_K$	0	0	$\frac{2}{3}$	$-\frac{2}{9}$	0	$-\frac{4}{3}$
$\left\langle \frac{\partial^2 V_{SB}}{\partial S_2^3 \partial S_3^2} \right\rangle_0, \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^3 \partial \phi_3^2} \right\rangle_0$	0	0	$\frac{2}{3}$	$-\frac{2}{9}$	0	$\frac{4}{3}$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 = \left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0 = \tfrac{1}{2}\pi_0{}^2 + b^2\eta^2 + a^2\eta'^2, \tag{4.7}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0 = -\tfrac{1}{2}\pi_0{}^2 + b^2\eta^2 + a^2\eta'^2, \tag{4.8}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0 = \left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0 = -\sqrt{2}\, ab(\eta^2 - \eta'^2), \tag{4.9}$$

and

$$\left\langle \frac{\partial^2 V}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0 = 2a^2\eta^2 + 2b^2\eta'^2, \tag{4.10}$$

where

$$a = \frac{1}{\sqrt{6}}(\sin\theta_P + \sqrt{2}\cos\theta_P)$$

and (4.11)

$$b = \frac{1}{\sqrt{6}}(\cos\theta_P - \sqrt{2}\sin\theta_P).$$

(A similar identification can be made for the scalar mesons, σ and σ', with the appropriate combinations of θ_S denoted by a' and b'.)

Using these relations, Eq. (2.17), and Table I, we can now write

$$\begin{aligned}
\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0 + \tfrac{1}{2}\pi_0{}^2 &= b^2\eta^2 + a^2\eta'^2 \\
&= \frac{g_0}{2\alpha} + h_0\omega - 12V_4\alpha_3, \\
\left\langle \frac{\partial^2 V}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0 &= 2a^2\eta^2 + 2b^2\eta'^2 \\
&= \frac{(g_0+g_3)}{\alpha_3} + \frac{4h_0}{\omega} - \frac{12V_4\alpha}{\omega}, \\
\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0 &= -\sqrt{2}\, ab(\eta^2 - \eta'^2) \\
&= -2h_0 - 12V_4\alpha.
\end{aligned} \tag{4.12}$$

The above expressions can be used to relate η', θ_P, and V_4 to η and the symmetry-breaking parameters of $V_{SB}^{I=0}$. These relations can be obtained from the more general analysis of Appendix A, in which the π^0-η-η' system is analyzed in the isospin-violating case.

The only scalar meson mass which can be calculated in a straightforward fashion is that of the κ. Using the above techniques leads to

$$\kappa^2 = \left\langle \frac{\partial^2 V}{\partial S_1^3 \partial S_3^1} \right\rangle_0 = \frac{-g_3/\alpha + \frac{2}{3}h_3(4+\omega)}{1-\omega}. \quad (4.13)$$

This mass is obtained using only the chiral invariance of V_0. For information about the σ and σ' masses we must invoke scale invariance.[23] Using Eq. (2.21) we can get two independent equations involving σ, σ', and the scalar mixing angle θ_S (the third equation obtained is related to the others by isospin invariance.) We have

$$(1+2\omega)(\sigma^2-\sigma'^2)\cos 2\theta_S - \frac{1}{\sqrt{2}}(\omega-4)(\sigma^2-\sigma'^2)\sin 2\theta_S = p + 3(\sigma^2+\sigma'^2) \quad (4.14)$$

and

$$(\omega-4)(\sigma^2-\sigma'^2)\cos 2\theta_S + \sqrt{2}\,(1+2\omega)(\sigma^2-\sigma'^2)\sin 2\theta_S = q - 3\omega(\sigma^2+\sigma'^2), \quad (4.15)$$

where

$$p = -6[3g_0/\alpha + \tfrac{4}{3}h_0(5+3\omega) + \tfrac{52}{9}h_3],$$
$$q = 6[3(g_0+g_3)/\alpha + \tfrac{8}{3}h_0(3+\omega) - \tfrac{8}{9}h_3\omega]. \quad (4.16)$$

If we choose a value for σ, we can solve these equations for σ' and θ_S. In particular, we have for σ'

$$\sigma'^2 = \frac{6(\omega q - 2p)\sigma^2 - (2p^2+q^2)}{6[6(2+\omega^2)\sigma^2 - (\omega q - 2p)]} \quad (4.17)$$

and for $\tan 2\theta_S$

$$\tan 2\theta_S = \frac{\sqrt{2}\,[(1+2\omega)q - (\omega-4)p - 6(\omega-1)(\omega+2)(\sigma^2+\sigma'^2)]}{2(1+2\omega)p + (\omega-4)q - 3(\omega^2-8\omega-2)(\sigma^2+\sigma'^2)}. \quad (4.18)$$

At this stage the ϵ meson mass cannot be calculated. As we will see in Sec. V, it can be determined in terms of the isospin-violating parameters if $\alpha_1 \neq \alpha_2$. It can also be determined in the renormalizable model.

Our predictions for scalar and pseudoscalar meson masses in the various symmetry-breaking schemes of interest are shown for the general models without and with scale invariance of V_0 in Tables II and III, respectively. Underlined entries throughout the tables denote quantities used as input. For the computations given in the tables, π, K, and sometimes η and η' are always used as input to determine the symmetry-breaking parameters that are not fixed. A detailed discussion of the numerical analysis for all of our results is deferred until Sec. VII. Since the σ mass (and, in the case of the general model without scale invariance, the σ' mass and θ_S) cannot be predicted, it (they) must be chosen for the subsequent calculations. Note that we have considered two possible assignments for the η', at 958 MeV and at 1450 MeV.

2. Strong decays

We will first discuss the scalar meson decays into two pseudoscalar mesons. These are governed, to lowest order, by the $S\phi\phi$ couplings which may be written in isospin-invariant form as[23]

$$-\mathcal{L}(S\phi\phi) = \frac{1}{\sqrt{2}} g_{\epsilon K\bar{K}} \bar{K}\vec{\tau}K\cdot\vec{\epsilon} + g_{\epsilon\pi\eta}\vec{\epsilon}\cdot\vec{\pi}\eta + g_{\epsilon\pi\eta'}\vec{\epsilon}\cdot\vec{\pi}\eta' + \tfrac{1}{2}g_{\sigma\pi\pi}\sigma\vec{\pi}\cdot\vec{\pi} + \tfrac{1}{2}g_{\sigma'\pi\pi}\sigma'\vec{\pi}\cdot\vec{\pi}$$
$$+ g_{\sigma K\bar{K}}\sigma\bar{K}K + g_{\sigma' K\bar{K}}\sigma'\bar{K}K + \left(\frac{1}{\sqrt{2}} g_{\kappa K\pi}\bar{K}\vec{\tau}\kappa\cdot\vec{\pi} + g_{\kappa K\eta}\bar{K}\kappa\eta + g_{\kappa K\eta'}\bar{K}\kappa\eta' + \text{H.c.}\right)$$
$$+ \tfrac{1}{2}g_{\sigma\eta\eta}\sigma\eta\eta + \tfrac{1}{2}g_{\sigma'\eta\eta}\sigma'\eta\eta + \tfrac{1}{2}g_{\sigma\eta'\eta'}\sigma\eta'\eta' + \tfrac{1}{2}g_{\sigma'\eta'\eta'}\sigma'\eta'\eta' + g_{\sigma\eta\eta'}\sigma\eta\eta' + g_{\sigma'\eta\eta'}\sigma'\eta\eta'. \quad (4.19)$$

TABLE II. Tree-diagram calculations in the general model for various combinations of (3, 3*)⊕(3*,3)⊕(8, 8) chiral symmetry breaking; $V_{SB} = -c_0 u_0 - c_8 u_8 - d_0 z_0 - d_8 z_8$. Also $g_0 = (\sqrt{2}c_0 + c_8)/\sqrt{6}$. For these calculations we have $f_\pi = 135$ MeV and $f_K/f_\pi = 1.28$ ($\alpha = 0.50\,\pi$, $\alpha_3 = 0.78\,\pi$, and $\omega = 1.56$). The fixed masses used in the strong-interaction calculations are $\pi = 135$, $K = 495.8$, $\eta = 548$, $\sigma = 660$, $\sigma' = 997$, and $\epsilon = 970$ (MeV). Other quantities in the table used as input are underlined.

	$d_0=d_8=0$	$d_8=0$	$d_0=0$	$g_0=d_8=0$	$g_0=d_0=0$	$g_0=0$	$g_0=0.3$	$g_0=-0.5$	$g_0=-1.5$	$d_8=0$
c_0 (π^3)	10.26	3.45	9.91	9.39	10.52	8.30	8.72	4.78	3.19	70.27
c_8 (π^3)	−13.28	−13.28	−11.17	−13.28	−14.88	−11.74	−11.59	−7.98	−8.19	−13.28
d_0 (π^2)	0.0	1.91	0.0	0.24	0.0	0.48	0.35	1.29	1.75	−16.86
d_8 (π^2)	0.0	0.0	0.66	0.0	−0.50	0.49	0.53	1.66	1.60	0.0
c_8/c_0	−1.30	−3.85	−1.13	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	−1.33	−1.67	−2.57	−0.19
η' (MeV)	1019	<u>958</u>	<u>958</u>	1011	1067	<u>958</u>	<u>958</u>	<u>1450</u>	<u>1450</u>	<u>1450</u>
θ_P (deg)	−0.3	−0.4	1.0	−0.3	−0.9	0.6	0.7	9.8	9.8	−0.1
κ (MeV)	1029	1029	994	1029	1055	1004	1001	939	943	1029
θ_S (deg)	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>120.0</u>	<u>0.0</u>
$\Gamma(\kappa \to K\pi)$ (MeV)	778	778	654	778	877	687	678	483	493	778
$\Gamma(\epsilon \to \eta\pi)$ (MeV)	201	201	225	201	179	219	221	189	189	201
$\Gamma(\sigma \to \pi\pi)$ (MeV)	781	1137	645	823	894	757	727	647	732	262
$\Gamma(\sigma' \to \pi\pi)$ (MeV)	25	5	17	23	33	14	14	~1	~0	17
$\Gamma(\sigma' \to K\bar{K})$ (MeV)	68	82	74	68	62	74	74	92	96	18
a_0 (π^{-1})	0.168	0.790	−0.057	0.234	0.367	0.118	0.071	−0.054	0.083	0.408
a_2 (π^{-1})	−0.038	0.164	−0.109	−0.017	0.026	−0.054	−0.069	−0.108	−0.064	0.128
$a_{1/2}$ (π^{-1})	0.152	0.197	0.154	0.156	0.151	0.161	0.159	0.181	0.191	0.323
$a_{3/2}$ (π^{-1})	−0.050	−0.005	−0.050	−0.046	−0.050	−0.042	−0.044	−0.028	−0.017	0.121
$\Gamma(K_L \to \pi^+\pi^-\pi^0)$ ($10^6\,\text{sec}^{-1}$)	5.4	69.4	0.8	8.8	17.6	3.6	2.0	0.1	2.7	0.6

TABLE III. Tree-diagram calculations in the general model with a scale-invariant V_0 for the various forms of $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ chiral symmetry breaking We have $V_{SB}=-c_0u_0-c_8u_8-d_0z_0-d_8z_8$ and $g_0=(\sqrt{2}c_0+c_8)/\sqrt{6}$. We set $f_\pi=135$ MeV and the pion, kaon, η, and ϵ masses at 135, 495.8, 548, and 970 MeV, respectively. Additional inputs are underlined in the table. $\Gamma(\eta'-\eta\pi\pi)$ is calculated for Rittenberg's (Ref. 33) value of $\mathfrak{a}=-0.11$.

	$d_0=d_8=0$	$d_8=0$	$d_0=0$	$g_0=d_8=0$	$g_0=d_0=0$	$g_0=0$	$g_0=0.3$	$g_0=-0.5$	$g_0=-1.5$	$d_8=0$
ω	1.66	1.66	1.66	1.66	1.66	1.56	1.56	1.56	1.56	1.56
c_0 (π^3)	10.65	7.23	10.38	9.78	10.97	8.30	8.72	4.78	3.19	70.28
c_8 (π^3)	−13.83	−13.83	−12.44	−13.83	−15.52	−11.74	−11.59	−7.98	−8.19	−13.28
d_0 (π^2)	0.0	0.93	0.0	0.24	0.0	0.48	0.35	1.29	1.75	−16.86
d_8 (π^2)	0.0	0.0	0.43	0.0	−0.52	0.49	0.53	1.66	1.60	0.0
c_8/c_0	−1.30	−1.91	−1.20	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	−1.33	−1.67	−2.57	−0.189
η' (MeV)	988	958.0	958.0	981	1030	958.0	958.0	1450.0	1450.0	1450.0
θ_P (deg)	0.7	0.8	1.9	0.8	−0.2	0.6	0.7	9.8	9.8	−0.1
κ (MeV)	967	967	947	967	991	1004	1001	939	943	1029
σ (MeV)	650.0	600.0	660.0	640.0	620.0	630.0	630.0	620.0	590.0	580.0
σ' (MeV)	1336	1342	1313	1342	1336	1350	1327	1334	1335	1411
θ_S (deg)	102	101	101	101	105	101	102	95	95	156
$\Gamma(\kappa\to K\pi)$ (MeV)	565	565	505	565	645	687	678	483	493	778
$\Gamma(\epsilon\to\eta\pi)$ (MeV)	191	191	202	191	172	219	221	189	189	201
$\Gamma(\sigma\to\pi\pi)$ (MeV)	631	623	589	634	642	564	544	472	473	36
$\Gamma(\sigma'\to\pi\pi)$ (MeV)	1176	1266	1129	1235	1008	1254	1095	1514	1507	96
$\Gamma(\sigma'\to K\bar{K})$ (MeV)	596	588	540	596	658	662	646	496	500	684
a_0 (π^{-1})	0.165	0.467	0.015	0.232	0.374	0.121	0.072	−0.063	0.100	0.083
a_2 (π^{-1})	−0.038	0.052	−0.088	−0.018	0.029	−0.056	−0.072	−0.124	−0.076	0.024
$a_{1/2}$ (π^{-1})	0.158	0.172	0.160	0.161	0.155	0.162	0.161	0.180	0.189	0.231
$a_{3/2}$ (π^{-1})	−0.048	−0.034	−0.047	−0.045	−0.050	−0.041	−0.043	−0.029	−0.019	0.029
$\Gamma(\eta'\to\eta\pi\pi)$ (MeV)	6.0	6.7	3.6	6.1	9.4	4.9	4.5	...	...	...
$\Gamma(K_L\to\pi^+\pi^-\pi^0)$ (10^6sec^{-1})	6.5	37.5	1.1	10.9	23.0	5.0	2.9	0.6	6.9	1.3

Inspection of Eq. (2.12) shows that these couplings may be determined from the appropriate third derivatives of the Lagrangian. Thus, for the $\kappa K\pi$ coupling one finds

$$g_{\kappa K\pi} = \left\langle \frac{\partial^3 V}{\partial S_1^3 \partial \phi_2^1 \partial \phi_3^2} \right\rangle_0 \tag{4.20}$$

$$= \left\langle \frac{\partial^3 V_0}{\partial S_1^3 \partial \phi_2^1 \partial \phi_3^2} \right\rangle_0 \tag{4.21}$$

as

$$\left\langle \frac{\partial^3 V_{SB}}{\partial S \partial \phi \partial \phi} \right\rangle_0 = 0. \tag{4.22}$$

Thus, from Eq. (2.18) and Table I

$$g_{\kappa K\pi} = \frac{1}{\alpha + \alpha_3}\left(\left\langle \frac{\partial^2 V_0}{\partial S_1^3 \partial S_3^1} \right\rangle_0 - \left\langle \frac{\partial^2 V_0}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 \right) \tag{4.23}$$

$$= \frac{1}{\alpha + \alpha_3}\left[\left(\left\langle \frac{\partial^2 V}{\partial S_1^3 \partial S_3^1} \right\rangle_0 - \left\langle \frac{\partial^2 V}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 \right) - \left(\left\langle \frac{\partial^2 V_{SB}}{\partial S_1^3 \partial S_3^1} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0 \right) \right]$$

$$= \frac{1}{\alpha + \alpha_3}\left[\kappa^2 - \pi^2 - (-\tfrac{4}{3} h_3) \right]. \tag{4.24}$$

Hence

$$g_{\kappa K\pi} = \frac{1}{\alpha + \alpha_3}(\kappa^2 - \pi^2 + \tfrac{4}{3} h_3). \tag{4.25}$$

This can be expressed in terms of other masses by using the mass formulas. For example

$$g_{\kappa K\pi} = -\frac{1}{\alpha - \alpha_3}(K^2 - \pi^2 + \tfrac{4}{3} h_3). \tag{4.26}$$

In summary, the other $S\phi\phi$ coupling constants obtained using chiral invariance are

$$g_{\kappa K\eta} = \frac{(b - \sqrt{2}a)(\kappa^2 - \eta^2) - \frac{8}{3}(3h_0 + h_3)b + 2(\sqrt{2}/3)(6h_0 - h_3)a}{\alpha + \alpha_3}, \tag{4.27}$$

$$g_{\kappa K\eta'} = \frac{(a + \sqrt{2}b)(\kappa^2 - \eta'^2) - \frac{8}{3}(3h_0 + h_3)a - 2(\sqrt{2}/3)(6h_0 - h_3)b}{\alpha + \alpha_3}, \tag{4.28}$$

$$g_{\epsilon K\bar{K}} = (\epsilon^2 - K^2 - \tfrac{4}{3} h_3)/(\alpha + \alpha_3), \tag{4.29}$$

$$g_{\epsilon\pi\eta} = [(\epsilon^2 - \eta^2)b - 4(h_0 + h_3)b + 2\sqrt{2}h_0 a]/\alpha, \tag{4.30}$$

$$g_{\epsilon\pi\eta'} = [(\epsilon^2 - \eta'^2)a - 4(h_0 + h_3)a - 2\sqrt{2}h_0 b]/\alpha, \tag{4.31}$$

$$g_{\sigma\pi\pi} = [(\sigma^2 - \pi^2)b' + 4(h_0 + h_3)b' - 2\sqrt{2}h_0 a']/\alpha, \tag{4.32}$$

$$g_{\sigma'\pi\pi} = [(\sigma'^2 - \pi^2)a' + 4(h_0 + h_3)a' + 2\sqrt{2}h_0 b']/\alpha, \tag{4.33}$$

$$g_{\sigma K\bar{K}} = \frac{(\sigma^2 - K^2)(b' - \sqrt{2}a') + \frac{8}{3}(3h_0 + h_3)b' - 2(\sqrt{2}/3)(6h_0 - h_3)a'}{\alpha + \alpha_3}, \tag{4.34}$$

and

$$g_{\sigma' K\bar{K}} = \frac{(\sigma'^2 - K^2)(a' + \sqrt{2}b') + \frac{8}{3}(3h_0 + h_3)a' + 2(\sqrt{2}/3)(6h_0 - h_3)b'}{\alpha + \alpha_3}. \tag{4.35}$$

Again, the above coupling constants can all be calculated using only the chiral invariance of V_0. As $g_{\sigma\eta\eta'}$ and $g_{\sigma'\eta\eta'}$ both involve three isoscalars, we do not obtain any useful information from chiral invariance and we must impose scale invariance on V_0 once more. Using Eq. (2.22) one can derive the relation

$$Xg_{\sigma\eta\eta'} + Yg_{\sigma'\eta\eta'} = \frac{2}{\alpha}\left[\cos 2\theta_P \left(\frac{10\sqrt{2}}{9} h_3 \right) - \sin 2\theta_P (3h_0 + \tfrac{5}{9} h_3) \right], \tag{4.36}$$

where

$$X = (\tfrac{2}{3})^{1/2}(1 - \omega)\cos\theta_S - \frac{1}{\sqrt{3}}(2 + \omega)\sin\theta_S$$

and (4.37)

$$Y = (\tfrac{2}{3})^{1/2}(1 - \omega)\sin\theta_S + \frac{1}{\sqrt{3}}(2 + \omega)\cos\theta_S.$$

Thus, we still need to specify either $g_{\sigma\eta\eta'}$ or $g_{\sigma'\eta\eta'}$. Our predictions for the allowed scalar meson decays in the two versions of the general model are shown in Tables II and III for the various symmetry-breaking schemes of interest. These results will be discussed in Sec. VII.

The final strong decay we will discuss is $\eta' \to \eta\pi\pi$. The lowest-order contributions to this process are shown in Fig. 3. In addition to $S\phi\phi$ couplings, which were dealt with above, the decay amplitude depends on the four-point $\eta'\eta\pi\pi$ vertex. If we define the four-point couplings of pseudoscalars to be

$$-\mathfrak{L}(\phi^4)=\tfrac{1}{2}g_{\eta'}^{(4)}\eta'\eta\vec{\pi}\cdot\vec{\pi}+\tfrac{1}{16}g_{\pi}^{(4)}(\vec{\pi}\cdot\vec{\pi})^2+\tfrac{1}{2}g_{K}^{(4)}\bar{K}K\vec{\pi}\cdot\vec{\pi}+\cdots\,, \tag{4.38}$$

then for $g_{\eta'}^{(4)}$ we have, using Eqs. (2.18) and (2.19) and Table I,

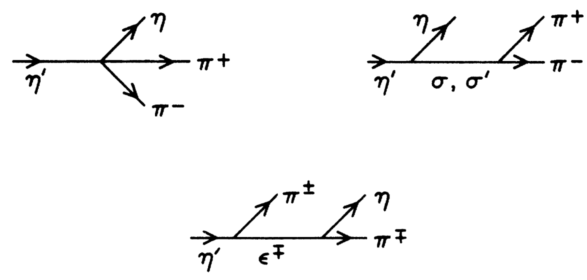

FIG. 3. Contributions to $\eta'\to\eta\pi^+\pi^-$ decay.

$$g_{\eta'}^{(4)}=\left\langle\frac{\partial^4 V}{\partial\eta'\partial\eta\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 \tag{4.39}$$

$$=\frac{1}{6\alpha^2}\{(2\sqrt{2}\,\cos2\theta_P-\sin2\theta_P)[\epsilon^2-\tfrac{1}{2}(\eta^2+\eta'^2)-4(h_0+h_3)]+(2\sqrt{2}\,\sin2\theta_P+\cos2\theta_P)(2\sqrt{2}\,h_0)\}$$

$$+\frac{1}{\alpha}(b' g_{\sigma\eta\eta'}+a' g_{\sigma'\eta\eta'}). \tag{4.40}$$

The decay amplitude calculated according to the diagrams of Fig. 3 is

$$T(\eta'(p)\to\eta(q)+\pi^+(k_+)+\pi^-(k_-))=g_{\eta'}^{(4)}-\frac{g_{\sigma\pi\pi}g_{\sigma\eta\eta'}}{\sigma^2-(p-q)^2}-\frac{g_{\sigma'\pi\pi}g_{\sigma'\eta\eta'}}{\sigma'^2-(p-q)^2}-\frac{g_{\epsilon\pi\eta}g_{\epsilon\pi\eta'}}{\epsilon^2-(p-k_+)^2}-\frac{g_{\epsilon\pi\eta}g_{\epsilon\pi\eta'}}{\epsilon^2-(p-k_-)^2}. \tag{4.41}$$

If we impose scale invariance on V_0 we can get enough information to calculate the width for this decay. Let us first go to the η' rest frame with $k_+^0=k_-^0$ and write, to first order,

$$T=A+BT_\eta, \tag{4.42}$$

where T_η is the η kinetic energy. Then

$$A=g_{\eta'}^{(4)}-\frac{g_{\sigma\pi\pi}g_{\sigma\eta\eta'}}{\sigma^2-(\eta'-\eta)^2}-\frac{g_{\sigma'\pi\pi}g_{\sigma'\eta\eta'}}{\sigma'^2-(\eta'-\eta)^2}-\frac{2g_{\epsilon\pi\eta}g_{\epsilon\pi\eta'}}{\epsilon^2-\pi^2-\eta\eta'} \tag{4.43}$$

and

$$B=2\eta'\left\{\frac{g_{\sigma\pi\pi}g_{\sigma\eta\eta'}}{[\sigma^2-(\eta'-\eta)^2]^2}+\frac{g_{\sigma'\pi\pi}g_{\sigma'\eta\eta'}}{[\sigma'^2-(\eta'-\eta)^2]^2}-\frac{g_{\epsilon\pi\eta}g_{\epsilon\pi\eta'}}{(\epsilon^2-\pi^2-\eta\eta')^2}\right\}. \tag{4.44}$$

The Dalitz-plot parametrization for the matrix element is of the form

$$T\propto 1+\mathfrak{a}\,y, \tag{4.45}$$

where

$$y=\frac{\eta+2\pi}{\pi}\frac{T_\eta}{Q}-1, \tag{4.46}$$

with $Q=\eta'-\eta-2\pi$. (4.47)

The experimental value[33] of $\mathfrak{a}$ can be used to furnish one condition on the amplitude. Another can be obtained from the scale invariance relation, Eq. (4.36). If we choose a value of the ϵ mass (a tentative value[34] will usually be 970 MeV) and a value of σ [thus determining σ' and θ_S from Eqs. (4.17) and (4.18)], we can predict a decay width using[35]

$$\Gamma(\eta'\to\eta\pi^+\pi^-)=\frac{1}{64(3.14159)^3\eta'}\int\int dk_+^0 dk_-^0|T|^2. \tag{4.48}$$

Our results for the width in the general model are given in Table III. The prediction is based on Rittenberg's value[33] of $\mathfrak{a}=-0.11$.

3. $\pi\pi$ and πK scattering lengths

We will now investigate the $\pi\pi$ and πK S-wave scattering lengths. For the $\pi\pi$ scattering lengths we consider the amplitude for the s-channel process $\pi^i\pi^j\to\pi^k\pi^l$ ($i,j,k,l=1,2,3$). The lowest-order contributions to $\pi\pi$ scattering from the model Lagrangian are shown in Fig. 4. The contributing four-pion vertex, defined in Eq. (4.38), is found to be

$$g_{\pi}^{(4)}=\left\langle\frac{\partial^4 V}{\partial\phi_1^2\partial\phi_2^1\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 \tag{4.49}$$

$$=\frac{1}{\alpha^2}[2b'^2\sigma^2+2a'^2\sigma'^2-\pi^2+4(h_0+h_3)]. \tag{4.50}$$

The S-wave scattering lengths a_I are defined by

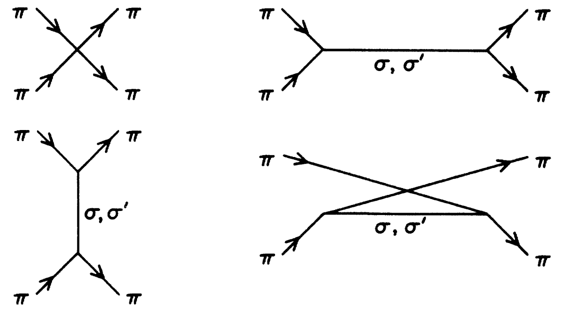

FIG. 4. Contributions to $\pi\pi$ scattering.

$$a_I = \frac{-1}{32(3.14159)\pi}\, T^I(s=4\pi^2,\ t=u=0). \tag{4.51}$$

Evaluating the $I=0$ and $I=2$ amplitudes corresponding to the diagrams in Fig. 4 we obtain

$$T^0(s=4\pi^2,\ t=u=0)=\tfrac{5}{2}g_\pi^{(4)} - g_{\sigma\pi\pi}^2\left(\frac{3}{\sigma^2-4\pi^2}+\frac{2}{\sigma^2}\right) - g_{\sigma'\pi\pi}^2\left(\frac{3}{\sigma'^2-4\pi^2}+\frac{2}{\sigma'^2}\right) \tag{4.52}$$

and

$$T^2(s=4\pi^2,\ t=u=0)=g_\pi^{(4)} - \frac{2g_{\sigma\pi\pi}^2}{\sigma^2} - \frac{2g_{\sigma'\pi\pi}^2}{\sigma'^2}. \tag{4.53}$$

Following a similar program for the πK scattering lengths we consider the amplitude for the s-channel process $\pi^i K \to \pi^j K$ which may be written in the form

$$T = A\delta_{ij} + B\tfrac{1}{2}[\tau_j, \tau_i]. \tag{4.54}$$

The contributions to these amplitudes are shown in Fig. 5. The relevant four-point coupling constant [see Eq. (4.38)] is found to be

$$g_K^{(4)} = \left\langle \frac{\partial^4 V}{\partial\phi_1^3\partial\phi_3^1\partial\phi_1^2\partial\phi_2^1}\right\rangle_0 \tag{4.55}$$

$$= \frac{1}{2\alpha(\alpha+\alpha_3)} \times[2b'^2\sigma^2+2a'^2\sigma'^2-2\sqrt{2}\,a'b'(\sigma^2-\sigma'^2) +\kappa^2-\pi^2-K^2+4(2h_0+h_3)]. \tag{4.56}$$

The amplitudes of definite s-channel isospin are given by

$$T^{I=1/2}=A+2B$$

and

$$T^{I=3/2}=A-B. \tag{4.57}$$

The πK S-wave scattering lengths are defined in the usual manner:

$$a_I = \frac{-1}{8(3.14159)(\pi+K)} \times T^I(s=(\pi+K)^2,\ t=0,\ u=(K-\pi)^2). \tag{4.58}$$

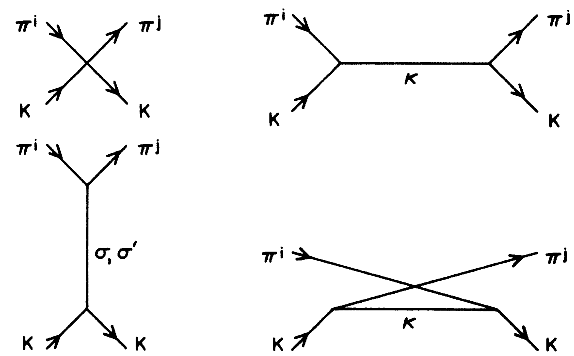

FIG. 5. Contributions to πK scattering.

Explicit evaluation of the diagrams in Fig. 5 gives

$$A(s=(\pi+K)^2,\ t=0,\ u=(\pi-K)^2) = g_K^{(4)} - \tfrac{1}{2}g_{\kappa K\pi}^2\left[\frac{1}{\kappa^2-(\pi+K)^2}+\frac{1}{\kappa^2-(\pi-K)^2}\right] - \frac{g_{\sigma\pi\pi}g_{\sigma K\bar{K}}}{\sigma^2} - \frac{g_{\sigma'\pi\pi}g_{\sigma' K\bar{K}}}{\sigma'^2} \tag{4.59}$$

and

$$B(s=(\pi+K)^2,\ t=0,\ u=(K-\pi)^2) = \tfrac{1}{2}g_{\kappa K\pi}^2\left[\frac{1}{\kappa^2-(\pi-K)^2}-\frac{1}{\kappa^2-(\pi+K)^2}\right]. \tag{4.60}$$

The predictions of the $\pi\pi$ and πK S-wave scattering lengths for the general models are shown in Tables II and III.

B. Renormalizable model

We will now point out the additional predictions which can be obtained from the renormalizable model. The calculational details are left to Appendix B.

1. Mass spectrum

We, of course, obtain the same expressions for the pseudoscalar meson masses in terms of the parameters of $V_{SB}^{I=0}$ as were found in Sec. IV A for the general model. These masses are again used as input to determine the symmetry-breaking parameters.

With respect to the scalar meson masses, the σ' (σ) mass and the σ-σ' mixing angle θ_S can be obtained once the σ (σ') mass is specified (see Appendix B). This is similar to the situation in the general model with a scale-invariant V_0. In addition, the ϵ mass can be determined in the renormalizable model. We have

$$\epsilon^2 = \left\langle\frac{\partial^2 V_0}{\partial S_1^2\partial S_2^1}\right\rangle_0 + \left\langle\frac{\partial^2 V_{SB}}{\partial S_1^2\partial S_2^1}\right\rangle_0 \tag{4.61}$$

and from the form of V_0 given in Eq. (2.25) one gets

$$\left\langle\frac{\partial^2 V_0}{\partial S_1^2 \partial S_2^1}\right\rangle_0 = 2V_1 + 12V_2\alpha^2 - 12V_4\alpha_3$$

and thus, with the help of Table I, we find

$$\epsilon^2 = 2V_1 + 12V_2\alpha^2 - 12V_4\alpha_3 + \tfrac{2}{3}h_0 + \tfrac{10}{9}h_3. \tag{4.62}$$

It is shown in Appendix B how the parameters of V_0 can be determined. Our results for the renormalizable model are shown in Table IV.

2. Strong decays

Previously we had to resort to scale invariance and input the slope parameter to find the decay width for $\eta' \to \eta\pi\pi$. This was necessary as both $g_{\sigma\eta\eta'}$ and $g_{\sigma'\eta\eta'}$ involve only $I=0$ fields. We can now evaluate these coupling constants directly and evaluate both $\Gamma(\eta' \to \eta\pi\pi)$ and the slope parameter $\mathfrak{a}$.

First, we calculate the coupling constants. From the form of V_0, Eq. (2.25), they can be evaluated directly and we find, using previous techniques,

$$g_{\sigma\eta\eta'} = \left\langle\frac{\partial^3 V}{\partial\sigma\partial\eta\partial\eta'}\right\rangle_0 \tag{4.63}$$

$$= 16ab\,V_2(\alpha b' + \sqrt{2}\,\alpha_3 a') - 24\sqrt{2}\;V_4[b'(b^2 - a^2) - a'ab] \tag{4.64}$$

and

$$g_{\sigma'\eta\eta'} = \left\langle\frac{\partial^3 V}{\partial\sigma'\partial\eta\partial\eta'}\right\rangle_0 \tag{4.65}$$

$$= 16ab\,V_2(\alpha a' - \sqrt{2}\,\alpha_3 b') - 24\sqrt{2}\;V_4[a'(b^2 - a^2) + b'ab]. \tag{4.66}$$

With these couplings and the ϵ mass determined above, we can evaluate the amplitudes A and B of Eqs. (4.43) and (4.44) and thus determine both the slope and width of the $\eta' \to \eta\pi\pi$ decay. These predictions are given in Table IV.

An analysis of all the results obtained in this section will be given in Sec. VII.

V. ELECTROMAGNETIC CALCULATIONS

We will now discuss electromagnetic corrections to some of the results presented above. For this purpose we will add to the Lagrangian of Sec. IV the isospin-violating terms $-V_{SB}^{I=1}$ and $-V_{SB}^{\gamma}$ given by Eqs. (3.23) and (3.25), respectively. Also, in the following α_1 will not be equal to α_2, thus resulting in an SU(2)-noninvariant ground state.

We will consider mass splitting within the π, K, and κ isospin multiplets as well as electromagnetic mixing between the $I=0$ and $I=1$ members in both the pseudoscalar and scalar nonets. The π^0-η-η' system is analyzed in Appendix A, where η', θ_P, V_4, and the π^0-η and π^0-η' mixing angles ψ_1 and ψ_3, respectively, are determined in terms of V_{SB} and η. The ϵ^0-σ and ϵ^0-σ' mixing angles χ_1 and χ_2, respectively, are determined for the renormalizable model in Appendix B. For the purposes of Sec. VI, where we study violation of the $\Delta I = \frac{1}{2}$ rule in the $K \to 2\pi$ decays, we will also calculate the changes in the $\kappa K\pi$ coupling constants due to isospin violation. In addition, a value for the ϵ mass can be obtained in the general model in terms of the isospin-violating parameters, since $\alpha_1 \neq \alpha_2$. Finally, we predict the width and slope of the decay $\eta \to \pi^+\pi^-\pi^0$ in the renormalizable model. The general model will be covered below in Sec. V A and the renormalizable model in Sec. V B.

A. The general model

1. Mass splitting

The techniques used in Sec. IV for calculating masses are applicable here. Using Eqs. (2.16), (2.17), (3.21), (3.23), and (3.25) and Table I we find

$$\pi_+^{\,2} = \frac{2[g_0 + 2h_0(\alpha_1 + \alpha_2 + \alpha_3) + 2h_3(\alpha_1 + \alpha_2)]}{\alpha_1 + \alpha_2} + d_\pi\,, \tag{5.1}$$

$$K_+^{\,2} = \frac{2g_0 + g_3 + 4h_0(\alpha_1 + \alpha_2 + \alpha_3) + \frac{2}{3}h_3(\alpha_1 + 3\alpha_2 - \alpha_3) + g_1 - \frac{2}{3}h_1(4\alpha_1 + 5\alpha_3)}{\alpha_1 + \alpha_3} + d_K\,, \tag{5.2}$$

$$K_0^{\,2} = \frac{2g_0 + g_3 + 4h_0(\alpha_1 + \alpha_2 + \alpha_3) + \frac{2}{3}h_3(3\alpha_1 + \alpha_2 - \alpha_3) - g_1 + \frac{2}{3}h_1(4\alpha_2 + 5\alpha_3)}{\alpha_2 + \alpha_3}\,, \tag{5.3}$$

TABLE IV. Tree-diagram calculations in the renormalizable model for various combinations of (3, 3*) ⊕ (3*, 3) ⊕ (8, 8) chiral symmetry breaking. We have $V_{SB} = -c_0 u_0 - c_8 u_8 - d_0 z_0 - d_8 z_8$ and $g_0 = (\sqrt{2}\, c_0 + c_8)/\sqrt{6}$. We set $f_\pi = 135$ MeV and fix π, K, and η at 135, 495.8, and 548 MeV, respectively, in the strong-interaction calculations. Other quantities used as input are underlined. At the end of the table we present the best results of the electromagnetic calculations for the cases $d_0 = d_8 = 0$ and $g_0 = d_8 = 0$. The mass of ϵ^+, calculated from Eq. (5.6), is 958 MeV for $d_3 = 0$ and 970 MeV for $d_3 = -0.0062\ \pi^2$.

	$d_0 = d_8 = 0$	$d_8 = 0$	$d_0 = 0$	$g_0 = d_8 = 0$	$g_0 = d_0 = 0$	$g_0 = 0$	$g_0 = 0.3$	$g_0 = -0.5$	$g_0 = -1.5$
ω	1.66	1.66	1.66	1.66	1.56	1.56	1.56	1.56	1.56
c_0 (π^3)	10.65	7.23	10.38	9.78	10.52	8.30	8.72	4.78	3.19
c_8 (π^3)	−13.83	−13.83	−12.44	−13.83	−14.88	−11.74	−11.59	−7.98	−8.19
d_0 (π^2)	0.0	0.93	0.0	0.24	0.0	0.48	0.35	1.29	1.75
d_8 (π^2)	0.0	0.0	0.43	0.0	−0.50	0.49	0.53	1.66	1.60
c_8/c_0	−1.30	−1.91	−1.20	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	−1.33	−1.67	−2.57
η' (MeV)	988	958.0	958.0	981	1067	958.0	958.0	1450.0	1450.0
θ_P (deg)	0.7	0.8	1.9	0.8	−0.9	0.6	0.7	9.8	9.8
κ (MeV)	967	967	947	967	1055	1004	1001	939	943
ϵ (MeV)	952	952	921	952	1039	956	952	1196	1204
σ (MeV)	640.0	620.0	500.0	640.0	450.0	660.0	640.0	550.0	700.0
σ' (MeV)	1123	1123	1032	1126	1129	1128	1115	864	909
θ_S (deg)	123	123	112	124	110	118	117	100	119
$\Gamma(\eta' \to \eta\pi\pi)$ (MeV)	5.4	5.5	6.0	5.4	3.1	3.9	3.9	81.9	1.9
$\mathfrak{a}$	−0.03	−0.03	−0.19	−0.03	−0.45	−0.01	−0.02	0.85	1.0
$\Gamma(\kappa \to K\pi)$ (MeV)	565	565	505	565	877	687	678	483	493
$\Gamma(\epsilon \to \eta\pi)$ (MeV)	169	170	148	169	273	201	198	460	472
$\Gamma(\sigma \to \pi\pi)$ (MeV)	709	784	226	505	240	756	648	319	901
$\Gamma(\sigma' \to \pi\pi)$ (MeV)	5	2	150	2	361	44	72	180	~0
$\Gamma(\sigma' \to K\bar{K})$ (MeV)	506	538	158	534	336	530	460	0	0
a_0 (π^{-1})	0.170	0.454	0.031	0.235	0.462	0.118	0.072	−0.032	0.090
a_2 (π^{-1})	−0.038	0.050	−0.081	−0.018	0.026	−0.055	−0.070	−0.107	−0.062
$a_{1/2}$ (π^{-1})	0.157	0.175	0.167	0.161	0.148	0.162	0.160	0.180	0.190
$a_{3/2}$ (π^{-1})	−0.049	−0.031	−0.041	−0.045	−0.053	−0.041	−0.043	−0.029	−0.019
$\Gamma(K_L \to \pi^+\pi^-\pi^0)$ (10^6 sec^{-1})	5.7	28.8	2.2	9.3	62.6	3.6	2.2	1.4	3.0
$\alpha_1 - \alpha_2$ (π)	−0.0148	···	···	−0.0148	···	···	···	···	···
d_K (π^2)	0.21			0.21					
c_3 (π^3)	−0.527			−0.527					
d_3 (π^2)	0.0			0.0					
ψ_1 (deg)	1.1			1.1					
ψ_3 (deg)	−0.1			−0.1					
χ_1 (deg)	1.1			1.1					
χ_3 (deg)	−1.5			−1.5					
$\Gamma(\eta \to \pi^+\pi^-\pi^0)$ (eV)	114			116					
β	−0.477			−0.477					
$\frac{T(K^+ \to \pi^+\pi^0)}{T(K_1^0 \to \pi^+\pi^-)}$	0.011			0.011					
$\alpha_1 - \alpha_2$ (π)	−0.0147			−0.0147					
d_K (π^2)	0.21			0.21					
c_3 (π^3)	−0.551			−0.551					
d_3 (π^2)	−0.0062			−0.0062					
ψ_1 (deg)	1.1			1.1					
ψ_3 (deg)	−0.1			−0.1					
χ_1 (deg)	1.0			1.1					
χ_3 (deg)	−1.5			−1.5					
$\Gamma(\eta \to \pi^+\pi^-\pi^0)$ (eV)	119			121					
β	−0.477			−0.477					
$\frac{T(K^+ \to \pi^+\pi^0)}{T(K_1^0 \to \pi^+\pi^-)}$	0.0111			0.0111					

$$\kappa_+{}^2 = \frac{-g_3 + \frac{2}{3}h_3(\alpha_1 + 3\alpha_2 + \alpha_3) + g_1 - \frac{2}{3}h_1(4\alpha_1 + \alpha_3)}{\alpha_1 - \alpha_3}, \tag{5.4}$$

$$\kappa_0{}^2 = \frac{-g_3 + \frac{2}{3}h_3(3\alpha_1 + \alpha_2 + \alpha_3) - g_1 + \frac{2}{3}h_1(4\alpha_2 + \alpha_3)}{\alpha_2 - \alpha_3}, \tag{5.5}$$

$$\epsilon_+{}^2 = \frac{2[g_1 - \frac{2}{3}h_1(\alpha_1 + \alpha_2 + 3\alpha_3)]}{\alpha_1 - \alpha_2}. \tag{5.6}$$

Note that $\pi_+{}^2 - \pi_0{}^2 = d_\pi$.

We now turn our attention to the combination $\alpha_1 - \alpha_2$. Since $\alpha_1 - \alpha_2 \neq 0$, we can calculate the ϵ mass in terms of the isospin-violating parameters g_1, h_1, and $\alpha_1 - \alpha_2$ using Eq. (2.16). As an estimate of the magnitude of $\alpha_1 - \alpha_2$ when $\alpha_1 \neq \alpha_2$, we will consider the value of Schechter and Ueda[22,24].

$$\alpha_1 - \alpha_2 = \alpha(\omega - 1)\frac{\Sigma^+ - \Sigma^-}{\Xi - N}. \tag{5.7}$$

2. Coupling constants

In our estimate of the first-order electromagnetic corrections to the $\kappa K\pi$ vertex, we neglect the one-photon emission and absorption contribution and, to be consistent, we then set $\pi_+ = \pi_0$.[25] (Consequently, in this calculation, we set $d_\pi = d_K = 0$.) We use this vertex in our calculation of the decay rates for the processes $K^+ \to \pi^+\pi^0$ and $K^0 \to \pi^+\pi^-$ in Sec. VI.

Using our standard techniques we obtain

$$g_{K^0\kappa^-\pi^+} = \frac{1}{\alpha_2 + \alpha_3}(\kappa_+{}^2 - \pi^2 + \tfrac{4}{3}h_3 + \tfrac{4}{3}h_1), \tag{5.8}$$

$$g_{K^+\bar{\kappa}^0\pi^-} = \frac{1}{\alpha_1 + \alpha_3}(\kappa_0{}^2 - \pi^2 + \tfrac{4}{3}h_3 - \tfrac{4}{3}h_1), \tag{5.9}$$

and

$$g_{K^+\kappa^-\pi^0} = \frac{(\kappa_+{}^2 - \pi^2)}{\sqrt{2}\,(\alpha_1 + \alpha_3)}[1 + \sqrt{2}\,\psi_1(b - \sqrt{2}\,a) + \sqrt{2}\,\psi_3(a + \sqrt{2}\,b)] - \frac{4\sqrt{2}\,h_0}{\alpha_1 + \alpha_3}[\psi_1(\sqrt{2}\,b - a) + \psi_3(\sqrt{2}\,a + b)]$$
$$+ \frac{2\sqrt{2}}{3(\alpha_1 + \alpha_3)}h_3[1 - \psi_1(a + 2\sqrt{2}\,b) + \psi_3(b - 2\sqrt{2}\,a)] + \frac{10\sqrt{2}}{3(\alpha_1 + \alpha_3)}h_1. \tag{5.10}$$

B. Renormalizable model

1. $\eta \to 3\pi$

We will now apply the renormalizable model to the decay

$$\eta(p) \to \pi^+(q_+) + \pi^-(q_-) + \pi^0(q_0). \tag{5.11}$$

The decay amplitude is calculated to first order in the isospin-violating interaction $V_{SB}^{I=1}$ of Eq. (3.23). The contributions to the amplitude are illustrated in Fig. 6.

The terms of the total Lagrangian which are relevant to the calculation are the strong couplings

$$g_{\epsilon\pi\eta}\eta\vec{\epsilon}\cdot\vec{\pi} + \tfrac{1}{2}g_{\sigma\pi\pi}\sigma\vec{\pi}\cdot\vec{\pi} + \tfrac{1}{2}g_{\sigma'\pi\pi}\sigma'\vec{\pi}\cdot\vec{\pi} \tag{5.12}$$

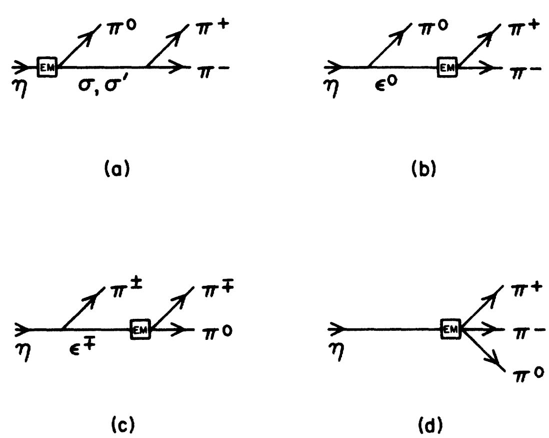

FIG. 6. Contributions to $\eta \to \pi^+\pi^-\pi^0$ decay. EM denotes an effective electromagnetic vertex.

and the isospin-violating couplings

$$f_{\sigma\pi^0\eta}\sigma\pi^0\eta+f_{\sigma'\pi^0\eta}\sigma'\pi^0\eta+f_{\epsilon 0\pi^+\pi^-}\epsilon^0\pi^+\pi^-$$
$$+f_{\epsilon^-\pi^+\pi^0}(\epsilon^-\pi^+\pi^0+\epsilon^+\pi^-\pi^0)+f^{(4)}_{\eta\pi^+\pi^-\pi^0}\eta\pi^+\pi^-\pi^0\,. \qquad (5.13)$$

The strong couplings are given by Eqs. (4.30), (4.32), and (4.33). The relevant isospin-violating couplings are given in Appendix B [Eqs. (B24)–(B32)].

Using Eqs. (5.12) and (5.13) the contributions to the decay amplitude $T(\eta(p)\to\pi^+(q_+)\pi^-(q_-)\pi^0(q_0))$, are found to be

$$T_{6(a)}=-i\,\frac{g_{\sigma\pi\pi}f_{\sigma\pi^0\eta}}{(q_0-p)^2\sigma^2-\sigma^2}-i\,\frac{g_{\sigma'\pi\pi}f_{\sigma'\pi^0\eta}}{(q_0-p)^2-\sigma'^2}\,, \qquad (5.14)$$

$$T_{6(b)}=-i\,\frac{f_{\epsilon 0\pi^+\pi^-}g_{\epsilon\pi\eta}}{(q_0-p)^2-\epsilon^2}\,, \qquad (5.15)$$

$$T_{6(c)}=-i\,\frac{f_{\epsilon^-\pi^+\pi^0}g_{\epsilon\pi\eta}}{(q_+-p)^2-\epsilon^2}-i\,\frac{f_{\epsilon^-\pi^+\pi^0}g_{\epsilon\pi\eta}}{(q_--p)^2-\epsilon^2}\,, \qquad (5.16)$$

$$T_{6(d)}=-if^{(4)}_{\eta\pi^+\pi^-\pi^0}\,, \qquad (5.17)$$

where the subscripts on the T's refer to the particular diagram in Fig. 6 according to which the contribution was calculated.

In the η rest frame, with $q_+^0=q_-^0$, we may write, to first order,

$$T(\eta\to\pi^+\pi^-\pi^0)=A+BT_{\pi^0}\,, \qquad (5.18)$$

where T_{π^0} is the π^0 kinetic energy. From Eqs. (5.14)–(5.17) we then have

$$A=f_{\sigma\pi^0\eta}\left[\frac{b'}{\alpha}+\frac{g_{\sigma\pi\pi}}{(\pi-\eta)^2-\sigma^2}\right]+f_{\sigma'\pi^0\eta}\left[\frac{a'}{\alpha}+\frac{g_{\sigma'\pi\pi}}{(\pi-\eta)^2-\sigma'^2}\right]+\frac{b}{\alpha}f_{\epsilon^-\pi^+\pi^0}$$
$$+g_{\epsilon\pi\eta}\left(\frac{\chi_1 b'+\chi_3 a'+\psi_1 b+\psi_3 a}{\alpha}+\frac{f_{\epsilon 0\pi^+\pi^-}}{(\pi-\eta)^2-\epsilon^2}+\left\{\frac{2(\pi^2-5\eta\pi+2\eta^2-\epsilon^2)}{[(\pi-\eta)^2-\epsilon^2]^2}\right\}f_{\epsilon^-\pi^+\pi^0}\right) \qquad (5.19)$$

and

$$B=2\eta\left(\frac{f_{\sigma\pi^0\eta}g_{\sigma\pi\pi}}{[(\eta-\pi)^2-\sigma^2]^2}+\frac{f_{\sigma'\pi^0\eta}g_{\sigma'\pi\pi}}{[(\pi-\eta)^2-\sigma'^2]^2}+g_{\epsilon\pi\eta}\left\{\frac{f_{\epsilon 0\pi^+\pi^-}-f_{\epsilon^-\pi^+\pi^0}}{[(\eta-\pi)^2-\epsilon^2]^2}\right\}\right)\,. \qquad (5.20)$$

The Dalitz-plot parameterization of the amplitude is of the form

$$T\propto 1+\beta y\,, \qquad (5.21)$$

where

$$y=\frac{3T_{\pi^0}}{Q}-1 \qquad (5.22)$$

with

$$Q=\eta-3\pi\,. \qquad (5.23)$$

Our calculated value of β is related to the amplitudes A and B via

$$\beta=\frac{1}{1+(3/Q)(A/B)}\,. \qquad (5.24)$$

Owing to the large slope, we cannot realistically neglect the B term in our calculation of the partial width of $\eta\to\pi^+\pi^-\pi^0$. Consequently, we take $|T|^2=A^2+2ABT_{\pi^0}$ from Eq. (5.18) and evaluate the partial width using the formula analogous to Eq. (4.48).

VI. WEAK-INTERACTION CALCULATIONS

We will now describe the calculation of various weak-interaction processes, including the $K\to 2\pi$ and $K\to 3\pi$ decays. We include some of the possible first-order electromagnetic effects in our treatment of the $K\to 2\pi$ decays.

The general form of the weak amplitudes, in terms of masses and coupling constants, will be the same in both the general and renormalizable models. The predictions will differ in the two models because of differences in certain masses and couplings.

Following Goswami, Schechter, and Ueda[25] we now calculate the $K\to 2\pi$ amplitudes to first order in the weak interaction. We neglect, as do Goswami *et al.*, the one-photon emission and absorption contributions and hence, to be consistent, we then set $\pi_+=\pi_0$ in our estimate of the electromagnetic corrections. The diagrams relevant to $K_1^0\to\pi^+\pi^-$ are given in Fig. 7. Those for $K^+\to\pi^+\pi^0$ are shown in Fig. 8. Here,

$$K_1=\frac{\bar{K}_0-K_0}{\sqrt{2}\,i}\,. \qquad (6.1)$$

From the structure of the weak interaction in Eq. (3.26), the form of the currents in Eqs. (3.28) and (3.29) and the form of the electromagnetic π^0-η-η' mixing in Eq. (A1), the relevant parts of the two- and three-point weak vertices are

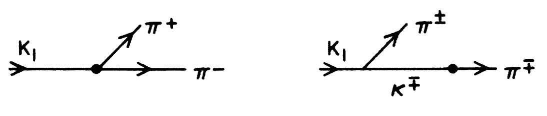

FIG. 7. Contributions to $K_1 \to \pi^+\pi^-$ decay. The dot denotes a weak-interaction vertex.

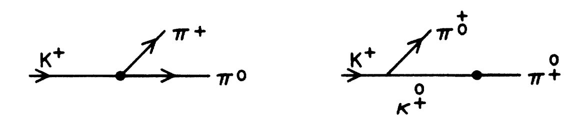

FIG. 8. Contributions to $K^+ \to \pi^+\pi^0$ decay. The dot denotes a weak-interaction vertex.

$$\mathcal{L}_{wk}(S\phi) = i\frac{GX}{\sqrt{2}}\left\{\tfrac{2}{3}(\alpha_3-\alpha_2)\left[\frac{2\alpha_1+\alpha_2}{\sqrt{2}} + \alpha\psi_1(b+\sqrt{2}\,a\omega) + \alpha\psi_3(a-\sqrt{2}\,b\omega)\right]\partial_\mu\pi^0\partial^\mu(\kappa^0-\bar{\kappa}^0) + (\alpha_1+\alpha_2)(\alpha_3-\alpha_1)(\partial_\mu\kappa^-\partial^\mu\pi^+ - \partial_\mu\kappa^+\partial^\mu\pi^-) + \cdots\right\} \tag{6.2}$$

and

$$\mathcal{L}_{wk}(\phi^3) = GX\Bigg(\frac{2}{3}\left\{-\frac{(2\alpha_1+\alpha_2)}{2} + \alpha\psi_1[\sqrt{2}\,b+(3-\omega)a] + \alpha\psi_3[\sqrt{2}\,a-(3-\omega)b]\right\}\partial_\mu\pi^0(K_1\overleftrightarrow{\partial}^\mu\pi^0) - (\alpha_1+\alpha_2)K_1\partial_\mu\pi^-\partial^\mu\pi^+ + \tfrac{1}{2}(\alpha_1+\alpha_2)\partial_\mu K_1(\partial^\mu\pi^-\pi^+ + \partial^\mu\pi^+\pi^-)\Bigg)$$
$$+ iGX\left\{(\pi^-\overleftrightarrow{\partial}_\mu\pi^0)\partial^\mu K^+(\alpha_1+\alpha_3) + \partial_\mu\pi^0(\pi^-\overleftrightarrow{\partial}^\mu K^+)\frac{\sqrt{2}}{3}\left[-\frac{(2\alpha_1+\alpha_2)}{\sqrt{2}} - \alpha\psi_1(\sqrt{2}\,a\omega+b) + \alpha\psi_3(\sqrt{2}\,b\omega-a)\right] + \partial_\mu\pi^-(\pi^0\overleftrightarrow{\partial}^\mu K^+)\left[\frac{\alpha_1+\alpha_2}{2} + \sqrt{2}\alpha\psi_1(b+\sqrt{2}\,a) + \sqrt{2}\,\alpha\psi_3(a-\sqrt{2}\,b)\right]\right\}, \tag{6.3}$$

where ψ_1 and ψ_3 are given by Eqs. (A57) and (A58), respectively. a and b are defined in Eq. (4.11).

Evaluating the contributions to $K_1^0 \to \pi^+\pi^-$ according to the diagrams of Fig. 7 and using Eq. (5.8) for $g_{K^0\kappa^-\pi^+}$ gives

$$T(K_1\to\pi^+\pi^-) = -GX(\alpha_1+\alpha_2)\left\{K_0^2-\pi^2-\pi^2\left(\frac{\alpha_3-\alpha_1}{\alpha_2+\alpha_3}\right)\left[1+\frac{\frac{4}{3}(h_3+h_1)}{\kappa_+^2-\pi^2}\right]\right\}. \tag{6.4}$$

A similar calculation for $K_+\to\pi^+\pi^0$ using Eqs. (5.9) and (5.10) for $g_{K^+\bar{\kappa}_0\pi^-}$ and $g_{K^+\kappa^-\pi^0}$ and the diagrams in Fig. 8 gives

$$T(K^+\to\pi^+\pi^0) = i\frac{GX}{6}(K_+^2-\pi^2)\{\alpha_1-\alpha_2-4\alpha\psi_1[\sqrt{2}\,b+(3-\omega)a]-4\alpha\psi_3[\sqrt{2}\,a-(3-\omega)b]\}$$
$$+iGX\frac{\sqrt{2}}{3}\left(\frac{\alpha_3-\alpha_2}{\alpha_1+\alpha_3}\right)\left\{\frac{2\alpha_1+\alpha_2}{\sqrt{2}} + \alpha\psi_1[b+\sqrt{2}\,\omega a] + \alpha\psi_3[a-\sqrt{2}\,\omega b]\pi^2\left[1+\frac{\frac{4}{3}(h_3-h_1)}{\kappa_0^2-\pi^2}\right]\right\}$$
$$+i\frac{GX}{\sqrt{2}}(\alpha_1+\alpha_2)\frac{\pi^2}{\pi^2-\kappa_+^2}\left(\frac{\alpha_3-\alpha_1}{\alpha_3+\alpha_1}\right)\left\{\frac{\kappa_+^2-\pi^2}{\sqrt{2}}[1+\sqrt{2}\,\psi_1(b-\sqrt{2}\,a)+\sqrt{2}\,\psi_3(a+\sqrt{2}\,b)] - 4\sqrt{2}\,h_0[\psi_1(\sqrt{2}\,b-a)+\psi_3(\sqrt{2}\,a+b)] + \frac{2\sqrt{2}\,h_3}{3}[1-\psi_1(a+2\sqrt{2}\,b)+\psi_3(b-2\sqrt{2}\,a)] + \frac{10\sqrt{2}}{3}h_1\right\}. \tag{6.5}$$

We proceed now to the calculation of the $K\to 3\pi$ amplitude and again follow Goswami, Schechter, and Ueda.[25] In this part the electromagnetic corrections are ignored and the various amplitudes are related by the $\Delta I=\frac{1}{2}$ rule. We shall calculate, to first order in the weak interaction, the amplitude for the decay

$$K^0(p)\to\pi^+(q^+)+\pi^-(q^-)+\pi^0(q^0). \tag{6.6}$$

The lowest-order diagrams for this process are shown in Fig. 9. The relevant parts of the weak Lagrangian, calculated as above, are

$$\mathcal{L}_{wk}(\phi^2) = \sqrt{2}\,GX\alpha^2(1+\omega)\left(\partial_\mu\pi^-\partial^\mu K^+ - \frac{1}{\sqrt{2}}\partial_\mu\pi^0\partial^\mu K^0\right) + \text{H.c.}, \tag{6.7}$$

$$\mathcal{L}_W(S\phi^2) = GX\alpha\left\{-\partial_\mu \kappa^0\left[\frac{(1-\omega)}{\sqrt{2}}(\pi^-\partial^\mu\pi^+ + \tfrac{1}{2}\pi^0\partial^\mu\pi^0) + \left(\frac{1+\omega}{\sqrt{2}}\right)(\partial^\mu\pi^-\pi^+ + \tfrac{1}{2}\partial^\mu\pi^0\pi^0)\right] + \sqrt{2}\,\kappa^0(\partial_\mu\pi^-\partial^\mu\pi^+ + \tfrac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0)\right.$$
$$\left. + (1+\omega)\partial_\mu K^0[b'(\pi^0\overleftrightarrow{\partial}^\mu\sigma) + a'(\pi^0\overleftrightarrow{\partial}^\mu\sigma')] + \partial_\mu\pi^0[(b' - \sqrt{2}\,a')(K^0\overleftrightarrow{\partial}^\mu\sigma) + (a' + \sqrt{2}\,b')(K^0\overleftrightarrow{\partial}^\mu\sigma')]\right\}$$
$$+ \mathrm{H.c.} + \cdots \tag{6.8}$$

(a' and b' are defined analogously to a and b [Eq. (4.11)] for θ_S) and

$$\mathcal{L}_{wk}(\phi^4) = GX[2(\pi^0\overleftrightarrow{\partial}_\mu\pi^-)\partial^\mu\pi^+ + (\pi^-\overleftrightarrow{\partial}_\mu\pi^+)\partial^\mu\pi^0]K^0 + \mathrm{H.c.} + \cdots\,. \tag{6.9}$$

The contributions corresponding to the diagrams of Fig. 9 are

$$T_{9(a)} = -GX(K^2 - 3p\cdot q^0)\,, \tag{6.10a}$$

$$T_{9(b)} = -GX\alpha^2(1+\omega)\frac{\pi^2}{K^2-\pi^2}\,T(K^0(p)\to\pi^+(q^+)\pi^-(q^-)K^0(q^0))\,, \tag{6.10b}$$

$$T_{9(c)} = \sqrt{2}\,GX\alpha^2(1+\omega)\frac{\pi^2}{K^2-\pi^2}\,T(K^0(p)\to\pi^-(q^-)\pi^0(q^0)K^+(q^+))\,, \tag{6.10c}$$

$$T_{9(d)} = GX\alpha^2(1+\omega)\frac{K^2}{K^2-\pi^2}\,T(\pi^0(p)\to\pi^+(q^+)\pi^-(q^-)\pi^0(q^0))\,, \tag{6.10d}$$

$$T_{9(e)} = GX\alpha g_{\kappa K\pi}\frac{\pi^2 + 2q^+\cdot q^-}{\kappa^2 - (q^+ + q^-)^2}\,, \tag{6.10e}$$

and

$$T_{9(f)} = GX\alpha\left\{\frac{g_{\sigma\pi\pi}}{\sigma^2 - (p-q^0)^2}[(1+\omega)b'(K^2 - 2p\cdot q^0) - (\sqrt{2}\,a' - b')(\pi^2 - 2p\cdot q^0)]\right.$$
$$\left. + \frac{g_{\sigma'\pi\pi}}{\sigma'^2 - (p-q^0)^2}[(1+\omega)a'(K^2 - 2p\cdot q^0) + (a' + \sqrt{2}\,b')(\pi^2 - 2p\cdot q^0)]\right\}\,. \tag{6.10f}$$

The expressions for the off-shell $\pi\pi$ and πK scattering amplitudes can be calculated from Figs. 4 and 5 using the three- and four-point couplings determined in Sec. IV. One finds

$$T(\pi^0(p)\to\pi^+(q^+)\pi^-(q^-)\pi^0(q^0)) = \tfrac{1}{2}g_\pi^{(4)} + \frac{g_{\sigma\pi\pi}^{\;2}}{(p-q^0)^2 - \sigma^2} + \frac{g_{\sigma'\pi\pi}^{\;2}}{(p-q^0)^2 - \sigma'^2}\,, \tag{6.11}$$

$$T(K^0(p)\to\pi^-(q^-)\pi^0(q^0)K^+(q^+)) = -\frac{1}{\sqrt{2}}g_{\kappa K\pi}^{\;2}\left[\frac{1}{(p-q^0)^2-\kappa^2} - \frac{1}{(p-q^-)^2-\kappa^2}\right]\,, \tag{6.12}$$

and

$$T(K^0(p)\to\pi^+(q^+)\pi^-(q^-)K^0(q^0)) = g_K^{(4)} + \frac{g_{\kappa K\pi}^{\;2}}{(p-q^-)^2-\kappa^2} + \frac{g_{\sigma\pi\pi}g_{\sigma K\bar{K}}}{(p-q^0)^2-\sigma^2} + \frac{g_{\sigma'\pi\pi}g_{\sigma'K\bar{K}}}{(p-q^0)^2-\sigma'^2}\,. \tag{6.13}$$

We determine the value of the parameter X from the $K_1^0\to\pi^+\pi^-$ decay rate and calculate the decay rate of $K_L^0\to\pi^+\pi^-\pi^0$ using the analog of Eq. (4.48). The results are listed in Tables II-IV for the var-

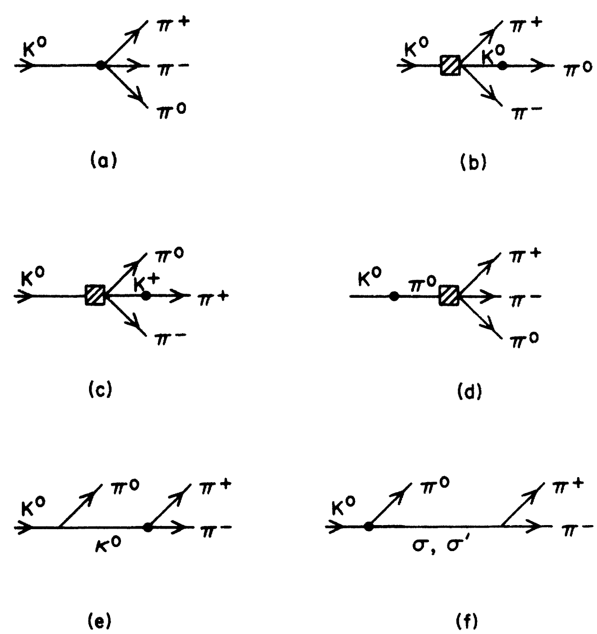

FIG. 9. Contributions to $K^0\to\pi^+\pi^-\pi^0$ decay. The dot denotes a weak-interaction vertex. The crosshatched boxes denote off-shell $\pi\pi$ or πK scattering amplitudes.

ious models. The predicted ratios of the $K^+ \to \pi^+ \pi^0$ to $K_1^0 \to \pi^+ \pi^-$ amplitudes are given in Table IV.

VII. THE NUMERICAL ANALYSIS AND DISCUSSION OF RESULTS

During the course of this work various types of $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry breaking were investigated. Tables II, III, and IV summarize the most interesting results for the models used. We now discuss these results and the assumptions that went into the numerical analysis.

We begin by considering the values of the π, K, η, and η' masses employed to determine the basic parameters (c_0, c_8, d_0, d_8) in our strong-interaction calculations. As we do not expect a shift in the π^0 mass to lowest order in the electromagnetic interaction, we identify the pion mass π with the π^0 mass. We use this as our basic mass unit.

Both the K^+ and K^0 masses will shift to lowest order in the electromagnetic interaction. Thus, we average the K^+ and K^0 masses to obtain the kaon mass for the strong-interaction calculations. Naturally, the parameters are sensitive to the value of K chosen, but the effect is not severe for small changes in the value of K.

The η mass is used in our determination of the η-η' system as discussed in Appendix A. Depending on the type of symmetry breaking being considered, η' is either calculated or fitted. The value of η' can be quite sensitive to that of η. For example, with $(3,3^*)\oplus(3^*,3)$ symmetry breaking one gets a value of 1019 MeV for η' when $\eta = 548$ MeV. If we set $\eta = 543$ MeV, then the η' mass shifts to 960 MeV. Nevertheless, we use $\eta = 548$ MeV throughout our calculations. In summary we use $\pi = \pi_0 = 135$ MeV, $K = \frac{1}{2}(K_+ + K_0) = 495.8$ MeV, and $\eta = 548$ MeV.

Let us next consider the values of α and α_3. With a value of 135 MeV for f_π we have $\alpha = 0.5\pi$ from Eq. (3.31). This value is employed throughout the calculations. The value of α_3 is then determined by f_K/f_π from Eqs. (3.31) and (3.32). For $f_K/f_\pi = 1.28$ we have $\alpha_3 = 0.78\pi$. As there is still some uncertainty in this value, f_K/f_π is allowed to be as large as 1.33. This corresponds to $\alpha_3 = 0.83\pi$.

Since the variable α_3 appears throughout all our calculations, we must consider the sensitivity of our results to the value of f_K/f_π. As an example, consider again the $(3,3^*)\oplus(3^*,3)$ case. With $f_K/f_\pi = 1.28$ (and $\eta = 548$ MeV) we have $\eta' = 1019$ MeV. If we set $f_K/f_\pi = 1.4$, we then have $\eta' = 958$ MeV. (Naturally this result is dependent on the value of K used). This latter value of f_K/f_π is close to those values chosen by Chan and Haymaker,[28] who input π^2, η^2, η'^2, f_π, and f_K/f_π for their tree solution to the GMOR model. They exploited the insensitivity of K^2 on (essentially) f_K/f_π with the other inputs fixed to obtain acceptable results for the scalar meson masses. A value of $f_K/f_\pi = 1.4$ is probably too large; thus, in general, we prefer to set $f_K/f_\pi = 1.28$ and deviate from this choice only when a change clearly improves the calculations as a whole.

Tables II, III, and IV contain all the interesting combinations of $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry breaking except the case which involves only $(8,8)$ terms. This latter case gives an unphysical solution for η' and is consequently rejected. As an indication of the magnitude of the $(8,8)$ contribution relative to other types of symmetry breaking, we compare the size of the $(3,3^*)\oplus(3^*,3)$ and the $(8,8)$ terms in the evaluation of K^2. These values are given in Table V. In all acceptable cases (except the $g_0 = -1.5$ case) note that the $(8,8)$ contribution is much smaller than that of the $(3,3^*)\oplus(3^*,3)$ part.

With some types of symmetry breaking the parameter solutions are not unique. The rejected solutions involve a larger contribution from the $(8,8)$ terms in $\mathcal{L}_{SB}$ and the calculations as a whole

TABLE V. Contributions to the kaon mass squared (in π^2) from the c_0, c_8, d_0, and d_8 terms in the kaon mass formula

$$K^2 = \frac{(2g_0+g_3)/\alpha + 4h_0(2+\omega) + \frac{2}{3}h_3(4-\omega)}{1+\omega}$$

for the various types of symmetry breaking in all models. Note $K = 3.67\pi$ and $K^2 = 13.49\pi^2$. We set $\omega = 1.56$. In the second, third, sixth, and seventh cases $\eta' = 958$ MeV.

	$d_0=d_8=0$	$d_8=0$	$d_0=0$	$g_0=d_8=0$	$g_0=d_0=0$	$g_0=0$	$g_0=0.3$	$g_0=-0.5$ $\eta'=1450$	$g_0=-1.5$ $\eta'=1450$	$d_8=0$ $\eta'=1450$
c_0	9.25	3.11	8.94	8.47	9.49	7.49	7.86	4.31	2.88	63.40
c_8	4.24	4.24	3.56	4.24	4.75	3.74	3.70	2.54	2.61	4.24
d_0	0.0	6.14	0.0	0.78	0.0	1.54	1.14	4.15	5.61	−54.14
d_8	0.0	0.0	0.99	0.0	−0.75	0.72	0.79	2.48	2.39	0.0

are less satisfactory than those given in the tables. Consequently, our calculations favor small (8,8) corrections to the dominant (3,3*)⊕(3*,3) contribution.

The particular forms of symmetry breaking for which results are presented in Tables II, III, and IV were chosen on the basis of both the above numerical results and of the ideas presented in the Introduction. The GMOR case $d_0 = d_8 = 0$ [i.e., no (8,8) contribution] is that studied by Schechter *et al.*[21-26] The case $g_0 = d_8 = 0$ is the Okubo case,[16,17] which has only the (8,8) part of $\mathcal{L}_{SB}$ breaking SU(2)×SU(2). The Sirlin-Weinstein model[18] corresponds to $g_0 = 0$. The other cases are included for interest and comparison. We also include some interesting calculations with $\eta' = 1450$ MeV, near the mass of the E meson,[34] another possible choice for the ninth member of the pseudoscalar nonet.[36]

The masses of the pseudoscalar meson nonet are accurately known (except for a possible uncertainty with regard to the choice of η'). The scalar nonet masses, on the other hand, are subject to a great deal of uncertainty. First, we identify the isovector δ(970) with the ϵ.[34] Secondly, we identify the broad $K\pi$ signal in the 1200–1400 MeV region[34] with the κ. Finally, we use the results of Protopopescu *et al.*[37] to complete the nonet with the σ around 660 MeV and the σ' at approximately 997 MeV. In our calculations the above masses are chosen if possible. If the model being considered applies constraints to these masses, we try to fit them while maintaining reasonable results for the other quantities of interest. Such a compromise is not always possible.

We now consider the strong, electromagnetic and weak interaction calculations separately in Secs. VII A, VII B, and VII C, respectively.

A. The strong-interaction calculations

Before considering the actual results we recall that the values of f_π, f_K/f_π, π, K, and possibly η and η' are used as input to determine the fundamental parameters in the model. Some of the results of Table II will now be discussed in detail.

First, we consider the mass spectra and decay widths. The only comment we make on the pseudoscalar mass spectrum is that, when the η' mass is not used as an input, the Okubo and GMOR cases give the best results. It is also interesting that with $\eta' = 958$ MeV θ_P is small, whereas for $\eta' = 1450$ MeV θ_P is an order of magnitude larger.

The κ mass generally prefers a value around 1000 MeV with $\Gamma(\kappa \to K\pi)$ around 400–500 MeV for all forms of symmetry breaking. Cicogna *et al.*[27] comment that by shifting the input masses by several MeV, they can get a large value for κ. However, it is not clear that the over-all results improve. If one tries to force the κ mass to a larger value, the width increases too rapidly to be acceptable. This is a consequence of the form of $g_{\kappa K\pi}$, which varies as $(\kappa^2 - \pi^2)$. These results hold for all models.

In the general model, we have no constraints on ϵ, σ, and σ' and, hence, choose them at 970, 660, and 997 MeV, respectively. We are also free to choose the σ-σ' mixing angle (θ_S), and, on the basis of the results for the σ and σ' widths, we tend to favor a value around 120°. This choice generally minimizes the calculated widths and puts them in an accepted region.[37] Our best width for the ϵ is ~200 MeV in the model calculations. This is much larger than the experimental width of 50 ± 30 MeV.[34,38]

If we choose to impose scale invariance on V_0, this places a constraint on σ, σ', and θ_S. This restriction is not particularly pleasing as it does not allow a value of σ' around 1000 MeV. One can adjust the value of σ; however, in the region of interest, the σ' mass is quite insensitive to that of the σ. In addition, not all values of σ and σ' are allowed. This is demonstrated in Table VI for the case with $g_0 = d_8 = 0$ (the Okubo case). The resulting widths in all cases are undesirably large. Moreover, with this large value of σ', we cannot identify it with the particle[37] at approximately

TABLE VI. The σ' mass for various σ masses in the general model with scale invariance. We set $g_0 = d_8 = 0$ and $\omega = 1.66$, corresponding to the values given in Table III. There are no solutions for σ' for σ^2 between 38 and 60 π^2.

σ^2 (π^2)	σ (MeV)	σ'^2 (π^2)	σ' (MeV)
6	331	65.7	1094
8	382	67.7	1111
10	427	70.0	1129
12	468	72.7	1151
14	505	75.8	1175
16	540	80.0	1204
18	573	84.1	1234
20	604	89.7	1279
22	633	96.8	1328
24	661	106.2	1391
26	688	119.0	1473
28	714	137.6	1583
30	739	167.0	1745
32	764	220.9	2006
34	787	350.7	2528
36	810	1101.7	4480
38	832	...	...
62	1063	1.5	164
64	1080	4.1	272
66	1097	6.3	339
68	1113	8.3	388

1000 MeV. Consequently, this modification of the general model is not of particular interest.

In passing we also note that scale invariance gives us enough information to calculate $\Gamma(\eta' \to \eta\pi\pi)$. We find that for small, negative values[33] of the slope parameter $\mathcal{A}$ ($-0.28 \le \mathcal{A} \le 0$), the width is larger than the experimental limit of 0.8 MeV.[39] The width increases as $\mathcal{A}$ becomes more negative.

In the renormalizable model there is also a constraint on the σ-σ' system. Again, the same difficulties occur as mentioned above, although, in this case, we can get the σ' mass down to about 1100 MeV. With this model we can also calculate the ϵ mass. This mass tends to be close to the "physical" one,[34] but, as in the general case, the width is usually around 200 MeV.

In addition to being able to calculate the $\eta' \to \eta\pi\pi$ width, which turns out to be $\gtrsim 2$ MeV, one can calculate the slope $\mathcal{A}$ in the renormalizable model. $\mathcal{A}$ is found to be small and negative, in accord with the latest data.[33]

Finally, we should point out that the imposition of scale invariance on the renormalizable form of V_0 [i.e., setting $d=0$ in Eq. (2.23)] leads to unsatisfactory results.

The scattering lengths, in particular the π-π S-wave scattering lengths a_0 and a_2, provide a useful criterion to assess the success of the various types of symmetry breaking. The requirements that $a_0 \gtrsim 0.05$ and $a_2 < 0$ imply that we should reject the three cases $d_8 = 0$, $d_0 = 0$, and $g_0 = d_0 = 0$. The scattering lengths arrived at in the various symmetry-breaking schemes may be compared with their current-algebra counterparts[40] which, for $\pi\pi$ scattering, are $a_0 \simeq 0.20\pi^{-1}$ and $a_2 \simeq -0.06\pi^{-1}$, and for πK scattering are $a_{1/2} \cong 0.17$ and $a_{3/2} \cong -0.09$.

One can also compare the $\pi\pi$ scattering amplitude at the Adler point[41] (T_A) to that at threshold (T_T). In the GMOR case the ratio T_A/T_T is approximately 10^{-3}. In all other cases the magnitude of T_A/T_T is around 1, varying from about 3 with $d_0 = 0$ to about 0.3 in the Okubo case. A similar calculation for πK scattering gives a ratio of about 10^{-2} in all cases except that with $d_8 = 0$, when it is around 10^{-1}.

B. The electromagnetic-interaction calculations

In our numerical analysis we considered three types of electromagnetic interactions. These include employing the $(3,3^*) \oplus (3^*,3)$ and the $(8,8)$ contributions individually and jointly. The computations of interest are given in Table IV. Most entries are omitted as they are completely unsatisfactory. In general, we find that the case using the $(8,8)$ term alone gives unacceptable results. At this point we note that the tabled electromagnetic-interaction results were not subjected to fine tuning, as it was felt that this refinement was not justified considering the approximations used.

In order to determine the model parameters in this case we extended the techniques employed in the strong-interaction calculations. Consequently, to evaluate the α's we use f_π and f_K/f_π as above, and, in addition, consider many values for the difference, $\alpha_1 - \alpha_2$. From Eq. (5.7) we expect this difference to be negative. It is also related to ϵ mass through Eq. (5.6). In order to evaluate the c's and d's in these computations we use, in order of their usage (the number employed depending on the number of parameters to be determined) K_0, π_0, η and η', and ϵ. Some masses not used as input can then be calculated.

The value of d_π is given directly by the physical π^+-π^0 mass difference; it is $0.069\pi^2$. From the analysis of Socolow[42] we expect a value for d_K of about $0.15\pi^2$. This corresponds to a K^+-K^0 self-energy mass difference of 2.8 MeV. From our calculations we prefer $d_K \approx 0.21\pi^2$. With the model parameters determined, one gets a κ^+-κ^0 mass difference of several MeV; however, the masses are still in the region around 1000 MeV. The above remarks apply to all three types of models being considered for V_0.

In the renormalizable model we have enough constraints to allow further analysis. In particular, we are interested in the decay $\eta \to \pi^+\pi^-\pi^0$. The results for both the partial width $\Gamma(\eta \to \pi^+\pi^-\pi^0)$ and the Dalitz-plot slope parameter β are presented in Table IV. β has been determined[43] to be -0.478 ± 0.038. We see that it is possible to have β in the acceptable range for a reasonable value of ϵ. (Recall that these results have not been finely tuned.) The partial width has generally been quoted[34] as 605 ± 150 eV. However, more recently, Browman *et al.*[44] have determined a value of 324 ± 46 eV. Our results are closer to the latter determination.

We again find that, in general, the Okubo and GMOR forms of symmetry breaking give the best numerical results. The Sirlin-Weinstein scheme fairs less well in the electromagnetic calculations. We have thus chosen to present the results only of the Okubo and GMOR cases in Table IV.

C. The weak-interaction calculations

In this subsection we briefly discuss our calculation of the partial width $\Gamma(K_L^0 \to \pi^+\pi^-\pi^0)$ and the

$K\to 2\pi$ decays. The $K\to 3\pi$ partial width affords us another criterion to evaluate the symmetry-breaking forms. Comparing the tabled values with the experimental value[34] of $(2.31\pm 0.07)\times 10^6$ sec^{-1}, we find that, of the three types of symmetry breaking preferred on theoretical grounds, the Sirlin-Weinstein form gives the best result, with the GMOR model next, although none of these predictions is close to the experimental value.

We conclude by considering the $K\to 2\pi$ decays with first-order electromagnetic corrections. From Table IV one sees that in both the Okubo and GMOR cases, the ratio $T(K^+\to\pi^+\pi^0)/T(K_1^0\to\pi^+\pi^-)$ is approximately 0.01. The experimental value[34] is 0.044 and consequently our ratio is in error by a factor of about 4.

VIII. CONCLUSIONS

In the foregoing we have studied in some detail a number of chiral $SU(3)\times SU(3)$ symmetry-breaking schemes within the context of a simple linear σ model. This has included consideration of a large number of possible symmetry-breaking combinations[45] belonging to the $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ representations of $SU(3)\times SU(3)$. While a number of the conclusions that can be drawn from the present work are model-dependent, it is felt that some of our results may have a more general validity. In particular, several of the simpler symmetry-breaking forms seem to provide a competitive alternative to the GMOR[2,3] $(3,3^*)\oplus(3^*,3)$ model, which was previously studied in the context of the linear σ model by Schechter *et al.*[21-26]

Perhaps it is a bit surprising to find another simple, two-parameter symmetry-breaking scheme which works as well as the GMOR one does. However, this is indeed the case; the Okubo form[16] of symmetry breaking [Eq. (1.5)] gives results across the board which are at least as acceptable as those of the GMOR form [of Eq. (1.1)], with the exception of the predictions for $\Gamma(K_L^0\to\pi^+\pi^-\pi^0)$, which neither model succeeds in predicting too well. The other theoretically appealing symmetry-breaking form (among those considered here), due to Sirlin and Weinstein,[18] [Eq. (1.4)] is almost as successful as those of GMOR and Okubo; its prediction of $\Gamma(K_L^0\to\pi^+\pi^-\pi^0)$ is better, but its performance in the electromagnetic calculations is poorer. Of course, it might be argued that since the Sirlin-Weinstein model contains three parameters, we should require from it a much better fit to the data than we do for the other two.

While we prefer the above models on theoretical grounds and because they appear to do a better job with fewer parameters, we must point out to the reader at this point the rather striking results achieved in one of the four-parameter cases. For an η' mass of 1450 MeV [near the $E(1420)$ (Ref. 34)] the renormalizable model with general $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry breaking gives very acceptable results for most of the quantities of interest. Of more interest, however, is the fact that this is the only case which can accommodate a very narrow σ' (at ~900 MeV) together with a broad σ (at 700 MeV)[37]; also, its value for $\Gamma(K_L^0\to\pi^+\pi^-\pi^0)$ is quite close to the experimental one.

One discovery which did not come as too much of a surprise is that the symmetry-breaking Lagrangian cannot be pure $(8,8)$ or even dominantly so. In view of our assignment of the scalar and pseudoscalar mesons to the $(3,3^*)\oplus(3^*,3)$ representation of $SU(3)\times SU(3)$, pure $(8,8)$ symmetry-breaking would force a complete abandonment of operator PCAC in even an approximate sense. This could lead to mass extrapolation difficulties in the current-algebra low-energy theorems which seem to be belied by the general success of these calculations. This suppression of $(8,8)$ symmetry breaking is quite possibly a model-dependent result. However, because of the success of the Okubo and GMOR models, we feel that it might be true generally.

Finally, the imposition of scale invariance on the chiral-symmetric part of the Lagrangian proved unacceptable in all cases. Also, the renormalizable model, with the exception of the case mentioned above in which $\eta'=1450$ MeV, is unable, at least in the tree approximation, to accommodate a narrow σ' at ~1 GeV.

APPENDIX A: SOLUTION OF THE π^0-η-η' SYSTEM

We present here the solutions for the masses and mixing angles of the π^0-η-η' system in the general case where both SU(3) and isospin are violated. The corresponding results for the isospin-conserving case, which are needed in Sec. IV, may be obtained by letting $\alpha_1=\alpha_2$ and setting to zero all parameters describing isospin-violating effects in what follows. Here, and in Appendix B, the analysis will not depend on the chiral decomposition of V_{SB}.

In the presence of SU(3) symmetry breaking alone there is mixing between the η and η' mesons characterized by an angle θ_P [see Eq. (4.6)]. When isospin violation is included, there is additional η-η' mixing, which changes θ_P slightly by an angle ψ_2, as well as mixing between π^0 and η and π^0 and η' with the small angles ψ_1 and ψ_3, respectively. This additional mixing can be represented by the rotation

$$\begin{pmatrix} 1 & \psi_1 & \psi_3 \\ -\psi_1 & 1 & \psi_2 \\ -\psi_3 & -\psi_2 & 1 \end{pmatrix},$$

which, when performed after the η-η' rotation of Eq. (4.6), leads to the field transformation

$$\begin{bmatrix} \pi^0 \\ \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \psi_1 b + \psi_3 a & -\frac{1}{\sqrt{2}} + \psi_1 b + \psi_3 a & \sqrt{2}(-\psi_1 a + \psi_3 b) \\ -\frac{1}{\sqrt{2}}\psi_1 + b & \frac{1}{\sqrt{2}}\psi_1 + b & -\sqrt{2}\, a \\ -\frac{1}{\sqrt{2}}\psi_3 + a & \frac{1}{\sqrt{2}}\psi_3 + a & \sqrt{2}\, b \end{bmatrix} \begin{bmatrix} \phi_1^1 \\ \phi_2^2 \\ \phi_3^3 \end{bmatrix}, \quad \text{(A1)}$$

where a and b are defined in Eq. (4.11) in terms of θ_P. Note that ψ_2 has been absorbed into θ_P.

First, we note that from Eq. (A1) we can write

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 = \pi_0{}^2(\tfrac{1}{2} + \sqrt{2}\,\psi_1 b + \sqrt{2}\,\psi_3 a) + \eta^2(b^2 - \sqrt{2}\, b\psi_1) + \eta'^2(a^2 - \sqrt{2}\, a\psi_3), \quad \text{(A2)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0 = \pi_0{}^2(\tfrac{1}{2} - \sqrt{2}\,\psi_1 b - \sqrt{2}\,\psi_3 a) + \eta^2(b^2 + \sqrt{2}\, b\psi_1) + \eta'^2(a^2 + \sqrt{2}\, a\psi_3), \quad \text{(A3)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0 = 2a^2\eta^2 + 2b^2\eta'^2, \quad \text{(A4)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0 = -\tfrac{1}{2}\pi_0{}^2 + b^2\eta^2 + a^2\eta'^2, \quad \text{(A5)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0 = \pi_0{}^2(-\psi_1 a + \psi_3 b) + \eta^2(-\sqrt{2}\, ab + \psi_1 a) + \eta'^2(\sqrt{2}\, ab - \psi_3 b), \quad \text{(A6)}$$

and

$$\left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0 = \pi_0{}^2(\psi_1 a - \psi_3 b) + \eta^2(-\sqrt{2}\, ab - \psi_1 a) + \eta'^2(\sqrt{2}\, ab + \psi_3 b). \quad \text{(A7)}$$

Next we use Eq. (2.17) to express these matrix elements of the mass squared matrix in terms of the symmetry-breaking parameters and V_4. This gives

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 = -\frac{1}{\alpha_1}\left\langle \frac{\partial V_{SB}}{\partial S_1^1} \right\rangle_0 - \frac{12V_4\alpha_2\alpha_3}{\alpha_1} + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0, \quad \text{(A8)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0 = -\frac{1}{\alpha_2}\left\langle \frac{\partial V_{SB}}{\partial S_2^2} \right\rangle_0 - \frac{12V_4\alpha_1\alpha_3}{\alpha_2} + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0, \quad \text{(A9)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0 = -\frac{1}{\alpha_3}\left\langle \frac{\partial V_{SB}}{\partial S_3^3} \right\rangle_0 - \frac{12V_4\alpha_1\alpha_2}{\alpha_3} + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_3^3 \partial \phi_3^3} \right\rangle_0, \quad \text{(A10)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0 = -12V_4\alpha_3 + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0, \quad \text{(A11)}$$

$$\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0 = -12V_4\alpha_2 + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0, \quad \text{(A12)}$$

and

$$\left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0 = -12V_4\alpha_1 + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0. \quad \text{(A13)}$$

Using Eqs. (A2), (A3), and (A5) $\pi_0{}^2$ can now be expressed as

$$\pi_0{}^2 = \frac{1}{2}\left(\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 + \left\langle \frac{\partial^2 V}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0 - 2\left\langle \frac{\partial^2 V}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0\right). \quad \text{(A14)}$$

Rewriting this in terms of the symmetry-breaking parameters and V_4 we have, using Eqs. (A8), (A9), and (A10),

$$\pi_0{}^2 = \frac{1}{2}\left[-\frac{1}{\alpha_1}\left\langle \frac{\partial V_{SB}}{\partial S_1^1} \right\rangle_0 - \frac{1}{\alpha_2}\left\langle \frac{\partial V_{SB}}{\partial S_2^2} \right\rangle_0 - 12V_4\alpha_3\left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_1}{\alpha_2} - 2\right) + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0 - 2\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0\right]. \quad \text{(A15)}$$

As we are interested in calculations only to first order in the isospin-violating interaction we can neglect the V_4 term

$$V_4\alpha_3\frac{(\alpha_1-\alpha_2)^2}{\alpha_1\alpha_2}\ .$$

Finally, to first order

$$\pi_0{}^2=\frac{1}{2}\left(\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0+\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0-2\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_2^2}\right\rangle_0-\frac{1}{\alpha_1}\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0-\frac{1}{\alpha_2}\left\langle\frac{\partial V_{SB}}{\partial S_2^2}\right\rangle_0\right). \qquad \text{(A16)}$$

Using our form for V_{SB} [Eqs. (3.1), (3.21), (3.23), and (3.25)] we have

$$\pi_0{}^2=\frac{\alpha_1+\alpha_2}{2\alpha_1\alpha_2}[g_0+2h_0(\alpha_1+\alpha_2+\alpha_3)+2h_3(\alpha_1+\alpha_2)]. \qquad \text{(A17)}$$

We now turn our attention to the η-η' system. From Eqs. (A2), (A3), and (A5)

$$4b^2\eta^2+4a^2\eta'^2=\left\langle\frac{\partial^2 V}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0+\left\langle\frac{\partial^2 V}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0+2\left\langle\frac{\partial^2 V}{\partial\phi_1^1\partial\phi_2^2}\right\rangle_0. \qquad \text{(A18)}$$

Also from Eqs. (A6) and (A7) we have

$$2\sqrt{2}\,ab\eta^2-2\sqrt{2}\,ab\eta'^2=-\left\langle\frac{\partial^2 V}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V}{\partial\phi_2^2\partial\phi_3^3}\right\rangle_0. \qquad \text{(A19)}$$

We define the variable

$$\delta=\frac{\cos2\theta_P}{2}+\sqrt{2}\,\sin2\theta_P. \qquad \text{(A20)}$$

Then

$$a^2=\tfrac{1}{4}+\tfrac{1}{6}\delta, \qquad \text{(A21)}$$

$$b^2=\tfrac{1}{4}-\tfrac{1}{6}\delta, \qquad \text{(A22)}$$

and

$$ab=\pm(\tfrac{1}{16}-\tfrac{1}{36}\delta^2)^{1/2}. \qquad \text{(A23)}$$

Using these definitions, Eq. (A4), and Eqs. (A8) to (A13) then gives

$$\eta^2+\eta'^2+\tfrac{2}{3}\delta(\eta'^2-\eta^2)=\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0+\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0+2\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_2^2}\right\rangle_0-\frac{1}{\alpha_1}\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0-\frac{1}{\alpha_2}\left\langle\frac{\partial V_{SB}}{\partial S_2^2}\right\rangle_0-12V_4\alpha_3\frac{(\alpha_1+\alpha_2)^2}{\alpha_1\alpha_2}, \qquad \text{(A24)}$$

$$\eta^2+\eta'^2-\tfrac{2}{3}\delta(\eta'^2-\eta^2)=2\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_3^3\partial\phi_3^3}\right\rangle_0-\frac{2}{\alpha_3}\left\langle\frac{\partial V_{SB}}{\partial S_3^3}\right\rangle_0-24V_4\frac{\alpha_1\alpha_2}{\alpha_3}, \qquad \text{(A25)}$$

and

$$\pm\frac{\sqrt{2}}{3}(\tfrac{9}{4}-\delta^2)^{1/2}(\eta^2-\eta'^2)=-\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_3^3}\right\rangle+12V_4(\alpha_1+\alpha_2). \qquad \text{(A26)}$$

We rewrite these equations in the form

$$\tfrac{3}{2}(\eta^2+\eta'^2)+\delta(\eta'^2-\eta^2)+AV_4=F_1, \qquad \text{(A27)}$$

$$\tfrac{3}{2}(\eta^2+\eta'^2)-\delta(\eta'^2-\eta^2)+BV_4=F_2, \qquad \text{(A28)}$$

and

$$\pm(\tfrac{9}{4}-\delta^2)^{1/2}(\eta^2-\eta'^2)+CV_4=F_3, \qquad \text{(A29)}$$

where

$$A=18\alpha_3\frac{(\alpha_1+\alpha_2)^2}{\alpha_1\alpha_2}, \qquad \text{(A30)}$$

$$B=36\frac{\alpha_1\alpha_2}{\alpha_3}, \qquad \text{(A31)}$$

$$C=-18\sqrt{2}(\alpha_1+\alpha_2), \qquad \text{(A32)}$$

$$F_1=\frac{3}{2}\left(\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0+\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0+2\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_2^2}\right\rangle_0-\frac{1}{\alpha_1}\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0-\frac{1}{\alpha_2}\left\langle\frac{\partial V_{SB}}{\partial S_2^2}\right\rangle_0\right), \qquad \text{(A33)}$$

$$F_2=3\left(\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_3^3\partial\phi_3^3}\right\rangle_0-\frac{1}{\alpha_3}\left\langle\frac{\partial V_{SB}}{\partial S_3^3}\right\rangle_0\right), \qquad \text{(A34)}$$

and

$$F_3=-\frac{3}{\sqrt{2}}\left(\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0+\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_3^3}\right\rangle_0\right). \qquad \text{(A35)}$$

F_1, F_2, and F_3 can be evaluated using Table I, giving

$$F_1 = \frac{3(\alpha_1+\alpha_2)}{2\alpha_1\alpha_2}(g_0+2h_0\alpha_3), \tag{A36}$$

$$F_2 = \frac{3}{\alpha_3}[g_0+g_3+2h_0(\alpha_1+\alpha_2)], \tag{A37}$$

$$F_3 = 6\sqrt{2}\,h_0. \tag{A38}$$

To solve this set of equations we first add Eqs. (A27) and (A28) to give

$$3(\eta^2+\eta'^2)+(A+B)V_4 = F_1+F_2. \tag{A39}$$

We define the variable G in terms of given quantities

$$G = F_1+F_2-3\eta^2$$
$$= 3\eta'^2+(A+B)V_4. \tag{A40}$$

Next we subtract Eq. (A28) from Eq. (A27) to obtain

$$2\delta(\eta'^2-\eta^2)+(A-B)V_4 = F_1-F_2. \tag{A41}$$

We define another known quantity

$$H = F_1-F_2. \tag{A42}$$

We now square Eq. (A29) and multiply by 4 to obtain

$$9(\eta^2-\eta'^2)^2-4\delta^2(\eta^2-\eta'^2)^2 = 4F_3^2+4C^2V_4^2-8F_3CV_4. \tag{A43}$$

Using Eq. (A40) to substitute for η'^2 in the final term and Eq. (A41) to remove the δ^2 term, we finally have (after rearranging terms)

$$4(AB-C^2)V_4^2+2[E(A+B)+H(A-B)+4F_3C]V_4 + (E^2-H^2-4F_3^2) = 0, \tag{A44}$$

where

$$E = 3\eta^2-G. \tag{A45}$$

We solve this quadratic equation for V_4, then calculate η'^2 and δ using

$$\eta'^2 = \tfrac{1}{3}[G-(A+B)V_4], \tag{A46}$$

$$\delta = \frac{\tfrac{1}{2}[H-(A-B)V_4]}{\eta^2-\eta'^2}. \tag{A47}$$

As we squared Eq. (A29) we must check to see if our solution obeys the original set of equations, with either a plus or minus sign being allowed in Eq. (A29).

Once δ is known we may solve for $\tan 2\theta_P$ using (A20). This gives

$$\tan 2\theta_P = \frac{2\{F_3+\sqrt{2}(F_1-F_2)-[C+\sqrt{2}(A-B)]V_4\}}{F_1-F_2-4\sqrt{2}F_3-(A-B-4\sqrt{2}C)V_4}. \tag{A48}$$

Again the value of θ_P determined from the above equation must be checked in the original system of equations to ensure that it is the correct one. This concludes our determination of η', θ_P, and V_4.

In the final part of this appendix we indicate the solution for ψ_1 and ψ_3. Equations (A2) and (A3) give

$$2\sqrt{2}\,\pi_0^2(\psi_1 b+\psi_3 a)-2\sqrt{2}\,\eta^2 b\psi_1-2\sqrt{2}\,\eta'^2 a\psi_3 = \left\langle\frac{\partial^2 V}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0-\left\langle\frac{\partial^2 V}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0. \tag{A49}$$

From Eqs. (A6) and (A7) one finds

$$2\pi_0^2(-\psi_1 a+\psi_3 b)+2\eta^2\psi_1 a-2\eta'^2\psi_3 b = \left\langle\frac{\partial^2 V}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V}{\partial\phi_2^2\partial\phi_3^3}\right\rangle_0. \tag{A50}$$

We rewrite these equations to give

$$2\sqrt{2}\,\psi_1 b(\pi_0^2-\eta^2)+2\sqrt{2}\,\psi_3 a(\pi_0^2-\eta'^2) = M \tag{A51}$$

and

$$2\sqrt{2}\,\psi_1 a(\pi_0^2-\eta^2)-2\sqrt{2}\,\psi_3 b(\pi_0^2-\eta'^2) = N, \tag{A52}$$

where

$$M = \left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_1^1}\right\rangle_0-\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_2^2}\right\rangle_0-\frac{1}{\alpha_1}\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0 + \frac{1}{\alpha_2}\left\langle\frac{\partial V_{SB}}{\partial S_2^2}\right\rangle_0-12V_4\alpha_3\frac{(\alpha_2^2-\alpha_1^2)}{\alpha_1\alpha_2} \tag{A53}$$

and

$$N = -\sqrt{2}\left[\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_1^1\partial\phi_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V_{SB}}{\partial\phi_2^2\partial\phi_3^3}\right\rangle_0-12V_4(\alpha_2-\alpha_1)\right]. \tag{A54}$$

With our choice of V_{SB}

$$M = \frac{1}{\alpha_1\alpha_2}\{(\alpha_2-\alpha_1)[g_0+2h_0(\alpha_1+\alpha_2+\alpha_3)+2h_3(\alpha_1+\alpha_2)] + g_1(\alpha_1+\alpha_2)-2h_1\alpha_3(\alpha_1+\alpha_2) - 12V_4\alpha_3(\alpha_2^2-\alpha_1^2)\} \tag{A55}$$

and

$$N = -4\sqrt{2}\,h_1+12\sqrt{2}\,V_4(\alpha_2-\alpha_1). \tag{A56}$$

This linear set of equations can easily be solved to give

$$\psi_1 = \frac{bM+aN}{\sqrt{2}(\pi_0^2-\eta^2)} \tag{A57}$$

and

$$\psi_3 = \frac{aM-bN}{\sqrt{2}(\pi_0^2-\eta'^2)}. \tag{A58}$$

This concludes the determination of η', θ_P, V_4, ψ_1, and ψ_3 in terms of V_{SB} and η.

APPENDIX B: DETERMINATION OF PARAMETERS IN THE RENORMALIZABLE MODEL

1. V_0

It can be seen from Eq. (2.25) that the renormalizable model for V_0 contains four parameters: V_1, V_{11}, V_2, and V_4. We will now solve for these constants in terms of the symmetry-breaking parameters, assuming that the latter have been obtained independently.

The imposition of the extremum conditions gives two relations (in the isospin-symmetric limit) between V_0 and V . Thus one needs two more conditions to completely determine V_0. Perhaps the simplest conditions are to choose the value of V_4 calculated in the η-η' system evaluation in which the η mass is used as input and to use the σ-σ' system in which the σ mass is input.

We first consider the extremum condition

$$\left\langle\frac{\partial V_0}{\partial S_b^a}\right\rangle_0+\left\langle\frac{\partial V_{SB}}{\partial S_b^a}\right\rangle_0=0\ . \tag{B1}$$

Evaluating the derivatives in terms of the V's and noting that

$$\left\langle\frac{\partial I_1}{\partial S_b^a}\right\rangle_0=2\alpha_a\delta_a^b\ ,\qquad \left\langle\frac{\partial I_2}{\partial S_b^a}\right\rangle_0=4\alpha_a{}^3\delta_a^b\ , \tag{B2}$$

and

$$\left\langle\frac{\partial I_4}{\partial S_b^a}\right\rangle_0=\frac{12\alpha_1\alpha_2\alpha_3}{\alpha_a}\delta_a^b\ ,$$

one obtains

$$2\alpha_a V_1+4\alpha_a{}^3V_2+12\frac{\alpha^2\alpha_3}{\alpha_4}V_4=-\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0 \tag{B3}$$

$(a=1,2,3)$.

From this equation

$$2\alpha V_1+4\alpha^3V_2+12\alpha\alpha_3V_4=-\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0\ ,$$

$$2\alpha_3V_1+4\alpha_3{}^3V_2+12\alpha^2V_4=-\left\langle\frac{\partial V_{SB}}{\partial S_3^3}\right\rangle_0\ . \tag{B4}$$

Assuming a value of V_4 obtained from the η-η' system we have

$$V_2=\frac{3V_4}{\alpha_3}+\frac{1}{4\alpha^2\alpha_3(1-\omega^2)}\left(\left\langle\frac{\partial V_{SB}}{\partial S_3^3}\right\rangle_0-\omega\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0\right) \tag{B5}$$

and

$$V_1=-2\alpha^2V_2-6\alpha_3V_4-\frac{1}{2\alpha}\left\langle\frac{\partial V_{SB}}{\partial S_1^1}\right\rangle_0\ . \tag{B6}$$

To obtain a value for V_{11} we now consider the σ-σ' system (in the isospin-conserving limit). Consider the quantities

$$m_{ab}=\left\langle\frac{\partial^2V}{\partial S_a^a\partial S_b^b}\right\rangle_0=\left\langle\frac{\partial^2V_0}{\partial S_a^a\partial S_b^b}\right\rangle_0+\left\langle\frac{\partial^2V_{SB}}{\partial S_a^a\partial S_b^b}\right\rangle_0\ . \tag{B7}$$

We have, from the scalar analog of Eqs. (4.6) to (4.11) and Eq. (B2)

$$m_{11}=\tfrac{1}{2}\epsilon_0{}^2+b'^2\sigma^2+a'^2\sigma'^2=2V_1+4\alpha^2V_{11}+12V_2\alpha^2+\left\langle\frac{\partial^2V_{SB}}{\partial S_1^1\partial S_1^1}\right\rangle_0\ , \tag{B8}$$

$$m_{12}=-\tfrac{1}{2}\epsilon_0{}^2+b'^2\sigma^2+a'^2\sigma'^2=4\alpha^2V_{11}+12V_4\alpha_3+\left\langle\frac{\partial^2V_{SB}}{\partial S_1^1\partial S_2^2}\right\rangle_0\ , \tag{B9}$$

$$m_{13}=\sqrt{2}\,a'b'(\sigma'^2-\sigma^2)=4\alpha\alpha_3V_{11}+12V_4\alpha+\left\langle\frac{\partial^2V_{SB}}{\partial S_1^1\partial S_3^3}\right\rangle_0\ , \tag{B10}$$

and

$$m_{33}=2a'^2\sigma^2+2b'^2\sigma'^2=2V_1+4V_{11}\alpha_3{}^2+12V_2\alpha_3{}^2+\left\langle\frac{\partial^2V_{SB}}{\partial S_3^3\partial S_3^3}\right\rangle_0\ , \tag{B11}$$

where

$$a'=\frac{1}{\sqrt{6}}(\sin\theta_S+\sqrt{2}\cos\theta_S)$$

and

$$b'=\frac{1}{\sqrt{6}}(\cos\theta_S-\sqrt{2}\sin\theta_S)\ . \tag{B12}$$

From Eqs. (B8) to (B11) we get immediately

$$m_{11}+m_{12}+m_{33}=\sigma^2+\sigma'^2\ . \tag{B13}$$

We can now derive an expression relating V_{11} to σ^2 and known quantities (V_1, V_2, V_4 and the symmetry-breaking parameters). We define the known quantities

$$n_{11}=m_{11}-4\alpha^2 V_{11}$$
$$=2V_1+12V_2\alpha^2+\left\langle\frac{\partial^2 V_{SB}}{\partial S_1^1\partial S_1^1}\right\rangle_0 ,$$
$$n_{12}=m_{12}-4\alpha^2 V_{11}$$
$$=12V_4\alpha_3+\left\langle\frac{\partial^2 V_{SB}}{\partial S_1^1\partial S_2^2}\right\rangle_0 ,$$
$$n_{13}=m_{13}-4\alpha\alpha_3 V_{11} \tag{B14}$$
$$=12V_4\alpha+\left\langle\frac{\partial^2 V_{SB}}{\partial S_1^1\partial S_3^3}\right\rangle_0 ,$$

and

$$n_{33}=m_{33}-4V_{11}\alpha_3{}^2$$
$$=2V_1+12V_2\alpha_3{}^2+\left\langle\frac{\partial^2 V_{SB}}{\partial S_3^3\partial S_3^3}\right\rangle_0 .$$

Then

$$V_{11}=\frac{(\sigma^2-n_{11}-n_{12})(\sigma^2-n_{33})-2n_{13}{}^2}{4\alpha^2[4\omega n_{13}-2n_{33}-\omega^2(n_{11}+n_{12})+\sigma^2(2+\omega^2)]} . \tag{B15}$$

Thus we can relate V_{11} to σ^2 by Eq. (B15) and then can get σ'^2 using Eq. (B13). We also obtain

$$\tan 2\theta_S=2\sqrt{2}\left(\frac{m_{11}+m_{12}-m_{33}-m_{13}}{m_{11}+m_{12}-m_{33}+8m_{13}}\right) . \tag{B16}$$

Thus, given a value for σ, we can calculate V_{11}, σ'^2, and θ_S. This completes our determination of the parameters in V_0.

2. χ_1 and χ_3

We now calculate χ_1, the ϵ_0-σ electromagnetic mixing angle and χ_3, the ϵ_0-σ' electromagnetic mixing angle. The calculation parallels that of ψ_1 and ψ_3 in Appendix A.

First, we note

$$\left\langle\frac{\partial^2 V}{\partial S_1^1\partial S_1^1}\right\rangle_0-\left\langle\frac{\partial^2 V}{\partial S_2^2\partial S_2^2}\right\rangle_0=2\sqrt{2}\,\chi_1 b'(\epsilon_0{}^2-\sigma^2)+2\sqrt{2}\,\chi_3 a'(\epsilon_0{}^2-\sigma'^2)$$

and (B17)

$$\left\langle\frac{\partial^2 V}{\partial S_1^1\partial S_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V}{\partial S_2^2\partial S_3^3}\right\rangle_0=-2\chi_1 a'(\epsilon_0{}^2-\sigma^2)+2\chi_3 b'(\epsilon_0{}^2-\sigma'^2) .$$

Evaluating the V_0 contribution directly and defining

$$p'=\left\langle\frac{\partial^2 V_{SB}}{\partial S_1^1\partial S_1^1}\right\rangle_0-\left\langle\frac{\partial^2 V_{SB}}{\partial S_2^2\partial S_2^2}\right\rangle_0$$

and (B18)

$$q'=\left\langle\frac{\partial^2 V_{SB}}{\partial S_1^1\partial S_3^3}\right\rangle_0-\left\langle\frac{\partial^2 V_{SB}}{\partial S_2^2\partial S_3^3}\right\rangle_0 ,$$

leads to

$$2\sqrt{2}\,\chi_1 b'(\epsilon_0{}^2-\sigma^2)+2\sqrt{2}\,\chi_3 a'(\epsilon_0{}^2-\sigma'^2)=8\alpha(\alpha_1-\alpha_2)(V_{11}+3V_2)+p' ,$$
$$2\chi_1 a'(\epsilon_0{}^2-\sigma^2)-2\chi_3 b'(\epsilon_0{}^2-\sigma'^2)=-4(\alpha_1-\alpha_2)(\alpha_3 V_{11}-3V_4)-q' . \tag{B19}$$

Solving these equations for χ_1 and χ_3 gives

$$\chi_1=\frac{4(\alpha_1-\alpha_2)[\sqrt{2}\,\alpha b'(V_{11}+3V_2)+a'(3V_4-\alpha_3 V_{11})]+p'b'/\sqrt{2}-q'a'}{\epsilon_0{}^2-\sigma^2} \tag{B20}$$

and

$$\chi_3=\frac{4(\alpha_1-\alpha_2)[\sqrt{2}\,\alpha a'(V_{11}+3V_2)-b'(3V_4-\alpha_3 V_{11})]+a'p'/\sqrt{2}+b'q'}{\epsilon_0{}^2-\sigma'^2} . \tag{B21}$$

With $(3,3^*)\oplus(3^*,3)\oplus(8,8)$ symmetry breaking

$$p'=\tfrac{8}{3}h_1 \text{ and } q'=4h_1 . \tag{B22}$$

3. Calculation of some electromagnetic coupling constants

In this subsection we outline the calculation of the coupling constants that are required for our calculation of the $\eta\to\pi^+\pi^-\pi^0$ decay rate. These include $f_{\sigma\pi^0\eta}$, $f_{\sigma'\pi^0\eta}$, $f_{\epsilon^0\pi^+\pi^-}$, $f_{\epsilon^-\pi^+\pi^0}$, and $f^{(4)}_{\eta\pi^+\pi^-\pi^0}$. The calculations are to first order in the isospin-violating interaction.

First we note that the inverse of Eq. (A1) is

$$\begin{bmatrix} \phi_1^1 \\ \phi_2^2 \\ \phi_3^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \psi_1 b + \psi_3 a & -\frac{\psi_1}{\sqrt{2}} + b & -\frac{\psi_3}{\sqrt{2}} + a \\ -\frac{1}{\sqrt{2}} + \psi_1 b + \psi_3 a & \frac{\psi_1}{\sqrt{2}} + b & \frac{\psi_3}{\sqrt{2}} + a \\ \sqrt{2}(-\psi_1 a + \psi_3 b) & -\sqrt{2}\, a & \sqrt{2}\, b \end{bmatrix} \begin{bmatrix} \pi_0 \\ \eta \\ \eta' \end{bmatrix}, \tag{B23}$$

with a similar relation for the scalar case. We evaluate the coupling constants directly from V_0 [Eq. (2.25)]. Using our previous techniques we have

$$\begin{aligned} f_{\sigma\pi^0\eta} &= \left\langle \frac{\partial^3 V}{\partial\sigma\partial\pi^0\partial\eta} \right\rangle_0 \\ &= \frac{8}{\sqrt{2}}(\alpha_1 - \alpha_2)V_2 bb' + \frac{8\chi_1}{\sqrt{2}}(3aV_4 - \sqrt{2}\,\alpha b V_2) + 8b'\{2\psi_1(-\alpha V_2 a^2 + 3\sqrt{2}\, ab V_4) + \psi_3[2\alpha ab V_2 + 3\sqrt{2}\,(a^2 - b^2)V_4]\} \\ &\quad - 4\sqrt{2}\, a'\{\psi_1[4\alpha_3 a^2 V_2 - 3V_4(1 + 2b^2)] - 2ab\psi_3(2\alpha_3 V_2 + 3V_4)\} \end{aligned} \tag{B24}$$

and

$$\begin{aligned} f_{\sigma'\pi^0\eta} &= \left\langle \frac{\partial^3 V}{\partial\sigma'\partial\pi^0\partial\eta} \right\rangle_0 \\ &= \frac{8}{\sqrt{2}} a'b V_2(\alpha_1 - \alpha_2) + \frac{8}{\sqrt{2}}\chi_3(3aV_4 - \sqrt{2}\,\alpha b V_2) + 8a'\{2\psi_1(-\alpha a^2 V_2 + 3\sqrt{2}\, ab V_4) + \psi_3[2\alpha ab V_2 + 3\sqrt{2}\,(a^2 - b^2)V_4]\} \\ &\quad + 4\sqrt{2}\, b'\{\psi_1[4\alpha_3 a^2 V_2 - 3(1 + 2b^2)V_4] - 2ab\psi_3(2\alpha_3 V_2 + 3V_4)\}\ . \end{aligned} \tag{B25}$$

Again using the standard techniques, one finds

$$f_{\epsilon^0\pi^+\pi^-} = \frac{1}{\alpha}\left[(\epsilon^2 - \pi^2)(\chi_1 b' + \chi_3 a') - \frac{(A' - B')}{2\sqrt{2}} - \frac{(A' + B')}{2}(\chi_1 b' + \chi_3 a') + \frac{C'}{\sqrt{2}}(\chi_1 a' - \chi_3 b')\right], \tag{B26}$$

where

$$A' = \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_1^1} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_2^2} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0, \quad B' = \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_2^2} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial S_2^2 \partial S_2^2} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^2 \partial \phi_2^1} \right\rangle_0,$$

and (B27)

$$C' = \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^1 \partial S_3^3} \right\rangle_0 + \left\langle \frac{\partial^2 V_{SB}}{\partial S_2^2 \partial S_3^3} \right\rangle_0 .$$

With (3, 3*)⊕ (3*, 3)⊕ (8, 8) symmetry breaking

$$A' = -4h_0 - 4h_3 + \tfrac{4}{3}h_1, \quad B' = -4h_0 - 4h_3 - \tfrac{4}{3}h_1, \quad C' = -4h_0\ . \tag{B28}$$

Also

$$f_{\epsilon^-\pi^+\pi^0} = \frac{1}{\alpha}\left[(\epsilon^2 - \pi^2)(\psi_1 b + \psi_3 a) - \frac{(A'' - B'')}{2\sqrt{2}} - \frac{(A'' + B'')}{2}(\psi_1 b + \psi_3 a) + \frac{C''}{\sqrt{2}}(\psi_1 a - \psi_3 b)\right], \tag{B29}$$

where

$$A'' = \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^2 \partial S_2^1} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_1^1} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0, \quad B'' = \left\langle \frac{\partial^2 V_{SB}}{\partial S_1^2 \partial S_2^1} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_2^2} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_2^2} \right\rangle_0,$$

and (B30)

$$C'' = -\left\langle \frac{\partial^2 V_{SB}}{\partial \phi_1^1 \partial \phi_3^3} \right\rangle_0 - \left\langle \frac{\partial^2 V_{SB}}{\partial \phi_2^2 \partial \phi_3^3} \right\rangle_0 .$$

With (3, 3*)⊕ (3*, 3)⊕ (8, 8) symmetry breaking

$$A'' = 4h_0 + 4h_3 - \tfrac{4}{3}h_1, \quad B'' = 4h_0 + 4h_3 + \tfrac{4}{3}h_1, \quad C'' = 4h_0\ . \tag{B31}$$

Finally,

$$f^{(4)}_{\eta\pi^+\pi^-\pi^0} = \frac{1}{\alpha}\left[bf_{\epsilon^-\pi^+\pi^0} + b'f_{\sigma\pi^0\eta} + a'f_{\sigma'\pi^0\eta} + (\chi_1 b' + \chi_3 a' + \psi_1 b + \psi_3 a)g_{\epsilon\pi\eta}\right] . \qquad \text{(B32)}$$

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[28]Among those who have worked with renormalizable σ models (besides the authors of Refs. 20 and 26) are M. Lêvy [Nuovo Cimento 52A, 23 (1967)], B. W. Lee [Nucl. Phys. B9, 649 (1969)], H. W. Crater [Phys. Rev. D 1, 3313 (1970)], J. L. Basdevant and B. W. Lee [*ibid.* 2, 1680 (1970)], P. Carruthers [*ibid.* 2, 2265 (1970)], P. Carruthers and R. W. Haymaker [*ibid.* 4, 1808 (1971); 6, 1528 (1972)], and L. Chan and R. W. Haymaker [*ibid.* 7, 402 (1973); 7, 415 (1973). Of the above investigations those of Lee, Crater, Basdevant and Lee, and Chan and Haymaker have dealt with higher-order (typically one-loop) corrections. In particular, Lee showed that the SU(2) σ model was renormalizable in the presence of spontaneous symmetry breaking.

[29]In addition to Mathur, Altarelli *et al.*, and Crewther (Ref 12), others who have explored this possibility include J. Ellis [Phys. Lett. 33B, 591 (1970)], S. P. de Alwis and P. J. O'Donnell [Phys. Rev. D 2, 1023 (1970)], P. J. O'Donnell [*ibid.* 3, 1021 (1971)], J. Ellis, P. Weisz, and B. Zumino [Phys. Lett. 34B, 91 (1971)], and R. Jackiw [Phys. Rev. D 3, 1356 (1971)].

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[32]Chan and Haymaker (Ref. 28) have found that one-loop corrections to the tree-approximation values of masses, f_π, and f_K are of the order of 10% or less. We have extended their analysis to the K_{l3} form factors and also find the corrections to be small. The details of this calculation will be presented elsewhere.

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