Dynamics of light and heavy bound quarks*

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A simple relativistic potential model is constructed in order to describe meson spectroscopy in a unified way, encompassing both light- and heavy-quark systems. A linear potential, transforming as the fourth component of a vector potential, is assumed in accordance with quark-confinement mechanisms appropriate to gauge theories. Scaling arguments are used to derive the Regge behavior of the model. We present evidence that the forces which confine quarks are essentially independent of quark species. The over-all systematics of our predictions for meson energy levels and leptonic decay widths give support to the usefulness of the model. We comment on the application of asymptotic freedom as a means of estimating decay rates which violate the Okubo - Zweig - Iizuka rule.

I. INTRODUCTION

Discovery of narrow resonances at 3.1 (Ref. 1) and 3.7 GeV,[2] and a broad enhancement at 4.2 GeV[3] in e^+e^- annihilation has caused intense interest[4] in the particle physics community. Several possible interpretations of the phenomena have been offered,[4] with an extremely attractive explanation[4,5] being that these are the $^3S_1$ ground state and radial excitations of a heavy quark-antiquark pair. It has been further suggested that these heavy quarks are in fact the charmed quarks[4-6] needed for the suppression of strangeness-changing neutral currents in weak decays.[7] In the simplest models, the charmed quark is the fourth quark required for an SU(4) (Refs. 6–8) classification of the strong interactions, although more elaborate models with additional quarks may be considered.[9]

The bound-charm explanation of the new resonances is far from confirmed, with many problems remaining to be solved by both theorists and experimentalists before the charmonium picture can be fully accepted. Although we have no instant answers to these questions, we still regard the bound-charmed-quark picture as the best working hypothesis until a more compelling framework supersedes this explanation.

The bound-charmed-quark picture is of interest for studies of strong interaction dynamics, as it may provide a fertile proving ground for ideas concerning quark binding and confinement. It is tempting to interpolate between the nonrelativistic bound charmed quark states and their analogs in other systems in order to gain further insights into the more difficult problems encountered in the relativistic domain. In particular, a fruitful comparison can be made between the 1^- state $\rho(770)$, considered to be the lowest $^3S_1$ bound state of light ($\mathfrak{N}$ and $\mathcal{P}$) quarks, and the $\phi_c(3100)$ state, assumed to be the lowest $^3S_1$ bound state of heavy (charmed) quark pairs. (Indirect evidence[10] suggests that the effective mass of the charmed quark is of the order of 1–3 GeV.) The first $^3S_1$ *radial* excitation of the ρ system should be identified with $\rho'(1600)$, although there may be some admixture of $^3D_1$ in ρ'. Analogously, $\phi_c'(3700)$ should be the first radial excitation of bound charmed quarks in a $^3S_1$ state. The radial excitation energy of the ρ, ρ' pair is $\Delta s = 1.96$ GeV^2, but $\phi_c'(3700)$ is separated from $\phi_c(3100)$ by $\Delta s = 4$ GeV^2. [It is relevant to discuss *radial* excitations in (mass)^2 for the same reason that one discusses orbital or Regge recurrences in (mass)^2.] Furthermore, typical quark models[11] suggest the sequence of states: $1S, 2P, (2S, 3D), \ldots$; which means the spacing typical of Regge trajectories should be roughly half that of radial recurrences. Noting that $\Delta s = 4$ GeV^2 for the ϕ_c, ϕ_c' pair, one could then conclude anyone of the following: (a) the interaction strength among charmed quarks is significantly different from that of the light quarks, (b) there is a missing ϕ_c state to be interpolated between the two discovered, (c) ϕ_c' is not a radially excited ϕ_c, or (d) the large spacing between ϕ_c and ϕ_c' is a *kinematical* effect due to the large mass of the charmed quarks. It is one of the purposes of this paper to present model calculations to support the idea that the large ϕ_c, ϕ_c' spacing is in fact a kinematical effect,[12] with the *dynamics* of light and heavy quarks comparable.

Attempts have been made[5] to explain the binding of charmed quarks near the charmed particle threshold on the basis of a Coulomb-type mechanism based on a Yang-Mills theory of strong colored gluons, with the predominate contributions coming when the quark separation is small. However, for reasonable parameters[5] this mechanism underestimates the ϕ_c-ϕ_c' splitting (in mass) by roughly a factor of 10, and further suggests that the quarks are far apart in the ϕ_c' state, a regime where the forces that confine the quarks play an important role. Several independent lines of argument[13,14] suggest that the confinement of quarks is by means of an interaction which increases for large quark separation in proportion to the separation itself. It is likely that this is the dominant mechanism for explaining the excitations in the quark-antiquark systems, with the Coulomb-type interactions negligible except perhaps for forming the ground state.

It is the purpose of this paper to explore a possible unifying quark dynamics of bound heavy- and light-quark systems based on the above ideas. Since the methods employed will be dynamical (albeit crude), and not group-theoretic, we are not committed to models with any particular number of quarks, so that our results may be applied to charmed quark models more elaborate[9] than the original SU(4) models.[6-8]

In the absence of detailed field theoretic methods applicable to the relativistic region, we propose to study the quark spectrum by means of a static linear potential acting as the fourth component of a four-vector, as might be expected from a confinement mechanism based on gauge theories.[13] We choose a formulation which reduces to the Schrödinger equation in the nonrelativistic limit, and yields a spectrum similar to that of the massless string model in the extreme relativistic limit. Detailed numerical studies of the meson spectrum with our model allow us to conclude that the linear potential, with more or less universal strength, can be used for both light- and heavy-quark systems in a unified way. In particular the large splitting [in $(\text{mass})^2$] of the ϕ_c-ϕ_c' pair as compared to the ρ-ρ' pair is a kinematical effect.[12]

II. THE MODEL

We are interested in constructing a bound-state model which is capable of interpolating between the nonrelativistic and relativistic domains so that we compare the spectroscopy of bound charmed quarks with that of mesons built of light quarks. On the purely classical level, the string model with massive quarks[15] is a reasonable candidate; however, a classical model cannot provide us with wave functions, which would limit the applications of the model. On the other hand, practical calculations in a field-theoretic context encounter a number of difficulties in the relativistic domain. For example, the Bethe-Salpeter equation can be formulated, but not solved except in the simplest of cases. A single-time relativistic Schrödinger equation would be a useful tool; however, there is no formulation which follows from the first principle of field theory.

In the absence of a satisfactory field-theoretic framework in which to imbed our problem, we consider a simplified single-time two-body equation based on classical considerations. We assume that quarks and antiquarks are confined via a potential which behaves as the fourth component of a four-vector, as might be expected from a confinement mechanism arising from gauge theories.[13] Retardation and spin-dependent corrections will be neglected as negligible compared to the effects we wish to consider.

The *total* energy E in the center of mass of a classical system of a quark and an antiquark interacting by means of this potential is[16]

$$E - V = (\vec{p}^2 + m_1^2)^{1/2} + (\vec{p}^2 + m_2^2)^{1/2}\,, \tag{1}$$

where $\vec{p}$ is the three momentum of the quark. For equal masses $m_1 = m_2 = m$,

$$\tfrac{1}{4}(E - V)^2 = \vec{p}^2 + m^2\,. \tag{2}$$

Making the usual quantum identifications, we arrive at the Klein-Gordon equation for the energy eigenstates

$$[\nabla^2 + \tfrac{1}{4}(E - V)^2 - m^2]\,\phi(\vec{r}) = 0\,, \tag{3}$$

which reduces to the Schrödinger equation in the nonrelativistic limit, with the correct reduced mass dependence. Based on theoretical speculations,[13] we consider the potential[12,17,18]

$$V = ar + b\,, \tag{4}$$

where the constant b subsumes effects not described by the confinement mechanism, and indicates that one cannot calculate absolute energies, but only energy differences. Negative values of E[19] should be omitted from consideration in accordance with Eq. (1). If we rewrite (3) as a Schrödinger-type equation, translating E by b,

$$\left[-\frac{1}{2\mu}\nabla^2 + \mathcal{U}(r)\right]\phi(\vec{r}) = \frac{\frac{1}{4}E^2 - m^2}{2\mu}\,\phi(\vec{r})\,, \tag{5}$$

with $\mu = \frac{1}{2}m$ being the reduced mass and

$$2\mu\,\mathcal{U}(r) = \tfrac{1}{2}aEr - \tfrac{1}{4}a^2r^2\,, \tag{6}$$

we expose a basic inconsistency of our approach.[12,19] The equivalent (energy-dependent) potential $\mathcal{U}(r)$ has no lower bound, but has a barrier whose maximum grows as $E^2/4m$, and whose width [measured where $\mathcal{U}(r) = 0$, $r \neq 0$] increases as $E/2a$. Therefore, (3) or (5) do not have genuine bound states with real eigenvalues. They can give, to a good approximation, (quasi-) stationary states if the probability of tunneling through $\mathcal{U}(r)$ from $r = 0$ to $r = \infty$ is sufficiently small. A rough estimate shows that the transmission coefficient $\mathcal{T}$ appropriate to (3)–(6) is

$$\mathcal{T} \sim \exp\left(-\frac{2\pi m^2}{a}\right), \tag{7}$$

which if small, makes (3)–(6) a useful way of estimating the excitation energies of the system, and computing an approximate wave function. As $\mathcal{T}$ approaches unity, (7) ceases to be valid; nonetheless it indicates where our approach breaks down. Equation (7) indicates that $\mathcal{T}$ is large when the constant a is large, which means that the model is not valid for quarks confined close together.

The origin of this tunneling problem is the presence of the positive-definite term V^2 in Eq. (3), which dominates for large r for any confining potential which increases with r. The tunneling through the barrier produces complex energy eigenvalues, a difficulty which can be associated with the neglect of the possibility of creating pairs from the vacuum. It has been emphasized[20] that the vacuum polarization can lower the energy of the bound quark-antiquark pair by the screening effects of other pairs which emerge from the vacuum, and can allow the bound state to decay into ordinary particles. [The imaginary part of the energy eigenvalues appropriate to (3) should *not* be identified with the physical lifetime of the bound state, but should only be used as a measure of the applicability of the model.]

A very good approximate solution to the (quasi-) stationary-state problem is obtained by neglecting the quantum-mechanical tunneling through $\mathcal{U}(r)$, defined by (6). This approach can be compared with the exact solution of Eqs. (3)–(6) for S waves, which we do in Sec. IV. The approximate solutions compare very favorably with the exact results. As discussed in Sec. III the spectrum of this model is rather similar, but not identical to that of the classical string model.

A possible alternative strategy is to neglect the V^2 term completely in Eqs. (3)–(6) as a higher-order effect to be associated with the neglected vacuum polarization. This prescription gives a spectrum which is also qualitatively like that of the string model, but the dependence of the spectrum on the parameter a does not agree well with the string model. For this reason, we choose the first prescription and retain the V^2 term, so that we can use the (open-) string model for comparison purposes whenever possible. We suggest that this is the appropriate classical model to be considered for comparison, since it describes the confinement of quarks, neglecting the decay of the bound states to first approximation. Furthermore, in our model, as well as the string model, the *rest* energy per unit length is a constant.

III. WKB APPROXIMATION

A. Spectroscopy

Given the uncertainties inherent in our approach, a detailed analytic or numerical solution to the problem hardly seems justified. However, a very good approximate solution to the (quasi-) stationary-state eigenvalue problem presented by (3)–(6) is given by the WKB method.[21] Consider the radial equation appropriate to (3)–(6),

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{1}{4}(E-ar)^2+m^2\right]u_{nl}(r)=0\,, \tag{8}$$

where $u_{nl}(r)$ is the usual reduced wave function.

S waves ($l=0$). For S waves the WKB quantization appropriate to (8) gives[21]

$$2\int_0^{r_1} dr\, p_r = 2(n'+\tfrac{3}{4})\pi\hbar\,, \tag{9}$$

where

$$p_r=\tfrac{1}{2}[(E^2-4m^2)-2aEr+a^2r^2]^{1/2}\,, \tag{10}$$

where $n=n'+1$ is the principal quantum number of the $l=0$ system, and r_1 is the first classical turning point ($p_r=0$). Note that we have neglected the effect of tunneling [cf. (7)] by not including the region r_2 to ∞ in the quantization condition (9), as discussed in Sec. II. This exercise gives the result

$$\tfrac{1}{2}(E(E^2-4m^2)^{1/2}+4m^2\ln\{[E-(E^2-4m^2)^{1/2}]/2m\}) = 2(n'+\tfrac{3}{4})a\pi\,. \tag{11}$$

[Note we have absorbed the unknown constant term b in the potential (4) by making the translation $E \to E-b$, so that only energy *differences* are computable by these methods.] For $E-2m \ll 2m$, one obtains

$$(E-2m)^{3/2}=\frac{3a\pi}{2\sqrt{m}}(n'+\tfrac{3}{4}) \tag{12}$$

while if $E \gg m$

$$E^2=4a\pi(n'+\tfrac{3}{4})\,. \tag{13}$$

These nonrelativistic and relativistic limits will be useful in giving a qualitative understanding of the meson spectrum, as well as allowing a comparison with the string model. Applications will be made by means of numerical integration of Eqs. (9) and (10), which will be discussed in Sec. V. Comparison with the exact solutions presented in Sec. IV indicates that the WKB quantization leads to typical errors for radially excited states of the order of (1–2)%.

Higher partial waves ($l\neq 0$). The WKB quantization condition[21] appropriate to $l \neq 0$ is

$$2\int_{r_0}^{r_1} dr\, p_r = 2(n'+\tfrac{1}{2})\pi\hbar\,, \tag{14}$$

with $n'=n-l-1$, and

$$p_r=\left[-\frac{(l+\frac{1}{2})^2}{r^2}+\tfrac{1}{4}(E-ar)^2-m^2\right]^{1/2}, \tag{15}$$

where the replacement $l(l+1)\to(l+\frac{1}{2})^2$ has been made in accordance with the WKB method. One can also use (14)–(15) for S waves by setting $l=0$ in (15), with negligible numerical difference from Eqs. (9)–(11). The advantage of using (9)–(11) for the S waves is that one can perform the integration analytically.

Equations (14) and (15) cannot be integrated analytically; however, the integration can be represented in terms of sums of elliptic integrals of the first, second, and third kind, which we have done. In actual applications the integrations are performed by computer, with the evaluation in terms of the elliptic integrals being used to check the results.

B. Scaling theorems

It is unfortunate that one cannot give an analytic integration of Eqs. (14) and (15) in terms of elementary functions, since this would give an explicit expression for the Regge trajectories of the model. However, one can derive scaling theorems appropriate to the WKB integrals which will yield the behavior of the Regge trajectories in the nonrelativistic and relativistic limits. We consider these two cases separately.

The nonrelativistic limit of the WKB integral (14) and (15) is

$$I=\int_{r_0}^{r_1} dr\left[-\frac{(l+\frac{1}{2})^2}{r^2}+m(T-ar)\right]^{1/2}, \tag{16}$$

where $E-2m=T$.

If we make a change to dimensionless variables

$$x=\sqrt{a}\; r\,,$$
$$T=\sqrt{a}\;\tau\,,$$
and
$$\sqrt{a}\;M=m\,, \tag{17}$$

then

$$I=(2M)^{1/2}\int_{x_0}^{x_1}\frac{dx}{x}\left[-\frac{(l+\frac{1}{2})^2}{2M}+\frac{x^2}{2}(\tau-x)\right]^{1/2} \tag{18a}$$

$$=(2M)^{1/2}F\left(\frac{(l+\frac{1}{2})^2}{2M},\tau\right), \tag{18b}$$

where the functional dependence expressed by (18b) follows from the fact that the limits of integration are determined from the zeros of the cubic equation

$$x^3-\frac{\tau}{2}x^2+\frac{(l+\frac{1}{2})^2}{2M}=0\,. \tag{19}$$

Let us rescale our variables as

$$x=\lambda y\,,$$
$$\tau=\lambda\epsilon\,,$$
$$(l+\tfrac{1}{2})=\lambda^{3/2}(L+\tfrac{1}{2})\,,$$

from which one obtains

$$F\left(\frac{(l+\frac{1}{2})^2}{2M},\tau\right)=\lambda^{3/2}\int_{y_0}^{y_1}\frac{dy}{y}\left[-\frac{(L+\frac{1}{2})^2}{2M}+y^2(\epsilon-y)\right]^{1/2} \tag{20a}$$

$$=\lambda^{3/2}F\left(\frac{(L+\frac{1}{2})^2}{2M},\epsilon\right) \tag{20b}$$

where (20b) follows from the fact that y_0 and y_1 are the *same* functions of L and ϵ as x_0 and x_1 are of l and τ. Therefore, we have the scaling law

$$\lambda^{3/2}F\left(\frac{(l+\frac{1}{2})^2}{2M},\tau\right)=F\left(\frac{\lambda^3(l+\frac{1}{2})^2}{2M},\lambda\tau\right). \tag{21}$$

This assures us that

$$F\left(\frac{(l+\frac{1}{2})^2}{2M},\tau\right)=G\left(\frac{(l+\frac{1}{2})^2}{2M},\tau^3\right), \tag{22}$$

where $G(u,v)$ is a *homogeneous* function of u and v satisfying $G(\lambda u,\lambda v)=\lambda^{3/2}G(u,v)$. This proves that

$$F\left(\frac{(l+\frac{1}{2})^2}{2m},\tau\right)=\tau^{3/2}H\left(\frac{l+\frac{1}{2}}{(2M)^{1/2}\tau^{3/2}}\right), \tag{23}$$

where H is a function which can be determined only by numerical integration. The WKB quantization (14) combined with (16) and (23) implies that

$$\left(\frac{T}{m}\right)^{3/2}H\left(\frac{a(l+\frac{1}{2})}{(2m)^{1/2}T^{3/2}}\right)=\left(\frac{a}{m^2}\right)\frac{\pi(n'+\frac{1}{2})}{\sqrt{2}}, \tag{24}$$

where we have restored the original variables. (Recall that $E=2m+T$.) We arrive at a scaling law for the nonrelativistic spectrum. Comparison with (12), where $l=0$, gives the approximate relations

$$H\left(\frac{a}{2(2m)^{1/2}T^{3/2}}\right)\simeq H\left(\frac{1}{3\pi\sqrt{2}\,(n'+\frac{3}{4})}\right)$$
$$\simeq H(0)$$
$$\simeq\frac{\sqrt{2}}{3}\,. \tag{25}$$

The Regge trajectories are described by the condition $n'=$constant. Let us consider the range of

l along the Regge trajectory for which the argument of H is slowly varying. The trajectories then satisfy (24) expressed as

$$\left(\frac{T}{m}\right)^{3/2}\left[H(0)+\frac{a(l+\frac{1}{2})}{(2m)^{1/2}T^{3/2}}H'(0)\right]\simeq\frac{a}{m^2}\,\frac{\pi(n'+\frac{1}{2})}{\sqrt{2}} = \text{const}, \tag{26}$$

where we have assumed that H can be well approximated by the first two terms of the Taylor expansion. The nonrelativistic Regge trajectory therefore must satisfy

$$\left(\frac{T}{m}\right)^{3/2}\simeq A+B(l+\tfrac{1}{2}), \tag{27}$$

where A and B are dimensionless constants fixed by (26). Using (25) we estimate

$$A\simeq\left(\frac{a}{m^2}\right)\frac{3\pi}{2}\left(n'+\frac{1}{2}\right). \tag{28}$$

We have no general argument which estimates B aside from numerical computations. We only note that in the linear string model, identifying the constant with the rest tension, one obtains from the *leading* Regge trajectory ($n'=0$) [see Eq. (41)] in the nonrelativistic limit the estimate[15]

$$B(n'=0)=\sqrt{2}\left(\frac{a}{m^2}\right).$$

If we had considered the more general homogeneous potential of the form

$$V(r)=ar^k, \tag{29}$$

we would have found the scaling law

$$F\left(\frac{(l+\frac{1}{2})^2}{2M},\tau\right)=\tau^{(k+2)/2k}H\left(\frac{l+\frac{1}{2}}{(2M)^{1/2}\tau^{(k+2)/2k}}\right), \tag{30}$$

with radial excitations behaving as

$$\left(\frac{T}{m}\right)^{(k+2)/2k}\sim\left(\frac{a}{m^{k+1}}\right)^{1/k}(n'+\tfrac{1}{2}). \tag{31}$$

Proceeding in analogy with Eqs. (24)–(27) one obtains Regge trajectories which satisfy

$$\left(\frac{T}{m}\right)^{(k+2)/2k}\sim A+B(l+\tfrac{1}{2}), \tag{32}$$

with A and B being the appropriate dimensionless constants.

It is possible to derive scaling laws appropriate to the relativistic region as well. Returning to the linear potential, one must evaluate

$$J=\int_{r_0}^{r_1}\frac{dr}{r}\left[-(l+\tfrac{1}{2})^2+\tfrac{1}{4}(E-ar)^2r^2-m^2r^2\right]^{1/2}. \tag{33}$$

The change of variables

$$r=\frac{1}{\sqrt{a}}x,\quad \epsilon=\frac{E}{\sqrt{a}},\quad M=\frac{m}{\sqrt{a}} \tag{34}$$

shows that

$$J=F((l+\tfrac{1}{2})^2,\epsilon,(\epsilon^2-M^2)). \tag{35}$$

The scalings

$$x=\lambda y,\quad \epsilon=\lambda\bar{E},\quad (l+\tfrac{1}{2})=\lambda^2(L+\tfrac{1}{2}) \tag{36}$$

enable one to demonstrate that F is a homogeneous function of $(l+\frac{1}{2})^2$, ϵ^4, and $(\epsilon^2-M^2)^2$, with the property

$$F((l+\tfrac{1}{2})^2,\epsilon,(\epsilon^2-4M^2))=\epsilon^2G\left(\frac{(l+\frac{1}{2})^2}{\epsilon^4},\frac{\epsilon^2-4M^2}{\epsilon^2}\right), \tag{37}$$

where F and G are functions to be determined numerically (and not to be confused with the functions appearing earlier). The WKB condition implies that

$$\epsilon^2G\left(\frac{(l+\frac{1}{2})^2}{\epsilon^4},\frac{\epsilon^2-4M^2}{\epsilon^2}\right)=(n'+\tfrac{1}{2})\pi, \tag{38}$$

with the Regge trajectory again described by the condition $n'=$ constant. In the relativistic limit $\epsilon^2\gg M^2$,

$$G\left(\frac{(l+\frac{1}{2})^2}{\epsilon^4},1\right)=\int_{x_0}^{x_1}\frac{dx}{x}\left[-\frac{(l+\frac{1}{2})^2}{\epsilon^4}+\frac{1}{4}\,\frac{x^2(\epsilon-x)^2}{\epsilon^4}\right]^{1/2}\simeq 0 \tag{39}$$

describes the Regge trajectories in the extreme relativistic domain. Recall that x_0 and x_1 are the zeros of the integrand, which is positive. Therefore, Eq. (39) implies that x_0 and x_1 are a *double* root of the integrand, providing us with the equation of the Regge trajectories in the relativistic limit,

$$\epsilon^2=8(l+\tfrac{1}{2})$$

or (40)

$$E^2=8a(l+\tfrac{1}{2})$$

in terms of the original variables. Therefore, our model leads to straight-line Regge trajectories in the relativistic limit, with inverse slope numerically $\alpha^{-1}=8a$. Equation (40) has also been verified by computer calculations, by evaluating Eqs. (14) and (15) for a large (fixed) value of n' and several (large) values of l. Note that the homogeneous potential (29) gives rise to Regge trajectories behaving as $E^2\propto l^{2k/k+1}$, so that if $V\propto r^2$, then $E^2\propto l^{4/3}$ in the relativistic region (in this model).

C. String model

It is useful to compare the spectrum of our model with that of the classical string model.[15] Consider two quarks of mass m_1 and m_2, attached to a massless string, whose rest tension is T_0, and consider the rigid rotation of the system, which gives rise to the leading Regge trajectory ($n'=0$) if quantized. It is easy to show[15] that in the nonrelativistic limit this trajectory satisfies

$$E-(m_1+m_2)=\left(\frac{T_0^{\,2}}{\mu}\right)^{1/3}L^{2/3}, \tag{41}$$

where μ is the reduced mass of the two quarks, while in the relativistic limit

$$E^2=2\pi T_0 L, \tag{42}$$

which gives the Regge slope

$$\alpha=(2\pi T_0)^{-1}, \tag{43}$$

which is identical to that of the massless limit. Recall that the quantized version of the string model (with massless quarks) gives rise to the spectrum of the dual resonance model in which both ghosts and *odd daughters* are *absent*. Thus the *radial* excitations of this string model behave as

$$E^2=4\pi T_0 n' \tag{44}$$

in the relativistic region, which is identical to (13).

One is tempted to identify the rest tension T_0 of the string model with the parameter a since they both correspond to a constant interaction energy per unit length in the rest system [cf. Eq. (4)]. Comparison of Eq. (13) with (44) supports this identification. However, comparison of (40) with (42) shows (to our disappointment) that

$$\alpha^{-1}=2\pi T_0 \quad \text{(string model)} \tag{45a}$$

and

$$\alpha^{-1}=8a \quad \text{(potential model)} \tag{45b}$$

in the two models. Thus our potential model does not correspond to the string model in detail, although the two models are similar in a number of features.

IV. EXACT SOLUTION FOR S WAVES

In this section we will present an exact solution of Eq. (8) for the case $l=0$. The equation to be solved is

$$\frac{d^2u(r)}{dr^2}+\left[\tfrac{1}{4}(E-ar)^2-m^2\right]u(r)=0. \tag{46}$$

The exact solution of this equation allows us to test the validity of the WKB approximation and our treatment of the instability problem. In general, the WKB method provides an excellent approximation for states with a large number of radial nodes, i.e., for n' large. However, for low excitations the accuracy depends on the particular problem being considered. (For example, this approximation gives correct energy eigenvalues for the nonrelativistic harmonic oscillator.)

To solve (46) we make the change of variables

$$x=\frac{i}{2a}(E-ar)^2, \qquad u(r)=e^{-x/2}v(x), \tag{47}$$

which transforms (46) to the confluent hypergeometric differential equation

$$x\frac{d^2v}{dx^2}+\left(\frac{1}{2}-x\right)\frac{dv}{dx}-\left(\frac{1}{4}-\frac{im^2}{2a}\right)v=0 \tag{48}$$

whose general solution is[22]

$$v(x)=c_1\,{}_1F_1\left(\frac{1}{4}-\frac{im^2}{2a};\frac{1}{2};x\right)+c_2x^{1/2}\,{}_1F_1\left(\frac{3}{4}-\frac{im^2}{2a};\frac{3}{2};x\right). \tag{49}$$

Therefore the solution of the S-wave radial equation (46) is

$$u(r)=c_1e^{-x/2}\,{}_1F_1\left(\frac{1}{4}-\frac{im^2}{2a};\frac{1}{2};x\right)+c_2e^{-x/2}x^{1/2}\,{}_1F_1\left(\frac{3}{4}-\frac{im^2}{2a};\frac{3}{2};x\right), \tag{50}$$

where the constants c_1 and c_2 are to be determined by the boundary conditions.

Since the wave number in Eq. (46) becomes real for large r [cf. (5) and (6)], the boundary condition at infinity is such that only outgoing waves should be present asymptotically. This necessarily leads to complex energy eigenvalues. Furthermore, we only consider those states with $\mathrm{Re}E>0$ as appropriate to Eq. (1). In other words, a wave packet confined inside the potential barrier at time $t=0$ has a finite probability of tunneling through the barrier at later times, which would be a purely outgoing wave sufficiently far from the origin.

The wave function $u(r)$ has a branch cut in the x plane due to the presence of $x^{1/2}$ in Eq. (50). We shall choose a cut from 0 to ∞ so that

$$\arg x=-\tfrac{3}{2}\pi \quad \text{when } ar>E$$

and (51)

$$\arg x=\tfrac{1}{2}\pi \quad \text{when } ar<E.$$

The wave function for large r can be readily obtained from the asymptotic behavior of the individual confluent hypergeometric functions in Eq.

(50).[23] The appropriate limit for large r is

$$ {}_1F_1(a;b;x)\xrightarrow[x\to\infty]{}\frac{\Gamma(b)}{\Gamma(b-a)}\,e^{-i\pi a}x^{-a}+\frac{\Gamma(b)}{\Gamma(a)}e^{x}x^{a-b}, $$

with (52)

$$ \arg x=-\tfrac{3}{2}\pi . $$

The boundary condition at infinity then fixes the constants c_1 and c_2 up to an over-all normalization, so that

$$ u(r)=i\,\frac{\Gamma(\frac{1}{4}+im^2/2a)}{\Gamma(\frac{1}{2})}\,e^{-x/2}\,{}_1F_1\left(\frac{1}{4}-\frac{im^2}{2a};\frac{1}{2};x\right) + \frac{\Gamma(\frac{3}{4}+im^2/2a)}{\Gamma(\frac{3}{2})}\,e^{-x/2}x^{1/2}\,{}_1F_1\left(\frac{3}{4}-\frac{im^2}{2a};\frac{3}{2};x\right), \tag{53} $$

with x related to r by (47).

The energy eigenvalues can now be determined by the boundary condition at the origin $u(0)=0$. This usually requires a numerical calculation; however, a check of the WKB approximation to the exact solution can be made analytically in two limits: nonrelativistic and extreme relativistic limits. Before doing this, first consider the high-energy limit of the confluent hypergeometric functions appearing in Eq. (53). For large E one must choose $\arg x=\frac{1}{2}\pi$, as indicated by (51), so that in this limit

$$ {}_1F_1(a;b;x)\xrightarrow[x\to\infty]{}\frac{\Gamma(b)}{\Gamma(b-a)}\,e^{i\pi a}x^{-a}+\frac{\Gamma(b)}{\Gamma(a)}\,e^{x}x^{a-b}. \tag{54} $$

This shows that $u(0)$ is an elementary transcendental function of E. Assuming that E has a small imaginary part, $(-i/2)\Gamma$, we can readily solve the eigenvalue problem, with the result

$$ E^2-4m^2\ln\left(\frac{E}{m}\right)\simeq 4a\pi n' \tag{55a} $$

and

$$ E\Gamma\simeq a\exp\left(\frac{-2\pi m^2}{a}\right). \tag{55b} $$

The equation for the real part of energy is the same as the large-E limit of the WKB condition, Eq. (11), up to an additive constant in E. The imaginary part of the energy level Γ, has the exponential behavior as estimated in Eq. (7).

Next consider two limits where the *exact* wave function can be expressed in terms of more familiar functions:

(i) *Nonrelativistic limit.* In the large mass limit, each confluent hypergeometric function becomes a linear combination of two Airy functions Ai and Bi.[24] However, the coefficient of Bi is suppressed by a factor of $\exp(-\pi m^2/a)$ compared with that of Ai. Discarding the contribution of Bi we have the eigenvalue condition

$$ \mathrm{Ai}\left[-\left(\frac{m}{a}\right)^{1/3}E_n\right]=0, \tag{56} $$

which gives *real* eigenvalues. Since the transmission probability, (7) or (55b), becomes zero in the large-mass limit the particle is completely confined inside the potential barrier, and the eigenvalues must be real, in accord with (56).

(ii) *Extreme relativistic limit.* In the massless limit, the wave function becomes the sum of two Bessel functions of order $\frac{1}{4}$,[25] with the wave function at the origin satisfying

$$ u(0)\propto J_{1/4}\left(\frac{E^2}{4a}\right)-\frac{2+i}{5}\,Y_{1/4}\left(\frac{E^2}{4a}\right)=0. \tag{57} $$

The solution of Eq. (57) now requires complex energies.

In Table I we display the first few eigenvalues obtained from Eqs. (56) and (57) and compare them with the WKB values. As can be seen from the table, the WKB method is better in the nonrelativistic limit than in the extreme relativistic one. Nevertheless, in the latter case our approximation is good within 2% for low radial excitations. Furthermore, one should note that the *imaginary* parts of the energy levels are quite small compared to the real parts, and are expected to be even smaller for massive quark systems due to the exponential factor in Eqs. (7) and (55b). Thus the exact solution for the $l=0$ radial equation justifies our treatment of the tunneling effect in Sec. II and allows us to trust the WKB solution of our model. Furthermore, the exact solution for $l=0$ suggests that the WKB approximation is probably rather accurate even for $l\neq 0$.

V. APPLICATIONS

A. Energy levels

In order to apply these considerations to meson spectroscopy, one must be able to fix the free parameters of the model: the constants a and b of Eq. (4), as well as the quark masses. If quark confinement is due to strong long-range interactions of Yang-Mills gluons, in the approximation in which closed fermion loops are neglected one expects the constant a to be universal, i.e., independent of the meson system being considered. On this basis we propose to apply our model to meson spectroscopy with the assumption that the constant a does not depend on the nature of the quark-antiquark pairs being bound. In doing so

TABLE I. A comparison of exact and WKB approximate energy levels obtained from Eq. (8) for S waves ($l=0$). The range of eigenvalues of Eq. (57) is somewhere between the practical limits of the power series expansion and the asymptotic expansion. We used the Taylor expansion of the Whittaker function for the first three eigenvalues. For $n=2,3,4$, Eq. (57) is expanded near Im$E=0$ up to the quadratic terms and numerical tables are used for the solution. Therefore the eigenvalue for $n=2$ is checked by two methods. For numerical values of the Bessel functions, see Ref. 25.

	Nonrelativistic limit		Extreme relativistic limit	
n	$\left(\frac{m}{a^2}\right)^{1/3} E_n$ (exact)	$\left(\frac{m}{a^2}\right)^{1/3} E_n$ (WKB)	$\frac{1}{\sqrt{a}} E_n$ (exact)	$\frac{1}{\sqrt{a}} E_n$ (WKB)
0	2.338	2.349	$3.096-0.110i$	3.151
1	4.088	4.091	$4.697-0.074i$	4.717
2	5.521	5.519	$5.883-0.059i$	5.890
3	6.787	6.781	$6.867-0.051i$	6.868
4	7.944	7.935	$7.728-0.045i$	7.724

we obtain a satisfactory description of meson spectroscopy with "reasonable" quark masses, and claim support for the idea that the forces which bind quarks are more or less universal, and that bound light- and heavy-quark systems are distinguished to a large degree by kinematical considerations.[26] We treat the constant a as a free parameter of the model, rather than one to be obtained from comparison with the string model or from other theoretical considerations. (Our model is sufficiently different from the string-model to justify this approach.)

In some sense the constant b must account for all other interactions omitted from consideration, such as the shorter-range spin-independent interactions, as well as spin-dependent corrections, both of which depend on the masses of the quarks being bound. Therefore, we cannot assume that b is a universal constant. Furthermore, it is to be expected that the effective value of b will depend on the spin-state being considered, even for the same quark content. One can ignore the parameter b if one is only interested in energy differences, and not in an absolute energy scale. However, it will be interesting to evaluate b for each bound quark system, as an estimate of the energies associated with the omitted interactions.

We may fix the constant a of the potential by choosing as input the masses of three different meson states composed of $\mathfrak{N}$ and $\mathcal{P}$ quarks in an $I=1$ spin-triplet state, which seems to be better understood than the $I=0$ or singlet systems. To avoid the seemingly anomalous behavior of the pionlike states,[27] we take as input $\rho(770)$, $\rho'(1600)$, and $A_2(1310)$, which we assume to be the $^3S_1$ ground state, the first ($^3S_1$) radial excitation, and the first Regge ($^3P_1$) excitation of the ground state, respectively. Fitting the masses of these states with our model yields

$$a=0.30\text{ GeV}^2, \tag{58}$$

$$b_\rho=-1.13\text{ GeV} \tag{59}$$

for the potential of the ρ-like states, and

$$m_{\mathcal{P}}=260\text{ MeV} \tag{60}$$

for the mass of the nucleon-type quarks.

Now consider the triplet bound states of $\bar{\lambda}\lambda$ quarks. We assume that $\phi(1019)$ is the $^3S_1$ ground state of this system, and that $f'(1516)$ is the first Regge ($^3P_2$) excitation. Given the value of a from (58), we find

$$b_\phi=-1.16\text{ GeV} \tag{61}$$

and

$$m_\lambda=475\text{ MeV} \tag{62}$$

for the λ quark mass.

We are encouraged by the reasonable values obtained for the quark masses[11] to apply our model to the bound-charm system. Using the value for the constant a given by (58), we assume that $\phi_c'(3700)$ is the first radial excitation of the $^3S_1$ $\phi_c(3100)$ ground state, and we find

$$b_\psi=-1.72\text{ GeV} \tag{63}$$

and

$$m_c=2.0\text{ GeV}. \tag{64}$$

Again, a plausible value for the quark mass emerges.[10] Given these parameters we predict the $^3S_1$ $\phi_c(3100)$ ground state and radially excited partners to have masses

$$E=3.1, 3.7, 4.2, 4.6, 4.9, \ldots\text{ GeV}. \tag{65}$$

All free parameters of the model have been determined for the triplet systems, so that we can predict the other excitations of $\rho(770)$, $\omega(783)$, $\phi(1019)$, and $\phi_c(3100)$ within the accuracy ex-

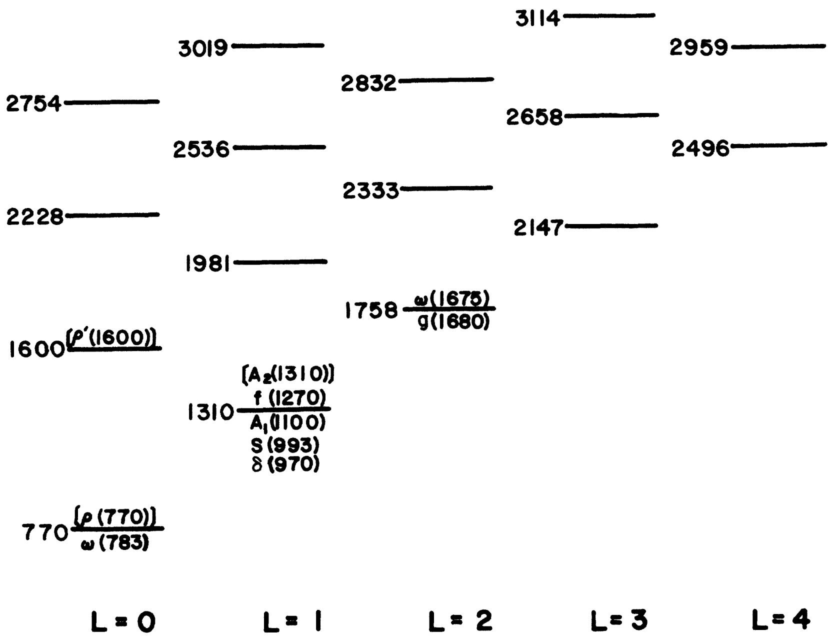

FIG. 1. Predictions for mean energy levels (in MeV) for $I=0$ and $I=1$ mesons with $\mathcal{N}$- and $\mathcal{P}$-type quark pairs in a triplet state (^3L_J). States used as input are $\rho(770)$, $\rho'(1600)$, and $A_2(1310)$. Relevant experimentally confirmed (Ref. 28) states are indicated in the level diagram.

pected, given our neglect of spin-orbit and tensor couplings. The level scheme that emerges is shown in Figs. 1, 2, and 3, with *confirmed*[28] meson states indicated in each of these figures. One is struck by the scarcity of solid experimental information on meson spectroscopy.

Let us now turn to a discussion of the singlet states.[29] The spectroscopy of the $I=0$ singlet levels is complicated by the annihilation of quark-antiquark pairs into gluons, which makes a *strongly energy-dependent*[29] contribution to the meson mass matrix. As a consequence, the pseudoscal-

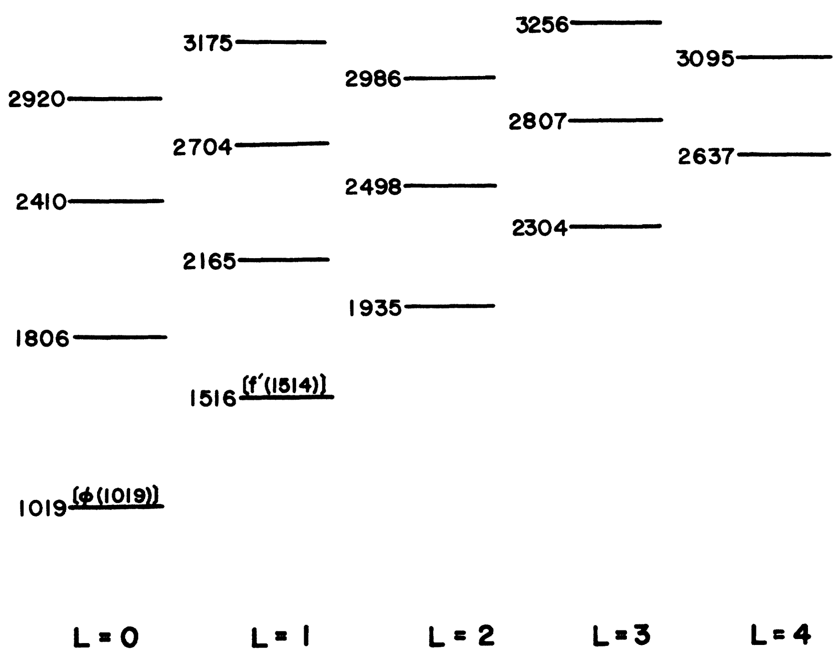

FIG. 2. Predictions for mean energy levels for $I=0$ mesons with λ-type quark pairs in a triplet state (^3L_J). States used as input are $\phi(1019)$ and $f'(1514)$.

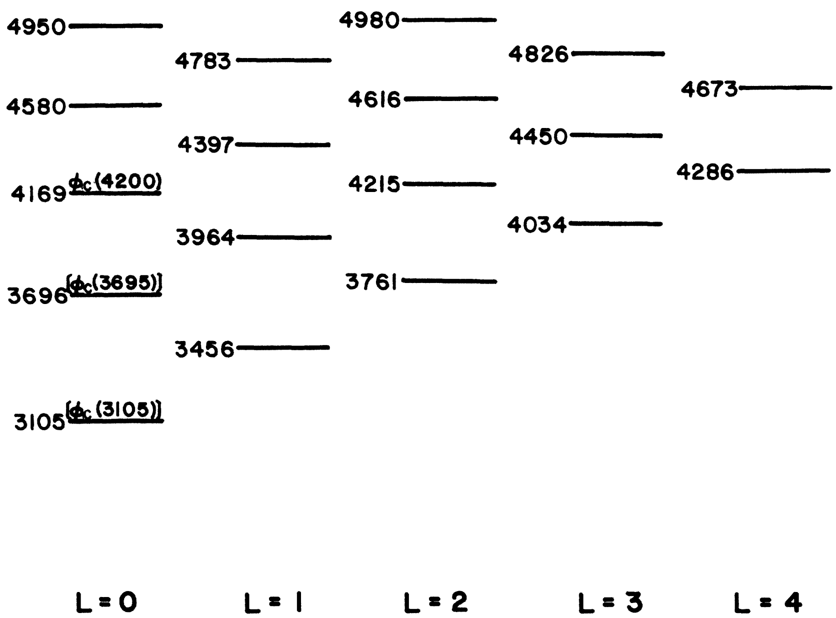

FIG. 3. Predictions for mean energy levels for $I=0$ mesons with charmed quark pairs. States used as input are $\phi_c(3105)$ and $\phi_c(3695)$.

ar meson states η, η', and η'' need not be orthogonal.[29,30] Arguments based on asymptotic freedom[29,31] suggest that this annihilation term is negligible for the charmed quarks, so that η'' can be considered to be entirely bound *charmed* quarks. Therefore the level structure of the singlet $\bar{c}c$ states is essentially that of the corresponding triplet levels, and hence can be obtained from Fig. 3 with a downward shift in energy of roughly 25–100 MeV appropriate to the hyperfine interaction. Because of the annihilation term, the η and η' contain nonorthogonal combinations[29,30] of $\mathfrak{N}$-, $\mathcal{P}$-, and λ-type quarks. Furthermore, the hyperfine interaction is strongly mass- and angular-momentum-dependent, since it behaves as[29,32]

$$\frac{1}{6m_1m_2}\vec{\sigma}_1\cdot\vec{\sigma}_2\nabla^2 V(r)\,. \tag{66}$$

We believe these complications place the discussion of the η and η' outside the scope of our model, so that we shall not discuss these $I=0$ singlet states or their excitations further.

We now consider the $I=1$ singlet states. It has been emphasized that the spin-spin interaction (66) makes its most important contribution to $l=0$ states,[29] so that with this hyperfine interaction omitted in our model, $\pi(140)$ does not seem to be well suited as an input for fixing the parameter b_π. Instead we choose $B(1235)$ as the input, determining

$$b_\pi=-1.21\text{ GeV}\,, \tag{67}$$

with the resulting spectroscopy shown in Fig. 4. It is not suprising that our prediction for the pion mass is very much too large. [The pion always presents difficulties for this kind of spectroscopy,[11] if one ignores the contribution of Eq. (66).] Our satisfactory prediction of 1683 MeV for $A_3(1640)$ leads us to concur with the opinion[29] that the anomalous behavior of the pion is due to the hyperfine interaction.

It should be noted that we are unable to treat meson systems with unequal-mass quarks without generalizing Eqs. (2) and (3) to the appropriate quartic differential equation.[16] Thus, we cannot discuss K^*, K, or charmed meson states without extending our analysis, although we expect the resulting spectroscopy to be of the same qualitative nature as the states discussed here.

Let us now comment briefly on the values obtained for the constant b which appears in the potential. Comparing Eqs. (59), (61), (63), and (67), we see that the value of b is essentially independent of the meson system for noncharmed-quark systems, and is somewhat larger (in absolute value) for the charmed-quark system. We can offer no theoretical explanation for this aspect of our fit to the data, although the large value for the constant in the potential leads us to question the validity of omitting such a constant[33] in fits to the meson spectra with linear potential models.

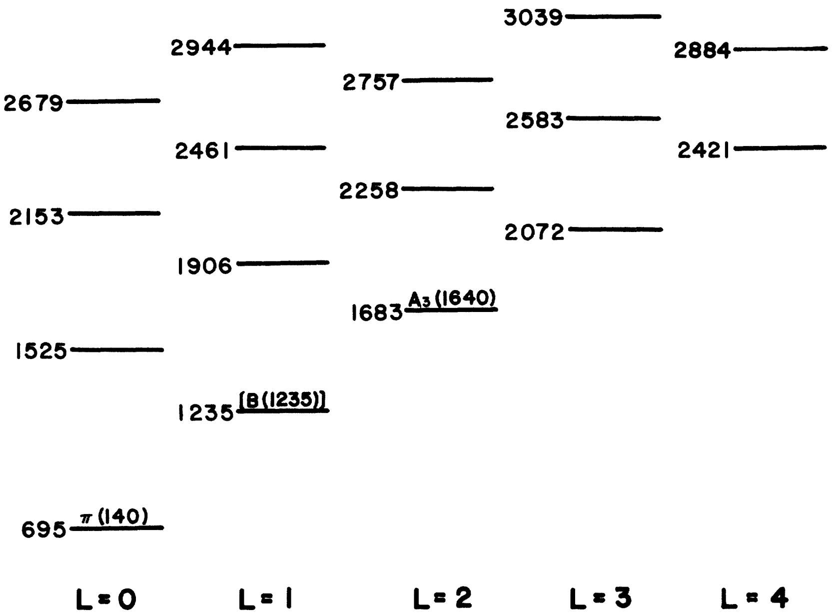

FIG. 4. Predictions for mean energy levels for $I=1$ mesons with $\mathfrak{N}$- and $\mathcal{P}$-type quark pairs in a singlet state (^1L_l). The state used as input was $B(1235)$.

B. Leptonic widths

In Sec. IV we found the S-wave wave functions of the model in terms of confluent hypergeometric functions. From the wave functions of the model, one can estimate a number of hadronic and leptonic widths, and radiative transitions of the meson states. To do so here would require a computer calculation of the wave functions overlap integrals appropriate to our model; results which we are not prepared to present at this time. In lieu of this, we appeal to the nonrelativistic limit presented in Sec. IV [e.g., Eq. (56)] in terms of Airy functions.

As an illustration, we estimate various vector-meson decay widths into lepton pairs, $V\to l^+l^-$, whose width is given by,[34] assuming that there are three colors,

$$\Gamma(V\to l^+l^-)=16\pi\alpha^2\frac{|\phi(0)|^2}{m_V^{\,2}}e_Q^{\,2}\,, \tag{68}$$

neglecting the lepton mass. Here α is the fine-structure constant, m_V is the vector-meson mass, and $e_Q^{\,2}$ is the square of the quark charges, with[8,11]

$$e_Q^{\,2}=\tfrac{1}{2}\,;\ \tfrac{1}{18}\,;\ \tfrac{1}{9}\,;\ \tfrac{4}{9}$$

for the ρ, ω, ϕ, and ϕ_c systems, respectively. We (overoptimistically) use the nonrelativistic limit to estimate the wave function at the origin, i.e.,

$$|\phi(0)|^2=\frac{ma}{4\pi} \tag{69}$$

where m is the appropriate quark mass. Using our values for the parameters, we obtain the results shown in Table II, with the agreement with experiment[35] better than we have a right to expect. (Our predictions for the leptonic widths of the ρ,ω,ϕ states are roughly a factor of 2 too large.)

C. Hadron decays

Attempts have been made to describe hadron decays which violate the Okubo-Zweig-Iizuka rule in terms of decays into colored gluons.[36,37] Thus, in analogy with the three-photon decay of orthopositronium, one has[38]

$$\Gamma(\phi_c(3100)\to\text{hadrons})=\tfrac{160}{81}(\pi^2-9)[\alpha_s(3.1)]^3\times\frac{|\phi(0)|^2}{m_{\phi_c}^{\,2}} \tag{70}$$

and[39]

$$\Gamma(\phi\to\rho\pi)=(\tfrac{2}{3})(\tfrac{160}{81})(\pi^2-9)[\alpha_s(1)]^3\frac{|\phi(0)|^2}{m_\phi^{\,2}}\,, \tag{71}$$

where the extra factor $\frac{2}{3}$ accounts for the fact that the λ quark contributes to Okubo-Zweig-Iizuka-rule *allowed* decays of the ϕ meson,[40] and where $\phi(0)$ is the appropriate wave function at the origin. Since the value of $\phi(0)$ is model-dependent, it is desirable to avoid a specific model by relating it to the leptonic widths by means of Eq. (68).

TABLE II. Predictions for leptonic decays and wave functions at the origin for vector mesons. Experimental values are obtained from Refs. 28 and 35.

State	$\|\phi(0)\|^2$ (MeV)^3 theory	$\Gamma(V\to l^+l^-)$ (keV) theory	$\Gamma(V\to l^+l^-)$ (keV) experiment
$\rho(770)$	6.27×10^6	14.1	6.5 ± 0.5
$\omega(783)$	6.27×10^6	1.53	0.76 ± 0.17
$\phi(1020)$	11.4×10^6	3.27	1.34 ± 0.084
$\phi_c(3100)$	48.1×10^6	5.97	4.8 ± 0.6
$\phi_c(3700)$	48.1×10^6	4.20	2.2 ± 0.6
$\phi_c(4200)$	48.1×10^6	3.24	4.0 ± 1.2
$\rho'(1600)$	6.27×10^6	3.27	1.02 to 2.65

Consider the two ratios[29]

$$R_1=\frac{\Gamma(\phi\to\rho\pi)}{\Gamma(\phi\to l^+l^-)}=\frac{20(\pi^2-9)}{27\pi\alpha^2}[\alpha_s(1)]^3 \qquad (72)$$

and

$$R_2=\frac{\Gamma(\phi_c\to \text{hadron})}{\Gamma(\phi_c\to l^+l^-)}=\frac{5(\pi^2-9)}{18\pi\alpha^2}[\alpha_s(3)]^3, \qquad (73)$$

which should be compared with experimental values

$$(R_1)_{\rm exp}=\frac{663}{1.34}=495 \qquad (74a)$$

and

$$(R_2)_{\rm exp}=\frac{59}{4.8}=12.3\,. \qquad (74b)$$

These experimental ratios can be combined with Eqs. (72) and (73) to give

$$[\alpha_s(1)]_{\rm exp}=0.51 \qquad (75a)$$

and

$$[\alpha_s(3)]_{\rm exp}=0.21\,, \qquad (75b)$$

values which are similar to those previously reported.[5,29]

If these values of $\alpha_s(m_v)$ are connected by asymptotic freedom, they should satisfy

$$\alpha_s(3)=\alpha_s(1)\left[1+\frac{25}{12\pi}\alpha_s(1)\ln 9\right]^{-1}. \qquad (76)$$

Using (75a) as input, one predicts

$$[\alpha_s(3)]_{\rm theory}=0.29\,. \qquad (77)$$

Perhaps more relevant is the comparison of

$$\left[\frac{\alpha_s(1)}{\alpha_s(3)}\right]^3_{\rm exp}=\left(\frac{0.51}{0.21}\right)^3=15.2\,, \qquad (78)$$

with

$$\left[\frac{\alpha_s(1)_{\rm exp}}{\alpha_s(3)_{\rm theory}}\right]^3=\left(\frac{0.51}{0.29}\right)^3=5.44\,, \qquad (79)$$

which indicates that the asymptotic freedom mechanism is qualitatively correct, although quantitatively it may overestimate the decay width $\Gamma(\phi_c\to\text{hadrons})$ by a factor of three.

It is essential to note that this test of the gluon annihilation model has made no use of an *explicit* calculation of the bound-state wave function. All that is required is that the annihilation into gluons take place at short distances. Thus, if quantitative understanding of decay rates suppressed by the Okubo-Zweig-Iizuka rule is to be obtained, one should perhaps consider corrections to the simplest estimates (70) and (71). This is especially true for the rate for $\phi\to\rho\pi$, since the effective gluon coupling constant is not particularly small.

V. CONCLUSIONS

We have presented a simple relativistic potential model to describe meson spectroscopy in a sufficiently unified way so as to encompass both light- and heavy-quark systems. Totality of our results supports the view expressed in the Introduction that the large $\phi_c(3100)$, $\phi_c'(3700)$, $\phi_c''(4200)$ spacing [in (mass)^2] as compared to level spacings in the ρ-ρ' system was in fact a kinematical effect due to the large mass of the charmed quark. The evidence we collect here is consistent with the premise that forces which confine quarks are essentially independent of quark species. A particular feature which characterizes the spectra of our model is the prediction that parent and daughter energy levels are not degenerate, as is obvious from Figs. 1–4, as well as from Eqs. (44) and (45), which is to be contrasted with the string model.[15]

The over-all systematics of our predictions for the meson energy levels and leptonic decay widths give support to the usefulness of our model and the parameters and wave functions that emerge from analysis of existing data. We look forward to further tests of our simple picture of quark binding.

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[40]We thank T. Appelquist for a discussion of this point.