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Exact spectral-function sum rules*

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A general procedure for extracting exact spectral-function sum rules is presented. The short-distance behavior of products of vector and axial-vector currents is related to the convergence (or superconvergence) of the original first and second spectral sum rules together with a third sum rule involving only the spin-0 spectral function. The operator-product expansion is then applied to determine all (and only) those linear combinations of current propagators for which the short-distance behavior is sufficiently soft to yield superconvergent sum rules for the corresponding combinations of spectral functions. Our method is applied to determine the complete set of sum rules for a theory defined by a global chiral $SU(4)\times SU(4)$ symmetry, broken (a) explicitly by hadron (quark) masses and (b) by dynamical symmetry breaking to any subgroup containing the symmetry group of the mass matrix. Our derivation is strictly true only for asymptotically free theories, but the results are expected to apply for a range of other theories. The method is easily extended to deal with current propagators involving scalar and pseudoscalar densities (*not* necessarily divergences of vector or axial-vector currents)—the relevant sum rules in the context of the $SU(4)\times SU(4)$ model are derived. Finally, we compare our approach and results to those of several recent studies of the spectral-function sum rules. An appendix presents a proof that Wilson functions exhibit the full symmetries of any theory, whether or not these are spontaneously broken.

I. INTRODUCTION

Spectral-function sum rules for vector and axial-vector current propagators were introduced some time ago,[1] and have since found a number of applications in current-algebra calculations— most notably, the derivation of relations among masses and couplings of vector and Goldstone bosons[2] (in the context of a meson dominance hypothesis), and the PCAC computation of the electromagnetic pion mass splitting,[3] as well as several recent applications to chiral symmetry breaking.[4] For some time after their discovery, however, the general criteria for the validity of the sum rules, especially the second, remained unclear. It was first pointed out by Wilson[5] that the operator-product expansion (OPE) provides an ideal tool for examining the question of validity of the spectral-function sum rules. Indeed, the short-distance singularities in current propagators turn out to be directly relevant to the convergence (or superconvergence) of the corresponding spectral integrals.

In studying the spectral-function sum rules, Wilson considered certain specific combinations of various current propagators, and concluded that (in the context fo chiral symmetry breaking by hadron mass terms) neither the first nor second original sum rule was valid except in the limit of exact chiral $SU(2)\times SU(2)$, assuming the scale dimension of the quark mass term to be three or larger. Furthermore, in the absence of an assumption of exact chiral symmetry, Wilson was able to find only one combination of current propagators for which the first sum rule was valid.

In this paper we show that the above conclusions are unduly pessimistic, and that several exact sum rules can, in fact, be derived even in the presence of explicit chiral symmetry breaking, assuming that the strong interactions are asymptotically free. Instead of restricting our attention *ab initio* to specific combinations of current commutators, we employ the OPE analysis to *determine* those linear combinations for which the leading short-distance singularities are sufficiently soft and the corresponding sum rules valid. Our method is based on a modified spurion analysis which carefully treats the infrared singularities which arise in a naive mass expansion of Wilson coefficient functions. The coefficients of the linear combinations which we find are generally functions of the symmetry-breaking parameters. Specifically, in a quark-gluon theory with chiral symmetry breaking by a quark mass term, the coefficients are functions of ratios of quark masses. Such ratios are determined in a wide variety of current-algebra and phenomenological quark-model calculations.[6]

The outline of the paper is as follows. In Sec. II we describe the connections between the short-distance expansion of current products and various spectral-function sum rules. In cases in which the second sum rule is valid, we find an additional third sum rule involving only the spin-zero spectral function. The sum rules for the exact chiral SU(3)×SU(3) limit of an SU(4)×SU(4) symmetric theory with explicit quark-mass breaking are discussed briefly as illustrative examples. Our results are valid in asymptotically free theories. In Sec. III we present the complete set of sum rules for a theory with an initial chiral SU(4)×SU(4) broken (a) explicitly by quark masses to SU(2)×U(1)×U(1) (isospin, strangeness, and charm), and (b) spontaneously to any subgroup of SU(4)×SU(4) containing the SU(2)×U(1)×U(1) symmetry of the quark mass matrix [say, nonchiral SU(4) or SU(3)]. In particular, we obtain a pair of exact sum rules which go over to the original first and second sum rules[1] in the limit of vanishing proton and neutron quark masses, as well as several additional sum rules. Saturation of the latter with low-lying meson states unfortunately does not appear to yield any phenomenologically interesting relations, owing to the paucity of experimental information concerning the couplings of observed mesons to the various currents, and the masses and couplings of hitherto unobserved charmed mesons. In Sec. IV, we extend our methods to discuss the sum rules for propagators involving general scalar and pseudoscalar densities (*not* necessarily divergences of vector or axial-vector currents). We here consider both mixed propagators, involving vector and scalar currents, and pure scalar propagators. Finally, in Sec. V we discuss the relation of our work to other recent attempts[4] to derive rigorously exact spectral-function sum rules. The Appendix contains a proof that spontaneous symmetry-breaking effects never appear in Wilson coefficient functions, a fact crucial to our analysis.

II. DERIVATION OF SPECTRAL-FUNCTION SUM RULES; THE SU(3)-SYMMETRIC LIMIT

In this section we explain the relevance of the short-distance expansion of current operator products to the extraction of valid spectral-function sum rules. Our calculations are performed for a vector-gluon strong-interaction theory with an initial chiral U(4)×U(4) symmetry broken (a) explicitly in the Hamiltonian by quark masses and (b) spontaneously to some subgroup containing the symmetry group of the mass matrix. After presenting the general theory, we apply it to the situation in which only the charmed quark mass is nonzero and derive exact sum rules in this limit. The correct sum rules for the more general case of nonvanishing $\mathcal{P}$, $\mathfrak{N}$, Λ quark masses are derived in Sec. III.

We begin with the Källén-Lehmann spectral representation for the product of vector or axial-vector currents:

$$\langle 0|J_A^\mu(x)J_B^\nu(0)|0\rangle = \int d\mu^2\left[g^{\mu\nu}\rho_{AB}^{(1)}(\mu^2) - \left(\rho_{AB}^{(0)}(\mu^2)+\frac{\rho_{AB}^{(1)}(\mu^2)}{\mu^2}\right)\partial^\mu\partial^\nu\right]\Delta^{(+)}(x;\mu^2). \tag{2.1}$$

In (2.1), $\rho_{AB}^{(j)}(\mu^2)$ are the spin-j spectral functions, and $\Delta^{(+)}(x;\mu^2)$ is the positive-frequency Green's function with the short-distance expansion

$$\Delta^{(+)}(x;\mu^2)\underset{x\to 0}{\simeq} -\frac{1}{4\pi^2}\,\frac{1}{x^2-i\eta\epsilon(x^0)}+\frac{\mu^2}{8\pi^2}\left(\ln\frac{\gamma}{2}(\mu^2|x^2|)^{1/2}-\tfrac{1}{2}+i\pi\epsilon(x^0)\theta(-x^2)\right)+O(x^2) \tag{2.2}$$

(η is a positive infinitesimal; γ is the Euler constant). We now perform a short-distance expansion on both sides of (2.1), using a Wilson expansion on the left-hand side, and Eq. (2.2) *inside* the spectral integral on the right-hand side. This procedure fails (and, presumably, fails only) if the coefficient spectral integrals arising on the right-hand side diverge. Knowledge of the leading singularities in the operator-product expansion (OPE) thus places convergence (and superconvergence) constraints on various spectral integrals. The combinations

$$\int d\mu^2\left(\rho^{(0)}(\mu^2)+\frac{\rho^{(1)}(\mu^2)}{\mu^2}\right),$$

$$\int d\mu^2\rho^{(1)}(\mu^2), \tag{2.3}$$

$$\int d\mu^2\rho^{(0)}(\mu^2)\mu^2,$$

will be referred to in the following as the first, second, and third spectral integrals, respectively, and the statement that these integrals vanish for some particular linear combination of current products will be referred to as the first, second, and third spectral-function sum rules for that combination.

The following cases arise.

(a) The leading singularity in the OPE for $\langle 0|J_A^\mu(x)J_B^\nu(0)|0\rangle$ as $x\to 0$ is stronger than $1/x^4$. In this case, the first spectral integral, and at least one of the second or third spectral integrals, must diverge. Our analysis thus fails to yield any sum rules.

(b) If the leading singularity in the OPE is precisely $1/x^4$, the first spectral integral must converge to some finite value:

$$\int d\mu^2\left(\rho^{(0)}(\mu^2)+\frac{\rho^{(1)}(\mu^2)}{\mu^2}\right)=c_1. \tag{2.4}$$

The contributions from the second and third spectral integrals are subdominant, and we can say nothing of their possible divergence or convergence by this analysis.

(c) If the leading singularity is stronger than $1/x^2$ (but weaker than $1/x^4$), the first spectral sum rule holds:

$$\int\left(\rho^{(0)}(\mu^2)+\frac{\rho^{(1)}(\mu^2)}{\mu^2}\right)d\mu^2=0. \tag{2.5}$$

In this case, either the second or third spectral integrals (or both) diverge. (Knowledge of the tensor structure of the leading singularity would allow us to isolate the divergent combination, but this is of little interest.)

(d) If the leading singularity is precisely $1/x^2$, we have, in addition to (2.5), that the second and third spectral integrals converge:

$$\int d\mu^2\rho^{(1)}(\mu^2)=c_2, \tag{2.6}$$

$$\int d\mu^2\mu^2\rho^{(0)}(\mu^2)=c_3. \tag{2.7}$$

(e) Finally, if the leading singularity is weaker than $1/x^2$, we obtain two additional spectral-function sum rules, as all three spectral integrals must now converge to zero:

$$\int d\mu^2\rho^{(1)}(\mu^2)=0, \tag{2.8}$$

$$\int d\mu^2\mu^2\rho^{(0)}(\mu^2)=0. \tag{2.9}$$

Equations (2.5) and (2.8) are just the original first and second spectral-function sum rules.[1] Equation (2.9) can alternatively be derived by considering the spectral representation for $\langle 0|\partial_\mu J_A^\mu(x)J_B^\nu(0)|0\rangle$.

Since the operator-product expansion for the product of two currents usually begins with the unit operator multiplying a coefficient function with singularity $1/x^6$, the utility of the above categorization may seem questionable. Nevertheless, we shall see that specific linear combinations of various spectral functions can be chosen to eliminate the troublesome leading singularities in the short-distance expansion. In fact, nontrivial sum rules can be found in this way, even for the most general case to be considered, in which the quark masses break chiral $U(4)\times U(4)$ down to $SU(2)\times U(1)\times U(1)$ (isospin, strangeness, and charm).

The study of the operator-product expansion proceeds by a modified spurion analysis in which the coefficient functions are expanded in powers of the quark mass matrix. It is well known[7] that such an expansion eventually encounters infrared singularities owing to the piling up of mass insertions on a single internal fermion line. Such singularities give rise to nonanalytic behavior of the coefficient functions in the chiral limit. This problem is symptomized by the appearance of terms behaving like $m^3\ln m$ in the mass expansion.

To avoid the infrared singularities mentioned above, we will modify the bare fermion propagators as follows:

$$\Delta^\psi(p)=\frac{1}{i\not p+m}\to\Delta_\lambda^\psi(p)\equiv\frac{-i\not p+m}{p^2+m^2+\lambda^2}. \tag{2.10}$$

This modification eliminates the infrared singularities arising from multiple mass insertions in fermion-loop propagators. Furthermore, it is actually a "chiral symmetric" cutoff in the sense that arguments relying on γ_5 bookkeeping are still valid. In the Appendix we demonstrate, using "soft-pion" techniques, that evidence of spontaneous symmetry breaking never appears in Wilson coefficient functions. We can, therefore, carry out a mass expansion of the coefficient functions, using the above cutoff: The only possible terms appearing can be determined simply by constructing $U(4)\times U(4)$ invariants, employing the quark mass matrix m, and the matrices defining the currents

$$J_A^\mu(x)\equiv -i\bar\psi(x)\gamma^\mu A\psi(x).$$

From the point of view of spectral-function sum rules, the only relevant terms in the short-distance expansion of $\langle 0|J_A^\mu(x)J_B^\nu(0)|0\rangle$ are those at least as singular as $1/x^2$ for $x\to 0$, or equivalently, in momentum space, those terms which fall off no more rapidly than $1/p^2$ as $p\to\infty$ (in a fixed Euclidean direction). The only relevant operators are 1 and $\psi\bar{\psi}$, with canonical dimensions 0 and 3, respectively. Furthermore, the relevant portion of the mass expansion of the coefficient function $U_{\psi\bar{\psi}}(p)$ consists of just the terms linear in m. Any higher terms fall off at least as rapidly as $1/p^4$ in perturbation theory (we discuss the question of logarithms below: For the time being, "$1/p^n$" should be interpreted as "$1/p^n$ ×powers of log$p$.")

Unfortunately, successively higher terms in the mass expansion of $U_1(p)$ (the coefficient function of the unit operator) do *not* necessarily fall off faster than $1/p^2$ as $p\to\infty$. In fact, a straightforward bridge analysis[8] shows that the graph indicated in Fig. 1, which is evidently proportional to $\mathrm{Tr}(Am^5Bm)$, gives a contribution of order $1/p^2$ (for $p\to\infty$) to the propagator

$$\Sigma^-_{AB,\mu\nu}(p)\equiv\int dx\, e^{-ip\cdot x}\langle 0|T\{\bar{\psi}(x)(1+\gamma_5)\gamma_\mu A\psi(x)\times\bar{\psi}(0)(1-\gamma_5)\gamma_\nu B\psi(0)\}|0\rangle\,. \tag{2.11}$$

Furthermore, additional mass insertions do not lower the asymptotic behavior in p—instead, higher inverse powers of λ appear. Intuitively, the asymptotic momentum is "forced" to flow through the lower fermion line. On the other hand, the graph indicated in Fig. 2, which is proportional to $\mathrm{Tr}(Am^3Bm^3)$, is suppressed and behaves asymptotically as $1/p^4$. In fact, a bridge analysis readily shows that the only graphs contributing to $\Sigma^-_{AB\mu\nu}(p)$ which behave asymptotically as $1/p^2$, which contain more than four mass insertions, and in which the currents insert on the same fermion loop correspond to invariant structures of the generic form $\mathrm{Tr}(Am^{2l+1}Bm)+A\leftrightarrow B$, with $l\geq 2$.[9]

We are now in a position to summarize the general results of the bridge analysis of the mass expansion. Note that since we assume that the vacuum does not break parity, only the spectral functions of two vector or two axial-vector currents are nonvanishing. It is convenient to consider separately the sum and difference of the vector and corresponding axial-vector spectral functions—henceforth referred to as $V+A$, $V-A$ spectral functions, respectively. We have, neglecting all terms less singular than $1/x^2$ (or $1/p^2$ in momentum space), the following.

(a) In the $V+A$ sector (Lorentz indices μ,ν will be suppressed)

$$\begin{aligned}\Sigma^+_{AB}(x)&\equiv\langle 0|\bar{\psi}(x)\gamma_\mu(1+\gamma_5)A\psi(x)\bar{\psi}(0)\gamma_\nu(1+\gamma_5)B\psi(0)|0\rangle\\ &\xrightarrow[x\to 0]{}\sum_{l=0}^{\infty}f_l^+(x)\mathrm{Tr}(\{A,B\}m^{2l})+\sum_{l=0}^{\infty}g_l^+(x)[\mathrm{Tr}(Am^{2l})\mathrm{Tr}B+\mathrm{Tr}(Bm^{2l})\mathrm{Tr}A]\\ &\quad+h_1^+(x)\mathrm{Tr}(Am^2)\mathrm{Tr}(Bm^2)+h_2^+(x)\mathrm{Tr}(Am^2Bm^2)+h_3^+(x)[\mathrm{Tr}(ZmA)\mathrm{Tr}B+\mathrm{Tr}(ZmB)\mathrm{Tr}A]\\ &\quad+h_4^+(x)\mathrm{Tr}(Zm\{A,B\}).\end{aligned} \tag{2.12}$$

We have defined the matrix $Z_{nm}\equiv\langle 0|\psi_n(0)\bar{\psi}_m(0)|0\rangle$, and chosen for convenience Z, m simultaneously diagonal and γ_5-free (the possibility of this choice is guaranteed by the strong-interaction symmetries—isospin, strangeness, charm, and parity—together with Dashen's theorem,[10] which mutually constrains the "directions" of explicit and spontaneous symmetry breaking).

In perturbation theory, ignoring logarithms, $f_0^+(x)$ and $g_0^+(x)$ behave like $1/x^6$ for small x, $f_1^+(x)$ and $g_1^+(x)$ behave like $1/x^4$, and all other coefficient functions in (2.12) behave like $1/x^2$, despite the unlimited powers of m which can occur. [Graphs giving rise to f_l^+ and g_l^+ are exhibited in Figs. 3(a) and 3(b).] We now see immediately that the validity of the first sum rule [case (c) above] is established for any linear combination of spectral functions for which the coefficients of f_0^+, f_1^+, g_0^+, and g_1^+ vanish. For combinations eliminating all the (infinitely many) coefficient functions in (2.12), all three sum rules are valid [case (e) above].

A word or two is in order here concerning the

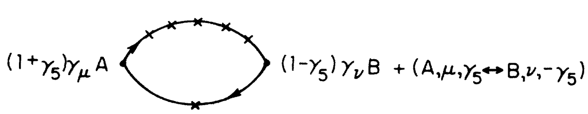

FIG. 1. Lowest-order graph proportional to $\mathrm{Tr}(AmBm^5)+(A\leftrightarrow B)$ (crosses denote zero-momentum $\bar{\psi}m\psi$ insertions).

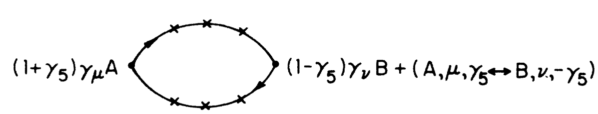

FIG. 2. Lowest-order graph proportional to $\mathrm{Tr}(Am^3Bm^3)$.

neglected logarithms. If we restrict ourselves to theories in which the strong interactions are asymptotically free, then we expect perturbation theory to be a reliable guide to the asymptotic behavior summed to all orders. Renormalization-group arguments show that the current propagator, with external momentum scaled up by a factor κ, is proportional to the same propagator at unscaled momentum, but evaluated in terms of the effective coupling constant and mass [$g(\kappa)$ and $m(\kappa)$] and a scaled-down infrared cutoff, $\lambda(\kappa)=\lambda_0/\kappa$. We believe that it can be shown [to all orders in $g(\kappa)$] that a finite power of $\ln\lambda(\kappa)$ accompanies any given power of $\lambda(\kappa)$. In that case, one can see that the terms we have omitted from (2.12) are asymptotically negligible compared to those we have kept. We also expect these terms to be negligible in theories with nonvanishing, but small, anomalous dimensions.

At first sight Eq. (2.12) may well seem useless insofar as the derivation of second and third sum rules is concerned. Indeed, for these sum rules to be valid, a (necessarily finite) linear combination of spectral functions must exist which simultaneously cancels an infinite number of independent invariant structures.

However, as explained in the following section, the cancellation of a finite *subset* of the invariants in Eq. (2.12) suffices to remove the entire set. This fact will enable us to derive a nontrivial, and presumably exhaustive, collection of first, second, and third sum rules for an (asymptotically free) U(4)×U(4) quark-gluon theory.

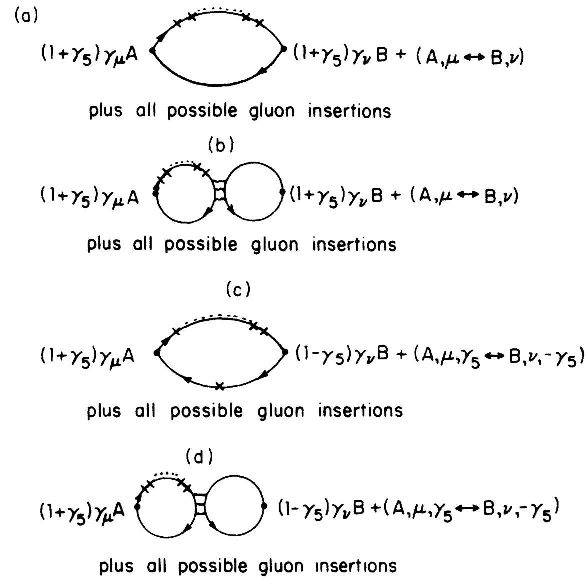

FIG. 3. (a) $V+A$ graphs proportional to $\mathrm{Tr}(\{A, B\}m^{2l})$; (b) $V+A$ graphs proportional to $\mathrm{Tr}(Am^{2l})\mathrm{Tr}B+(A\leftrightarrow B)$; (c) $V-A$ graphs proportional to $\mathrm{Tr}(AmBm^{2l-1})+(A\leftrightarrow B)$; (d) $V-A$ graphs proportional to $\mathrm{Tr}(Am^{2l})\mathrm{Tr}B+(A\leftrightarrow B)$.

(b) In the $V-A$ sector, the short-distance expansion analogous to (2.12) is found to be

$$\Sigma^-_{AB}(x)\equiv\langle 0|\bar\psi(x)\gamma_\mu(1+\gamma_5)A\psi(x)\bar\psi(0)\gamma_\nu(1-\gamma_5)B\psi(0)|0\rangle$$

$$\xrightarrow[x\to 0]{}\sum_{l=0}^{\infty}f_l^-(x)\mathrm{Tr}(AmBm^{2l+1}+BmAm^{2l+1})+\sum_{l=0}^{\infty}g_l^-(x)[\mathrm{Tr}(Am^{2l})\mathrm{Tr}B+\mathrm{Tr}(Bm^{2l})\mathrm{Tr}A]$$

$$+h_1^-(x)\mathrm{Tr}(Am^2)\mathrm{Tr}(Bm^2)+h_2^-(x)[\mathrm{Tr}(ZmA)\mathrm{Tr}B+\mathrm{Tr}(ZmB)\mathrm{Tr}A]+h_3^-(x)\mathrm{Tr}(ZAmB+ZBmA). \tag{2.13}$$

[Graphs giving rise to f_l^- and g_l^- are exhibited in Figs. 3(c) and 3(d).] The first spectral-function sum rule is found to hold for linear combinations eliminating f_0^-, g_0^-, and g_1^-. The second and third sum rules are valid only for linear combinations eliminating all the f_l^-, g_l^-, and h_l^-. Here again, however, the elimination of a finite subset is sufficient (cf. Sec. III).

As a simple application of the foregoing theory, consider a model characterized by a U(4)×U(4) chiral global symmetry of the Lagrangian, broken down (a) to U(3)×U(3) by the quark mass matrix, and (b) to SU(3) by dynamical symmetry breaking in the vacuum. Thus, the m and Z matrices take the form

$$m=\begin{bmatrix}0&&&\\&0&&\\&&0&\\&&&m_c\end{bmatrix},\quad Z=\begin{bmatrix}Z_0&&&\\&Z_0&&\\&&Z_0&\\&&&Z_c\end{bmatrix}. \tag{2.14}$$

One now searches for linear combinations which satisfy sum rules but do not vanish trivially by algebraic SU(3) symmetry. In the $(V+A)$ sector, one finds a single valid first sum rule, and no valid second or third sum rules. In terms of quark fields, the required linear combination of current products is found to be

$$\begin{aligned}
&\tfrac{2}{3}[\bar{\psi}_{\mathcal{P}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathcal{P}}(x)\bar{\psi}_{\mathcal{P}}(0)\gamma_\nu(1+\gamma_5)\psi_{\mathcal{P}}(0)+\bar{\psi}_{\mathfrak{N}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathfrak{N}}(x)\bar{\psi}_{\mathfrak{N}}(0)\gamma_\nu(1+\gamma_5)\psi_{\mathfrak{N}}(0)] \\
&\quad -\tfrac{1}{3}[\bar{\psi}_{\mathcal{P}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathcal{P}}(x)\bar{\psi}_{\mathfrak{N}}(0)\gamma_\nu(1+\gamma_5)\psi_{\mathfrak{N}}(0)+\bar{\psi}_{\mathfrak{N}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathfrak{N}}(x)\bar{\psi}_{\mathcal{P}}(0)\gamma_\nu(1+\gamma_5)\psi_{\mathcal{P}}(0)] \\
&\quad +\tfrac{1}{6}[\bar{\psi}_{\lambda}(x)\gamma_\mu(1+\gamma_5)\psi_{\lambda}(x)-3\bar{\psi}_c(x)\gamma_\mu(1+\gamma_5)\psi_c(x)][\bar{\psi}_{\mathcal{P}}(0)\gamma_\nu(1+\gamma_5)\psi_{\mathcal{P}}(0)+\bar{\psi}_{\mathfrak{N}}(0)\gamma_\nu(1+\gamma_5)\psi_c(0)] \\
&\quad +\tfrac{1}{6}[\bar{\psi}_{\mathcal{P}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathcal{P}}(x)+\psi_{\mathfrak{N}}(x)\gamma_\mu(1+\gamma_5)\psi_{\mathfrak{N}}(x)][\bar{\psi}_{\lambda}(0)\gamma_\nu(1+\gamma_5)\psi_{\lambda}(0)-3\bar{\psi}_c(0)\gamma_\nu(1+\gamma_5)\psi_c(0)] \\
&\quad +\tfrac{1}{6}[\bar{\psi}_{\lambda}(x)\gamma_\mu(1+\gamma_5)\psi_{\lambda}(x)-3\bar{\psi}_c(x)\gamma_\mu(1+\gamma_5)\psi_c(x)][\bar{\psi}_{\lambda}(0)\gamma_\nu(1+\gamma_5)\psi_{\lambda}(0)-3\bar{\psi}_c(0)\gamma_\nu(1+\gamma_5)\psi_c(0)] \\
&\quad -3\bar{\psi}\;(x)\gamma_\mu(1+\gamma_5)\psi_c(x)\bar{\psi}_c(0)\gamma_\nu(1+\gamma_5)\psi_{\mathcal{P}}(0).
\end{aligned} \tag{2.15}$$

In the $V-A$ sector, one finds a single set of valid first, second, and third sum rules. Setting

$$A=B=\frac{1}{\sqrt{2}}\begin{bmatrix}1&&&\\&-1&&\\&&0&\\&&&0\end{bmatrix}$$

we find immediately that the coefficients of all f_l^-, g_l^-, and h_l^- in (2.13) vanish. The first and second sum rules in this case are, in fact, just those originally proposed for the isospin-one zero-strangeness vector and axial-vector currents. We now turn to a consideration of the more realistic situation, in which the quark mass matrix preserves SU(2)×U(1)×U(1) only.

III. SUM RULES IN THE SU(2) × U(1) × U(1) LIMIT

We start with the expressions for the m and Z matrices in the SU(2)×U(1)×U(1) limit:

$$m=\begin{bmatrix}m_0&&&\\&m_0&&\\&&m_s&\\&&&m_c\end{bmatrix},\quad Z=\begin{bmatrix}Z_0&&&\\&Z_0&&\\&&Z_s&\\&&&Z_c\end{bmatrix}. \tag{3.1}$$

The invariance of the vacuum under isospin, strangeness, and charm transformations fixes the form of Z and also reduces considerably the number of independent vacuum expectation values of current products (and, hence, the number of independent spectral functions). We define Σ_1^+, $\Sigma_2^+, \ldots, \Sigma_{10}^+$ and $\Sigma_1^-, \Sigma_2^-, \ldots, \Sigma_{10}^-$ as the 10 independent nontrivial vacuum expectation values in the $V+A$ and $V-A$ sectors, respectively:

$$\Sigma_i^{\pm}=\langle 0|\bar{\psi}(x)\gamma^\mu(1+\gamma_5)A_i\psi(x)\times\bar{\psi}(0)\gamma^\nu(1\pm\gamma_5)B_i\psi(0)|0\rangle\,, \tag{3.2a}$$

where we have

$$A_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=B_1,$$

$$A_2=\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=B_2^T,\qquad A_3=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=B_3^T,$$

$$A_4=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}=B_4^T,$$

$$A_5=A_6=A_7=\frac{1}{\sqrt{6}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&0\end{bmatrix}=B_5,$$

$$A_8=A_9=\frac{1}{\sqrt{12}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{bmatrix}=B_6=B_8,$$

$$A_{10}=\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=B_7=B_9=B_{10}. \tag{3.2b}$$

Thus, Σ_1 corresponds to the "ρ sum rule"; Σ_2 corresponds to the K^*; Σ_3 and Σ_4 correspond to the charmed doublet and singlet, respectively; and Σ_5–Σ_{10} correspond to the various diagonal combinations which may mix. Using (2.12) and (2.13), we can then calculate the coefficients of f_i^+, g_i^+, and h_i^+ for $\Sigma_{1\text{-}10}^+$. Linear combinations of $\Sigma_{1\text{-}10}^+$ or $\Sigma_{1\text{-}10}^-$ which satisfy the first sum are then easily found since only a finite number of invariants need be eliminated (the coefficients of f_0^+, g_0^+, f_1^+, and g_1^+ for $V+A$, and the coefficients of f_0^-, g_0^-, and g_1^- for $V-A$).

To find linear combinations of $\Sigma_{1\text{-}10}^+$ or $\Sigma_{1\text{-}10}^-$ that satisfy all three sum rules is more difficult since an infinite number of invariants must be eliminated. However, an examination of (2.12) shows that all the invariants in the $V+A$ sector, except for those multiplying h_1^+ and h_2^+, have the form $\mathrm{Tr}(\{A,B\}X)$ or $\mathrm{Tr}(AX)\,\mathrm{Tr}B+\mathrm{Tr}(BX)\,\mathrm{Tr}A$, where X is an arbitrary matrix of the form

$$X=\begin{bmatrix}\alpha & & & \\ & \alpha & & \\ & & \beta & \\ & & & \gamma\end{bmatrix}.$$

Thus, linear combinations which satisfy all three sum rules may be found by requiring that the coefficients of α, β, and γ in $\mathrm{Tr}(\{A,B\}X)$ and $\mathrm{Tr}(AX)\,\mathrm{Tr}B+\mathrm{Tr}(BX)\,\mathrm{Tr}A$ (as well as the coefficients of h_1^+ and h_2^+) vanish. All told, this makes eight equations in ten unknowns, so we will find two nontrivial combinations of $\Sigma_{1\text{-}10}^+$ that satisfy all three sum rules. The coefficients of $\Sigma_{1\text{-}10}^+$ in such sum rules will automatically be independent of the matrix Z and will depend only on quark mass ratios. This is because Z appears only in those invariants which can be written, as above, in terms of the matrix X, and which we have eliminated for *arbitrary* X.

The analysis of the $V-A$ sector is exactly parallel. Examination of (2.13) shows that all the invariants, except $\mathrm{Tr}(Am^2Bm^2)$, have the form $\mathrm{Tr}(AmBX+BmAX)$ or $\mathrm{Tr}(AX)\,\mathrm{Tr}B+\mathrm{Tr}(BX)\,\mathrm{Tr}A$, with X as above. Requiring that these three invariants vanish (the latter two for arbitrary X) gives a total of seven equations, which means that there will be three nontrivial linear combinations of $\Sigma_{1\text{-}10}^-$ which satisfy all three sum rules. As before, the coefficients in such linear combinations will be independent of the Z matrix and will depend only on ratios of quark masses.

The results of the analysis are the following.

(a) In the $V+A$ sector, there are six independent combinations which satisfy the first sum rule alone (of course, any linear combinations of these also satisfy the sum rule):

$$\Sigma_1^+ - \frac{9m_c^2+m_s^2-10m_0^2}{3m_c^2+m_s^2-4m_0^2}\,\Sigma_3^+ + \frac{6(m_c^2-m_0^2)}{3m_c^2+m_s^2-4m_0^2}\,\Sigma_8^+, \tag{3.3a}$$

$$\Sigma_1^+ - 4\Sigma_2^+ + 3\Sigma_5^+, \tag{3.3b}$$

$$\Sigma_1^+ - \Sigma_2^+ - \Sigma_3^+ + \Sigma_4^+, \tag{3.3c}$$

$$\Sigma_1^+ - \frac{m_c^2-m_0^2}{m_c^2-m_s^2}\,\Sigma_2^+ + \frac{m_s^2-m_0^2}{m_c^2-m_s^2}\,\Sigma_3^+, \tag{3.3d}$$

$$\Sigma_1^+ - \Sigma_2^+ - \frac{3\sqrt{2}}{2}\,\Sigma_6^+, \tag{3.3e}$$

$$\Sigma_7^+ - \frac{2\sqrt{2}\,(m_s^2-m_0^2)}{3m_c^2-2m_0^2-m_s^2}\,\Sigma_9^+. \tag{3.3f}$$

Note that in the limit $m_0=m_s=0$, (3.3a) is just the sum rule obtained in Sec. II for the case of an SU(3)×SU(3) symmetric mass matrix Eq. (2.15), whereas Eqs. (3.3b)–(3.3f) just represent corrections to relations among Σ's which would be identically true for an SU(3) symmetric theory.

(b) In the $V+A$ sector, there are two independent combinations which satisfy the first, second, and third sum rules. These are complicated to write down in the general case, but if we neglect m_0^2 compared to m_s^2 or m_c^2 (one set of estimates[6] gives $m_0/m_s\sim 1/25$), they take a fairly simple form

$$\Sigma_1^+ - 4\Sigma_2^+ + 3\Sigma_5^+ + \left(\frac{m_s}{m_c}\right)^4\left(-\Sigma_1^+ + 4\Sigma_3^+ - \tfrac{1}{3}\Sigma_5^+ - \frac{4\sqrt{2}}{3}\Sigma_6^+ - \tfrac{8}{3}\Sigma_8^+\right), \tag{3.4a}$$

$$\Sigma_1^+ - 2\Sigma_2^+ - 2\Sigma_3^+ + 2\Sigma_4^+ + \Sigma_5^+ + 2\sqrt{2}\,\Sigma_6^+ + \left(\frac{m_s}{m_c}\right)^2\left(-\Sigma_1^+ + 4\Sigma_3^+ - \tfrac{1}{3}\Sigma_5^+ - \frac{4\sqrt{2}}{3}\Sigma_6^+ - \tfrac{8}{3}\Sigma_8^+\right). \tag{3.4b}$$

Equations (3.4a) and (3.4b) merely give corrections to sum rules which would be identically true in the case of SU(3)×SU(3) symmetry of the mass matrix; this corresponds to the fact that no nontrivial second and third $V+A$ sum rules could be found under the assumptions of Sec. II. Note that (3.3b) is just the 27-plet combination for the first sum rule that Wilson[5] obtains, and that (3.4a) shows the corrections to (3.3b) that are necessary for the second and third sum rules to be valid in addition.

(c) In the $V-A$ sector, there are seven combinations that make the first sum rule valid:

$$\Sigma_1^- - \frac{m_0}{m_s}\Sigma_2^-, \tag{3.5a}$$

$$\Sigma_1^- - \frac{m_0}{m_c}\Sigma_3^-, \tag{3.5b}$$

$$\Sigma_3^- - \frac{m_0}{m_s}\Sigma_4^-, \tag{3.5c}$$

$$\Sigma_1^- - \frac{3m_0^2}{m_0^2+2m_s^2}\Sigma_5^- \tag{3.5d}$$

$$\Sigma_1^- - \frac{3\sqrt{2}m_0^2}{m_0^2-m_s^2}\Sigma_6^-, \tag{3.5e}$$

$$\Sigma_1^- - \frac{12m_0^2}{9m_c^2+m_s^2+2m_0^2}\Sigma_8^-, \tag{3.5f}$$

$$\Sigma_7^- - \frac{2\sqrt{2}(m_s^2-m_0^2)}{3m_c^2-m_s^2-2m_0^2}\Sigma_9^-. \tag{3.5g}$$

(d) In the $V-A$ sector, there are three independent combinations which satisfy the first, second, and third sum rules.

$$\Sigma_1^- - \frac{m_0}{m_s}\Sigma_2^- - \frac{m_0}{m_c}\Sigma_3^- + \frac{m_0^2}{m_s m_c}\Sigma_4^-, \tag{3.6a}$$

$$\Sigma_2^- + \frac{1}{3m_0^2m_s^2 - m_0^2m_c^2 - 2m_s^2m_c^2}\left[(2m_s^2m_c^2 - m_0^2m_s^2 - m_0^2m_c^2)\left(\frac{m_s}{m_c}\Sigma_3^- - \frac{m_0}{m_c}\Sigma_4^-\right)\right.$$
$$\left. - (2m_0^2+m_s^2-3m_c^2)m_0m_s\Sigma_5^- + 2\sqrt{2}(m_0^2-m_s^2)m_0m_s\Sigma_6^-\right], \tag{3.6b}$$

$$\Sigma_2^- + \frac{(2m_0^2m_c^2 - m_s^2m_c^2 - m_0^2m_s^2)}{(m_s^2-m_0^2)m_c^2}\frac{m_s}{m_c}\Sigma_3^- + \frac{(m_0^2m_s^2 + m_0^2m_c^2 - 2m_s^2m_c^2)}{(m_s^2-m_0^2)m_c^2}\frac{m_0}{m_c}\Sigma_4^-$$
$$+ \frac{1}{\sqrt{2}}\frac{(2m_0^2+m_s^2-3m_c^2)}{(m_s^2-m_0^2)}\frac{m_0m_s}{m_c^2}\Sigma_6^- - \frac{2m_0m_s}{m_c^2}\Sigma_8^-. \tag{3.6c}$$

Equation (3.6a) shows the order m_0/m_s and higher corrections to the original[1] first and second sum rules for Σ_1^-. In the SU(2)×SU(2) limit where $m_0=0$, these sum rules are exact; this agrees with a result of Borchardt and Mathur.[11] Further, by letting $m_0=m_s=0$ in (3.5) and (3.6) [remembering, of course, that $(\Sigma_1^- - \Sigma_2^-)\sim(\Sigma_1^- - \Sigma_5^-)\sim(\Sigma_3^- - \Sigma_4^-)\sim\Sigma_6^-\sim\Sigma_7^-\sim m_s^2-m_0^2$ because of approximate SU(3)] we see that (3.5) and (3.6) represent corrections to the original Σ_1^- sum rules, plus corrections to sum rules that would be trivially true in the SU(3) limit. This is in agreement with the results of Sec. II.

IV. SUM RULES INVOLVING SCALAR AND PSEUDOSCALAR CURRENTS

The Källén-Lehmann representation for the product of two scalar or pseudoscalar currents has the following form:

$$\langle 0|J_A(x)J_B(0)|0\rangle = \int d\mu^2\hat{\rho}_{AB}^{(0)}(\mu^2)\Delta^{(+)}(x;\mu^2). \tag{4.1}$$

$\hat{\rho}_{AB}^{(0)}(\mu^2)$ is a spin-zero spectral function. $\Delta^+(x;\mu^2)$ is defined in Eq. (2.2). If we compare the short-distance behavior of both sides of (4.1) we find

$$\int d\mu^2\hat{\rho}_{AB}^{(0)}(\mu^2)=0 \tag{4.2}$$

if the current product is less singular than $1/x^2$.

We will again find it convenient to consider $S+P$ and $S-P$ combinations separately. A brief computation yields the following short-distance expansion:

$$\langle 0|\bar{\psi}(x)(1+\gamma_5)A\psi(x)\bar{\psi}(0)(1+\gamma_5)B\psi(0)|0\rangle \underset{x\to 0}{\longrightarrow} \sum_{l=0}^{\infty} p_l^+(x)\mathrm{Tr}(Am^{2l+1}Bm + AmBm^{2l+1})$$
$$+\sum_{l=0}^{\infty} q_l^+(x)[\mathrm{Tr}(Am^{2l+1})\mathrm{Tr}(Bm) + \mathrm{Tr}(Am)\mathrm{Tr}(Bm^{2l+1})$$
$$+k_1^+(x)\mathrm{Tr}(AZBm + BZAm) + k_2^+(x)[\mathrm{Tr}(AZ)\mathrm{Tr}(Bm) + \mathrm{Tr}(BZ)\mathrm{Tr}(Am)]$$
$$+\text{ less singular terms, in the } S+P \text{ case;} \tag{4.3}$$

$$\langle 0|\bar\psi(x)(1+\gamma_5)A\psi(x)\bar\psi(0)(1-\gamma_5)B\psi(0)|0\rangle \underset{x\to 0}{\longrightarrow} \sum_{l=0}^{\infty} p_l^-(x)\mathrm{Tr}(\{A,B\}m^{2l})$$
$$+\sum_{l=0}^{\infty} q_l^-(x)[\mathrm{Tr}(Am^{2l+1})\mathrm{Tr}(Bm)+\mathrm{Tr}(Bm^{2l+1})\mathrm{Tr}(Am)]$$
$$+k_1^-(x)\mathrm{Tr}(Am^2Bm^2)+k_2^-(x)\mathrm{Tr}(\{A,B\}mZ)$$
$$+k_3^-(x)[\mathrm{Tr}(Am)\mathrm{Tr}(BZ)+\mathrm{Tr}(Bm)\mathrm{Tr}(AZ)]$$
$$+\text{less singular terms, in the } S-P \text{ case.} \qquad (4.4)$$

As our arguments have indicated, sum rule (4.2) will be valid for those linear combinations which eliminate the dependence on all singular functions $p^{\pm}(x)$, $q^{\pm}(x)$, $k^{\pm}(x)$.

We define

$$\Sigma_i^{\pm} \equiv \langle 0|\bar\psi(x)(1+\gamma_5)A_i\psi(x)\bar\psi(0)(1\pm\gamma_5)B_i\psi(0)|0\rangle\,, \qquad (4.5)$$

where A_i and B_i ($i=1-10$) are defined in (3.2b). When we evaluate the expansion in the SU(2)×U(1)×U(1) limit of Sec. III we obtain four sum rules in the $S+P$ case and two in the $S-P$ case. In the $S+P$ case

$$\Sigma_1^+ - \frac{m_0}{m_s}\Sigma_2^+ - \frac{m_0}{m_c}\Sigma_3^+ + \frac{m_0^2}{m_s m_c}\Sigma_4^+, \qquad (4.6a)$$

$$6\sqrt{2}\,\Sigma_6^+ - \frac{12(m_0-m_s)}{2m_0+m_s-3m_c}\Sigma_8^+ - \frac{3(2m_0+m_s-3m_c)}{2(m_0-m_s)}\Sigma_5^+ + \left(\frac{(m_0^2-2m_s^2)(2m_0+m_s-3m_c)}{m_0-m_s} + \frac{2(m_0-m_s)(2m_0^2-m_s^2-9m_c^2)}{2m_0+m_s-3m_c}\right)\frac{1}{2m_0^2}\Sigma_1^+ + \frac{18m_c}{m_0}\frac{m_0-m_s}{2m_0+m_s-3m_c}\Sigma_3^+ + \frac{2m_s}{m_0}\left(\frac{2m_0+m_s-3m_c}{m_0-m_s}+\frac{m_0-m_s}{2m_0+m_s-3m_c}+2\right)\Sigma_2^+, \qquad (4.6b)$$

$$2\sqrt{6}\,\Sigma_7^+ - \frac{4(m_0-m_s)}{2m_0+m_s+m_c}\Sigma_{10}^+ - \frac{3(2m_0+m_s+m_c)}{2(m_0-m_s)}\Sigma_5^+ + \frac{2m_c}{m_0}\left(\frac{m_0-m_s}{2m_0+m_s+m_c}\right)\Sigma_3^+ + \frac{2m_s}{m_0}\left(\frac{2m_0+m_s+m_c}{m_0-m_s}+\frac{m_0-m_s}{2m_0+m_s+m_c}+2\right)\Sigma_2^+ + \left(\frac{(m_0^2-2m_s^2)(2m_0+m_s+m_c)}{m_0-m_s}+\frac{2(2m_0^2-m_s^2-m_c^2)(m_0-m_s)}{2m_0+m_s+m_c}-4(m_0^2+m_s^2)\right)\frac{1}{2m_0^2}\Sigma_1^+, \qquad (4.6c)$$

$$4\sqrt{3}\,\Sigma_9^+ - \frac{2m_0-m_s-3m_c}{2m_0+m_s+m_c}2\Sigma_{10}^+ - \frac{2m_0+3m_s+m_c}{2m_0+m_s-3m_c}6\Sigma_8^+ + \frac{2m_c}{m_0}\left(\frac{9(2m_0+3m_s+m_c)}{2(2m_0+m_s-3m_c)}+\frac{2m_0-m_s-3m_c}{2(2m_0+m_s+m_c)}+3\right)\Sigma_3^+ + \frac{2m_s}{m_0}\left(\frac{2m_0+3m_s+m_c}{2(2m_0+m_s-3m_c)}+\frac{2m_0-m_s-3m_c}{2(2m_0+m_s+m_c)}-1\right)\Sigma_2^+ + \left(\frac{(2m_0^2-m_s^2-9m_c^2)(2m_0+3m_s+m_c)}{(2m_0+m_s-3m_c)}+\frac{(2m_0^2-m_s^2-m_c^2)(2m_0-m_s-3m_c)}{(2m_0+m_s+m_c)}-2(2m_0^2-m_s^2+3m_c^2)\right)\frac{1}{2m_0^2}\Sigma_1^+. \qquad (4.6d)$$

In the $S-P$ case the sum rules are considerably more complicated and will not be displayed here.

One can also consider the product of a vector current with a scalar current.[12] The Källên-Lehmann representation is of the form

$$\langle 0|J_A^\mu(x)J_B(0)|0\rangle = i\int\bar\rho_{AB}^{(0)}(\mu^2)\partial^\mu\Delta^+(x;\mu^2)\,d\mu^2. \qquad (4.7)$$

To find good sum rules we must again examine the short-distance behavior of $J_A^\mu(x)J_B(0)$. If the expansion is less singular than $1/x^3$ our sum rule converges:

$$\int d\mu^2\bar\rho_{AB}^{(0)}(\mu^2)=0. \qquad (4.8)$$

Let us look at the expansion in the $VS+AP$ case.

$$\langle 0|\bar\psi(x)\gamma^\mu(1+\gamma_5)A\psi(x)\bar\psi(0)(1+\gamma_5)B\psi(0)|0\rangle \underset{x\to 0}{\rightarrow} \sum_{l=0}^{\infty} a_l^{+\,\mu}(x)\mathrm{Tr}(A)\mathrm{Tr}(Bm^{2l+1}) + \sum_{l=0}^{\infty} b_l^{+\,\mu}(x)\mathrm{Tr}(Am^{2l+1}B)$$
$$+ \sum_{l=0}^{\infty} c_l^{+\,\mu}(x)\mathrm{Tr}(AmBm^{2l}) + d_1^{+\,\mu}(x)\mathrm{Tr}(Am^2)\mathrm{Tr}(Bm)$$
$$+ d_2^{+\,\mu}(x)\mathrm{Tr}A\,\mathrm{Tr}(BZ) + d_3^{+\,\mu}(x)\mathrm{Tr}(AZB)$$
$$+\text{less singular terms.} \qquad (4.9)$$

We no longer have the symmetry $A \leftrightarrow B$ of our former sum rules. This extends the number of nontrivial vacuum expectation values (Σ's) from 10 to 16 and, consequently, gives us many more sum rules. We also note

$$\langle 0|\bar\psi(x)\gamma_\mu(1+\gamma_5)A\psi(x)\bar\psi(0)(1+\gamma_5)B\psi(0)|0\rangle^* = \langle 0|\bar\psi(0)(1-\gamma_5)B^\dagger\psi(0)\bar\psi(x)\gamma_\mu(1+\gamma_5)A^\dagger\psi(x)|0\rangle$$
$$= \langle 0|\bar\psi(x)\gamma_\mu(1+\gamma_5)A^\dagger\psi(x)\bar\psi(0)(1-\gamma_5)B^\dagger\psi(0)|0\rangle\,. \qquad (4.10)$$

The last step is clearly true if we take x spacelike. We conclude that the sum rules in the $VS-AP$ case are just the charge-conjugate sum rules of the $VS+AP$ case.

To get sum rules we need those linear combinations that eliminate all dependence on $a^+(x)$, $b^+(x)$, $c^+(x)$, and $d^+(x)$. We now define

$$\tilde\Sigma_i^\mu = \langle 0|\bar\psi(x)\gamma^\mu(1+\gamma_5)A_i\psi(x)\bar\psi(0)(1+\gamma_5)B_i\psi(0)|0\rangle\,,$$

where A_i, B_i ($i=1$–10) are defined in Eq. (3.2b) and

$$\begin{aligned} &A_{11}=B_2,\quad B_{11}=A_2,\\ &A_{12}=B_3,\quad B_{12}=A_3,\\ &A_{13}=B_4,\quad B_{13}=A_4,\\ &A_{14}=B_6,\quad B_{14}=A_6,\\ &A_{15}=B_7,\quad B_{15}=A_7,\\ &A_{16}=B_9,\quad B_{16}=A_9. \end{aligned} \qquad (4.11)$$

In the $SU(2)\times U(1)\times U(1)$ limit we find the following $VS+AP$ sum rules:

$$\tilde\Sigma_3-\tilde\Sigma_4+\frac{m_c}{m_0}\tilde\Sigma_{11}-\frac{m_c}{m_0}\tilde\Sigma_1, \qquad (4.12a)$$

$$\tilde\Sigma_1-\tilde\Sigma_{12}+\frac{m_0}{m_s}\tilde\Sigma_{13}-\frac{m_0}{m_s}\tilde\Sigma_2, \qquad (4.12b)$$

$$\tilde\Sigma_1+\left(\frac{m_c^2-m_0^2}{m_s^2-m_c^2}\right)\tilde\Sigma_{11}+\left(\frac{m_0^2-m_s^2}{m_s^2-m_c^2}\right)\tilde\Sigma_{12}, \qquad (4.12c)$$

$$3\tilde\Sigma_5-2\tilde\Sigma_2-\left(1+2\frac{m_s}{m_0}\right)\tilde\Sigma_1-\frac{2m_s}{m_0}\tilde\Sigma_{11}+\frac{m_0-m_s}{2m_c}(3\sqrt2\,\tilde\Sigma_6-\sqrt6\,\tilde\Sigma_7), \qquad (4.12d)$$

$$3\sqrt2\,\tilde\Sigma_{14}-\left(1+\frac{m_s}{m_0}\right)\tilde\Sigma_1+\tilde\Sigma_2+\frac{m_s}{m_0}\tilde\Sigma_{11}+\frac{m_0-m_s}{m_c}\left(6\tilde\Sigma_8-\sqrt{12}\tilde\Sigma_9-6\tilde\Sigma_4+\frac{6m_c}{m_0}\tilde\Sigma_{11}-\frac{6m_c}{m_0}\tilde\Sigma_{12}\right), \qquad (4,12e)$$

$$3\sqrt2\,\tilde\Sigma_6-\left(1+\frac{m_s}{m_0}\right)\tilde\Sigma_1+\tilde\Sigma_2+\frac{m_s}{m_0}\tilde\Sigma_{11}+\frac{2m_0+m_s-3m_c}{4m_c}(3\sqrt2\,\tilde\Sigma_6-\sqrt6\,\tilde\Sigma_7), \qquad (4.12f)$$

$$12\tilde\Sigma_8-\left(2-\frac{m_s}{m_0}\right)\tilde\Sigma_1-\tilde\Sigma_2-9\tilde\Sigma_4+\left(9\frac{m_c}{m_0}-\frac{m_s}{m_0}\right)\tilde\Sigma_{11}-9\frac{m_c}{m_0}\tilde\Sigma_{12}$$
$$+\frac{2m_0+m_s-3m_c}{4m_c}\left(12\tilde\Sigma_8-2\sqrt{12}\,\tilde\Sigma_9-12\tilde\Sigma_4+\frac{12m_s}{m_0}\tilde\Sigma_{11}-\frac{12m_s}{m_0}\tilde\Sigma_{12}\right), \qquad (4.12g)$$

$$3\sqrt{2}\,\tilde{\Sigma}_6-\sqrt{6}\,\tilde{\Sigma}_7-\frac{2(m_0^{\,2}-m_s^{\,2})}{2m_0^{\,2}+m_s^{\,2}-3m_c^{\,2}}\left(6\tilde{\Sigma}_8-\sqrt{12}\,\tilde{\Sigma}_9-6\tilde{\Sigma}_4+6\,\frac{m_c}{m_0}(\tilde{\Sigma}_{11}-\tilde{\Sigma}_{12})\right). \tag{4.12h}$$

V. DISCUSSION AND COMPARISON WITH OTHER WORK

Using Wilson's approach to spectral-function sum rules together with an explicit expansion of the Wilson coefficient functions in powers of mass, we have been able to calculate sum rules for the spectral functions that appear in propagators of currents and/or (pseudo)scalar densities in quark-gluon theories. These sum rules are true if two conditions are met: (1) the symmetry of the quark mass matrix is SU(2)×U(1)×U(1); (2) the theory is asymptotically free. Our conclusions definitely apply to asymptotically free theories since all anomalous dimensions are zero in such theories. In this we differ from some previous attempts to obtain sum rules using Wilson's methods. The price we pay is that the coefficients of the various terms in our sum rules contain quark masses explicitly since we have used the known mass expansion of the Wilson coefficient functions to cancel the leading singularities. Auvil and Deshpande[12] do not use the explicit form of the mass expansions to cancel singularities, so that the sum rules they obtain are true only if one assumes that the mass operator $\psi\bar{\psi}$ has anomalous dimension δ which is negative and large enough in magnitude to reduce the singularities ($\delta<-\frac{1}{3}$). Broadhurst,[13] by using some of the freedom inherent in the mass expansion, is able to write down two sum rules which are true under a milder restriction on δ ($\delta<0$). The sum rules he obtains are then used to show that even this milder condition is inconsistent with experiments on K_{l3} decay. Using our methods, on the other hand, no sum rule of the type Broadhurst considers [involving only Σ_1, Σ_2, Σ_5 in a broken SU(3)×SU(3) theory] can be obtained, so there are no further restrictions on δ.

One might be tempted to proceed further with asymptotically free theories and calculate in perturbation theory the logarithmic deviations from canonical scaling. If the deviations reduced the asymptotic singularities of current products, then there would be additional sum rules to the ones we found in Secs. III and IV. This is an approach used by Hagiwara and Mohapatra. We believe, however, that those sum rules which hold because invariants are merely logarithmically better than $1/x^4$ or $1/x^2$ will converge so slowly that they will be useless for any low-mass saturation scheme.

In deriving the sum rules of this paper, we have made no assumptions about the size of quark masses aside from setting the mass of the proton quark equal to that of the neutron quark. However, if one is willing to assume that quark masses are in some sense "small," many more sum rules are derivable. In the extreme case—neglecting quark masses altogether—sum rules of the type originally derived by Glashow, Schnitzer, and Weinberg[2] and Das, Mathur, and Okubo[15] are true:

$$\int d\mu^2\left(\rho^{(0)}_{\alpha\beta}(\mu^2)+\frac{\rho^{(1)}_{\alpha\beta}(\mu^2)}{\mu^2}\right)=S\,\delta_{\alpha\beta}, \tag{5.1}$$

$$\int d\mu^2\rho^{(1)}_{\alpha\beta}(\mu^2)=Z\delta_{\alpha\beta}, \tag{5.2}$$

where α, β are SU(4) indices, with $\alpha=\beta=0$ excluded.

One could also take the somewhat more conservative position that terms quadratic or higher in quark masses may be dropped, but linear terms must be kept. In that case (5.1) is still true but (5.2) is no longer valid. This is the condition for the results recently derived by Hagiwara and Mohapatra[16] and Mathur, Okubo, and Borchardt[17], to be valid in quark-gluon theories [both groups use sum rules (5.1)].[18]

By neglecting terms higher than linear in masses, one can also derive new second and third sum rules which, when saturated with low mass states, produce results similar to those of Gell-Mann, Oakes, and Renner[19] and Glashow and Weinberg.[20] The relation between these old results of current algebra and the new sum rules will be explored more fully in a forthcoming paper.

Another approach would be to expand not in quark masses but in quark mass differences. This is the technique used by Prasad.[21] By working to lowest order in octet and 15-plet breaking of SU(4), and without using Wilson's method, he is able to obtain asymptotic sum rules. These sum rules differ from the ones given here since we make no such assumptions about the size of the masses and must cancel singularities that appear in higher order to obtain valid sum rules. In addition, Prasad neglects the 27-plet contributions to the expansion, but this would exclude, for example, tr$(AmBm)$ which we are forced to cancel by choosing suitable linear combinations of spectral functions.

The problem of obtaining useful information by saturating the sum rules we have derived is a difficult one. There are several reasons for this:

(1) There are no combinations which hold both for $V+A$ and $V-A$ so that one cannot isolate the vector from the axial-vector parts. The axial couplings of particles which do not decay weakly via axial currents are unmeasurable until the day when hadron production in neutrino-electron scattering can be measured. (2) The fact that many of our sum rules merely imply small corrections to sum rules that were true in the SU(3) limit (like the $V-A$ rule for the ρ and A_1, Σ_1^-) means that slight inaccuracies introduced by the saturation scheme or by the uncertainties in measured constants (like g_ρ) will produce large inaccuracies in our predictions about the corrections. (3) Despite the fact that we have many sum rules available (especially in the case of the mixed, scalar-vector sum rules of Sec. IV, which can be saturated with the pseudoscalar and scalar mesons), there are still so many unknown parameters that we cannot predict anything without some additional assumptions about the mixing of the "diagonal" particles (the η, η', and "paracharmonium") and their couplings to currents. Work is in progress on this problem.

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APPENDIX: SYMMETRIES OF THE WILSON COEFFICIENT FUNCTIONS

We now present a simple proof that the Wilson coefficient functions obey all the symmetries of the Lagrangian, whether or not these symmetries are spontaneously broken.

Recall first the definition of the operator-product-expansion coefficient functions. For two "currents" J_α and J_β (which may be scalars, vectors, or what you will) the operator-product expansion reads (in the time-ordered version)

$$\int \langle F|T\{J_\alpha(x)J_\beta(0)\}|I\rangle e^{-ik\cdot x}d^4x \xrightarrow[k\to\infty]{} \sum_N U^N_{\alpha\beta}(k)\langle F|O_N|I\rangle , \qquad (A1)$$

where F and I are arbitrary "out" and "in" states; k is an arbitrary four-momentum which goes to infinity in a fixed direction; O_N is a product of field operators and their derivatives, all evaluated at the same point $x=0$; and $U^N_{\alpha\beta}$ is a c-number function of k, which is finite except for Z factors needed to cancel divergences in the matrix elements of O_N. In perturbation theory, the asymptotic behavior of $U^N_{\alpha\beta}(k)$ is

$$U^N_{\alpha\beta}(k)\sim k^D \times \text{powers of } \log k, \qquad (A2)$$

where

$$D\equiv -4+d(J_\alpha)+d(J_\beta)-d(O_N) \qquad (A3)$$

with $d(\mathcal{O})$ the naive dimension of any operator $\mathcal{O}$.

Now consider a symmetry group G of the Lagrangian, with conserved currents $\mathcal{J}^\mu_a(x)$. We may suppose that the operators J_α and O_N form G multiplets in the sense that

$$\int d^3x[\mathcal{J}^0_a(\vec{x},t),J_\alpha(\vec{y},t)]=-\sum_\beta(\mathfrak{T}^J_a)_{\alpha\beta}J_\beta(\vec{y},t), \qquad (A4)$$

$$\int d^3x[\mathcal{J}^0_a(\vec{x},t),O_N(\vec{y},t)]=-\sum_M(\mathfrak{T}^0_a)_{NM}O_M(\vec{y},t), \qquad (A5)$$

where $\mathfrak{T}^J_a$, $\mathfrak{T}^0_a$ form representations of the Lie algebra of G:

$$[\mathfrak{T}^J_a,\mathfrak{T}^J_b]=iC_{abc}\mathfrak{T}^J_c,$$

$$[\mathfrak{T}^0_a,\mathfrak{T}^0_b]=iC_{abc}\mathfrak{T}^0_c.$$

Our theorem states that the U functions are invariant under G in the sense that

$$0=\sum_\gamma(\mathfrak{T}^J_a)_{\alpha\gamma}U^N_{\gamma\beta}(k)+\sum_\gamma(\mathfrak{T}^J_a)_{\beta\gamma}U^N_{\alpha\gamma}(k) - \sum_M(\mathfrak{T}^0_a)_{MN}U^M_{\alpha\beta}(k). \qquad (A6)$$

When the symmetry associated with $\mathcal{J}^\mu_a$ is *not* spontaneously broken, (A6) is a trivial consequence of the G invariance of the theory and the G transformation rules (A4) and (A5). Let us therefore turn immediately to the case of a spontaneously broken symmetry. The current $\mathcal{J}^\mu_a$ will then couple to a massless Goldstone boson π_a. Our proof proceeds by an examination of the matrix elements

$$\langle F|T\{J_\alpha(x)J_\beta(0)\}|I+\pi_a\rangle ,$$

with π_a taken to have an infinitesimal four-momentum. We can first use the usual soft-pion technique to write

$$\langle F|T\{J_\alpha(x)J_\beta(0)|I+\pi_a\rangle = -i(2E_\pi)^{-1/2}(2\pi)^{-3/2}F_a^{-1}\int dz\frac{\partial}{\partial z^\mu}\langle F|T\{\mathcal{J}^\mu_a(z)J_\alpha(x)J_\beta(0)\}|I\rangle$$

$$=i(2E_\pi)^{-1/2}(2\pi)^{-3/2}F_a^{-1}\sum_\gamma[(\mathfrak{T}^J_a)_{\alpha\gamma}\langle F|T\{J_\gamma(x)J_\beta(0)\}|I\rangle + (\mathfrak{T}^J_a)_{\beta\gamma}\langle F|T\{J_\alpha(x)J_\gamma(0)\}|I\rangle]$$

and then apply the operator-product expansion (A1) to obtain

$$\int d^4x \langle F | T\{J_\alpha(x) J_\beta(0)\} | I + \pi_a \rangle e^{-ik\cdot x} \to i(2E_\pi)^{-1/2}(2\pi)^{-3/2} F_a^{-1} \sum_\gamma \sum_N [(\mathfrak{T}_a^J)_{\alpha\gamma} U^N_{\gamma\beta}(k) + (\mathfrak{T}_a^J)_{\beta\gamma} U^N_{\alpha\gamma}(k)] \langle F | O_N | I \rangle . \quad \text{(A7)}$$

Alternatively, we can first apply the operator-product expansion

$$\int d^4x \langle F | T\{J_\alpha(x) J_\beta(0)\} | I + \pi_\alpha \rangle e^{-ik\cdot x} \to \sum_M U^M_{\alpha\gamma}(k) \langle F | O_M | I + \pi_a \rangle$$

and then use the soft-pion technique

$$\begin{aligned} \int d^4x \langle F | T\{J_\alpha(x) J_\beta(0)\} | I + \pi_a \rangle e^{-ik\cdot x} &\to -i(2E_\pi)^{-1/2}(2\pi)^{-3/2} F_a^{-1} \sum_M U^M_{\alpha\beta}(k) \int dz \frac{\partial}{\partial z^\mu} \langle F | T\{\mathcal{J}^\mu_a(z), O_M\} | I \rangle \\ &= i(2E_\pi)^{-1/2}(2\pi)^{-3/2} F_a^{-1} \sum_{NM} U^M_{\alpha\beta}(k) (\mathfrak{T}^0_a)_{MN} \langle F | O_N | I \rangle . \end{aligned} \quad \text{(A8)}$$

The two formulas (A7) and (A8) must agree for all states F and I, so the coefficients of the matrix elements $\langle F | O_N | I \rangle$ must be equal, yielding the desired result (A6).

At this point the reader may feel that we have proved too much. Are there not contributions to the functions $U^N_{\alpha\beta}(k)$ arising from diagrams in which scalar particles, fermion-antifermion pairs, etc., disappear into the vacuum? If so, then such diagrams could infect the coefficient functions with whatever spontaneous symmetry breaking is taking place, though perhaps not in the leading order as $k \to \infty$.

The answer is that a diagram, in which some set of particle lines disappears into the vacuum, is not considered a correction to some U function, but instead regarded as a contribution to a matrix element $\langle F | O_N | I \rangle$, in which some proper subset of factors in O_N appears in a vacuum expectation value. Of course, the expansion (A1) could just as well have been performed in terms of Wick products $:O_N:$ defined to exclude such terms, but then the argument leading to (A8) would have failed, as the commutator of $\mathcal{J}^0_a$ with a Wick product is *not* a linear combination of Wick products.

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