Failure of inelastic N/D equations to generate the ρ resonance

E. P. Tryon

Department of Physics and Astronomy, Hunter College of The City University of New York, New York, New York 10021

(Received 7 October 1974)

The effective left cut of the $\pi\pi$ P wave is rigorously determined by comparing the usual partial-wave dispersion relation with a new dispersion relation of the kind derived by Roskies and Roy. A recent experimental result for the inelasticity below $M_{\pi\pi} = 1.9$ GeV is inserted into the Frye-Warnock N/D equations. The solutions are nonresonant, even when the inelasticity is increased by 50% over the experimental value. Because of the rigorous basis for the effective left cut, I conclude that the ρ resonance is not generated by forces in the $\pi\pi$ channel.

I. INTRODUCTION AND SUMMARY

In a recent Letter,[1] I argued that elastic N/D equations for the $\pi\pi$ P wave are incapable of generating the ρ resonance for any plausible choice of the left cut. I noted, however, that inelasticity at some energy above m_ρ would be conducive to a generation of the ρ. A recent experiment[2] indicates that the elasticity η_1^1 may fall to 0.5 near $M_{\pi\pi} = 1.6$ GeV. Hence I study here the possibility that inelastic N/D equations, as formulated by Frye and Warnock,[3] may be capable of generating the ρ resonance. The answer is negative, because the exchange forces are simply too weak.

The major innovation of this work is a clean and precise determination of the effective left cut, based on a dispersion relation of the type first studied by Roskies and Roy. The analysis proceeds as follows.

II. NOTATION AND CONVENTIONS

Let us use units wherein $m_\pi = \hbar = c = 1$. We denote the $\pi\pi$ elastic amplitude with isospin "I" in the direct, s channel by $A^I(s,t)$, and the amplitude with isospin I in the t channel by $T^I(s,t)$. The A^I and T^I are related by

$$T^I(s,t) = \sum_{I'=0}^{2} C_{II'} A^{I'}(s,t) , \tag{1}$$

where $C = C^{-1}$ denotes the s-t crossing matrix. The elements we shall need here are $C_{1I} = \frac{1}{3}, \frac{1}{2}$, and $-\frac{5}{6}$ for $I = 0$, 1, and 2, respectively.

We normalize the amplitudes such that

$$A^I(s,t) = \sum_{l=0}^{\infty} (2l+1) A^{(l)I}(s)\, P_l(z) , \tag{2}$$

where $z \equiv \cos\theta_s$ is given by

$$z = 1 + \frac{2t}{s-4} . \tag{3}$$

The $A^{(l)I}$ vanish when $(l+I)$ is odd. The nonzero partial waves satisfy

$$A^{(l)I}(s) = \tfrac{1}{2} i\, Q^{-1}(s) [1 - \eta_l^I \exp(2i\delta_l^I)] , \tag{4}$$

where

$$Q(s) \equiv \left(\frac{s-4}{s}\right)^{1/2} ,$$

η_l^I denotes the elasticity ($0 \le \eta_l^I \le 1$), and the phase shifts δ_l^I are real.

III. DISPERSION RELATIONS AND THE LEFT CUT

I assume that $A^{(1)1}$ satisfies

$$A^{(1)1}(s) = \frac{s-4}{\pi}\left(\int_{-\infty}^{0} ds' + \int_4^{\infty} ds'\right) \frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)(s'-s)}$$
$$\equiv A_L(s) + A_R(s) , \tag{5}$$

and also that A^1 satisfies[4]

$$A^1(s,t) = \frac{t-u}{\pi}\int_4^{\infty} \frac{ds'}{(s'-t)(s'-u)} \left(\mathrm{Im}T^1(s',t) + \frac{(s-t)(2s'+t-4)}{(s'-s)(s'+2t-4)}\, \mathrm{Im}A^1(s',t)\right) , \tag{6}$$

where

$$u \equiv 4 - s - t .$$

Equation (6) is valid for arbitrary s when t is real ($\pm i\epsilon$) and $-32 \leq t \leq 4$.[4]

Bose symmetry implies that $A^1(s,t)$ is an odd function of z, so we can write

$$A^{(1)1}(s) = \int_0^1 dz\, P_1(z) A^1(s,t) \ . \tag{7}$$

Equations (6) and (7) yield an $A^{(1)1}$ valid for $-4 \leq s \leq 68$, where the upper limit corresponds to $M_{\pi\pi} = 1.14$ GeV.

It will prove convenient to introduce a parameter Λ such that the $\mathrm{Im}A^{(l)I}$ with $l \geq 4$ are negligible when $s < \Lambda$. In practice, I will use $\Lambda^{1/2} = 1.9$ GeV (see Appendix A, where my input absorptive parts are enumerated). Then Eqs. (6) and (7) yield

$$A^{(1)1}(s) = \frac{1}{\pi}\int_4^{\Lambda} ds' \left[\frac{(s-4)\,\mathrm{Im}A^{(1)1}(s')}{(s'-4)(s'-s)} + \sum_{I=0}^{2} C_{1I} \sum_{l=0}^{3} M_l(s',s)\, \mathrm{Im}A^{(l)I}(s')\right] + A^{(1)1}_{\mathrm{HE}}(s) \ , \tag{8}$$

where the functions $M_l(s',s)$ are given in Appendix B, and

$$A^{(1)1}_{\mathrm{HE}}(s) \equiv \frac{s-4}{\pi}\int_0^1 z^2\, dz \int_{\Lambda}^{\infty} \frac{ds'}{(s'-t)(s'-u)}\left[\mathrm{Im}T^1(s',t) + \frac{(s-t)(2s'+t-4)}{(s'-s)(s'+2t-4)}\,\mathrm{Im}A^1(s',t)\right] \ . \tag{9}$$

Upon comparing Eqs. (5) and (8), we obtain

$$A_L(s) = \frac{1}{\pi}\sum_{I=0}^{2} C_{1I}\sum_{l=0}^{3}\int_4^{\Lambda} ds'\, M_l(s',s)\,\mathrm{Im}A^{(l)I}(s') + A^{(1)1}_{\mathrm{HE}}(s) - \frac{s-4}{\pi}\int_{\Lambda}^{\infty} ds'\, \frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)(s'-s)} \ , \tag{10}$$

which is valid for $-4 \leq s \leq 68$ [subject of course to our assumption that the $\mathrm{Im}A^{(l)I}(s)$ are negligible for $l \geq 4$ when $s < \Lambda$].

For $-32 \leq s \leq 0$, crossing and analyticity imply that[5]

$$\mathrm{Im}A^{(1)1}(s) = \frac{2}{s-4}\int_4^{4-s} ds'\, P_1\!\left(1 + \frac{2s'}{s-4}\right)\sum_{I=0}^{2} C_{1I}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}A^{(l)I}(s')\, P_l\!\left(1 + \frac{2s}{s'-4}\right) \ . \tag{11}$$

In order to determine the unknown distant left cut, we decompose $A_L(s)$ into parts coming from the distant and near sections of the left cut:

$$A_L(s) = \frac{s-4}{\pi}\left(\int_{-\infty}^{-32} ds' + \int_{-32}^{0} ds'\right)\frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)(s'-s)}$$
$$\equiv A_{DL}(s) + A_{NL}(s) \ . \tag{12}$$

Using Eq. (11) for $-32 \leq s \leq 0$, we can write

$$A_{NL}(s) = \frac{1}{\pi}\sum_{I=0}^{2} C_{1I}\sum_{l=0}^{\infty}\int_4^{36} ds'\, H_l(s',s)\,\mathrm{Im}A^{(l)I}(s') \ , \tag{13}$$

where the functions $H_l(s',s)$ are given in Appendix B for $l=0$ and 1. I shall assume in this work that the $\mathrm{Im}A^{(l)I}$ with $l \geq 2$ are negligible for $s < 36$ ($M_{\pi\pi} < 0.84$ GeV).[6]

Since $A_{DL} = A_L - A_{NL}$, Eqs. (10) and (13) yield $A_{DL}(s)$ for $-4 \leq s \leq 68$. This is the result for A_{DL} upon which I shall base my model for the distant left cut.

IV. RESONANCE GENERATION

A resonance is not likely to result from exchange forces unless A_L becomes more positive than the unitarity bound would permit for $\mathrm{Re}A^{(1)1}$, over some range of energy *above* the resonance:

$$A_L(s) > \tfrac{1}{2} Q^{-1}\eta_1^1 \ . \tag{14}$$

The inequality (14) implies (together with unitarity) that $A_R(s)$ is *negative*. Since $\mathrm{Im}A^{(1)1}$ is positive-definite along the right-hand cut, a negative value for A_R requires that $\mathrm{Im}A^{(1)1}$ have a relatively large value *below* the range of s where A_R is negative. This is the mechanism of resonance generation: $\mathrm{Im}A^{(1)1}$ develops a peak at some *lower* energy in order to render A_R negative, thereby avoiding the violation of unitarity which would otherwise be implied by the inequality (14).

V. STRENGTH OF THE LEFT CUT

Let us now examine the $A_L(s)$ implied by Eq. (10). A_L was defined in such a way as to be analytic for $s > 0$, so the right-hand side of Eq. (10) must share this analyticity. It is readily verified that the functions $M_l(s',s)$ are analytic in s for $s > 0$. Furthermore, the function $A^{(1)1}_{\mathrm{HE}}(s)$ has only the right-hand cut of the P wave with $s > \Lambda$ if (a) the $\mathrm{Im}T^1(s',t)$ and $\mathrm{Im}A^1(s',t)$ used in Eq. (9) are analytic in t, and (b) $\mathrm{Im}A^1(s',t)$ satisfies Bose symmetry:

$$\mathrm{Im}A^1(s',t) = -\mathrm{Im}A^1(s',4-s'-t) \ . \tag{15}$$

Hence the right-hand cut of $A_{HE}^{(1)1}$ will be precisely canceled by that of the integral over $\text{Im}A^{(1)1}$ on the right-hand side of Eq. (10), provided that $\text{Im}A^{(1)1}$ is precisely the P-wave projection of the $\text{Im}A^1(s', t)$ used in Eq. (9). I shall use input absorptive parts which satisfy all of these conditions (see Appendix A).

Although Eq. (10) is only valid for $-4 \leq s \leq 68$, my choice of input absorptive parts renders the right-hand side of Eq. (10) analytic for arbitrarily large positive s, so the analytic continuation of $A_L(s)$ can be achieved simply by evaluating Eq. (10) for large s. There is, however, a limitation to this procedure: Our neglect of $\text{Im}A^{(l)I}$ with $l \geq 4$ for $s < \Lambda$ becomes a less good approximation as s increases above 68, for we are then using a truncated version of a formally divergent series.[4] Hence our results for $A_L(s)$ lose their rigor for $s > 68$ ($M_{\pi\pi} > 1.14$ GeV), but may well be a good approximation for substantially higher energies (i.e., the series may well be asymptotic).

In Table I are presented the individual contributions to, and total value of, the right-hand side of Eq. (10) for $M_{\pi\pi} = 0.5$, 1.0, 1.5, and 2.0 GeV. The latter two energies lie above $s = 68$, so the corresponding values for A_L can only be regarded as approximate, with errors which are, I would hope, negligible, but are not accessible to my powers of estimation.

The most significant feature of Table I is that $A_L(s)$ is small, relative to the unitarity bound on $\text{Re}A^{(1)1}$. For example, η_1^1 would have to fall as low as 0.10, 0.25, or 0.39 at $M_{\pi\pi} = 1.0$, 1.5, or 2.0 GeV, respectively, in order for the inequality (14) to be satisfied. There is no experimental evidence that η_1^1 reaches any of these small values at the required energies, so Table I indicates that the ρ resonance is *not* generated by exchange forces in the $\pi\pi$ channel.

Another significant feature of Table I is that no single contribution to $A_L(s)$ is dominant. Hence even a 100% increase in the largest contribution would be unlikely to result in a generation of the ρ. In reality, the experimental uncertainties in the input absorptive parts are estimated to be fairly modest—none more than 30% (see Appendix A).

VI. MODEL FOR DISTANT LEFT CUT

The remarks of the preceding section indicate that the ρ resonance is not likely to emerge from N/D equations in the $\pi\pi$ channel. An explicit calculation, however, remains desirable. Toward this end, I now propose a simple model for the effective left cut of $A^{(1)1}$.

For $-32 \leq s \leq 0$, we of course use Eq. (11). As mentioned earlier, absorptive parts with $l \geq 2$

TABLE I. Individual contributions to, and net values of, right-hand side of Eq. (10) for $A_L(s)$. The contribution from $\text{Im}A^{(1)1}(s')$ for $s' > \Lambda$ is included in A^1_{Regge}, since one must use the P-wave projection of A^1_{Regge} for $A^{(1)1}$ (see Sec. V).

$M_{\pi\pi}$ (GeV)	0.5	1.0	1.5	2.0
S_0	0.018	0.039	0.044	0.043
S_2	−0.006	−0.015	−0.020	−0.022
P	−0.019	−0.031	−0.010	0.021
D_0	0.006	0.014	0.010	0.006
D_2	−0.000	−0.000	−0.000	−0.000
F	0.002	0.016	0.037	0.056
T^1_{Regge}	0.011	0.035	0.040	0.035
A^1_{Regge}	−0.003	0.001	0.026	0.060
Total	0.010	0.057	0.128	0.199

below $M_{\pi\pi} = 0.84$ GeV are neglected (which is certainly a good approximation). The resulting $\text{Im}A^{(1)1}$ is shown in Fig. 1.

For $s < -32$, let us represent $\text{Im}A^{(1)1}$ by an expression of the form

$$\text{Im}A^{(1)1}(s) = \frac{a + bs}{s^2} + \sum_m c_m \cos(m\pi x) , \qquad (16)$$

where a and b will be chosen to reproduce the values implied by Eq. (11) for the zeroth and first derivatives of $\text{Im}A^{(1)1}$ at $s = -32$, while

$$x \equiv \left(\frac{\sigma - 32}{\sigma + s}\right)^\tau$$

for some $\sigma < 32$ and $\tau > 0$ remaining to be selected. Hence x ranges from 0 to 1 as s varies from $-\infty$ to -32.

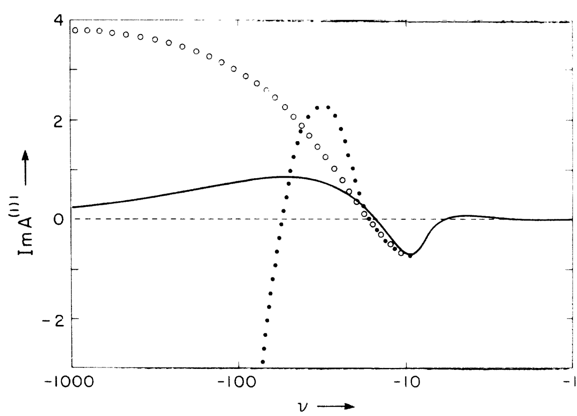

FIG. 1. $\text{Im}A^{(1)1}$ as a function of $\nu \equiv \frac{1}{4}(s - 4)$. For $\nu \geq -9$ ($s \geq -32$), the solid curve displays the result of Eq. (11). For $\nu < -9$, the solid curve displays the result of Eq. (17). For the sake of comparison, open circles display the net contribution of low-energy ($M_{\pi\pi} \leq 1.5$ GeV) S waves and P wave to the right-hand side of Eq. (11) in its forbidden region, while closed circles display the net contribution of S, P, and D waves.

Since proper zeroth and first derivatives at $s=-32$ are being embodied in a and b, we impose the constraint

$$\sum_m (-1)\, c_m = 0 \ .$$

To avoid the generation of spurious poles in $A^{(1)1}$ when N/D equations are solved, we also constrain $\mathrm{Im}A^{(1)1}$ to vanish at $-\infty$, i.e.,

$$\sum_m c_m = 0 \ .$$

σ and τ are selected in such a way as to minimize the number of terms in the Fourier series required to reproduce, to good approximation, the $A_{DL}(s)$ implied by Eqs. (10), (12), and (13). A surprisingly small number of terms is sufficient. Specifically, I find that

$$\mathrm{Im}A^{(1)1}(s) = \frac{1351+64.4s}{s^2} + 0.609[1-\cos(2\pi x)] \ , \tag{17}$$

with

$$x \equiv \left(\frac{-17}{15+s}\right)^{0.36} \ ,$$

reproduces the $A_{DL}(s)$ implied by Eqs. (10), (12), and (13) within ± 0.0001 for $-4 \leqslant s \leqslant 68$, and within ± 0.0005 for $68 \leqslant s \leqslant 210$ ($M_{\pi\pi}=2.0$ GeV). Hence Eq. (17) provides a very precise representation of the *effective* distant left cut. The $\mathrm{Im}A_1^{(1)1}$ of Eq. (17) is shown in Fig. 1. I remark that the result seems highly plausible.

VII. SOLUTIONS TO INELASTIC N/D EQUATIONS

As formulated by Frye and Warnock,[3] inelastic N/D equations require a knowledge of the elasticity function $\eta_1^1(s)$ in addition to the left cut of $A^{(1)1}$. I have used the η_1^1 reported by Hyams *et al.*[2] below 1.9 GeV. This η_1^1 equals unity below 1.0 GeV, falls smoothly to 0.5 near 1.6 GeV, then rises smoothly to unity at 1.9 GeV. Above 1.9 GeV, I have assumed that η_1^1 is unity. (The latter assumption cannot be realistic to arbitrarily high energies, but should not affect the solution in the ρ region.)

For integrations over the distant left cut, x has been used as the integration variable; a finite range of integration was thereby obtained. The N/D equations were then solved by matrix inversion; 100 mesh points were used for the integrations. The resulting δ_1^1 is nonresonant, and remains smaller than 6° below 1 GeV.

To see if greater inelasticity would generate the ρ, I repeated the calculation with an η_1^1 given by

$$(1-\eta_1^1) = 1.5[1-\eta_1^1(\text{Ref. 2})] \ .$$

In this case, η_1^1 falls to 0.3 near 1.6 GeV. The resulting δ_1^1 remains less than 7° below 1 GeV. The small value of η_1^1 near 1.6 GeV does produce a broad ($\Gamma=0.24$ GeV), highly inelastic resonance at 1.57 GeV, but there is no hint of ρ generation. In both the preceding cases, I have established that D is free of zeros on the physical sheet, hence that the solutions $A^{(1)1}$ are free of spurious poles.

In view of the rigorous basis for my distant left cut, I regard the preceding two solutions as conclusive evidence that the ρ resonance is not generated by forces in the $\pi\pi$ channel.

APPENDIX A

Between threshold and $M_{\pi\pi}=0.9$ GeV, I assume elastic unitarity, and that[7]

$$Q\cot\delta_0^0 = \frac{16.4}{s-0.05} - 0.36 \ , \tag{A1}$$

$$Q\cot\delta_0^2 = \frac{-45.8}{s-2.04} - 0.97 \ , \tag{A2}$$

$$Q\cot\delta_1^1 = \frac{97.7}{s-4} - 2.79 - 0.0262s \ . \tag{A3}$$

Equation (A1) corresponds to an S-wave scattering length $a_0=0.26$, with $\delta_0^0=43°$, 73°, and 89° at $M_{\pi\pi}=0.50$, 0.70, and 0.90 GeV, respectively. Equation (A2) corresponds to $a_2=-0.041$, with $\delta_0^2=-9°$, $-18°$, and $-24°$ at 0.50, 0.70, and 0.90 GeV, respectively. Equation (A3) corresponds to $a_1=0.040$, with $m_\rho=0.770$ GeV, and $\Gamma_\rho=0.146$ GeV. All $\mathrm{Im}\,A^{(l)I}$ with $l\geqslant 2$ below 0.9 GeV are neglected.

Between 0.9 and 1.9 GeV, I use the S, P, D, and F-wave phase shifts and elasticities of Hyams *et al.*[2] for $I=0$ and 1, and the S- and D-wave phase shifts and elasticities of Durusoy *et al.*[8] for $I=2$. All $\mathrm{Im}\,A^{(l)I}$ with $l\geqslant 4$ below 1.9 GeV are neglected.

Above 1.9 GeV, Regge theory is used to evaluate the $\mathrm{Im}\,T^1(s,t)$ and $\mathrm{Im}\,A^1(s,t)$ in Eq. (9). It is assumed that

$$\mathrm{Im}\,T^0(s,t) = \gamma_P(t)(s/\bar{s})^{\alpha_P(t)} + \gamma_f(t)(s/\bar{s})^{\alpha_f(t)}, \tag{A4}$$

$$\mathrm{Im}\,T^1(s,t) = \gamma_\rho(t)(s/\bar{s})^{\alpha_\rho(t)} \ , \tag{A5}$$

$$\mathrm{Im}\,T^2(s,t) = 0 \ , \tag{A6}$$

where $\bar{s}\equiv 1$ GeV^2 defines the scale of the γ's.

For Pomeranchon exchange, I use the parametrization and experimental results of Robertson and Walker,[9]

$$\alpha_P(t)=1\,, \tag{A7}$$

$$\gamma_P(t)=1.2\exp[0.3(t/\bar{s})]\,, \tag{A8}$$

which corresponds to an asymptotic total cross section of 15 mb. The uncertainty in the cross section was estimated to be roughly 30%.

Assuming that

$$\alpha_\rho(t)=0.50+0.90(t/\bar{s})\,, \tag{A9}$$

a recent analysis[10] of $\pi\pi$ charge-exchange data led to a result for γ_ρ, which is denoted here by $\bar{\gamma}_\rho$, and which is valid within about ± 0.10 for $-1.0\text{ GeV}^2 \le t \le 0.1\text{ GeV}^2$:

$$\bar{\gamma}_\rho(t)=0.67+1.78(t/\bar{s})+0.41(t/\bar{s})^2-0.17(t/\bar{s})^3\,. \tag{A10}$$

When evaluating the $\mathrm{Im}\,T^1(s,t)$ in Eq. (9), I use $\bar{\gamma}_\rho$. When evaluating the $\mathrm{Im}\,A^1(s,t)$ in Eq. (9), I use a slightly modified γ_ρ, to be described later.

For the effect of f_0 exchange, I assume ρ-f_0 exchange degeneracy, which implies that

$$\alpha_f(t)=\alpha_\rho(t)\,, \tag{A11}$$

$$\gamma_f(t)=\tfrac{3}{2}\gamma_\rho(t)\,. \tag{A12}$$

Crossing symmetry implies that

$$A^1(s,t)=\sum_{I=0}^{2} C_{1I}T^I(s,t)\,. \tag{A13}$$

The Regge forms (A4) and (A5), however, are only valid in the forward hemisphere, and may in fact be poor approximations near $\theta=90°$. As was explained in Sec. V, it is important that the asymptotic expression for $\mathrm{Im}\,A^1(s,t)$ satisfy Bose symmetry:

$$\mathrm{Im}\,A^1(s,t)=-\mathrm{Im}\,A^1(s,u)\,. \tag{A14}$$

Let us therefore use the Regge forms (A4) and (A5) for $\mathrm{Im}\,T^0$ and $\mathrm{Im}\,T^1$, but modify the result of Eq. (A13) by using

$$\mathrm{Im}\,A^1(s,t)=\sum_{I=0}^{2} C_{1I}[\mathrm{Im}\,T^I(s,t)-\mathrm{Im}\,T^I(s,u)]\,. \tag{A15}$$

Equation (A15) will be a good approximation if the functions $\mathrm{Im}\,T^I(s,u)$ are negligible in the forward hemisphere.

The Pomeranchon term in $\mathrm{Im}\,T^0(s,u)$ is negligible in the forward hemisphere because of the exponential decrease of γ_P in Eq. (A8). Furthermore, the f_0 contribution to $\mathrm{Im}\,T^0(s,u)$ and the ρ contribution to $\mathrm{Im}\,T^1(s,u)$ tend to be small in the forward hemisphere because of the slope of α_f and α_ρ. The result (A10) for $\bar{\gamma}_\rho$, however, is a cubic function of t, so that $\bar{\gamma}_\rho(u)$ can be substantial for t near zero. Let us therefore add to $\bar{\gamma}_\rho(t)$ a quartic term which is chosen to make $\gamma_\rho(u)$ vanish for $s^{1/2}=1.9$ GeV, $t=0$:

$$\gamma_\rho(t)=\bar{\gamma}_\rho(t)-0.045(t/\bar{s})^4\,. \tag{A16}$$

The above modification is negligible near $t=0$ (forward scattering), and the $\mathrm{Im}\,A^1(s,t)$ of Eq. (A15) has the correct value at $\theta=90°$ by construction. Although $\gamma_\rho(u)$ does not vanish for higher values of s when $t=0$, the slopes of α_f and α_ρ are sufficient to render the errors inherent to Eqs. (A15) and (A16) completely negligible for all $s^{1/2}>1.9$ GeV. Hence I use Eq. (A15) to evaluate the $\mathrm{Im}\,A^1(s,t)$ in Eq. (9), while using Eq. (A16) for γ_ρ, and simultaneously Eq. (A12) for γ_f.

For the $\mathrm{Im}\,A^{(1)1}$ required above $s=\Lambda$ in Eq. (10), I use the P-wave projection of the asymptotic $\mathrm{Im}\,A^1(s,t)$ described above. As was explained in Sec. V, it is essential that this be done.

APPENDIX B

The functions $M_l(s',s)$ defined implicitly by Eqs. (6), (7), and (8) are given for $l=0$, 1, 2, and 3 by

$$M_0(s',s)=\frac{1}{s-4}[G^+-G^--4]\,,$$

$$M_1(s',s)=\frac{3}{(s-4)(s'-4)}[(s'+2s-4)G^++(3s'-4)G^-+2(2s'-s-4)]\,,$$

$$M_2(s',s)=\frac{5}{(s-4)(s'-4)^2}[(s'^2-8s'+6s's-24s+6s^2+16)G^+-(13s'^2-32s'+16)G^--2(14s'^2-52s'+9s's-20s+4s^2+48)]\,,$$

$$M_3(s',s)=\frac{7}{(s-4)(s'-4)^3}[(s'^3-12s'^2+48s'+12s'^2s+30s's^2-96s's+192s-120s^2+20s^3-64)G^+ +(63s'^3-228s'^2+240s'-64)G^- +\tfrac{1}{3}(372s'^3-1680s'^2+2336s'+24s'^2s-178s's^2+272s's-656s+508s^2-83s^3-960)]\,,$$

where

$$G^{\pm} \equiv 2\left(1+\frac{2s'}{s-4}\right)\ln\left(1\pm\frac{s-4}{2s'+s-4}\right).$$

The functions $H_l(s', s)$ defined implicitly by Eqs. (11), (12), and (13) are given for $l=0$ and 1 by

$$H_0(s', s)=\frac{1}{s-4}\left[L+\tfrac{1}{648}(s'-36)(76-s)\right],$$

$$H_1(s', s)=\frac{3}{(s-4)(s'-4)}\left[(s'+2s-4)L+\tfrac{1}{648}(s'-36)(76s'-s's+140s-272)\right],$$

where

$$L\equiv 2\left(1+\frac{2s'}{s-4}\right)\ln\left[\frac{36(s'+s-4)}{s'(s+32)}\right].$$

---

[1]E. P. Tryon, Phys. Lett. 38B, 527 (1972).

[2]B. Hyams *et al.*, Nucl. Phys. B64, 134 (1973). I use the energy-dependent results.

[3]G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).

[4]E. P. Tryon, Phys. Rev. D 8, 1586 (1973).

[5]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[6]Using an energy-dependent Breit-Wigner formula for the f_0 resonance, I find the f_0 contribution to the right-hand side of Eq. (11) to be less than 0.01% of the ρ contribution for $-32 \leq s \leq 0$. The f_0 contribution to A_{NL} must be comparable, and hence is quite negligible.

[7]E. P. Tryon, Phys. Rev. D 10, 1595 (1974).

[8]N. B. Durusoy *et al.*, Phys. Lett. 45B, 517 (1973).

[9]W. J. Robertson and W. D. Walker, Phys. Rev. D 7, 2554 (1973).

[10]E. P. Tryon, Phys. Rev. D 11, 698 (1975).