Vacuum polarization in a strong Coulomb field. II. Short-distance corrections*

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(Received 6 March 1975)

We continue our study of the behavior of the vacuum polarization near the nucleus, the region of importance for heavy muonic atoms. We show that the vacuum polarization potential induced by a point nuclear charge can be expanded in certain integral and nonintegral powers of the radius r, an expansion involving r^{-1}, r, $r^{2\lambda}$, and higher terms. The coefficient of the r^{-1} term was computed in the previous paper, analytically, to all orders of $Z\alpha$. In this paper we compute the coefficients of the r and $r^{2\lambda}$ terms analytically, to all orders in $Z\alpha$. Our results agree in third order with earlier calculations, which had been done only to this order. Parts of our calculation are considerably simplified by an expansion in powers of the electron mass using formal operator and determinantal techniques.

I. INTRODUCTION AND SUMMARY

Vacuum polarization in heavy muonic atoms is a large effect, and it must be treated to all orders in the coupling strength $Z\alpha$ between the virtual-electron loop and the nucleus. As we discussed in the previous paper,[1] only the short-distance limit of the higher-order terms $[\alpha(Z\alpha)^3, \ldots]$ is important. In that paper[1] we computed, analytically, and to all orders in Za, the higher-order vacuum-polarization point charge which is induced by a point nuclear charge. The induced point charge is the major effect of the higher-order vacuum polarization. In this paper we discuss the nature of smaller terms in the vacuum-polarization potential about a point nuclear charge. In particular, we compute, analytically, and to all orders in $Z\alpha$, the coefficients of the first two short-distance corrections to the induced point charge potential.[2] The following paper[3] discusses the corrections to the vacuum-polarization potential brought about by the finite nuclear size and surveys the experimental situation.

Before entering into the details of our calculation, we shall discuss our results. We develop a short-distance expansion of the vacuum-polarization potential energy, $V_{\rm pol}(r)$, in ascending powers of the radial distance r scaled by the electronic Compton wavelength $\lambda\!\!\!^{-}_e = m^{-1}$. The general form is

$$V_{\rm pol}(r) = m\left[\sum_{l=0}^{\infty} a_{2l-1}(mr)^{2l-1} + a_0 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} a_{2\lambda_k+n}(mr)^{2\lambda_k+n}\right], \qquad (1)$$

where the coefficients a_{2l-1} and $a_{2\lambda_k+n}$ are functions of $Z\alpha$. In the second double sum, $\lambda_k = [k^2-(Z\alpha)^2]^{1/2}$ is the effective angular momentum in the relativistic Coulomb problem for an electron with an angular momentum quantum number $k=j+\frac{1}{2}$. We have omitted from Eq. (1) the well-known portion of the Uehling potential, $\alpha Z\alpha(2\ln mr + 2\gamma + \frac{5}{3})/3\pi r$, which arises from the infinite charge renormalization in order $\alpha(Z\alpha)$. Equation (1) contains all the remaining portions of the Uehling potential, as well as all higher-order contributions. The coefficient of the leading term is, up to a factor, the induced point charge computed in paper I, $a_{-1} = -\alpha\delta Q'/e$. The constant term a_0 does not affect transition energies, and there is no need to compute it. Any of the other terms in the potential (1) can be calculated by the methods of this paper, but the evaluation of only the coefficients a_1 and $a_{2\lambda_1}$ is adequate to determine accurately the energy shifts in heavy muonic atoms.

The coefficients a_{2l-1} of the odd-integer powers in potential (1) are related to an expansion of the electron's Green's function in powers of the squared electron mass. These terms are related essentially to the high-energy behavior of the Green's function. They can be evaluated by formal operator and determinantal techniques. For the first term we find

$$a_1 = \alpha(Z\alpha)\sum_{k=1}^{\infty}\frac{4k}{\pi\lambda_k}\left[\frac{k^2}{4\lambda_k^2-1}\,{\rm Re}\,\psi'(\lambda_k+iZ\alpha) - \frac{\lambda_k+k^2}{2k^2(2\lambda_k+1)}\right], \qquad (2)$$

where $\psi'(z) = (d^2/dz^2)\ln\Gamma(z)$. The first two terms of the series expansion of a_1 in powers of $Z\alpha$ are computed in the Appendix, with the result that

$$a_1 = \frac{\alpha}{\pi}\left\{(Z\alpha) + \left[\tfrac{1}{16}\pi^4 + \tfrac{1}{6}\pi^2 - 6\zeta(3)\right](Z\alpha)^3 + \cdots\right\}, \qquad (3)$$

where $\zeta(p) = \sum n^{-p}$ is the Riemann ζ function. The coefficients that appear here agree with the previous calculations to third order of Blomqvist[4] and Bell,[5] who worked from expressions obtained some time ago by Wichmann and Kroll.[6] Our methods are entirely different from those used in the earlier work.[4-6]

The coefficients $a_{2\lambda_k+n}$ of the noninteger powers in the potential (1) depend essentially only on the low-energy behavior of the electron's Green's function. They can be evaluated from the series development of the Green's function in powers of the radius r. We find that the first of these terms can be expressed as a limit of a regularized integral,

$$a_{2\lambda}=\frac{8\alpha(Z\alpha)}{2\lambda(2\lambda+1)}\frac{\Gamma(-2\lambda)}{\Gamma(2\lambda+1)}\frac{\cos\pi\lambda}{\pi}\times\lim_{\nu\to 0}\int_m^\infty\frac{dq}{m}\left(\frac{2q}{m}\right)^{2\lambda-\nu}\times\frac{\epsilon}{q}\left[\frac{\Gamma(\lambda+i(Z\alpha)\epsilon/q)}{\Gamma(1+i(Z\alpha)\epsilon/q)}\right]^2. \qquad (4)$$

Here $\epsilon^2=q^2-m^2$ and, for notational simplicity, we write $\lambda=\lambda_1=[1-(Z\alpha)^2]^{1/2}$. This limit is to be taken in the following manner: First, the integral is evaluated in a region $\mathrm{Re}(\nu-2\lambda)>1$ where it converges. The result is then analytically continued to the point[7] $\nu=0$ along a route in the complex ν plane that avoids the poles at $\nu-2\lambda=1$, $\nu-2\lambda=0$, and $\nu-2\lambda=-1$. The integral (4) can be expanded in powers of $Z\alpha$ in a straightforward fashion using a standard representation of the beta function.[8] We find

$$a_{2\lambda}=\alpha\{-\tfrac{2}{9}(Z\alpha)+\tfrac{2}{9}[\gamma+3\zeta(3)+2\ln 2-\tfrac{31}{6}](Z\alpha)^3+\cdots\}, \qquad (5)$$

where $\gamma=0.577\cdots$ is Euler's constant. The coefficients that appear here agree with those found previously.[9]

Both coefficients a_1 and $a_{2\lambda}$ are singular at $\lambda=\frac{1}{2}$. However, at this point both terms are of order $(mr)^1$, and the singularities cancel. The pole in a_1 at $\lambda=\frac{1}{2}$ is evident in the first term ($k=1$) in the k sum, Eq. (2). The pole in a_2 at $\lambda=\frac{1}{2}$ comes from the large-q range of the integral (4); at $\lambda=\frac{1}{2}$ the continuation is singular at $\nu=0$. The pole in Eq. (4) is easily extracted by expanding the integrand in the first few leading powers of m/q. We have

$$\lim_{\lambda\to 1/2}a_1=-\lim_{\lambda\to 1/2}a_{2\lambda}=\alpha(Z\alpha)\frac{2}{\pi}\mathrm{Re}\,\psi'\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\frac{1}{\lambda-\frac{1}{2}}=\alpha(Z\alpha)\frac{\pi}{(\cosh\frac{1}{2}\pi\sqrt{3}\,)^2}\frac{1}{\lambda-\frac{1}{2}}. \qquad (6)$$

Here the last equality follows from the identities

$$\mathrm{Re}\,\psi'(\tfrac{1}{2}-iZ\alpha)=-\left(\frac{d}{dZ\alpha}\right)^2\tfrac{1}{2}\ln|\Gamma(\tfrac{1}{2}-iZ\alpha)|^2=\frac{1}{2}\left(\frac{d}{dZ\alpha}\right)^2\ln\cosh\pi Z\alpha. \qquad (7)$$

It is convenient to separate this pole from the coefficients so as to obtain smooth, slowly varying functions of $Z\alpha$. We also separate the lowest-order $[\alpha(Z\alpha)]$ contribution. Thus, we define regular parts of the coefficients by writing

$$a_1=\frac{\alpha(Z\alpha)}{\pi}+\alpha(Z\alpha)^3\left[a_1^{\mathrm{reg}}+\frac{4}{3}\frac{\pi}{(\cosh\frac{1}{2}\pi\sqrt{3}\,)^2}\frac{1}{\lambda-\frac{1}{2}}\right] \qquad (8)$$

and

$$a_{2\lambda}=-\tfrac{2}{9}\alpha(Z\alpha)+\alpha(Z\alpha)^3\left[a_{2\lambda}^{\mathrm{reg}}-\frac{4}{3}\frac{\pi}{(\cosh\frac{1}{2}\pi\sqrt{3}\,)^2}\frac{1}{\lambda-\frac{1}{2}}\right]. \qquad (9)$$

At the point $\lambda=\frac{1}{2}$, one has $(Z\alpha)^{-2}=\frac{4}{3}$, so we have removed poles with the correct residues. Figures 1 and 2 display our numerical evaluations of the functions a_1^{reg} and $a_{2\lambda}^{\mathrm{reg}}$ computed from the sum (2), the integral (4), and the definitions (8) and (9). Accurate numerical values of a_1 and $a_{2\lambda}$ may be determined from these figures and Eqs. (8) and (9). The coefficients a_1^{reg} and $a_{2\lambda}^{\mathrm{reg}}$ are indeed smooth and slowly varying functions except for singularities as $Z\alpha\to 1$. It can be shown that these singularities cancel in the complete potential.

In order to assess the significance of our calculations[10] done to all orders in $Z\alpha$, let us consider the (not atypical) case of a Pb nucleus where $Z\alpha\simeq 0.6$, $\lambda\simeq 0.8$. Now

$$a_1^{\mathrm{reg}}+\frac{4}{3}\frac{\pi}{(\cosh\frac{1}{2}\pi\sqrt{3}\,)^2}\frac{1}{\lambda-\frac{1}{2}}=0.036+0.240=0.276 \qquad (10a)$$

and

$$a_{2\lambda}^{\mathrm{reg}}-\frac{4}{3}\frac{\pi}{(\cosh\frac{1}{2}\pi\sqrt{3}\,)^2}\frac{1}{\lambda-\frac{1}{2}}=0.198-0.240=-0.042. \qquad (11a)$$

These numbers should be compared to the corresponding coefficients previously calculated[4,5] for the order-$(Z\alpha)^3$ terms. Referring to Eqs. (3) and (5), we see that these are

$$\frac{1}{\pi}\left[\frac{\pi^4}{16}+\frac{\pi^2}{6}-6\zeta(3)\right]=0.166 \qquad (10b)$$

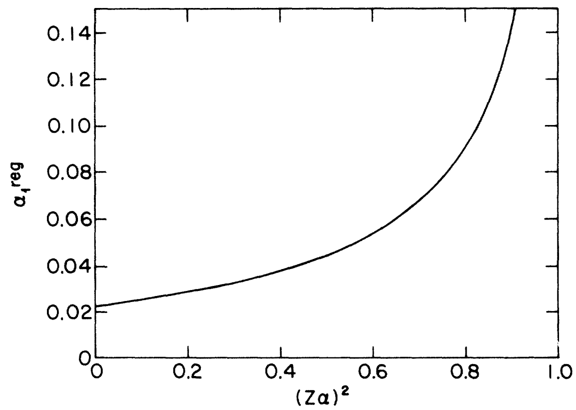

FIG. 1. The coefficient a_1^{reg} as a function of $(Z\alpha)^2$ [Eq. (8)].

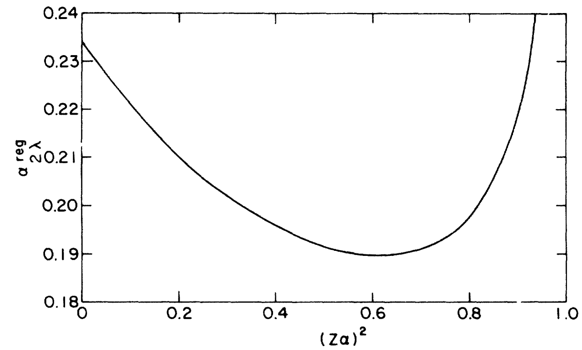

FIG. 2. The coefficient $a_{2\lambda}^{reg}$ as a function of $(Z\alpha)^2$ [Eq. (9)].

and

$$\tfrac{2}{9}[\gamma + 3\zeta(3) + 2\ln 2 - \tfrac{31}{6}] = +0.090\,. \tag{11b}$$

There are very large differences between Eqs. (10a) and (10b) and between Eqs. (11a) and (11b). Thus, the higher-order corrections are very important in these coefficients. However, when all the contributions are added up, including the third- and higher-order contributions to the potential from the difference

$$-m\tfrac{2}{9}\,\alpha(Z\alpha)[(mr)^{2\lambda} - (mr)^2]$$

[cf. Eqs. (1) and (9)], the higher-order corrections have little effect on transition energies. Substantial cancellations occur. For example,[2] in the $5g \to 4f$ transition in muonic Pb, the terms which we have calculated to all orders in $Z\alpha$ raise the transition energy by 7.9 eV. The previous calculations[4,5] of these terms to only order $(Z\alpha)^3$ gave a transition energy shift of 6.6 eV. Since large cancellations do occur, small errors could change results considerably. Our confirmation of previous work is thus not without significance.

The plan of this paper is as follows: In Sec. II we develop the general formulation of the problem, deriving an expansion of the vacuum-polarization potential in ascending powers of mr. The general method for calculating the odd-integer terms is given in Sec. III, and the leading odd-integer term is explicitly evaluated. The general method for calculating the remaining, noninteger terms is given in Sec. IV, and the leading noninteger term is explicitly evaluated. The sums needed for the expansion (3) are worked out in the Appendix.

II. GENERAL FORMULATION

In the notation of paper I, the vacuum-polarization charge density

$$\rho_{\text{vacpol}}(r) = -e\langle\psi^\dagger(\vec{r},t)\psi(\vec{r},t)\rangle \tag{I.84}$$

may be written in the form

$$\rho_{\text{vacpol}}(r) = -ie\int_{-i\infty}^{i\infty}\frac{dE}{2\pi}\,\text{tr}G(\vec{r},\vec{r};E)\gamma^0\,. \tag{I.85}$$

An angular momentum decomposition of the Dirac-Coulomb Green's function G yields

$$4\pi r^2\rho_{\text{vacpol}}(r) = e\sum_{k=1}^{\infty}\sigma_k(r)\,. \tag{12}$$

Changing variables from $E = i\epsilon$ to $q = (\epsilon^2 + m^2)^{1/2}$ and making use of Eqs. (I.89) and (I.90) gives

$$\sigma_k(r) = \int_m^\infty dq\, f_k\left(qr;\,\frac{\epsilon}{q}\right), \tag{13}$$

with

$$f_k\left(qr;\,\frac{\epsilon}{q}\right) = \frac{2k}{\pi}\,\text{Re}\frac{q}{\epsilon}\left\{2\left(i\epsilon + \frac{Z\alpha}{r}\right)[\mathcal{G}_\lambda(r,r;i\epsilon) + \mathcal{G}_{-\lambda}(r,r;i\epsilon)] + \frac{Z\alpha}{\lambda}\frac{d}{dr}[\mathcal{G}_\lambda(r,r;i\epsilon) - \mathcal{G}_{-\lambda}(r,r;i\epsilon)]\right\}. \tag{14}$$

We now write, as we often shall, $\lambda = \lambda_k = [k^2 - (Z\alpha)^2]^{1/2}$ to achieve some notational simplicity. Here $\mathcal{G}_{\pm\lambda}$ are radial Green's functions for the second-order Dirac-Coulomb equation. They may be expressed as

$$\mathcal{G}_{\pm\lambda}(r,r';E) = A_{\pm\lambda}(r_>;E)B_{\pm\lambda}(r_<;E)\,, \tag{I.33}$$

with $A_{\pm\lambda}$ and $B_{\pm\lambda}$ proportional to Whittaker functions [(I.35) and (I.36)] which are regular at infinity and the origin, respectively. It is convenient to define the functions

$$\Gamma_\lambda^{(\pm)}\left(qr;\ \frac{Z\alpha\epsilon}{q}\right)=q[\mathcal{G}_\lambda(r,r;i\epsilon)\pm\mathcal{G}_{-\lambda}(r,r;i\epsilon)] \tag{15}$$

and

$$H_\lambda\left(qr;\ \frac{Z\alpha\epsilon}{q}\right)=q\left\{A_\lambda(r;i\epsilon)B_{-\lambda}(r;i\epsilon)-\left[\lambda^2+\left(\frac{Z\alpha\epsilon}{q}\right)^2\right]A_{-\lambda}(r;i\epsilon)B_\lambda(r;i\epsilon)\right\}. \tag{16}$$

It follows from Eqs. (I.35) and (I.36) that, if λ is considered an independent parameter, then these functions depend upon the variables q, r, $Z\alpha$, and ϵ only in the combination displayed in Eqs. (15) and (16). This simplicity will be of importance in the next section. Now, using Eqs. (I.56) and (I.57),

$$\frac{d}{dr}\Gamma_\lambda^{(-)}=2\left(\frac{iZ\alpha\epsilon}{\lambda}-\frac{\lambda}{r}\right)\Gamma_\lambda^{(+)}+\frac{2q}{\lambda}H_\lambda. \tag{17}$$

This enables us to write Eq. (14) as

$$f_k\left(qr;\ \frac{\epsilon}{q}\right)=\frac{4k}{\pi\lambda^2}\,\mathrm{Re}\left[ik^2\Gamma_\lambda^{(+)}\left(qr;\frac{Z\alpha\epsilon}{q}\right)+Z\alpha\frac{q}{\epsilon}H_\lambda\left(qr;\frac{Z\alpha\epsilon}{q}\right)\right]. \tag{18}$$

We turn now to the development of $\sigma_k(r)$ in ascending powers of mr. The Whittaker functions which appear in the integrand (18) can be expanded as a sum of powers of qr, $(qr)^\beta$, which includes noninteger as well as integer powers. We can use these expansions to decompose the integrand $f_k(qr;\epsilon/q)$ into a finite power series and a remainder, $\bar{f}_k^{(N)}(qr;\epsilon/q)$, which is of order r^N:

$$f_k\left(qr;\ \frac{\epsilon}{q}\right)=\sum_{\beta<N}C_k^\beta\left(\frac{\epsilon}{q}\right)(qr)^\beta+\bar{f}_k^{(N)}\left(qr;\ \frac{\epsilon}{q}\right). \tag{19}$$

However, we cannot insert this decomposition into the integral (13) and integrate the sum and the remainder separately because the two integrals would not converge at large q values. To remedy this situation, we return to the original integral (13) and note that it can be written as a limit of a regularized integral

$$\sigma_k(r)=\lim_{\nu\to 0}\sigma_k^{(\nu)}(r), \tag{20}$$

where

$$\sigma_k^{(\nu)}(r)=\int_m^\infty dq\, f_k\left(qr;\frac{\epsilon}{q}\right)\left(\frac{m}{q}\right)^\nu. \tag{21}$$

Since the original integral (13) converges, the new regularized integral (21) defines an analytic function of ν for $\mathrm{Re}\,\nu\geq 0$. Now, if $\mathrm{Re}\,\nu>N+1$, the separate regularized integrals converge, and we have

$$\sigma_k^{(\nu)}(r)=\sum_{\beta<N}m(mr)^\beta\int_m^\infty\frac{dq}{m}C_k^\beta\left(\frac{\epsilon}{q}\right)\left(\frac{m}{q}\right)^{\nu-\beta}+\bar{\sigma}_k^{(\nu,N)}(r), \tag{22}$$

with

$$\bar{\sigma}_k^{(\nu,N)}(r)=\int_m^\infty dq\,\bar{f}_k^{(N)}\left(qr;\frac{\epsilon}{q}\right)\left(\frac{m}{q}\right)^\nu. \tag{23}$$

The integrals in the β sum in Eq. (22) have only simple poles in ν in the interval $0\leq\mathrm{Re}\,\nu\leq N+1$. These poles arise from the large-q region of the integration where $C_k^\beta(\epsilon/q)$ can be expanded in powers of m^2/q^2 since $C_k^\beta(\epsilon/q)$ is analytic in $\epsilon/q=(1-m^2/q^2)^{1/2}$ when $|\epsilon/q|$ is less than $\lambda/Z\alpha$. This expansion generates poles at $\nu=\beta-$(odd integers). We show in Sec. IV that the powers β are either even integers, $\beta=2n$, or nonintegral, $\beta=2\lambda_k+n$. Hence, there is no ambiguity in analytically continuing any of the integrals in the β sum in Eq. (22) from the region $\mathrm{Re}\,\nu>N+1$ to the point $\nu=0$. Since $\sigma_k^{(\nu)}(r)$ is analytic in ν for $0\leq\mathrm{Re}\,\nu\leq N+1$, the remaining term in Eq. (22), $\bar{\sigma}_k^{(\nu,N)}(r)$, must have only simple poles in this region which precisely cancel the poles in the β sum. Hence, $\bar{\sigma}_k^{(\nu,N)}(r)$ can be analytically continued to the point $\nu=0$ without ambiguity. We can, therefore, compute the expansion of $\sigma_k(r)$ in powers of mr up to terms of order $(mr)^N$ by separately computing the β sum and the remainder $\bar{\sigma}_k^{(\nu,N)}(r)$ in Eq. (22) for $\mathrm{Re}\,\nu>N+1$ and then analytically continuing the two parts to the point $\nu=0$. We show in Sec. IV that the even-integer terms $\beta=2n$ vanish when they are analytically continued to the point $\nu=0$. Thus, the β sum in Eq. (22), the sum coming from a simple power-series expansion of the integrand, gives only the nonintegral powers $\beta=2\lambda_k+n$ in the series expansion (1) of the vacuum-polarization potential. We show now that the remainder term $\bar{\sigma}_k^{(\nu,N)}(r)$ gives the odd-integer powers in the series expansion (1) of the vacuum-polarization potential.

This can be done by the use of power-series expansions in the squared electron mass. Since this mass appears only in the variable $\epsilon/q=(1-m^2/q^2)^{1/2}$, such an expansion gives ascending powers of m^2/q^2. We expand the original integrand

$$f_k\left(qr;\ \frac{\epsilon}{q}\right)=\sum_{l=0}^{\infty} f_{k,l}(qr)\left(\frac{m^2}{q^2}\right)^l, \tag{24}$$

and also each of the coefficients of its power-series development in qr,

$$C_k^\beta\left(\frac{\epsilon}{q}\right)=\sum_{l=0}^{\infty} c_{k,l}^\beta\left(\frac{m^2}{q^2}\right)^l. \tag{25}$$

We now write the remainder in Eq. (19) as

$$\bar{f}_k^{(N)}\left(qr;\ \frac{\epsilon}{q}\right)=\sum_{l=0}^{L} f_{k,l}(qr)\left(\frac{m^2}{q^2}\right)^l - \sum_{\beta<N}(qr)^\beta\sum_{l=0}^{L} c_{k,l}^\beta\left(\frac{m^2}{q^2}\right)^l + \bar{f}_k^{(N,L)}\left(qr;\ \frac{\epsilon}{q}\right). \tag{26}$$

We choose the upper limit on the l sum to satisfy $2L+1>N$ so that $\bar{f}^{(N,L)}(qr;\epsilon/q)$ is smaller than $O(q^{-1})$ for large q. This ensures that the contribution of $\bar{f}^{(N,L)}(qr;\epsilon/q)$ to $\bar{\sigma}_k^{(\nu,N)}(r)$ at $\nu=0$ converges:

$$R_k^{(N,L)}(mr)=\int_m^\infty dq\,\bar{f}_k^{(N,L)}\left(qr;\ \frac{\epsilon}{q}\right). \tag{27}$$

We now demonstrate that $R_k^{(N,L)}(mr)$ is of order $m(mr)^N$ for $r\to 0$. Since $\bar{f}_k^{(N,L)}(qr;\epsilon/q)$ has the powers $(qr)^\beta$ deleted for $\beta<N$, $\bar{f}_k^{(N,L)}(qr;\epsilon/q)=O(r^N)$ as $r\to 0$ at fixed q. It follows that the integral (27) up to $q=M\gg m$ is $O(r^N)$. For the interval $M<q<\infty$ we exploit the analyticity of $\mathfrak{F}_k(qr;\epsilon/q)$, the complex function whose real part is $f_k(qr;\epsilon/q)$ [cf. Eq. (14)]. We shall consider this a function of two independent variables, qr and ϵ/q, with ϵ/q extended to complex values. This function is also analytic regarded as a function of m/q for $|m/q|\le m/M$ since $\mathfrak{F}_k(qr;\epsilon/q)$ is analytic in ϵ/q for $0<|\epsilon/q|<\lambda/Z\alpha$, while ϵ/q is analytic in m/q for $|m/q|<1$. Let $\bar{\mathfrak{F}}_k^{(N,L)}(qr;m/q)$ represent the corresponding analytic function of m/q, whose real part is $\bar{f}_k^{(N,L)}(qr;\epsilon/q)$. Below we establish the uniform bound

$$\left|\bar{\mathfrak{F}}_k^{(N,L)}\left(qr;\ \frac{m}{q}\right)\right|<K(qr)^N\left(\frac{m}{q}\right)^{2L+2}, \tag{28}$$

where K is some constant. From this follows the bound

$$\left|\int_M^\infty dq\,\bar{f}_k^{(N,L)}\left(qr;\ \frac{\epsilon}{q}\right)\right|\le\int_M^\infty dq\left|\bar{\mathfrak{F}}_k^{(N,L)}\left(qr;\ \frac{m}{q}\right)\right| < Km(mr)^N\frac{(M/m)^{N-2L-1}}{2L+1-N}. \tag{29}$$

Thus both portions of the integral (27) are indeed of order $m(mr)^N$ as $r\to 0$.

The bound for $\bar{\mathfrak{F}}_k^{(N,L)}(qr;m/q)$ may be demonstrated by considering

$$H(qr;m/q)=(qr)^{-N}(m/q)^{-2L-2}\bar{\mathfrak{F}}_k^{(N,L)}(qr;m/q).$$

Now $H(qr;m/q)$ is analytic in m/q within the circle $|m/q|<m/M$, with qr fixed. From the maximum-modulus theorem, if $|m/q|<m/M$, then

$$|H(qr;m/q)|<\sup_{|m/q|=m/M}|H(qr;m/q)|=K_{qr}.$$

We shall have our bound if K_{qr} is a bounded function of qr. As $qr\to\infty$, $H(qr;m/q)\to 0$ since $\bar{\mathfrak{F}}_k^{(N,L)}(qr;m/q)$ goes as $(qr)^\beta$, $\beta<N$. In particular, on the circle $|m/q|=m/M$, $H(qr;m/q)\to 0$ and consequently

$$K_{qr}\xrightarrow[qr\to\infty]{}0.$$

For $qr\to 0$, $H(qr;m/q)$ approaches a finite limit, implying that K_{qr} does as well. It follows that K_{qr} is bounded as a function of qr, $K_{qr}<K$. This establishes the bound, Eq. (28).

We may now insert the decomposition (26) in the remainder integral (23) and compute

$$\bar{\sigma}_k^{(\nu,N)}(r)=\sum_{l=0}^{L}\int_m^\infty dq\,f_{k,l}(qr)\left(\frac{m}{q}\right)^{2l+\nu} - \sum_{\beta<N}\sum_{l=0}^{L}c_{k,l}^\beta\frac{m(mr)^\beta}{2l+\nu-\beta-1} + O((mr)^N). \tag{30}$$

The sum of integrals which appears here is just the first L terms of the expansion of the convergent integral (21) in powers of the squared electron mass, and there is no difficulty in analytically continuing it to the point $\nu=0$. The analytic continuation of the β sum appearing in Eq. (29) is manifest, and so

$$\bar{\sigma}_k^{(\nu=0,N)}(r)=\sum_{l=0}^{L}\int_m^\infty dq\,f_{k,l}(qr)\left(\frac{m}{q}\right)^{2l} + \sum_{\beta<N}\sum_{l=0}^{L}c_{k,l}^\beta\frac{m(mr)^\beta}{\beta+1-2l} + O((mr)^N). \tag{31}$$

This result can be simplified by observing that

$$\frac{m(mr)^{\beta}}{\beta+1-2l}=\int_{``0"}^{m}dq(qr)^{\beta}\left(\frac{m}{q}\right)^{2l}, \tag{32}$$

where quotation marks on the lower limit indicate that if $\beta-2l\leq 1$, the divergent lower limit of the indefinite integral is to be deleted. Using this prescription, the β sum occurring in Eq. (31) may be written as

$$\sum_{\beta<N}c_{k,l}^{\beta}\frac{m(mr)^{\beta}}{\beta+1-2l}=\int_{``0"}^{m}dq\,f_{k,l}(qr)\left(\frac{m}{q}\right)^{2l}+O((mr)^{N}). \tag{33}$$

Accordingly, we can rewrite Eq. (31) in the simple form

$$\bar{\sigma}_k^{(\nu=0,N)}(r)=\sum_{l=0}^{L}m(mr)^{2l-1}\int_{``0"}^{\infty}dx\,f_{k,l}(x)x^{-2l}+O((mr)^{N}). \tag{34}$$

Let us recapitulate. We have shown that the vacuum-polarization charge density can be expanded in a power series of mr, the sum containing odd-integer powers and certain noninteger powers. The odd-integer terms are obtained from an expansion of the integral (13) in powers of the squared electron mass, Eq. (24), with the result given in Eq. (34). Recalling that

$$4\pi r^2\rho_{\text{vac pol}}(r)=e\sum_{k=1}^{\infty}\sigma_k(r), \tag{12}$$

we see that each term in the vacuum-polarization potential energy obeys the Poisson equation in the form

$$-r\frac{d^2}{dr^2}rV_k(r)=-\alpha\sigma_k(r), \tag{35}$$

where $e^2/4\pi=\alpha$ is the fine-structure constant. It is straightforward to insert Eq. (34) into Eq. (35) and to solve for the odd-integer terms in $V_k(r)$. Comparing this with the general form (1), we find that the coefficients of the odd-integer powers are given by

$$a_{2l-1}=\sum_{k=1}^{\infty}\frac{\alpha}{2l(2l-1)}\int_{``0"}^{\infty}dx\,f_{k,l}(x)x^{-2l}. \tag{36}$$

The noninteger terms in the potential are obtained from an expansion of the integral (13) in powers of the radius using Eq. (19). We have already noted that these powers are of the form $\beta=2\lambda_k+n$. Thus, referring to Eq. (22), we see that the coefficients of the noninteger powers are given by

$$a_{2\lambda_k+n}=\frac{\alpha}{(2\lambda_k+n+1)(2\lambda_k+n)}\times\lim_{\nu\to 0}\int_m^{\infty}\frac{dq}{m}C_k^{2\lambda_k+n}\left(\frac{\epsilon}{q}\right)\left(\frac{q}{m}\right)^{2\lambda_k+n-\nu}, \tag{37}$$

where the limit is actually the analytic continuation to $\nu=0$. The constant term in the potential, a_0, cannot be determined by our methods since it involves an integration of the charge density over all space. The constant term is, however, irrelevant in the calculation of transition energy differences.

III. ODD-INTEGER POWERS

The odd-integer terms in the vacuum-polarization potential are obtained from an expansion of the integrand

$$f_k\left(qr;\frac{\epsilon}{q}\right)=\frac{4k}{\pi\lambda^2}\operatorname{Re}\left[ik^2\Gamma_\lambda^{(+)}\left(qr;Z\alpha\frac{\epsilon}{q}\right)+Z\alpha\frac{q}{\epsilon}H_\lambda\left(qr;Z\alpha\frac{\epsilon}{q}\right)\right] \tag{18}$$

in powers of the squared electron mass. This mass enters only through the ratio

$$\epsilon/q=[1-m^2/q^2]^{1/2}.$$

The term of zero order in the electron mass gives rise to the induced point charge: a δ function in the charge density. This term must be separated and computed as a generalized function, as done in paper I. In view of Eq. (18), the term of first order in the squared electron mass can be expressed [in the notation of Eq. (24)] as

$$f_{k,1}(x)=\frac{2k}{\pi\lambda^2}\operatorname{Re}\left\{-ik^2(Z\alpha)\frac{\partial}{\partial(Z\alpha)}\Gamma_\lambda^{(+)}(x;Z\alpha)+Z\alpha\left[1-(Z\alpha)\frac{\partial}{\partial(Z\alpha)}\right]H_\lambda(x;Z\alpha)\right\}. \tag{38}$$

Here λ is considered an independent parameter, and it is not to be differentiated. The higher-order terms in the expansion can be expressed similarly by differential operators of $Z\alpha$ applied to the two functions $\Gamma_\lambda^{(+)}(x;Z\alpha)$ and $H_\lambda(x;Z\alpha)$, but there is no need to work them out explicitly. We shall, however, outline the calculation to arbitrary order. This we see, according to Eq. (36), involves the computation of the parameters

$$\gamma_{\lambda,n}^{(\pm)}=\int_{``0"}^{\infty}dx\,\Gamma_\lambda^{(\pm)}(x;Z\alpha)x^{-n} \tag{39}$$

and

$$h_{\lambda,n}=\int_{``0"}^{\infty}dxH_\lambda(x;Z\alpha)x^{-n}\ , \quad (40)$$

with n even.

These parameters can be computed recursively. In the previous section we observed that Eqs. (I.56) and (I.57) enabled us to express a derivative of $\Gamma_\lambda^{(-)}$ in terms of $\Gamma_\lambda^{(+)}$ and H_λ, Eq. (17). Specializing this result to the case $\epsilon/q=1$ gives

$$\frac{d}{dx}\Gamma_\lambda^{(-)}(x;Z\alpha)=2\left(\frac{iZ\alpha}{\lambda}-\frac{\lambda}{x}\right)\Gamma_\lambda^{(+)}(x;Z\alpha)+\frac{2}{\lambda}H_\lambda(x;Z\alpha)\ . \quad (41)$$

The same technique can also be employed to derive

$$\frac{d}{dx}\Gamma_\lambda^{(+)}(x;Z\alpha)=2\left(\frac{iZ\alpha}{\lambda}-\frac{\lambda}{x}\right)\Gamma_\lambda^{(-)}(x;Z\alpha) \quad (42)$$

and

$$\frac{d}{dx}H_\lambda(x;Z\alpha)=\frac{2}{\lambda}(\lambda^2+(Z\alpha)^2)\Gamma_\lambda^{(-)}(x;Z\alpha)\ . \quad (43)$$

We write $x^{-n}=-(n-1)^{-1}(d/dx)x^{-n+1}$, integrate by parts, and use Eq. (43) to obtain our first recursion relation:

$$h_{\lambda,n}=\frac{1}{n-1}\ \frac{2}{\lambda}(\lambda^2+(Z\alpha)^2)\gamma_{\lambda,n-1}^{(-)}\ . \quad (44)$$

There are no end-point contributions at $x=0$ by virtue of the "0" prescription introduced in the previous section. In an entirely similar fashion we find that

$$\gamma_{\lambda,n}^{(+)}=\frac{2}{n-1}\left(\frac{iZ\alpha}{\lambda}\gamma_{\lambda,n-1}^{(-)}-\lambda\gamma_{\lambda,n}^{(-)}\right)\ . \quad (45)$$

Finally, the same technique relates $\gamma_{\lambda,n}^{(-)}$ to $\gamma_{\lambda,n}^{(+)}$, $\gamma_{\lambda,n-1}^{(+)}$, and $h_{\lambda,n-1}$. Using Eqs. (44) and (45) for these latter parameters, all quantities may be expressed in terms of $\gamma_\lambda^{(-)}$'s to secure the second-order recursion relation

$$\gamma_{\lambda,n}^{(-)}=\frac{1}{n-2}\ \frac{1}{\frac{1}{4}(n-1)^2-\lambda^2}\times[(n-1)\gamma_{\lambda,n-2}^{(-)}-iZ\alpha(2n-3)\gamma_{\lambda,n-1}^{(-)}]\ . \quad (46)$$

Thus, all the various parameters are determined uniquely by two parameters which start the iteration sequence.

Since Eq. (44) directly relates $h_{\lambda,2l}$ to $\gamma_{\lambda,2l-1}^{(-)}$, we find from Eq. (18) that a term in the potential of order $2l-1$ involves $\mathrm{Im}\gamma_{\lambda,2l}^{(+)}$ and $\mathrm{Re}\gamma_{\lambda,2l-1}^{(-)}$. Moreover, Eqs. (45) and (46) relate the imaginary parts of $\gamma_{\lambda,n}^{(\pm)}$'s with n even to the real parts of $\gamma_{\lambda,n}^{(\pm)}$'s with n odd. Thus, we may start with $\mathrm{Re}\gamma_{\lambda,1}^{(-)}$ and $\mathrm{Im}\gamma_{\lambda,2}^{(+)}$ to determine a term in the potential of arbitrary order. Recalling the definition, (15), of $\Gamma_\lambda^{(-)}$, we see that

$$\mathrm{Re}\gamma_{\lambda,1}^{(-)}=\mathrm{Re}\int_0^\infty dr[\mathcal{G}_\lambda(r,r;i\epsilon)-\mathcal{G}_{-\lambda}(r,r;i\epsilon)]\ \frac{q}{r}\ , \quad (47)$$

with the Green's functions evaluated at zero electron mass. Here we have dispensed with the "0" notation since the integral converges at its lower limit. This integral can be viewed as the trace of an operator. It was evaluated in paper I, Eq. (I.77):

$$\mathrm{Re}\gamma_{\lambda,1}^{(-)}=-\tfrac{1}{2}\mathrm{Re}[\psi(\lambda+1-iZ\alpha)-\psi(\lambda-iZ\alpha)]=-\tfrac{1}{2}\mathrm{Re}\ \frac{1}{\lambda-iZ\alpha}\ , \quad (48)$$

with the last form following from the relationship between ψ functions

$$\psi(z+1)=\frac{1}{z}+\psi(z)\ . \quad (49)$$

There remains the determination of

$$\mathrm{Im}\gamma_{\lambda,2}^{(+)}=\mathrm{Im}\int_{``0"}^{\infty}dr[\mathcal{G}_\lambda(r,r;i\epsilon)+\mathcal{G}_{-\lambda}(r,r;i\epsilon]\ \frac{1}{r^2}\ . \quad (50)$$

In this case, the integral does not converge at its lower limit, and it is necessary to use the "0" prescription. However, this divergence is a purely real quantity, and it disappears when the imaginary part is taken. We can dispose of the "0" prescription by subtracting free-particle Green's functions $\mathcal{G}_{\pm\lambda}^{(0)}(r,r;i\epsilon)$ which are purely real and have the same limiting behavior for small r as $\mathcal{G}_{\pm\lambda}(r,r;i\epsilon)$. Thus

$$\mathrm{Im}\gamma_{\lambda,2}^{(+)}=\mathrm{Im}\int_0^\infty dr[\mathcal{G}_\lambda(r,r;i\epsilon)-\mathcal{G}_\lambda^{(0)}(r,r;i\epsilon)+\mathcal{G}_{-\lambda}(r,r;i\epsilon)-\mathcal{G}_{-\lambda}^{(0)}(r,r;i\epsilon)]\ \frac{1}{r^2}\ , \quad (51)$$

and we again obtain the trace of an operator evaluated in paper I, Eq. (I.31),

$$\mathrm{Im}\gamma_{\lambda,2}^{(+)}=-\mathrm{Im}[(2\lambda+1)^{-1}\psi(\lambda+1-iZ\alpha)+(2\lambda-1)^{-1}\psi(\lambda-iZ\alpha)]\ . \quad (52)$$

Using the relationship between ψ functions, Eq. (49), this result may be expressed as

$$\mathrm{Im}\gamma_{\lambda,2}^{(+)} = -\mathrm{Im}\left[\frac{4\lambda}{4\lambda^2-1}\psi(\lambda - iZ\alpha) + \frac{1}{2\lambda+1}\,\frac{1}{\lambda - iZ\alpha}\right]. \tag{53}$$

We specialize now to the leading correction with coefficient [Eq. (36)]

$$a_1 = \tfrac{1}{2}\alpha \sum_{k=1}^{\infty} \int_{\text{“}0\text{”}}^{\infty} dx f_{k,1}(x)x^{-2}. \tag{54}$$

Using Eq. (38) and the results of the paragraphs above, we find that

$$a_1 = \sum_{k=1}^{\infty} \frac{\alpha k}{\pi\lambda^2}\,\mathrm{Re}\left\{-ik^2(Z\alpha)\frac{\partial}{\partial(Z\alpha)}\gamma_{\lambda,2}^{(+)} + Z\alpha\left[1-(Z\alpha)\frac{\partial}{\partial(Z\alpha)}\right]\frac{2}{\lambda}\left[\lambda^2+(Z\alpha)^2\right]\gamma_{\lambda,1}^{(-)}\right\}. \tag{55}$$

Inserting the evaluations (52) and (53) of the γ parameters into Eq. (55), we achieve the result (2) quoted in Sec. I.

IV. NONINTEGER TERMS

The noninteger terms in the vacuum-polarization potential are obtained by an expansion of the integrand $f_k(qr;\epsilon/q)$ in powers of the radius. Such an expansion can be carried out if the integrand is written entirely in terms of the regular confluent hypergeometric function

$$\Phi(a,c;z) = 1 + \frac{a}{c}\,\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\,\frac{z^2}{2!} + \cdots. \tag{56}$$

It is convenient to make use of Kummer's transformation[11]

$$\Phi(a,c;z) = e^{z}\Phi(c-a,c;-z) \tag{57}$$

to remove an overall exponential factor from the integrand. We use this transformation and write our Whittaker function (I.36), which is regular at the origin as

$$B_{\pm\lambda}(r;i\epsilon) = \Gamma(2\lambda+1\pm1)^{-1}(2q)^{-1}(2qr)^{\lambda+1/2\pm1/2}e^{+qr} \times\Phi(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2},\,2\lambda+1\pm1;\,-2qr), \tag{58}$$

where

$$\eta = Z\alpha\,\frac{i\epsilon}{q} \tag{59}$$

is purely imaginary. The other Whittaker function which we employ, the function which is regular at infinity (I.35), can be expressed by the Φ functions according to the formula[12]

$$\Psi(a;c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\Phi(a-c+1,2-c;z). \tag{60}$$

We obtain

$$A_{\pm\lambda}(r;i\epsilon) = \Gamma(2\lambda\pm1)(2qr)^{-\lambda+1/2\mp1/2}e^{-qr}\Phi(-\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2};1-2\lambda\mp1;2qr) + \Gamma(-2\lambda\mp1)\frac{\Gamma(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2})}{\Gamma(-\lambda-\eta+\tfrac{1}{2}\mp\tfrac{1}{2})}(2qr)^{\lambda+1/2\pm1/2}e^{-qr}\Phi(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2};2\lambda+1\pm1;2qr). \tag{61}$$

The integrand $f_k(qr;\epsilon/q)$ is a bilinear form in $A_{\pm\lambda}B_{\pm\lambda}$. By virtue of Eqs. (58) and (61), it is of the form

$$f(qr;\epsilon/q) = E_k(qr;\epsilon/q) + (2qr)^{2\lambda}F_k(qr;\epsilon/q), \tag{62}$$

where both $E_k(qr;\epsilon/q)$ and $F_k(qr;\epsilon/q)$ possess a series development in integer powers of qr.

We refer to Eqs. (15), (16), (18), and (58), (61), (62) to compute

$$\begin{aligned}E_k\left(qr;\frac{\epsilon}{q}\right) = \frac{4k}{\pi\lambda^2}\,\mathrm{Re}\Bigg\{ & ik^2(qr)\bigg[\frac{1}{2\lambda+1}\Phi(-\lambda-\eta,-2\lambda;2qr)\Phi(\lambda-\eta+1,2\lambda+2;-2qr) \\ & + \frac{1}{2\lambda-1}\Phi(-\lambda-\eta+1,2-2\lambda;2qr)\Phi(\lambda-\eta,2\lambda;-2qr)\bigg] \\ & + Z\alpha\frac{q}{\epsilon}\bigg[\lambda\Phi(-\lambda-\eta,-2\lambda;2qr)\Phi(\lambda-\eta,2\lambda;-2qr) \\ & - \left(\lambda^2+\left(\frac{Z\alpha\epsilon}{q}\right)^2\right)\frac{(qr)^2}{(2\lambda+1)\lambda(2\lambda-1)} \\ & \times\Phi(-\lambda-\eta+1,2-2\lambda;2qr)\Phi(\lambda-\eta+1,2\lambda+2;-2qr)\bigg]\Bigg\}.\end{aligned} \tag{63}$$

We shall need only the general, qualitative features of this gross expression. First, by virtue of Kummer's transformation (57), none of the products of the two Φ functions is altered by the simultaneous substitutions $\eta \to -\eta$, $qr \to -qr$. Since all the quantities in the curly bracket in Eq. (63) are real except for the purely imaginary parameter η and the factor $i(qr)$ in front of the first square brackets, complex conjugation of the curly brackets is equivalent to the reflection $(qr) \to -(qr)$. Since we are to take the real part of the curly brackets, the result is an even function of qr,

$$E_k\left(-qr;\frac{\epsilon}{q}\right)=E_k\left(qr;\frac{\epsilon}{q}\right). \tag{64}$$

We can now prove that the even powers of mr in the β sum in Eq. (22) vanish when the analytic continuation to $\nu=0$ is performed,

$$\lim_{\nu\to 0}\int_m^\infty \frac{dq}{m}\, C_k^{2n}\left(\frac{\epsilon}{q}\right)\left(\frac{m}{q}\right)^{\nu-2n}=0\,. \tag{65}$$

The coefficient $C_k^{2n}(\epsilon/q)$ is obtained from the terms of order $(qr)^{2n}$ in the power-series expansion of Eq. (63). Note that we are to take the imaginary part of the first square brackets in Eq. (63). Hence, it is odd in the imaginary parameter $\eta=i\epsilon/q$. Moreover, taking the real part of the second square brackets in Eq. (63) renders it even in ϵ/q, but there is an additional, overall multiplicative factor of q/ϵ. Thus $E_k(q,\epsilon/q)$ is an odd function of ϵ/q and the coefficients $C_k^{2n}(\epsilon/q)$ must be odd as well,

$$C_k^{2n}\left(-\frac{\epsilon}{q}\right)=-C_k^{2n}\left(\frac{\epsilon}{q}\right). \tag{66}$$

According to the structure of the power series (56), the coefficient of $(qr)^{2n}$ in Eq. (63) involves limited, non-negative, integer powers of η, with a maximum power η^{2n-1}. Hence, $C_k^{2n}(\epsilon/q)$ is a finite polynomial in ϵ/q, containing only odd powers $(\epsilon/q)^{2n'-1}$, with $0\le n'\le n$, and Eq. (65) involves a finite series of terms of the form[8]

$$\int_m^\infty \frac{dq}{m}\left(\frac{\epsilon}{q}\right)^{2n'-1}\left(\frac{m}{q}\right)^{\nu-2n} = \int_0^\infty dt(\sinh t)^{2n'}(\cosh t)^{2n+1-2n'-\nu} = \frac{1}{2}\,\frac{\Gamma(n'+\frac{1}{2})\Gamma(-n-\frac{1}{2}+\nu/2)}{\Gamma(n'-n+\nu/2)}\,. \tag{67}$$

Each of these vanishes when analytically continued to the point $\nu=0$.

We have now shown that only the noninteger terms $(2qr)^{2\lambda}F_k(qr;\epsilon/q)$ in Eq. (62) contribute to the β sum (22). The expansion of $F_k(qr;\epsilon/q)$ in integer powers of qr yields the functions $C_k^{2\lambda+n}(\epsilon/q)$ which enter in Eq. (37)—the formula which determines the noninteger coefficients $a_{2\lambda+n}$ of the vacuum-polarization potential. We can again use Eqs. (15), (16), (18) and (58), (61), (62) to express $F_k(qr;\epsilon/q)$ in terms of the regular Φ functions, functions which can be expanded in integer powers of qr. This expression can be simplified somewhat by using

$$-(\lambda^2-\eta^2)\frac{\Gamma(\lambda-n)}{\Gamma(1-\lambda-\eta)}=\frac{\Gamma(\lambda-\eta+1)}{\Gamma(-\lambda-\eta)}\,, \tag{68}$$

$$\frac{\Gamma(-2\lambda+1)}{\Gamma(2\lambda+2)}=\frac{\Gamma(-2\lambda-1)}{\Gamma(2\lambda)}\,, \tag{69}$$

and Kummer's transformation (57). We find

$$\begin{aligned} F_k\left(qr;\frac{\epsilon}{q}\right) = \frac{2k}{\pi\lambda^2}\,\mathrm{Re}\Bigg\{ & ik^2\left[\frac{\Gamma(-2\lambda-1)}{\Gamma(2\lambda+2)}\,\frac{\Gamma(\lambda-\eta+1)}{\Gamma(-\lambda-\eta)}(2qr)^2\Phi(\lambda-\eta+1,2\lambda+2;2qr)\Phi(\lambda+\eta+1,2\lambda+2;-2qr)\right.\\ & \left.+\frac{\Gamma(-2\lambda+1)}{\Gamma(2\lambda)}\,\frac{\Gamma(\lambda-\eta)}{\Gamma(1-\lambda-\eta)}\Phi(\lambda-\eta,2\lambda;2qr)\Phi(\lambda+\eta,2\lambda;-2qr)\right]\\ & +2Z\alpha\,\frac{q}{\epsilon}\,\frac{\Gamma(-2\lambda-1)}{\Gamma(2\lambda)}\,\frac{\Gamma(\lambda-\eta+1)}{\Gamma(-\lambda-\eta)}(2qr)\Phi(\lambda-\eta+1,2\lambda+2;2qr)\Phi(\lambda+\eta,2\lambda;-2qr)\Bigg\}\,. \end{aligned} \tag{70}$$

As discussed in Sec. I, we need work out explicitly only the first term in Eq. (70), $F_k(0,\epsilon/q)$, which gives

$$C_k^{2\lambda}\left(\frac{\epsilon}{q}\right)=2^{2\lambda}F_k\left(0,\frac{\epsilon}{q}\right)$$
$$=-\frac{2k^3}{\pi\lambda^2}2^{2\lambda}\,\mathrm{Im}\frac{\Gamma(-2\lambda+1)}{\Gamma(2\lambda)}\frac{\Gamma(\lambda-\eta)}{\Gamma(1-\lambda-\eta)}$$
$$=\frac{8k^3}{\pi}2^{2\lambda}\frac{\Gamma(-2\lambda)}{\Gamma(2\lambda+1)}\,\mathrm{Im}\frac{\Gamma(\lambda-\eta)}{\Gamma(1-\lambda-\eta)}\,. \quad (71)$$

We use

$$\mathrm{Im}\frac{\Gamma(\lambda-\eta)}{\Gamma(1-\lambda-\eta)}=|\Gamma(\lambda+\eta)|^2\,\mathrm{Im}\frac{1}{\pi}\sin\pi(\lambda+\eta)$$
$$=\left|\Gamma\left(\lambda+iZ\alpha\frac{\epsilon}{q}\right)\right|^2\frac{1}{\pi}\cos\pi\lambda\sinh\pi Z\alpha\frac{\epsilon}{q} \quad (72)$$

and

$$iZ\alpha\frac{\epsilon}{q}\Gamma\left(iZ\alpha\frac{\epsilon}{q}\right)\Gamma\left(1-iZ\alpha\frac{\epsilon}{q}\right)=\left|\Gamma\left(1+iZ\alpha\frac{\epsilon}{q}\right)\right|^2$$
$$=\pi Z\alpha\frac{\epsilon}{q}\left(\sinh\pi Z\alpha\frac{\epsilon}{q}\right)^{-1} \quad (73)$$

to write

$$C_k^{2\lambda}\left(\frac{\epsilon}{q}\right)=\frac{8k^3}{\pi}2^{2\lambda}\frac{\Gamma(-2\lambda)}{\Gamma(2\lambda+1)}(\cos\pi\lambda)Z\alpha\frac{\epsilon}{q}$$
$$\times\left|\frac{\Gamma(\lambda+iZ\alpha\epsilon/q)}{\Gamma(1+iZ\alpha\epsilon/q)}\right|^2. \quad (74)$$

The lowest angular momentum value $k=1$ gives the leading power with $\lambda=[1-(Z\alpha)^2]^{1/2}$. Taking $k=1$ and inserting Eq. (74) into Eq. (37), we arrive at the formula (4) quoted in Sec. I.

ACKNOWLEDGMENTS

One of the authors (L.S.B.) is grateful for the hospitality of the Aspen Center for Physics and the Los Alamos Scientific Laboratory, where much of his work on this paper was performed. Another (L.D.M.) is grateful for the hospitality of the Stanford Linear Accelerator Center.

APPENDIX

We shall compute the first two members of the expansion of a_1 [Eq. (2)] in powers of $Z\alpha$,

$$a_1=\alpha(Z\alpha)\left[\sum_{k=1}^{\infty}\frac{4k}{\pi\lambda}\frac{k^2}{4\lambda^2-1}\mathrm{Re}\,\psi'(\lambda+iZ\alpha)-\frac{\lambda+k^2}{2k^2(2\lambda+1)}\right]$$
$$=\alpha(Z\alpha)a_1^{(1)}+\alpha(Z\alpha)^3a_1^{(3)}+\cdots\,. \quad (A1)$$

In addition to its explicit appearance in the ψ function, $Z\alpha$ also occurs in the parameter $\lambda=[k^2-(Z\alpha)^2]^{1/2}$, and so we have

$$a_1^{(1)}=\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{4k^2}{4k^2-1}\psi'(k)-2\frac{1+k}{k(2k+1)}\right] \quad (A2)$$

and

$$a_1^{(3)}=\frac{1}{\pi}\sum_{k=1}^{\infty}\left\{-\frac{2k^2}{4k^2-1}\left[\psi'''(k)+\frac{1}{k}\psi''(k)\right]+\left[\frac{16k^2}{(4k^2-1)^2}+\frac{2}{4k^2-1}\right]\psi'(k)-\frac{1}{k^2}\frac{1}{(2k+1)^2}-\frac{2}{k^2}\frac{1}{2k+1}\right\}. \quad (A3)$$

Here a prime denotes a derivative. Note that

$$\psi'(z)=\sum_{l=0}^{\infty}\frac{1}{[l+z]^2}\,, \quad (A4)$$

which implies the recursion formula

$$\psi'(z)=\psi'(z+1)+1/z^2\,. \quad (A5)$$

We may write the first coefficient in the form

$$a_1^{(1)} = \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\left[\psi'(k) - \frac{1}{k}\right] + \frac{1}{2}\left[\frac{1}{2k-1} - \frac{1}{2k+1}\right]\psi'(k) - \frac{1}{k(2k+1)}\right\} . \tag{A6}$$

We use Eq. (A5) to calculate

$$\sum_{k=1}^{\infty}\frac{1}{2k+1}\psi'(k) = \sum_{k=1}^{\infty}\frac{1}{2k+1}\left[\psi'(k+1) + \frac{1}{k^2}\right]$$

$$= \sum_{k=1}^{\infty}\frac{1}{2k-1}\psi'(k) - \psi'(1) + \sum_{k=1}^{\infty}\frac{1}{k^2}\,\frac{1}{2k+1} . \tag{A7}$$

Expressing $\psi'(1)$ by the sum (A4), the last three terms in Eq. (A6) cancel and there remains only

$$a_1^{(1)} = \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\psi'(k) - \frac{1}{k}\right]. \tag{A8}$$

We represent

$$\frac{1}{k} = \sum_{l=0}^{\infty}\left[\frac{1}{k+l} - \frac{1}{k+l+1}\right], \tag{A9}$$

use the sum (A4), and change variables from k to $m=k+l$ to secure

$$a_1^{(1)} = \frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\frac{1}{m}\left[\frac{1}{m} - \frac{1}{m+1}\right] = \frac{1}{\pi} . \tag{A10}$$

This is the coefficient appearing in Eq. (3) of the text.

We turn now to evaluate the other sum, Eq. (A3). First we simplify the sums involving ψ's. We use Eq. (A7) to write

$$\sum_{k=1}^{\infty}\frac{2}{4k^2-1}\psi'(k) = \zeta(2) - \sum_{k=1}^{\infty}\frac{1}{k^2}\,\frac{1}{2k+1} , \tag{A11}$$

where we have identified $\psi'(1) = \zeta(2)$. We use Eq. (A5) to compute

$$\sum_{k=1}^{\infty}\frac{16k^2}{(4k^2-1)^2}\psi'(k) = \sum_{k=1}^{\infty}\left\{\left[\frac{1}{(2k-1)^2} + \frac{1}{2k-1}\right]\psi'(k) + \left[\frac{1}{(2k+1)^2} - \frac{1}{(2k+1)}\right]\left[\psi'(k+1) + \frac{1}{k^2}\right]\right\}$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\psi'(k) + \sum_{k=1}^{\infty}\frac{1}{k^2}\left[\frac{1}{(2k+1)^2} - \frac{1}{2k+1}\right] . \tag{A12}$$

We use the derivative of Eq. (A5) to compute

$$\sum_{k=1}^{\infty}\frac{2k}{4k^2-1}\psi''(k) = \tfrac{1}{2}\sum_{k=1}^{\infty}\left\{\frac{1}{2k-1}\psi''(k) + \frac{1}{2k+1}\left[\psi''(k+1) - \frac{2}{k^3}\right]\right\}$$

$$= \sum_{k=1}^{\infty}\frac{1}{2k-1}\psi''(k) + \zeta(3) - \sum_{k=1}^{\infty}\frac{1}{k^3}\,\frac{1}{2k+1} , \tag{A13}$$

where we have identified $\psi''(1) = -2\zeta(3)$. We use the second derivative of Eq. (A5) to compute

$$\sum_{k=1}^{\infty}\frac{2k^2}{4k^2-1}\psi'''(k) = \tfrac{1}{2}\sum_{k=1}^{\infty}\psi'''(k) + \tfrac{1}{4}\sum_{k=1}^{\infty}\left\{\frac{1}{2k-1}\psi'''(k) - \frac{1}{2k+1}\left[\psi'''(k+1) + \frac{6}{k^4}\right]\right\}$$

$$= \tfrac{1}{2}\sum_{k=1}^{\infty}\psi'''(k) + \tfrac{3}{2}\zeta(4) - \tfrac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^4}\,\frac{1}{2k+1} , \tag{A14}$$

where we have identified $\psi'''(1) = 6\zeta(4)$. The sum over $\psi'''(k)$ was calculated in paper I:

$$\sum_{k=1}^{\infty}\psi'''(k) = 6\zeta(3) . \tag{IA17}$$

Inserting these results in Eq. (A3) yields

$$a_1^{(3)} = \frac{1}{\pi}\left[\zeta(2) - 4\zeta(3) - \tfrac{3}{2}\zeta(4)\right] + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{3}{2k^4} - \frac{2}{k^3}\right] + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{2}{(2k-1)^2}\psi'(k) - \frac{1}{2k-1}\psi''(k)\right] \tag{A15}$$

$$= \frac{1}{\pi}\left[\zeta(2) - 6\zeta(3)\right] + \frac{1}{\pi}S_1 + \frac{1}{\pi}S_2 ,$$

with

$$S_1 = \sum_{k=1}^{\infty}\frac{2}{(2k-1)^2}\psi'(k) , \tag{A16}$$

$$S_2 = -\sum_{k=1}^{\infty}\frac{1}{2k-1}\psi''(k) . \tag{A17}$$

The sums S_1 and S_2 possess integral representations. If we change variables $x=e^{-t}$, expand the integrand, and integrate term by term, we find that

$$4\int_0^1 dx\,\frac{(\ln x)^2}{1-x^2}\ln\left[\frac{1+x}{1-x}\right] = \sum_{k=1}^{\infty}\frac{1}{2k-1}\sum_{l=0}^{\infty}\frac{2}{[k+l]^3} = S_2 . \tag{A18}$$

Similarly,

$$4\int_0^1 dx\,\frac{(-\ln x)}{1-x^2}\int_0^x dy\,\frac{1}{y}\ln\left[\frac{1+y}{1-y}\right] = \sum_{k=1}^{\infty}\frac{2}{(2k-1)^2}\sum_{l=0}^{\infty}\frac{1}{[k+l]^2}$$

$$= S_1 . \tag{A19}$$

An integration by parts,

$$\int_0^x dy\,\frac{1}{y}\ln\left[\frac{1+y}{1-y}\right] = \ln x \ln\left[\frac{1+x}{1-x}\right] + \int_0^x dy\,\frac{2(-\ln y)}{1-y^2} , \tag{A20}$$

yields

$$S_1 + S_2 = 8\int_0^1 dx\,\frac{(-\ln x)}{1-x^2}\int_0^x dy\,\frac{(-\ln y)}{1-y^2}$$

$$= 4\int_0^1 dx\,\frac{d}{dx}\left[\int_0^x dy\,\frac{(-\ln y)}{1-y^2}\right]^2$$

$$= 4\left[\int_0^1 dy\,\frac{(-\ln y)}{1-y^2}\right]^2 . \tag{A21}$$

Now

$$\int_0^1 dy\,\frac{(-\ln y)}{1-y^2} = \int_0^{\infty} dt\,e^{-t}\,\frac{t}{1-e^{-2t}}$$

$$= \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

$$= \tfrac{3}{4}\zeta(2)$$

$$= \tfrac{1}{8}\pi^2 , \tag{A22}$$

and so

$$S_1+S_2=\frac{\pi^4}{16}\,. \qquad \text{A23)}$$

Thus,

$$a_1^{(3)}=\frac{1}{\pi}\left[\frac{\pi^2}{6}+\frac{\pi^4}{16}-6\zeta(3)\right]\,. \qquad \text{(A24)}$$

This is the coefficient appearing in Eq. (3) of the text.

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*Work supported in part by the U. S. Atomic Energy Commission.

[1]L. S. Brown, R. N. Cahn, and L. D. McLerran, preceding paper, Phys. Rev. D 12, 581 (1975). We shall refer to this as paper I, quoting formulas in it using an additional roman numeral I, for example, Eq. (I.85).

[2]A brief account of this work appears in L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. Lett. 33, 1591 (1974). We have simplified our development and avoided the Mellin-Barnes technique mentioned there.

[3]L. S. Brown, R. N. Cahn, and L. D. McLerran, following paper, Phys. Rev. D 12, 609 (1975).

[4]J. Blomqvist, Nucl. Phys. B48, 95 (1972).

[5]T. L. Bell, Phys. Rev. A 7, 1480 (1973).

[6]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1965).

[7]It is easy to show that this method is equivalent to that of our letter, Ref. 2, where we evaluated the integral at $\nu=0$ by simply discarding the terms in the indefinite integral which diverge at the upper limit.

[8]For example, Eq. (23) on p. 11 of *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I.

[9]This is the work of Refs. 4 and 5. These authors, however, expand all quantities in powers of $Z\alpha$ and thus they have logarithmic terms brought about by the expansion of $r^{2\lambda}=r^{2[1-(Z\alpha)^2]^{1/2}}$.

[10]More detailed evaluations appear in Refs. 2 and 3.

[11]Ref. 8, p. 253, Eq. (7).

[12]Ref. 8, p. 257, Eq. (7).