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Bootstrap model with a bare Pomeron*

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We use the resolvent-kernel technique to obtain the inclusive distributions in a one-dimensional bootstrap model containing both pionization and diffraction. In the case where diffraction is described by a bare Pomeron with intercept less than one, we obtain an exact solution of the bootstrap equation for the single-particle distribution, which can display correct double-Regge behavior near $x = 0$. For a Pomeron with intercept exactly one, we obtain a multifireball expansion for the inclusive distribution in powers of the Pomeron coupling constant. Comparison of the multifireball expansion for the bare Pomeron, with the exact solution, indicates that the expansion does not correctly describe the distribution near $x = 0$. We discuss the solution of the n trajectory, one-dimensional bootstrap model and its relation to conventional treatments of the n-trajectory Chew-Pignotti model.

I. INTRODUCTION

The bootstrap model for inclusive reactions has been developed by a number of authors.[1-7] It is based on the bootstrap hypothesis of Krzywicki and Petersson,[1] the implications of which were first investigated in detail by Finkelstein and Peccei.[2] It is assumed that there is exactly one leading particle in each production event and that the inclusive distributions will factorize into products of the leading-particle distribution and the inclusive distribution of the other particles in the remaining fireball. The bootstrap hypothesis then states that the inclusive distributions inside the fireball, as functions of the variables appropriate to the fireball rest frame, are described by the same functions as described the inclusive distributions in the whole event as functions of the c.m. variables. In models where the leading particle is produced by the exchange of "something," the bootstrap hypothesis is equivalent to the statement that the "something"-particle cross sections have the same behavior as the particle-particle cross sections.

Since the integrated inclusive distributions are the multiplicity moments, it is an immediate consequence of the bootstrap hypothesis that the multiplicity distributions associated with a leading particle should depend only on the mass squared of the fireball, M^2, and not on s, and should, in fact, have the same behavior as functions of this variable, as the multiplicity distributions in the whole event as functions of s. This has been confirmed by the NAL bubble-chamber results for multiplicities associated with a slow proton in p-p collisions.[8] Similar results have been obtained for π-p collisions.[9] It has also been noted in the analysis[10] of the exclusive reaction $K^-p \to K^-(p\pi^+\pi^-)$ at 16 Gev/c that the $p\pi^+\pi^-$ system tends to decay longitudinally in its rest frame, as the bootstrap hypothesis would suggest, rather than isotropically.

The bootstrap hypothesis leads directly to a set of recursion relations in the form of integral equations for the multiparticle distributions[1,2,5] and for the generating function for the multiplicity moments.[5,7] The equations involve only the physical on-shell inclusive distributions and it has been shown[2,5] that the solutions of the bootstrap equations must satisfy the energy-momentum conservation sum rules.[11] The equations are written most simply in terms of the Feynman scaling variable x rather than the rapidity y and are thus best suited for the study of the inclusive distributions in the fragmentation regions where x is the appropriate variable and the energy-momentum constraints are important, rather than on the central plateau where the appropriate variable is the rapidity and the energy-momentum constraints can be neglected.

The bootstrap equations could provide a useful phenomenological tool[4] or they can be used to construct simple, soluble models that are capable of reproducing the gross features of the data. We shall adhere to the second philosophy and, for simplicity, we shall restrict ourselves to the case where all produced particles are identical and spinless, although quantum numbers can be included in the model.[3]

In a previous paper,[6] which we shall refer to as paper I, we studied the properties of a simple one-dimensional model for which the bootstrap equations are soluble. The model, which is closely related to the one-trajectory Chew-Pignotti model,[12] depends on a single parameter β, is Regge-behaved in the triple-Regge region, and can exhibit the properties of either the diffractive or pionization component, depending on the value

of β. Since *both* pionization and diffraction must be present in any realistic model, our object in this paper is to study the properties of models containing both. Since the behavior of the multiplicity distributions and correlation coefficients has been studied in such models by Morel and Petersson,[7] we shall concentrate on the inclusive distributions themselves.

We shall concentrate on the "bare-Pomeron" approach,[13] where the Pomeron is assumed to have an intercept slightly less than 1 and can be interchanged an arbitrary number of times, as it is simplest to handle in the bootstrap context. We shall, however, also discuss the case where the Pomeron intercept is exactly 1. There are several possible approaches to the inhomogeneous Volterra equations that arise in the bootstrap model, and while Finkelstein and Peccei[2] have emphasized the Mellin-transform technique, we shall find the reciprocal or resolvent-kernel technique to be more useful for our purposes.

In Sec. II we shall discuss the relationship between the inclusive and exclusive cross sections in the bootstrap model. We will show that if the square of the T matrix is a sum over all permutations of the outgoing momenta of q factorizable terms, then the bootstrap equations are satisfied, and, conversely, that if *all* of the bootstrap equations are satisfied by the inclusive distributions, then the square of the T matrix must be the sum of such q factorizable terms. In Sec. III we will use the resolvent kernel to obtain the solution for the model discussed in paper I. We shall discuss the properties of the multiparticle distributions and correlation functions in this model.

In Sec. IV we show how the resolvent identities can be used to obtain a perturbation expansion in powers of the Pomeron coupling constant, which is assumed to be small. Because we will obtain an expansion for the inclusive distribution rather than the inclusive cross section, the nth term in the expansion cannot be identified simply as the contribution from diagrams with n Pomerons on the multiperipheral chain. Nevertheless, for simplicity, we shall refer to the expansion as the "multifireball expansion." We shall discuss both the bare-Pomeron case and the case of unit Pomeron intercept.

In Sec. V we show that the two-trajectory Chew-Pignotti model (of which the ordinary trajectory plus bare-Pomeron model is a special case) is also soluble and we compare the results of those of Sec. IV. We shall see that the multifireball expansion provides a good approximation in the region near $x=1$ but can provide incorrect results in the neighborhood of $x=0$, although the combination of bare Pomeron and ordinary trajectory can result in reasonable double-Regge behavior in the neighborhood of $x=0$ in the exact solution. We indicate how the n-trajectory problem could be solved and speculate as to how this could be used to effect a multi-Regge bootstrap.

II. THE T MATRIX

The invariant inclusive cross section for the process $a+b\to k$ identical, spinless particles plus anything is given by

$$\frac{d\hat{\sigma}}{dp_1\cdots dp_k}=\tfrac{1}{2}(2\pi)^4\lambda^{-1}(s)\sum_{n=k}^{\infty}\frac{1}{(n-k)!}\int dp_{k+1}\cdots dp_n\,|\langle p_a,p_b\,|\,T\,|\,p_1\cdots p_n\rangle|^2\delta^4\Big(p_a+p_b-\sum_{i=1}^{n}p_i\Big). \tag{2.1}$$

The corresponding distribution, $N_{(k)}(p_a,p_b\,|\,p_1\cdots p_k)$, is obtained by dividing by the total cross section $\sigma(s)$. As in paper I, we assume that by specifying a leading particle i, we can write the square of the T matrix as a sum of q factorizable terms,

$$\begin{aligned}&|\langle p_a,p_b\,|\,T\,|\,p_1\cdots p_n\rangle|^2=\sum_{i=1}^{n}|\langle p_a,p_b\,|\,T_i\,|\,p_1\cdots p_n\rangle|^2,\\&|\langle p_a,p_b\,|\,T_i\,|\,p_1\cdots p_n\rangle|^2=f^2(s,M_i^2,t_i)|\langle p_a-p_i,p_b\,|\,T\,|\,p_1\cdots p_{i-1},p_{i+1}\cdots p_n\rangle|^2.\end{aligned} \tag{2.2}$$

This assumption of q factorizability of the T matrix is, of course, the fundamental assumption of multiperipheralism. If we were to take

$$f(s,M^2,t)=\Gamma(t)(s/M^2)^{\alpha(t)}, \tag{2.3}$$

then, in the strong-ordering limit, successive applications of Eq. (2.2) would give us a T matrix that is a sum of terms of the usual multi-Regge form

$$\prod_i\Gamma_i(t_i)(s_{i,i+1})^{\alpha(t_i)}. \tag{2.4}$$

However, it is essential that we do *not* make the strong-ordering assumption. In Eq. (2.2) we are, in effect, summing over all permutations of the outgoing momenta, and it is necessary to do so to obtain the bootstrap equations. Thus, as emphasized by Finkelstein and Peccei,[2] it is impossible to specify the leading particle in a given event as, for example, the particle with the largest longitudinal momentum. The leading particle is specified in the model as the first particle on the multiperipheral chain.

Substituting Eq. (2.2) into Eq. (2.1), making the additional assumption[14] that off-shell effects be either neglected or incorporated into the definition of f, we obtain

$$N_{(k)}(p_a, p_b | p_1 \cdots p_k) = \sum_{i=1}^{k} N_L(p_a, p_b | p_i) N_{(k-1)}(p_a - p_i, p_b | p_1 \cdots p_{i-1}, p_{i+1} \cdots p_k) + \int dp' N_L(p_a, p_b | p') N_{(k)}(p_a - p', p_b | p_1 \cdots p_k). \tag{2.5}$$

Equation (2.5) is the Krzywicki-Petersson–Finkelstein-Peccei bootstrap integral equation in its most general form. In the case where $k=1$, we have for the single-particle distribution

$$N(p_a, p_b | p) = N_L(p_a, p_b | p) + \int dp' N_L(p_a, p_b | p') N(p_a - p', p_b | p). \tag{2.6}$$

N_L is the leading-particle distribution, given by

$$N_L(p_a, p_b | p) = f^2(s, M^2, t) \frac{\lambda(M^2)\sigma(M^2)}{\lambda(s)\sigma(s)}. \tag{2.7}$$

Since there is one and only one leading particle, N_L must satisfy the normalization conditions

$$\int dp\, N_L(p_a, p_b | p) = 1. \tag{2.8}$$

f is an arbitrary function of the three independent invariants that we can form from the momenta p_a, p_b, and p_i,

$$s = (p_a + p_b)^2, \quad M_i^2 = (p_a + p_b - p_i)^2, \quad t_i = (p_a - p_i)^2. \tag{2.9}$$

We also have $dp = d^3p/[(2\pi)^3 2E_p]$, and λ^{-1} is the flux factor which satisfies $\lambda(s) \cong s$ for large s.

We can also argue that, conversely, if *all* of the bootstrap equations (2.5) are satisfied, then the square of the T matrix must be a sum of factorizable terms, as in Eq. (2.2). We can always write the inclusive density as a sum over semi-inclusive densities corresponding to states of fixed final-state multiplicity. If we define $\alpha_n(s) = \sigma_n(s)/\sigma(s)$, we have

$$N_{(k)}(p_a, p_b | p_1 \cdots p_k) = \sum_{n=k}^{\infty} \alpha_n(s) N_{(k)}^{\{n\}}(p_a, p_b | p_1 \cdots p_k) \tag{2.10}$$

and, also,

$$N_{(k-1)}(p_a - p_i, p_b | p_1 \cdots p_{i-1}, p_{i+1} \cdots p_k) = \sum_{n=k-1}^{\infty} \alpha_n(M_i^2) N_{(k-1)}^{\{n\}}(p_a - p_i, p_b | p_1 \cdots p_{i-1}, p_{i+1} \cdots p_k). \tag{2.11}$$

Since states of definite final-state multiplicity correspond to physically distinct processes, any equality between inclusive quantities expressed in this way must be valid term by term.[15] In particular, consider the contribution to $N_{(k)}$ in Eq. (2.5) from the state with k particles in the final state. There will be no contribution from the integral term which receives contributions only from states with at least $k+1$ final particles. Hence, substituting Eq. (2.10) and (2.11) into Eq. (2.5) and identifying the contribution from the state with k final particles we have

$$\alpha_k(s) N_{(k)}^{\{k\}}(p_a, p_b | p_1 \cdots p_k) = \sum_i N_L(p_a, p_b | p_i) \alpha_{k-1}(M_i^2) N_{(k-1)}^{\{k-1\}}(p_a - p_i, p_b | p_i \cdots p_{i-1}, p_{i+1} \cdots p_k). \tag{2.12}$$

Equation (2.2), in fact, relates the k-particle *exclusive* cross section at s to the $(k-1)$-particle exclusive cross section at M_i^2. In terms of the T matrix, we have

$$N^{(k)}_{(k)}(p_a, p_b | p_1 \cdots p_k) = \frac{\frac{1}{2}(2\pi)^4}{\lambda(s)\sigma_k(s)} |\langle p_a, p_b | T | p_1 \cdots p_k \rangle|^2, \tag{2.13}$$

and a similar expression for $N^{(k-1)}_{(k-1)}$.

So, Eq. (2.12) becomes

$$|\langle p_a, p_b | T | p_1 \cdots p_k \rangle|^2 = \sum_{i=1}^{k} N_L(p_a, p_b | p_i) \frac{\lambda(s)\sigma(s)}{\lambda(M_i^2)\sigma(M_i^2)} |\langle p_a - p_i, p_b | T | p_1 \cdots p_{i-1}, p_{i+1} \cdots p_k \rangle|^2. \tag{2.14}$$

Equation (2.14) is identical to Eq. (2.2), with $f^2(s, M^2, t)$ given in terms of $N_L(p_a, p_b | p)$ by Eq. (2.7).

III. THE RESOLVENT KERNEL FOR THE CHEW-PIGNOTTI MODEL

Assuming, as the data indicates, that transverse momentum is limited and that Feynman scaling holds at high energies, we integrate over the transverse momentum to obtain the one-dimensional bootstrap equations. Equations (2.6) and (2.5) become, respectively,

$$f(x) = g(x) + \int_0^{1-x} \frac{dy}{y}\, g(y) f\left(\frac{x}{1-y}\right), \tag{3.1}$$

$$f_k(x_1 \cdots x_k) = \sum_{i=1}^{k} g(x_i) f_{k-1}\left[\frac{x_1}{1-x_i} \cdots \frac{x_{i-1}}{1-x_i}, \frac{x_{i+1}}{1-x_i} \cdots \frac{x_k}{1-x_i}\right] + \int_0^{1-\Sigma_i x_i} \frac{dy}{y}\, g(y) f_k\left[\frac{x_1}{1-y} \cdots \frac{x_k}{1-y}\right], \tag{3.2}$$

where x is the Feynman scaling variable, $x = 2p_{\parallel}/\sqrt{s}$, and

$$g(x) = \int \frac{d^2p_\perp}{2(2\pi)^3} N_L(p_a, p_b | p), \tag{3.3}$$

$$f_k(x_1 \cdots x_k) = \int \frac{d^2p_{1\perp}}{2(2\pi)^3} \cdots \times \frac{d^2p_{k\perp}}{2(2\pi)^3} N(p_a, p_b | p_1 \cdots p_k).$$

The normalization condition, Eq. (2.8), becomes

$$\int_0^1 \frac{dx}{x}\, g(x) = 1. \tag{3.4}$$

Since, in the bootstrap model, in the scaling limit, we have complete factorization between hemispheres, Eq. (3.2) is valid for $x_1 \cdots x_k > 0$. For the case $x_1 \cdots x_l > 0$, $x_{l+1} \cdots x_k < 0$, the distribution will be a product of terms, f_l and f_{k-l}, each of which satisfies an equation of the form of Eq. (3.2).

Equation (3.1) is an inhomogeneous Volterra equation of the second kind. It can be put into standard Volterra form by making the substitution $z = x/(1-y)$ and we obtain

$$f(x) = g(x) + \int_x^1 dz\, K(x, z) f(z), \tag{3.5}$$

with

$$K(x, z) = \frac{x}{z(z-x)} g\left(1 - \frac{x}{z}\right). \tag{3.6}$$

Iterating, we obtain the Neumann series,

$$f(x) = g(x) + \int_x^1 dz\, K(x, z) g(z) + \int_x^1 dz \int_z^1 dz' K(x, z) K(z, z') g(z) + \cdots. \tag{3.7}$$

If the kernel has a singularity at $z = 0$, as will be the case in the model we are interested in, the Neumann series will converge uniformly to a unique solution on any interval $\epsilon \leq x \leq 1$ that excludes a neighborhood of $x = 0$. The fact that the Neumann series will converge more rapidly in the region near $x = 1$ than in the region near $x = 0$ is consistent with the interpretation of the nth term in the series as the contribution to the inclusive distribution from diagrams where the observed particle is the nth particle on the multiperipheral chain.

Since we could always write a Volterra equation as a Fredholm equation with a discontinuous kernel by defining $\bar{K}(x, z) = K(x, z)\theta(z - x)$, techniques that are applicable to Fredholm equations could also be useful in the case of Eq. (3.5). In particular, another way to obtain the Neumann series is to define the resolvent kernel to be

$$R(x, y) = K(x, y) + \int_x^y dz\, K(x, z) K(z, y) + \int_x^y dz \int_z^y dz' K(x, z) K(z, z') K(z', y) + \cdots. \tag{3.8}$$

The Neumann series is then given by $g + Rg$, so

$$f(x) = g(x) + \int_x^1 R(x, y) g(y)\, dy \tag{3.9}$$

is the solution to Eq. (3.5). Equation (3.8) is itself the Neumann series for the integral equation for the resolvent which can be written as

$$R(x,y)=K(x,y)+\int_x^y dz\,K(x,z)R(z,y) \tag{3.10}$$

or equivalently

$$R(x,y)=K(x,y)+\int_x^y dz\,R(x,z)K(z,y). \tag{3.11}$$

We can write these equations in a somewhat different form:

$$\int_{x_-}^{y_+} dz[\,\delta(x-z)+R(x,z)][\,\delta(z-y)-K(z,y)]$$

$$=\int_{x_-}^{y_+} dz[\,\delta(x-z)-K(x,z)][\,\delta(z-y)+R(z,y)]$$

$$=\delta(x-y). \tag{3.12}$$

Equation (3.12), which we can write in operator notation as

$$(1+R)(1-K)=(1-K)(1+R)=1, \tag{3.13}$$

is a statement of the fact that the operator $[1+R]$ is the inverse of the operator $[1-K]$. We shall refer to Eq. (3.12) or, equivalently, Eq. (3.13) as the resolvent identities. In Eq. (3.12), which is just a convenient shorthand notation for Eqs. (3.10) and (3.11), we have taken the limits of integration to be

$$\int_{x_-}^{y_+} dz=\lim_{\epsilon\to 0}\int_{x-\epsilon}^{y+\epsilon} dz \tag{3.14}$$

so that the arguments of the δ functions are always within the range of integration.

We now wish to consider a simple single-trajectory multi-Regge model with f given by Eq. (2.3). We will also assume that the total cross section in the model is Regge-behaved at large energies,

$$\sigma(s)\cong\sigma_0 s^{\alpha'-1}, \tag{3.15}$$

where α' is the intercept of the output trajectory. Then, if α is an average value for the input trajectory in Eq. (2.3), from Eq. (2.7), in the neighborhood of $x=1$, $g(x)$ must behave like $g^2(1-x)^\beta$, with g^2 being the coupling constant for the model and

$$\beta=\alpha'-2\alpha. \tag{3.16}$$

The behavior of $g(x)$ near $x=0$ is determined by the behavior of $\Gamma(t)$ which cuts off the amplitude when t becomes large. We must have $g(0)=0$ so that the normalization condition, Eq. (3.4), can be satisfied. A particularly simple choice for $g(x)$ which has the correct Regge behavior near $x=1$, which satisfies the normalization condition, and which has been shown by Jengo, Krzywicki, and Petersson[5] to reproduce the Poisson distribution for the multiplicity of the Chew-Pignotti model is

$$g_\beta(x)=(\beta+1)x(1-x)^\beta. \tag{3.17}$$

Since the coupling constant is then just

$$g^2=(\beta+1), \tag{3.18}$$

Eq. (3.16) becomes

$$\alpha'=2\alpha+g^2-1, \tag{3.19}$$

which is just the Chew-Pignotti relation for the output trajectory.

The kernel for this choice of $g(x)$ is then

$$K_\beta(x,y)=(\beta+1)\frac{x^{\beta+1}}{y^{\beta+2}}. \tag{3.20}$$

The Neumann series for the resolvent kernel, Eq. (3.8), becomes

$$R_\beta(x,y)=(\beta+1)\frac{x^{\beta+1}}{y^{\beta+2}}\sum_{n=0}^{\infty}\frac{[(\beta+1)\ln(y/x)]^n}{n!} \tag{3.21}$$

and, recognizing the power series for an exponential, we immediately have

$$R_\beta(x,y)=\frac{\beta+1}{y}. \tag{3.22}$$

Equation (3.9) then gives

$$f_\beta(x)=(\beta+1)(1-x)^\beta. \tag{3.23}$$

This is the solution that we found in paper I by trial and error. Since $f_\beta(0)=(\beta+1)=g^2$, we obtain another classic Chew-Pignotti result,

$$\langle n\rangle=g^2\ln(s). \tag{3.24}$$

Hence, since this model reproduces all the results of the Chew-Pignotti model wherever these models can be directly compared, we must conclude that this model *is* the Chew-Pignotti model in a somewhat unfamiliar form.

For the multiparticle distributions, the solution of Eq. (3.2) for the choice of g given by Eq. (3.17) is just

$$f_k(x_1\cdots x_k)=(\beta+1)^k\left(1-\sum_{i=1}^{k}x_i\right)^\beta. \tag{3.25}$$

We have just shown that Eq. (3.25) holds for $k=1$. Let us assume that it holds for $k-1$ and assume that f_k is a function only of the variable X,

$$X=\sum_{i=1}^{k}x_i. \tag{3.26}$$

It is then easy to see that Eq. (3.2) becomes

$$f_k(X) = (\beta+1)^{k-1} g_\beta(X) + \int_0^{1-X} \frac{dy}{y}\, g_\beta(y) f_k\left(\frac{X}{1-y}\right). \tag{3.27}$$

However, this equation is exactly the same as Eq. (3.1) except for the coefficient of the inhomogeneous term. The solution is obviously Eq. (3.25).

If $\beta > 0$, $f(x)$ will have a maximum at $x = 0$ and will vanish at $x = 1$, while $f_k(x_1 \cdots x_k)$ will have a maximum at the center and will vanish at the phase-space boundary. We have called such behavior "pionizationlike." Similarly, if $\beta < 0$, $f(x)$ has a minimum at $x = 0$ and diverges at $x = 1$, and $f_k(x_1 \cdots x_k)$ has a minimum at the center and diverges at the phase-space boundary. We have called this behavior "diffractionlike."

In the first quadrant of the Van Hove hexagon, where both particles are in the forward hemisphere and the two-particle distribution is nonvanishing, the two-particle correlation function has the form

$$c_2(x_1, x_2) = (\beta+1)^2 \times [(1-x_1-x_2)^\beta - (1-x_1)^\beta (1-x_2)^\beta]. \tag{3.28}$$

For $\beta > 0$ the correlation function is negative throughout this region and for $\beta < 0$ it is positive throughout the region and diverges at the phase-space boundary. Similar remarks are valid for the third quadrant, where both particles are in the backward hemisphere. In the second and fourth quadrants, factorization of the amplitude between hemispheres requires that the correlation function vanish identically. In the case where $\beta = 0$, all of the multiparticle amplitudes are constant and all of the correlation functions vanish identically.

We have previously defined a set of "kinematically corrected" correlation functions[16] which vanish identically in the Fermi model and uncorrelated jet model and should therefore be a measure of dynamical correlations free of kinematical effects. These "corrected" correlation functions $\tilde{c}$ also have a simple form in the above model:

$$\tilde{c}(x_1, x_2) = (\beta+1)^2 [(1-x_1-x_2)^{-2\alpha} - (1-x_1)^{-2\alpha}(1-x_2)^{-2\alpha}]. \tag{3.29}$$

Thus, as long as the intercept of the input trajectory is positive, the *dynamical* correlations will be positive. In the case where $\alpha = 0$, which would correspond to the uncorrelated jet model, of course, $\tilde{c} = 0$.

It is trivial to verify that the set of f_k, given by Eq. (3.25), satisfies the energy-momentum conservation sum rules and is, in fact, the simplest set of such functions that we can construct which does so. The analog of the energy sum rule for g determines the average fraction of the projectile energy carried away by the leading particle in the c.m. frame,

$$\left\langle \frac{E_L}{E_a} \right\rangle = \int_0^1 dy\, g(y) = \frac{1}{\beta+2}. \tag{3.30}$$

For $\beta = 0$ the leading particle carries off, on the average, $\frac{1}{2}$ of the projectile energy.

If we consider Eq. (3.2) with arbitrary $g(x)$, then it would also be advantageous to use the variable X, Eq. (3.26). We need $n-1$ other independent variables which we can take to be $\bar{x}_i = x_i/X$, which, of course, satisfies $\sum_i \bar{x}_i = 1$. Equation (3.2) will then reduce to a one-dimensional integral equation in the variable X. The equation will have the same kernel as Eq. (3.1) and, hence, the same resolvent. The inhomogeneous term will, in general, be a complicated function of X and the parameters $\bar{x}_i$, so that the resulting integrals will have to be done numerically. There are, however, two special cases where the two-particle distribution has a simple form independent of the choice of $g(x)$. Suppose that one of the particles is produced at $x = 0$. Since $g(0) = 0$, we get

$$f_2(0, x) = f_1(0) g(x) + \int_0^{1-x} \frac{dy}{y}\, g(y) f_2\left(0, \frac{x}{1-y}\right), \tag{3.31}$$

which is the same as Eq. (3.1) except that the inhomogeneous term is multiplied by $f_1(0)$. The solution is thus

$$f_2(0, x) = f_1(0) f_1(x). \tag{3.32}$$

There are thus no correlations between particles in the central plateau and particles in the fragmentation regions. This is a consequence of the fact that the bootstrap model in the scaling limit is a short-range correlation model. On the other hand, if we are at the phase-space boundary, $x_2 = 1 - x_1$, the integral term vanishes and we have

$$f_2(x_1, 1-x_1) = [g(x_1) + g(1-x_1)] f(1). \tag{3.33}$$

IV. RESOLVENT-IDENTITY PERTURBATION THEORY AND THE MULTIFIREBALL EXPANSION

Let us consider the case where the leading-particle distribution has the form

$$g(x) = g_0(x) + \epsilon g'(x), \tag{4.1}$$

where $g_0(x)$ is a leading-particle distribution, such as that considered in the previous section, for which we can obtain an exact solution. We assume that $\epsilon \ll 1$ so we will seek a solution in the form of a power series expansion in ϵ. Since g_0 satisfies the normalization condition, we must have

$$\int_0^1 \frac{dx}{x} g'(x) = 0 \tag{4.2}$$

so as to insure that the normalization condition is satisfied exactly and the model will produce physically meaningful results.

We can now use the resolvent-identity perturbation technique previously applied by the author[17] to the Faddeev equations. Writing $g = g_0 + \Delta g$, we can take the kernel to be $K_0 + \Delta K$ and let us call the resolvent R. The resolvent identity will then have the form

$$(1 - K_0 - \Delta K)(1 + R) = 1 \tag{4.3}$$

which we can rewrite as

$$(1 - K_0)(1 + R) = 1 + \Delta K(1 + R). \tag{4.4}$$

We can then multiply on the left by $(1 + R_0)$, and use the resolvent identities for the unperturbed problem to obtain

$$R = R_0 + (1 + R_0)\Delta K(1 + R). \tag{4.5}$$

Equation (4.5) is an integral equation for the resolvent which we can iterate to obtain the Neumann series,

$$\begin{aligned} R = {} & R_0 + (1 + R_0)\Delta K(1 + R_0) \\ & + (1 + R_0)\Delta K(1 + R_0)\Delta K(1 + R_0) + \cdots . \end{aligned} \tag{4.6}$$

Each term in the series is correct to all orders in g_0 and to a given order in Δg and hence in ϵ. The solution is given by $f = (1 + R)g$ and, using the fact that $f_0 = (1 + R_0)g_0$, we obtain, to first order,

$$f = f_0 + (1 + R_0)\Delta g + (1 + R_0)\Delta K f_0. \tag{4.7}$$

We know that both diffraction and pionization are present in the real world and, hence, that any realistic model must contain at least both a Pomeron and an ordinary trajectory. If we assume that the Pomeron is an isolated factorizable pole with intercept exactly unity, then the Finkelstein-Kajantie theorem[18] would require that there be no more than one Pomeron on a multiperipheral chain to avoid violation of the Froissart bound. This philosophy leads naturally to a two-component model where the pionization component arises from diagrams that contain no Pomerons and the diffractive component arises from diagrams having one Pomeron. An alternative philosophy that avoids the Finkelstein-Kajantie problem and associated decoupling theorems has recently been emphasized by Chew, Bishari, and Koplik.[13] One assumes that there is a "bare Pomeron" with intercept close to, but still less than, unity which can be exchanged an arbitrary number of times. Because the Pomeron coupling constant is much less than 1, the inclusive cross section can be expanded in powers of this parameter. The zeroth-order term gives the pionization component and the first-order term gives the diffractive component. Higher-order terms do not seem to be important, even at CERN-ISR energies. In what follows, we shall emphasize the bare-Pomeron approach simply because it is easier to handle, although a Pomeron with intercept exactly at 1 presents no essential difficulty in principle if one is careful to take into account threshold effects in the neighborhood of $x = 1$.

Let us consider a model that contains both an ordinary trajectory $\alpha(t)$ and a bare Pomeron $\alpha_P(t)$, both of which can be exchanged an arbitrary number of times. The T matrix will satisfy Eq. (2.2) and, if we assume that the total cross section is asymptotically constant in this model, instead of Eq. (2.3), we will have

$$f(s, M^2, t) = \Gamma_\alpha(t)(s/M^2)^{\alpha(t)} + \Gamma_P(t)(s/M^2)^{\alpha_P(t)}. \tag{4.8}$$

The bootstrap equations, (2.5) and (2.6), will then follow with the leading-particle distribution given by

$$\begin{aligned} & N = \bar{N}_{L\alpha} + N_{LP}, \\ & \bar{N}_{L\alpha}(p_a, p_b | p) = \Gamma_\alpha^2(t)(s/M^2)^{2\alpha(t)-1}, \\ & N_{LP}(p_a, p_b | p) = \Gamma_P^2(t)(s/M^2)^{2\alpha_P(t)-1}. \end{aligned} \tag{4.9}$$

If we were to turn off the bare-Pomeron coupling, we would have a one-trajectory model of the type discussed in the previous section. This model would describe the pionization component and we can assume that its cross section $\sigma_\alpha(s)$ falls asymptotically with s like $s^{-\epsilon}$. From Eq. (2.7) it is clear that the leading-particle distribution in this model would be

$$N_{L\alpha}(p_a, p_b | p) = \Gamma_\alpha^2(t)(s/M^2)^{2\alpha(t)+\epsilon-1}. \tag{4.10}$$

Since the normalization condition must be satisfied in both models, we have

$$\begin{aligned} \int dp(\bar{N}_{L\alpha} + N_{LP}) = {} & \int dp\, N_{L\alpha} \\ = {} & 1. \end{aligned} \tag{4.11}$$

If we were to set $\epsilon = 0$, we would have $N_{L\alpha} = \bar{N}_{L\alpha}$ and, since the inclusive distributions are positive-definite quantities, the only way that Eq. (4.11)

could be satisfied would be if N_{LP} were identically zero and there were no diffraction. Hence, both the diffractive and nondiffractive cross sections cannot be asymptotically constant in this type of model.

When we integrate over transverse momentum and make the same assumptions about the behavior of the cutoff function as in Sec. III, Eq. (4.10) becomes

$$g_\alpha(x) = (\beta + 1 - \epsilon)\, x(1-x)^{\beta - \epsilon}, \tag{4.12}$$

with α being an average value for the trajectory and $\beta = 1 - 2\alpha$. The coupling constant g_α^2 is thus determined. Integrating Eq. (4.9) over transverse momentum, we obtain, for the two-component model,

$$g(x) = (\beta + 1 - \epsilon)x(1-x)^{\beta} + g_P{}^2 x(1-x)^{1-2\alpha_P}. \tag{4.13}$$

We can set $\alpha_P = 1 - \delta$ and use the normalization condition for the two-component model to obtain $g_P{}^2$ and we have

$$\begin{aligned} g(x) &= g_\beta(x) + \Delta g(x), \\ \Delta g(x) &= -\epsilon x(1-x)^{\beta} + \frac{2\epsilon\delta}{\beta+1}\, x(1-x)^{-1+2\delta}. \end{aligned} \tag{4.14}$$

We assume that the parameter ϵ which determines the relative strengths of the coupling of the P and α trajectories is small, so that a perturbation expansion in powers of ϵ would be meaningful. Δg has been specifically constructed so that Eq. (4.2) is satisfied and hence contains a renormalization term as well as a Pomeron term.

In the simplest case, $\alpha = \frac{1}{2}$, $\beta = 0$, $g_0(x) = x$, $f_0(x) = 1$, it is a trivial matter to evaluate the first-order term in the multifireball expansion, Eq. (4.7), with Δg given by Eq. (4.14)

$$\begin{aligned} f(x) &= (1-\epsilon) + 2\epsilon\delta x(1-x)^{-1+2\delta} + \epsilon(1-x)^{2\delta} \\ &\quad + \epsilon\left(1 - \frac{1}{2\delta}\right)(1 - x^{2\delta}). \end{aligned} \tag{4.15}$$

It is easy to verify that, to first order in ϵ, this is indeed the solution of the bootstrap equation and that it satisfies the energy-momentum conservation sum rule.

The first term in Eq. (4.15) is just $(1-\epsilon)f_0(x)$ and the second term is $g_P(x)$ which dominates near $x = 1$. Since both ϵ and δ are small, the third term will be both small and relatively flat and, hence, can be considered negligible in comparison to the first term. It is the last term that creates problems. For small δ, $(1 - x^{2\delta})$ will be appreciably different from zero only in the neighborhood of $x = 0$, and the coefficient of this term will be negative. It thus tends to produce a "hole" in the distribution at $x = 0$, and if $\delta < \epsilon$ it can even lead to an $f(0)$ which is negative, which is absurd. Thus, the multifireball expansion gives reasonable results in the neighborhood of $x = 0$ only if $\epsilon \ll \delta$.

The perturbation technique is applicable whatever the shape of the diffractive term. We can allow a $(1-x)^{-1}$ behavior if we note that this behavior is valid only for $x < x_0$, $x_0 = 1 - M_0{}^2/s$. $M_0{}^2$ is a fixed mass above which the Pomeron-particle cross section is constant and we have high mass diffraction. For $x > x_0$, the diffractive term will depend on the assumptions made concerning the behavior of the Pomeron-particle cross section for small M^2, but the inclusive cross section must vanish as $x \to 1$ and the phase space available for the reaction vanishes. For simplicity we will neglect this low mass diffraction and consider a model with only high mass diffraction. For simplicity we shall also allow the Pomeron to be exchanged an arbitrary number of times. The first-order term in the multifireball expansion would correspond to the exact solution in a model where the Pomeron could be exchanged only once. We shall therefore assume that the leading-particle inclusive cross section has the form

$$\frac{d\hat{\sigma}_L}{dx} = \sigma_0 x[\,1 + \epsilon(1-x)^{-1}\theta(x_0 - x)]. \tag{4.16}$$

Again, for simplicity, we have taken the ordinary trajectory as having $\alpha = \frac{1}{2}$, $\beta = 0$. The normalization condition then gives the total cross section for the model,

$$\sigma = \sigma_0[\,1 + \epsilon \ln(1-x_0)^{-1}]. \tag{4.17}$$

So, to first order in ϵ, we can write

$$\begin{aligned} g(x) &= x + \Delta g(x), \\ \Delta g &= -\epsilon x \ln(1-x_0)^{-1} + \epsilon x(1-x)^{-1}\theta(x_0 - x). \end{aligned} \tag{4.18}$$

We can then substitute this expression for Δg in Eq. (4.7) and obtain

$$f(x) = 1 + \Delta f(x). \tag{4.19}$$

Because of the discontinuity of Δg, Δf will, in fact, consist of a sum of three terms which are nonvanishing in separate, overlapping regions of x, which we can write as

$$\Delta f(x) = \begin{cases} \epsilon x(1-x)^{-1} + \epsilon \ln\left(\dfrac{1-x}{1-x_0}\right), & 0 < x < x_0 \\ +\epsilon \ln(1-x_0) - \epsilon \ln x - \epsilon \ln(1-x_0)\ln x + \frac{1}{2}\epsilon \ln^2 x, & 1 - x_0 < x < 1 \\ -\frac{1}{2}\epsilon \ln^2(1-x_0), & 0 < x < 1 - x_0. \end{cases} \tag{4.20}$$

We should note that Δf is continuous at $x = 1 - x_0$, and the term $\epsilon x(1-x)^{-1}\theta(x_0 - x)$ dominates the amplitude for x near 1. We should also note that the solution will suffer from the same problems as in the bare-Pomeron case. In particular, as $s \to \infty$ and $x_0 \to 1$, the last term in Eq. (4.20) will become large and negative. The first term in the multifireball expansion will be a good approximation over the whole range of x only of $\epsilon \ln^2(s/M_0^2) \ll 1$, which is to say, only for those energies for which the diffractive cross section can be considered small compared to the total cross section.

If we turn on an additional interaction with strength ϵ, both the inclusive cross sections and the total cross sections must increase as ϵ increases. Hence if we expand the inclusive distribution $N = (1/\sigma)d\hat{\sigma}/dp$ in a power series in ϵ, the series will contain both positive and negative terms and there is no guarantee that the result will be positive-definite if we cut off the series after a finite number of terms. Thus as we have seen, the first order term in the expansion for $f(x)$ could be negative, near $x = 0$, for some values of the parameters. This problem could be avoided by expanding the inclusive *cross section* in powers of ϵ (in this case we could identify the nth term in the expansion with the contribution from diagrams containing n Pomerons) and normalizing with the total cross section, calculated to the same order in ϵ, at each stage. Such a procedure would be cumbersome in the context of the bootstrap model, and we indicated in paper I what form the first-order equations in such a model should take.

We should emphasize that for small ϵ the results are perfectly reasonable. We expect that the inclusion of diffraction in the model should add a small negative contribution to the inclusive distribution at $x = 0$, as the presence of diffraction should decrease the average multiplicity.

V. AN EXACT SOLUTION

The Chew-Pignotti model with an ordinary trajectory and a bare Pomeron is just a special case of a two-trajectory Chew-Pignotti model. We now will see that it is possible to obtain an analytic solution in this case. Let us assume we have two trajectories with average values α and α' and corresponding parameters $\beta = 1 - 2\alpha$, $\beta' = 1 - 2\alpha'$. The leading particle distribution will then have the form

$$g(x) = a g_\beta(x) + (1-a) g_{\beta'}(x) \tag{5.1}$$

where g_β and $g_{\beta'}$ have the form (3.17). The parameter a, $0 \leq a \leq 1$, determines the relative strengths of the couplings of the β and β' trajectories and it is obvious that the normalization condition is satisfied. The kernel of the equation will have the form

$$K(x,y) = aK_\beta(x,y) + (1-a)K_{\beta'}(x,y), \tag{5.2}$$

where, referring to Eq. (3.20), we can write

$$\begin{aligned} K_\beta(x,y) &= (\beta+1)\bar{K}_\beta(x,y), \\ \bar{K}_\beta(x,y) &= x^{\beta+1}/y^{\beta+2}. \end{aligned} \tag{5.3}$$

For $\beta \neq \beta'$, a useful relation is

$$\int_x^y \bar{K}_\beta(x,z)\bar{K}_{\beta'}(z,y)\,dz = (\beta - \beta')^{-1}[\bar{K}_{\beta'}(x,y) - \bar{K}_\beta(x,y)] \tag{5.4}$$

or, schematically,

$$\begin{aligned} \bar{K}_\beta \bar{K}_{\beta'} &= \bar{K}_{\beta'}\bar{K}_\beta \\ &= (\beta - \beta')^{-1}(\bar{K}_{\beta'} - \bar{K}_\beta). \end{aligned} \tag{5.5}$$

In the single-trajectory problem, the solution to Eq. (3.11) had the form $R_\beta = (\beta+1)\bar{K}_{-1}$. Hence, in the two-trajectory case, let us seek a solution of the form

$$R = c\bar{K}_{-1} + d\bar{K}_\gamma, \tag{5.6}$$

where c, d, and γ are parameters that we must now determine. We can then write Eq. (3.11) as

$$\begin{aligned} c\bar{K}_{-1} + d\bar{K}_\gamma = {} & a(\beta+1)\bar{K}_\beta + (1-a)(\beta'+1)\bar{K}_{\beta'} \\ & + (c\bar{K}_{-1} + d\bar{K}_\gamma) \\ & \times [a(\beta+1)\bar{K}_\beta + (1-a)(\beta'+1)\bar{K}_{\beta'}]. \end{aligned} \tag{5.7}$$

It is clear that the last term will also be a linear combination of $\bar{K}_{-1}$, $\bar{K}_\beta$, $\bar{K}_{\beta'}$, and $\bar{K}_\gamma$. Since these are linearly independent functions of x and y, the coefficients of each must be equated separately. Using Eq. (5.5), we obtain for the coefficients of $\bar{K}_{-1}$ the identity $c = ca + c(1-a)$. From the coefficients of $\bar{K}_\beta$, $\bar{K}_{\beta'}$, and $\bar{K}_\gamma$, we obtain respectively,

$$\frac{c}{\beta+1} + \frac{d}{\beta-\gamma} = 1, \tag{5.8}$$

$$\frac{c}{\beta'+1} + \frac{d}{\beta'-\gamma} = 1, \tag{5.9}$$

$$\frac{a(\beta+1)}{\beta-\gamma} + \frac{(1-a)(\beta'+1)}{\beta'-\gamma} = 1. \tag{5.10}$$

Equation (5.10) is a quadratic equation for γ. One solution is $\gamma = -1$, but this solution obviously leads to an inconsistent set of equations for c and

d. The other solution is

$$\gamma = a\beta' + (1-a)\beta. \tag{5.11}$$

We can then substitute this value of γ into Eqs. (5.8) and (5.9), solve the resulting set of simultaneous equations, and obtain

$$d = \frac{a(1-a)(\beta-\beta')^2}{(1-a)\beta + a\beta' + 1},$$
$$c = \frac{(\beta+1)(\beta'+1)}{(1-a)\beta + a\beta' + 1}. \tag{5.12}$$

Having found the resolvent, we can now obtain the solution $f = [1+R]g$,

$$f(x) = a(\beta+1)x(1-x)^{\beta} + (1-a)(\beta'+1)x(1-x)^{\beta'} + ac(1-x)^{\beta+1} + (1-a)c(1-x)^{\beta'+1}$$
$$+ ad(\beta+1)[B(-\gamma, \beta+1) - B_x(-\gamma, \beta+1)]x^{\gamma+1} + (1-a)d(\beta'+1)[B(-\gamma, \beta'+1) - B_x(-\gamma, \beta'+1)]x^{\gamma+1}, \tag{5.13}$$

where $B(-\gamma, \beta+1)$ is the beta function and $B_x(-\gamma, \beta+1)$ is the incomplete beta function. Since the incomplete beta function will behave[19] like $-x^{-\gamma}/\gamma$ as $x \to 0$, it is clear that only the second and third terms of Eq. (5.13) contribute to $f(0)$ and we simply have

$$f(0) = c, \tag{5.14}$$

which is both finite and positive-definite for all possible values of β, β' and a.

In the double-Regge region, $x \cong 0_+$, the distribution should have the form[2]

$$f(x) \cong \sum_i g_i x^{1-\tilde{\alpha}_i}, \tag{5.15}$$

where we can consider the $\tilde{\alpha}_i$ trajectories with coupling constants g_i to be the output trajectories of the model which, of course, are not necessarily identical to the input trajectories. As long as $f(0) \neq 0$ we will always get a Pomeron with $\tilde{\alpha} = 1$. In the one-component model discussed in Sec. III, expanding $(1-x)^{\beta}$ in a power series about $x = 0$ we can see that we also obtain contributions from an infinite number of Pomeron daughters. In the two-trajectory model, we can expand each term in Eq. (5.13) in a power series about $x = 0$. We can see that, in addition to the contributions from the Pomeron and its daughters, we will have additional contributions from a trajectory with

$$\tilde{\alpha} = -\gamma. \tag{5.16}$$

In order to compare the results of this section with that of Sec. IV, we take

$$a = \epsilon, \quad \beta = (-1+2\delta), \quad \beta' = 0, \tag{5.17}$$

and we have

$$\gamma = -(1-\epsilon)(1-2\delta),$$
$$c = \frac{2\delta}{\epsilon + 2\delta - 2\epsilon\delta}, \tag{5.18}$$
$$d = \frac{\epsilon(1-\epsilon)(1-2\delta)^2}{\epsilon + 2\delta - 2\epsilon\delta}.$$

The first thing to notice is that the combination of the ordinary trajectory and the bare Pomeron can produce reasonable double-Regge behavior in the central region. In particular, if we take $(1-\epsilon)(1-2\delta) = \frac{1}{2}$, the leading nonPomeron trajectory in the double-Regge region will have $\tilde{\alpha} = \frac{1}{2}$. In this case, of course, both ϵ and δ cannot be small.

If, on the other hand, we assume that ϵ, $\delta \ll 1$ and, in addition, that $\epsilon \ll \delta$, we will have $c \cong 1$, $d \cong 0$, the resolvent will be approximately equal to the unperturbed resolvent, and an expansion of the resolvent in powers of ϵ will converge. It is clear that under these assumptions Eq. (4.15) will result, but it is only correct to first order in (ϵ/δ). We, of course, have no exact solution to compare with the second example of Sec. IV. However, it is reasonable to conclude that in this case, also, the pathological behavior in the neighborhood of $x = 0$, for $\epsilon \ln^2(s/M_0^2)$ large, arises from the perturbation expansion and not from the model itself.

We can now speculate about the solution of the n-trajectory Chew-Pignotti model. Suppose we take

$$g(\lambda) = \sum_{i=1}^{n} a_i g_{\beta_i}(x) \tag{5.19}$$

and, to satisfy the normalization condition,

$$\sum_{i=1}^{n} a_i = 1. \tag{5.20}$$

The kernel is then

$$K = \sum_{i=1}^{n} a_i(\beta_i + 1)\bar{K}_{\beta_i}. \tag{5.21}$$

In analogy with Eq. (5.6), let us seek a solution of the form

$$R = c\bar{K}_{-1} + \sum_{i=1}^{n-1} d_i \bar{K}_{\gamma_i}. \tag{5.22}$$

We then substitute Eqs.(5.21) and (5.22) into Eq. (3.11). Using Eq. (5.5) and equating the coefficients of $\bar{K}_{-1}$, $\bar{K}_{\gamma i}$, and $\bar{K}_{\beta i}$, respectively, we obtain

$$c\left(1-\sum_i a_i\right)=0, \tag{5.23}$$

$$\sum_{i=1}^{n}\frac{a_i(\beta_i+1)}{\beta_i-\gamma_j}=1, \tag{5.24}$$

$$\frac{c}{\beta_i+1}+\sum_{j=1}^{n-1}\frac{d_j}{\beta_i-\gamma_j}=1, \tag{5.25}$$

in analogy with Eqs.(5.8)–(5.10). Equation (5.23) is automatically satisfied because of the normalization condition, (5.20). Equation (5.24) is an nth degree equation for γ_j. Because of the normalization condition, $\gamma_j=-1$ is a solution but it will produce an inconsistent set of equations for c and the d_j's. There are, however, $n-1$ other solutions. Substituting the $n-1$ values for γ_j into Eq. (5.25) we would have n simultaneous equations for the n parameters, c, d_j, $j=1,\ldots,n-1$. The solution will then have the form

$$f(x)=\sum_{i=1}^{n}a_i g_{\beta i}(x)+\sum_{i=1}^{n}ca_i(1-x)^{\beta_i+1}+\sum_{i=1}^{n}\sum_{j=1}^{n-1}a_i d_j(\beta_i+1)x^{\gamma_j+1}[B(-\gamma_j,\beta_i+1)-B_x(-\gamma_j,\beta_i+1)]. \tag{5.26}$$

In the double-Regge region, the leading output trajectories will be the Pomeron with $\alpha=1$ and the $n-1$ trajectories with

$$\tilde{\alpha}_i=-\gamma_i. \tag{5.27}$$

We are thus lead to the possibility of using the bootstrap model to effect a multi-Regge bootstrap (the multi-Regge bootstrap bootstrap?). If the input trajectories $\alpha_i=\frac{1}{2}(1-\beta_i)$, $i=1,\ldots,n-1$, are ordinary trajectories and $\alpha_n=\frac{1}{2}(1-\beta_n)=1-\delta$ is the bare Pomeron, then the bootstrap condition would require that the input and output trajectories be identical,

$$\tilde{\alpha}_i=\alpha_i,\quad i=1,\ldots,n-1. \tag{5.28}$$

We have seen in the two-trajectory case that we can have $\tilde{\alpha}=\alpha=\frac{1}{2}$ if the bare-Pomeron parameters have the appropriate values and the bootstrap can probably be effected in the general case.

Having obtained the resolvent and the single-particle distribution, it is possible to obtain the multiparticle distributions and correlation functions. However, without evaluating the integrals numerically, it is possible to make one statement concerning correlations in the presence of diffraction. Consider the two-particle correlation function in the scaling limit for both x_1 and $x_2>0$,

$$c_2(x_1,x_2)=f_2(x_1,x_2)-f_1(x_1)f_1(x_2). \tag{5.29}$$

If the model contains a bare Pomeron, $f_1(x)$ will diverge at $x=1$. From Eq. (3.33), it is clear that $f_2(x_1,x_2)$ will diverge at the line $x_1+x_2=1$, which is the phase-space boundary for the two particle production process. Since the product of single-particle distributions remains finite along this line, there should be large positive correlations in this region. This effect arises from diagrams where the second trajectory on the multiperipheral chain is the Pomeron. If the second particle on the multiperipheral chain is produced diffractively, it will tend to retain most of the remaining longitudinal momentum of the projectile, thus producing the large enhancement in the neighborhood of the boundary. Were this effect to be observed, it could be regarded as the signature for this process.

VI. DISCUSSION

In Sec. II we saw that if the square of the T matrix is a sum over all permutations of the outgoing momenta of q factorizable terms, then the bootstrap equations are satisfied, and, conversely, if all the bootstrap equations are satisfied, then the T matrix must be a sum of such q factorizable terms. Since this is the fundamental assumption of multiperipheralism, we can conclude that the bootstrap picture and multiperipheral picture are equivalent. This same conclusion is indicated by the results of Jengo, Krzywicki, and Petersson[5] for the multiplicity distributions and by the fact that in the bootstrap model in the scaling limit there are no correlations between opposite hemispheres or between either hemisphere and the point $x=0$. In the bootstrap model, however, all of the multiparticle distributions and the generating function for the multiplicity moments are determined by a set of integral equations once the leading particle distribution is given. The bootstrap model can be made to correspond to any particular multiperipheral model, such as the ABFST (Amati-Bertocci-Fubini-Stanghellini-Tonin) model or the Chew-Pignotti model, by the appropriate choice of the leading-particle distribution. The fact that the strong-ordering approximation is often made on conventional treatments of multiperipheral

models but cannot be made in the bootstrap model is due, must likely, to the fact that this may be a reasonable approximation for particles on the central plateau as a function of y, but not for particles in the fragmentation regions as a function of x.

In Sec. III we applied the resolvent-kernel technique to the one-dimensional bootstrap equations. With an appropriate choice of variables, all of the bootstrap equations for the multiparticle distributions can be represented as one-dimensional Volterra equations with the same kernel and, hence, with the same resolvent. We obtained the resolvent kernel, single-particle distribution, and multiparticle distributions for the leading-particle distribution corresponding to the single-trajectory Chew-Pignotti model and discussed the properties of the multiparticle distributions and correlation functions in this case.

In Sec. IV we used the resolvent identities to obtain a perturbation expansion for the inclusive distribution in the case where a small perturbation is added to a leading-particle distribution for which the solution is known. Treating the diffractive contribution to the leading-particle distribution as such a perturbation, we evaluated the first term in the multifireball expansion for both a bare Pomeron and a Pomeron with intercept 1. In both cases this term was well behaved in the neighborhood of $x=1$ but became large and negative in the neighborhood of $x=0$ when the Pomeron coupling constant became large.

We obtained an exact solution for the single-particle distribution in the two-trajectory Chew-Pignotti bootstrap model. We saw that a model with a bare Pomeron and ordinary trajectory could exhibit correct double-Regge behavior near $x=0$, indicating that the pathological behavior in this region, found in Sec. V, is the result of the convergence properties of the multifireball expansion and not of the model itself. We saw how a solution could be obtained for the n-trajectory model, which would have n output trajectories in the double-Regge region. This result is consistent with the observation, made by several authors[20] using conventional methods, that the n-trajectory multiperipheral model and the Mueller-Regge model are equivalent.

We have seen that it is possible to construct realistic bootstrap models, containing both pionization and diffraction, and to calculate the inclusive distributions and multiplicity moments in these models. Such models will have correct triple-Regge behavior by construction and can also exhibit reasonable double-Regge behavior. In a bootstrap model for the pionization component alone, double-Regge bahavior would have to be imposed by hand.[2,7] While in the Mueller-Regge model, the double-Regge region and the triple-Regge region must be considered as distinct limits, the bootstrap model provides a smooth interpolation between them.

The bootstrap equations could provide a useful phenomenological tool for the analysis of multiparticle distributions and correlation functions, in the fragmentation regions, now becoming available at Fermilab and the CERN ISR. Detailed comparison with the data would, however, necessitate further refinements of the model; in particular, the addition of low-mass diffraction in a realistic way and the use of the multichannel formalism of Kronenfeld and Peccei.[3]

Note added in proof. After the manuscript was submitted, the author became aware of a paper [J. DeBrion, C. Bromberg, T. Ferbel, P. Slattery, J. Cooper, A. Seidel, and J. Vander Velde, Phys. Lett. 52B, 477 (1974)] which presents a direct experimental confirmation of the bootstrap hypothesis. The above authors compared the characteristics of the remaining fireball in the process $p+p \to p +$ "anything" at 102 GeV/c to those of the whole event in $\gamma + p \to$ "anything" at $s = M^2$. They found striking similarities between the two processes, not only in the multiplicity moments, but also in the inclusive π^- distributions. (The author is grateful to Dr. Hannu Miettinen for bringing this paper to his attention and to Professor T. Ferbel for a short discussion.)

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[1]A. Krzywicki and B. Petersson, Phys. Rev. D 6, 924 (1972).

[2]J. Finkelstein and R. D. Peccei, Phys. Rev. D 6, 2606 (1972).

[3]R. Kronenfeld and R. D. Peccei, Phys. Rev. D 7, 1556 (1973).

[4]J. Finkelstein, Phys. Rev. D 8, 977 (1973).

[5]R. Jengo, A. Krzywicki, and B. Petersson, Nucl. Phys. B65, 319 (1973).

[6]R. J. Yaes, Phys. Rev. D 10, 941 (1974).

[7]A. Morel and B. Petersson, Nucl. Phys. B91, 109 (1975).
[8]J. Whitmore, Phys. Rep. 10C, 273 (1974); J. Whitmore and M. Derrick, Phys. Lett. 50B, 280 (1974).
[9]C. F. Chan and F. C. Winkelmann, Phys. Rev. D 10, 3645 (1974).
[10]G. Otter *et al.*, Nucl. Phys. B87, 1 (1975) (the author is grateful to Dr. J. Kasman for bringing this paper to his attention).
[11]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971); K. J. Biebl and J. Wolf, Phys. Lett. 37B, 197 (1971).
[12]G. F. Chew and A. Pignotti, Phys. Rev. 176, 2112 (1968).
[13]G. F. Chew, in *High Energy Collision—1973*, proceedings of the Fifth International Conference, Stony Brook, N. Y., edited by C. Quigg (A. I. P., N. Y., 1973), p. 98; M. Bishari, G. F. Chew, and J. Koplik, Nucl. Phys. B72, 61 (1974); M. Bishari and J. Koplik, Nucl. Phys. B72, 93 (1974): N. F. Bali and J. W. Dash, Phys. Lett. B51, 99 (1974).
[14]This assumption can be avoided by allowing the incoming particle momentum p_a to be off-shell, giving us an equation for the off-shell inclusive distribution. The physical, on-shell distribution is recovered by putting all external momenta on-shell after the equation is solved. However, besides introducing an additional variable in this approach, we would lose the direct connection of the bootstrap model with the physical, on-shell distributions. An additional justification for neglecting off-shell effects is the weak dependence of the associated multiplicity on the momentum transfer in Refs. 8 and 9.
[15]A. Ballestrero and E. Predazzi, Nuovo Cimento 13A, 470 (1973); R. J. Yaes, Nuovo Cimento Lett. 8, 365 (1973).
[16]R. J. Yaes, Nuovo Cimento Lett. 4, 611 (1972); Phys. Rev. D 7, 2161 (1973); T. L. Neff, SLAC Report No. 152, 1972 (unpublished).
[17]R. J. Yaes, Phys. Rev. 170, 1236 (1968); Nucl. Phys. A131, 623 (1969).
[18]J. Finkelstein and K. Kajantie, Nuovo Cimento 56A, 659 (1968).
[19]See, for example, G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970), p. 249.
[20]See, for example, S. Pinsky, D. Snider and G. Thomas, Phys. Lett. 47B, 505 (1973).