Vacuum polarization in a strong Coulomb field. I. Induced point charge*

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This is the first of a series of three papers concerned with the short-distance behavior of the vacuum polarization potential. It is this region that gives the principal energy level shift in heavy muonic atoms. In this paper we compute the induced vacuum polarization point charge that appears in the limit of a point nuclear charge source. Our result, computed analytically to all orders of $Z\alpha$, agrees with an earlier calculation of Wichmann and Kroll. We have been able to simplify the calculation considerably by taking advantage of a zero-electron-mass limit and by using formal operator and determinantal techniques.

I. INTRODUCTION AND SUMMARY

Recent measurements of energy-level differences in heavy muonic atoms have risen in accuracy to such an extent that they now probe high-order corrections to the polarization of the vacuum. These corrections are those that involve multiple virtual-photon exchange between the highly charged nucleus and the virtual electron. They are important because the nuclear charge, Z, is very large. Transitions between rather highly excited states of maximal angular momentum ("circular orbits") that are still inside the electronic K shell are of particular interest. The vacuum-polarization potential is the dominant correction to their unperturbed Dirac energy levels. Present measurements[1,2] of such transitions are in conflict with theory.

Although the muonic orbits of interest are large in the scale set by the first muonic Bohr orbit, they are very small in the scale set by the Compton wavelength of the electron, $\lambda\!\!\!^{-}_e$. Since the range of the vacuum-polarization potential is given by $\lambda\!\!\!^{-}_e/2$, we are interested in the short-distance limit of the induced vacuum-polarization charge density. In the idealization where the finite nucleus is replaced by a point charge, an induced vacuum-polarization point charge $\delta Q'$ appears in order $(Z\alpha)^3$ and higher.[3] The other portions of the vacuum-polarization charge density give much smaller effects, roughly of order $(2a_\mu/\lambda\!\!\!^{-}_e)^2$ and smaller, where a_μ is the radius of the muonic orbit. Thus, the induced point charge $\delta Q'$ is the major term. In this, the first in a sequence of three papers, we compute the induced point charge $\delta Q'$ to all orders of $Z\alpha$. Our result is in complete accord with that of Wichmann and Kroll.[3]

In the second paper of this sequence, we calculate explicitly (to all orders in $Z\alpha$) the next two corrections to the short-distance limit of the vacuum-polarization potential. We find that these terms, of order $(m_e r)$ and $(m_e r)^{2[1-(Z\alpha)^2]^{1/2}}$, do indeed make a very small contribution to the interesting muonic x-ray transitions. Finally, in the third paper of this sequence, we relax the previous idealization of a point nuclear charge and consider the short-distance effects of the finite nuclear size. As in all of our work, we do this analytically, including all orders in $Z\alpha$.[4]

Before entering into the details of the present calculation, we shall discuss the result for the induced point charge $\delta Q'$ and compare it with the important early work of Wichmann and Kroll, who made an extensive investigation of many aspects of the vacuum polarization in addition to calculating $\delta Q'$.

The induced vacuum-polarization charge-density terms that concern us are depicted in Fig. 1. As shown in the figure, the multiple-photon-exchange graphs sum to give the electron propagator (the Green's function) in the external Coulomb field. The lowest-order vacuum potential, the potential of order $\alpha Z\alpha$ first computed by Uehling,[5] $V_{\rm Ueh}(r)$, must be treated separately. It has a divergence that must be removed by a logarithmically infinite charge renormalization. Although the charge-renormalization procedure makes the potential finite, it alters its short-distance behavior, producing as $r \to 0$

$$V_{\rm Ueh}(r) \to \frac{2\alpha}{3\pi}\frac{Z\alpha}{r}\ln(mr)\,, \tag{1}$$

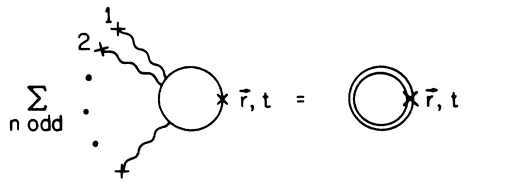

FIG. 1. Graphical representation of the vacuum-polarization induced charge density to all orders in $Z\alpha$. The double lines represent the electron propagating in the Coulomb field of the nucleus.

with m the electron mass. The charge density of this potential is more singular than the δ function at the origin. Thus, we must remove the lowest-order contribution and compute only the terms of order $(Z\alpha)^3$ and higher in $\delta Q'$.

A simple power-counting estimate shows that the amplitude corresponding to the graph of any order in Fig. 1 is superficially logarithmically divergent. However, because the theory is formally gauge-invariant, all of the graphs of order $(Z\alpha)^3$ and higher converge. The potential divergence is reflected, nonetheless, in the fact that the evaluation of the graphs is ambiguous, depending upon the order in which the integrations are performed. If the electron-loop integration is performed first, then the correct gauge-invariant result which has a vanishing total induced charge is obtained. This method of integration is, unfortunately, an impractical one, and, in order to get results in a closed analytical form, we are forced to do the external photon integrations first. This procedure yields a finite, but nonvanishing, total induced vacuum-polarization charge which must be subtracted from the original induced point charge to secure the correctly normalized value.

To be more specific, we compute

$$C_k(\rho, m) = -ie \int_{-i\infty}^{i\infty} \frac{dE}{2\pi} \int d\Omega \int_0^\rho r^2 dr \operatorname{tr} G_k(\vec{r}, \vec{r}; E) \gamma^0 , \tag{2}$$

where $G_k(\vec{r}, \vec{r}'; E)$ is the energy-dependent electron Green's function in the Coulomb field, projected into states that have angular momentum $j = k - \frac{1}{2}$. The energy integration produces the time-dependent Green's function evaluated at equal times. Thus, the coefficient $C_k(\rho, m)$ is a partial-wave component of the unrenormalized vacuum-polarization charge density integrated over the volume of a sphere of radius ρ (cf. the right-hand side of Fig. 1). It is crucial to do the integrations in the order indicated in Eq. (2) so as to pick up the δ function produced by the energy integration. We may write the renormalized induced point charge $\delta Q'$ as a partial-wave sum

$$\delta Q' = \sum_{k=1}^{\infty} \left[\lim_{\rho \to 0} C_k(\rho, m) - \lim_{\rho \to \infty} C_k(\rho, m) \right] . \tag{3}$$

Note that, since the electron mass m is the only parameter that bears a dimension, the limit $\rho \to 0$ with m fixed is equivalent to the limit $m \to 0$ with ρ fixed; thus

$$\delta Q' = \sum_{k=1}^{\infty} \left[C_k(\rho, m=0) - C_k(\rho=\infty, m) \right] . \tag{4}$$

The zero-mass limit is scale-invariant, and so the coefficients $C_k(\rho, m=0)$ are independent of the radius ρ. The use of this zero-mass limit greatly simplifies the calculation.

We employ operator and Fredholm determinant techniques that are very different from the methods of Wichmann and Kroll, who did not take advantage of the simplicity of the massless limit. We find that the renormalized induced point charge of order $(Z\alpha)^3$ and higher can be expressed as

$$\delta Q' = \sum_{k=1}^{\infty} \frac{4ek}{\pi} \operatorname{Im} \left[(\lambda - iZ\alpha)\psi(\lambda - iZ\alpha) - \ln\Gamma(\lambda - iZ\alpha) - \tfrac{1}{2}\ln(\lambda - iZ\alpha) + iZ\alpha k \psi'(k) - iZ\alpha/2k \right] . \tag{5}$$

Here we use the notation

$$\lambda = \left[k^2 - (Z\alpha)^2 \right]^{1/2} \tag{6}$$

and

$$\psi(z) = \frac{d}{dz} \ln\Gamma(z), \quad \psi'(z) = \frac{d}{dz} \psi(z) . \tag{7}$$

The charge of the electron is $-e$. Our result (5) should be compared to that found by Wichmann and Kroll[3]:

$$\begin{aligned} \delta Q' = \sum_{k=1}^{\infty} \frac{ek}{\pi} \Bigg\{ & -2\tan^{-1}\left(\frac{Z\alpha}{\lambda}\right) + \frac{2Z\alpha}{k} + \frac{(Z\alpha)^3}{3k^2} \\ & + 4 \sum_{n=1}^{\infty} \left[\frac{Z\alpha n}{(n+\lambda)^2 + (Z\alpha)^2} - \frac{Z\alpha n}{(n+k)^2} - \frac{(Z\alpha)^3 n^2}{k(n+k)^4} - \tan^{-1}\left(\frac{Z\alpha}{n+\lambda}\right) + \frac{Z\alpha}{(n+k)} - \frac{(Z\alpha)^3}{3(n+k)^3} + \frac{(Z\alpha)^3}{2k(n+k)^2} \right] \Bigg\} \\ & - \frac{e}{\pi} (Z\alpha)^3 \left[\frac{\pi^2}{6} - \frac{7}{9} - \frac{2}{3}\zeta(3) \right] , \end{aligned} \tag{8}$$

where

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \tag{9}$$

is the Riemann ζ function. An expansion of our result (5) produces precisely their result (8).

The behavior of $\delta Q'$ as a function of $Z\alpha$ is shown in Fig. 2. Although the point-charge idealization breaks down at $Z\alpha = 1$, a critical nuclear charge where the total energy of the lowest-lying bound state vanishes, the limit of $\delta Q'$ as $Z\alpha$ approaches unity from below is finite, with $-\delta Q'/e \simeq 0.0483$. A surprising feature of Eq. (5) is that the sum is dominated by the $k=1$ term, which gives more than 90% of the total for $0 \leq Z\alpha \leq 1$. The ratio of the $k=1$ term to the full sum of Eq. (5) is displayed in Fig. 3.

For small $Z\alpha$ we may expand $\delta Q'$ in a power series (see Appendix A), with the result

$$\delta Q' = \frac{e}{3\pi}\left\{(Z\alpha)^3\left[2\zeta(3)+\tfrac{7}{3}-\tfrac{1}{2}\pi^2\right]-(Z\alpha)^5\left[2\zeta(5)+\tfrac{71}{5}\zeta(3)-\tfrac{47}{240}\pi^4\right]+\mathcal{O}((Z\alpha)^7)\right\}$$

$$= -e\left[(Z\alpha)^3(0.020\,940)+(Z\alpha)^5(0.007\,121\,3)+\mathcal{O}((Z\alpha)^7)\right], \tag{10a}$$

which agrees with the expansion given by Wichmann and Kroll. Extraordinary cancellations are responsible for the small numerical coefficients which occur in Eq. (10a). If only the $k=1$ terms are retained in the expansion of the coefficients, the results are

$$\delta Q'_{k=1} = \frac{e}{3\pi}\left\{(Z\alpha)^3\left[6\zeta(2)-20\zeta(3)+12\zeta(4)+1\right]\right.$$

$$\left.+(Z\alpha)^5\left[\tfrac{3}{2}\zeta(2)+3\zeta(3)-27\zeta(4)+\tfrac{168}{5}\zeta(5)-12\zeta(6)+\tfrac{9}{20}\right]+\mathcal{O}((Z\alpha)^7)\right\}$$

$$= -e\left[(Z\alpha)^3(0.019\,486)+(Z\alpha)^5(0.007\,055\,8)+\mathcal{O}((Z\alpha)^7)\right]. \tag{10b}$$

Again, remarkable cancellations occur to produce very small coefficients for $(Z\alpha)^3$ and $(Z\alpha)^5$, nearly identical to those for the full k sum, Eq. (10a).

Following Wichmann and Kroll, we can display an accurate numerical evaluation of the induced point charge by writing

$$\delta Q' = -e\left[(Z\alpha)^3(0.020\,940)+(Z\alpha)^5(0.007\,121)\,F_0(Z\alpha)\right] \tag{10c}$$

Here, as shown in Fig. 4, $F_0(Z\alpha)$ rises slowly from $F_0 = 1$ at $Z\alpha = 0$ to $F_0 = 3.845$ at $Z\alpha = 1$. Our numerical results for $F_0(Z\alpha)$ are in approximate agreement with those of Wichmann and Kroll.

An approximate description of the size of the effects that we are considering is worthwhile. Consider, for example, the ciruclar orbit $5g \to 4f$ transition in muonic lead, where $Z\alpha \simeq 0.6$. The transition energy is about 400 000 eV. The lowest-order vacuum-polarization correction is about +2000 eV, and the energy shift brought about by $\delta Q'$ is about −50 eV. Finite-range ($m_e \neq 0$) corrections[4,6] to the higher-order vacuum-polarization potential amount to some +5 eV and are thus corrections on the order of 10%. This is only slightly larger than our estimate of $(2a_\mu/\lambda\!\!\!^{-}_e)^2 \sim 7\%$, although the coefficients in Eq. (10a) are anomalously small. This is so because cancellations also occur in the finite-range corrections.[4,6] Taking other small corrections into account, there remains a discrepancy between theory and experiment $\Delta E(\text{th.}-\text{exp.}) \simeq 54 \pm 17$ eV, which is typical

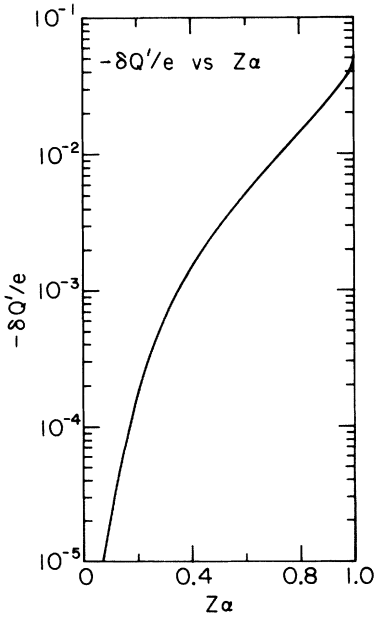

FIG. 2. The induced point charge, $\delta Q'$, as a function of $Z\alpha$.

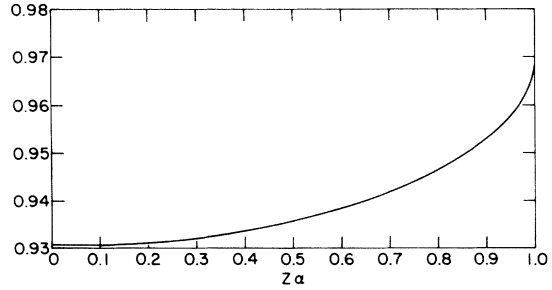

FIG. 3. The ratio of the $k=1$ term in the expansion of $\delta Q'$, the induced point charge, to the full value of $\delta Q'$, as a function of $Z\alpha$.

of a number of such transitions.[1,2] Note that the experimental precision is a factor of 2 smaller than the size of the effects of $\delta Q'$. Thus, our confirmation of the previous calculation[3] of this number is significant.

The plan of this paper is as follows: In Sec. II, in order to make this paper self-contained, we review and discuss the Green's function for a Dirac particle moving in a point Coulomb field, with particular emphasis on the formulation advocated by Martin and Glauber.[7] In Sec. III we discuss and evaluate an infinite determinantal representation of the Coulomb-Jost function using the method of Brown.[8] With these results in hand, we compute the induced point charge $\delta Q'$ in Sec. IV. Appendix A gives the derivation of sums necessary for the expansion (10a). Appendix B contains some minor details of the calculation.

II. GREEN'S FUNCTION

The Green's function for a Dirac field in an attractive point Coulomb field obeys[9]

$$\left[\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}-\gamma^0\left(E+\frac{Z\alpha}{r}\right)+m\right]G(\vec{r},\vec{r}';E)=\delta(\vec{r}-\vec{r}')\ . \tag{11}$$

It proves convenient to write this Green's function in terms of a second-order Green's function $\mathcal{G}(\vec{r},\vec{r}';E)$,

$$G(\vec{r},\vec{r}';E)=\left[m-\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}+\gamma^0\left(E+\frac{Z\alpha}{r}\right)\right]\mathcal{G}(\vec{r},\vec{r}';E)$$
$$=\mathcal{G}(r,r';E)\left[m-\vec{\gamma}\cdot i\vec{\nabla}'+\gamma^0\left(E+\frac{Z\alpha}{r'}\right)\right]\ . \tag{12}$$

But, before substituting Eq. (12) into Eq. (11) to secure the second-order Green's function equation, we note that a partial-wave decomposition must be performed eventually in order to secure a solution in closed form. A transformation to spherical variables is obtained if we use $\alpha^k\alpha^l=\delta^{kl}+i\epsilon^{klm}\sigma^m$ and

$$\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}=\beta r^{-2}\vec{\alpha}\cdot\vec{r}\,\vec{\alpha}\cdot\vec{r}\,\vec{\alpha}\cdot\frac{1}{i}\vec{\nabla}$$
$$=\beta\alpha_r\left[\frac{1}{i}\frac{\partial}{\partial r}+\frac{i}{r}\vec{\sigma}\circ\vec{L}\right], \tag{13}$$

where

$$\alpha_r=\vec{\alpha}\cdot\hat{r},\quad \vec{L}=\vec{r}\times\frac{1}{i}\vec{\nabla}\ . \tag{14}$$

It is advantageous to write this as

$$\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}=\beta\alpha_r\frac{1}{i}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)-\frac{i\alpha_r K}{r}\ , \tag{15}$$

where[10]

$$K=\beta(\vec{\sigma}\cdot\vec{L}+1)\ , \tag{16}$$

because the operator K, in addition to commuting with β, also commutes with α_r,

$$[K,\alpha_r]=0\ , \tag{17}$$

which one can verify with a little calculation. Moreover, the operator

$$\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}-\gamma^0\left(E+\frac{Z\alpha}{r}\right)$$
$$=\beta\alpha_r\frac{1}{i}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)-\frac{i\alpha_r K}{r}-\beta\left(E+\frac{Z\alpha}{r}\right) \tag{18}$$

now appears as the sum of three quantities which, except when a derivative acts, are mutually anticommutative. Hence,

$$\left[\vec{\gamma}\cdot\frac{1}{i}\vec{\nabla}-\gamma^0\left(E+\frac{Z\alpha}{r}\right)\right]^2=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)^2+\frac{\beta K}{r^2}+\frac{i\alpha_r Z\alpha}{r^2}$$
$$-\frac{K^2}{r^2}+\left(E+\frac{Z\alpha}{r}\right)^2\ . \tag{19}$$

The terms involving $1/r^2$ can be written in a simple form[7,11] using

$$K^2-\beta K-i\alpha_r(Z\alpha)-(Z\alpha)^2=\mathfrak{L}(\mathfrak{L}+1)\ , \tag{20}$$

with

$$\mathfrak{L}=-\beta K-i\alpha_r Z\alpha\ . \tag{21}$$

It is now a trivial matter to insert Eq. (12) into Eq. (11) and obtain the second-order Green's function equation:

$$\left[-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{\mathfrak{L}(\mathfrak{L}+1)}{r^2}+m^2-E^2-\frac{2EZ\alpha}{r}\right]\mathcal{G}(\vec{r},\vec{r}';E)$$
$$=\delta(\vec{r}-\vec{r}')\ . \tag{22}$$

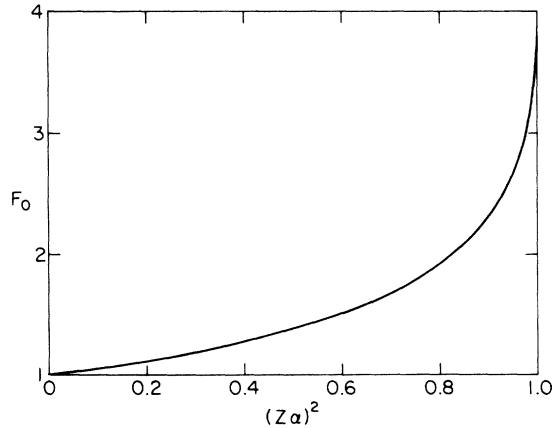

FIG. 4. The correction, $F_0(Z\alpha)$, to the $(Z\alpha)^5$ approximation to the induced point charge, Eq. (10c).

In a basis where $\mathfrak{L}$ is diagonal, this is essentially the equation for a nonrelativistic Coulomb Green's function with nonintegral orbital angular momentum. In order to discuss this basis, we must first return to the operator K defined in Eq. (16) and discuss its properties. It is Hermitian and commutes with the total angular momentum operator $\vec{J}=\vec{L}+\frac{1}{2}\vec{\sigma}$, and it can, therefore, be diagonalized in states of fixed total angular momentum j. Since

$$(\vec{\sigma}\cdot\vec{L})^2 = L^2 - \vec{\sigma}\cdot\vec{L} \ , \tag{23}$$

we find that

$$K^2 = J^2 + \tfrac{1}{4} \ . \tag{24}$$

Thus, the eigenvalues of K are $K' = \pm k$, where $k = j + \frac{1}{2} = 1, 2, 3, \ldots$. Since

$$K(K-\beta) = L^2 \ , \tag{25}$$

in the nonrelativistic limit where, effectively, $\beta \to 1$, positive eigenvalues of K correspond to orbital angular momentum $l = j - \frac{1}{2}$, negative eigenvalues to $l = j + \frac{1}{2}$. Now, $\mathfrak{L}$ commutes with K and

$$\mathfrak{L}^2 = K^2 - (Z\alpha)^2 \ . \tag{26}$$

Hence, $\mathfrak{L}$ can be diagonalized in eigenstates of K with a single K eigenvalue and has eigenvalues $\mathfrak{L}' = \pm\lambda$, where

$$\lambda = [k^2 - (Z\alpha)^2]^{1/2} \ . \tag{27}$$

In the limit $Z\alpha \to 0$, $\mathfrak{L} \to -\beta K$, and the two eigenvalues $\pm\lambda$ correspond to eigenstates of opposite parity.

Projection operators $P_{\pm\lambda}$ can be constructed that diagonalize $\mathfrak{L}$,

$$\mathfrak{L} P_{\pm\lambda}(\hat{r}, \hat{r}') = \pm\lambda P_{\pm\lambda}(\hat{r}, \hat{r}') \ , \tag{28}$$

and resolve the identity,

$$\sum_{\mathfrak{L}'} P_{\mathfrak{L}'}(\hat{r}, \hat{r}') \frac{1}{r}\delta(r-r')\frac{1}{r'} = \delta(\vec{r}-\vec{r}') \ . \tag{29}$$

Note that while a single projection operator $P_{\mathfrak{L}'}$ has a definite value of the total angular momentum, it does not diagonalize K but contains projections onto the two eigenvalues $K' = \pm k$. In this way, the formalism exploits the well-known doublet degeneracy of the Dirac-Coulomb problem. We may insert the decomposition

$$\mathcal{G}(\vec{r}, \vec{r}'; E) = \sum_{\mathfrak{L}'} P_{\mathfrak{L}'}(\hat{r}, \hat{r}') \frac{1}{r} \mathcal{G}_{\mathfrak{L}'}(r, r'; E) \frac{1}{r} \ , \tag{30}$$

in the second-order equation (22). From the orthonormality relation

$$\int d\Omega' \, P_{\mathfrak{L}'}(\hat{r}, \hat{r}') P_{\mathfrak{L}''}(\hat{r}', \hat{r}'') = \delta_{\mathfrak{L}'\mathfrak{L}''} P_{\mathfrak{L}'}(\hat{r}, \hat{r}'') \tag{31}$$

we infer that

$$\left[-\frac{d^2}{dr^2} + \frac{\lambda(\lambda\pm1)}{r^2} + m^2 - E^2 - \frac{2EZ\alpha}{r}\right] \mathcal{G}_{\pm\lambda}(r, r'; E) = \delta(r-r') \ . \tag{32}$$

This equation has the familiar, standard solution

$$\mathcal{G}_{\pm\lambda}(r, r'; E) = A_{\pm\lambda}(r_>; E) B_{\pm\lambda}(r_<; E) \ , \tag{33}$$

in which the functions A and B are two independent solutions to the homogeneous counterpart of Eq. (32), $r_>$ ($r_<$) is the larger (smaller) of r and r', and the solutions are normalized to have a Wronskian of unity:

$$A_{\pm\lambda}(r; E)\left[\frac{d}{dr}B_{\pm\lambda}(r; E)\right] - \left[\frac{d}{dr}A_{\pm\lambda}(r; E)\right]B_{\pm\lambda}(r; E) = 1 \ . \tag{34}$$

The functions A and B are confluent hypergeometric functions which are discussed at some length in Refs. 12 and 13.

We must specify boundary conditions to obtain a unique solution. Clearly, the function B, which involves the smaller coordinate $r_<$, must be the solution that is regular at the origin. In order to arrive at the correct boundary condition on the other function A, it is useful to recall the analytic nature of the Green's function as a function of its energy variable, E, which is illustrated in Fig. 5. We may assume that initially the energy lies on the real axis in the gap between the left-hand cut and the lowest-lying bound state, $-m < E < E_1$. The Green's function can then be obtained at general energy values by analytic continuation. In the gap region, one of the independent solutions to the homogeneous counterpart of Eq. (32) decreases exponentially as $r \to \infty$; the other increases exponentially as $r \to \infty$. Clearly, we must choose the former solution for the function A which involves the larger coordinate, $r_>$. Thus, in the notation of Whittaker and Watson[13] and Bateman,[12] respectively, we have

$$\begin{aligned} A_{\pm\lambda}(r; E) &= \Gamma(\lambda - \eta + \tfrac{1}{2} \pm \tfrac{1}{2}) \, W_{\eta, \lambda\pm1/2}(2qr) \\ &= \Gamma(\lambda - \eta + \tfrac{1}{2} \pm \tfrac{1}{2}) \, (2qr)^{\lambda+1/2\pm1/2} e^{-qr} \\ &\quad \times \Psi(\lambda - \eta + \tfrac{1}{2} \pm \tfrac{1}{2}; 2\lambda + 1 \pm 1; 2qr) \end{aligned} \tag{35}$$

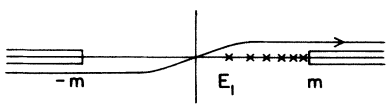

FIG. 5. The analytic structure of the Green's function as a function of energy. There are branch cuts along the $[-\infty, -m]$ and $[m, \infty]$ portions of the real axis. Poles appear at the bound-state energy eigenvalues which accumulate at the tip of the right-hand cut. The energy of the lowest-lying bound state is denoted by E_1. The contour shown is that which defines the time-ordered Green's function.

and

$$B_{\pm\lambda}(r;E) = \Gamma(2\lambda+1\pm1)^{-1}(2q)^{-1}M_{\eta,\lambda\pm1/2}(2qr)$$
$$= \Gamma(2\lambda+1\pm1)^{-1}(2q)^{-1}(2qr)^{\lambda+1/2\pm1/2}e^{-qr}$$
$$\times\Phi(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2};2\lambda+1\pm1;2qr)\ . \quad (36)$$

Here

$$q=(m^2-E^2)^{1/2} \quad (37)$$

and

$$\eta=\eta(E)=Z\alpha E/q\ . \quad (38)$$

We define the branch cuts of the square root in Eq. (37) to lie on the real axis in the intervals $[-\infty,-m]$ and $[m,\infty]$ in conformity with the cut structure of the Green's functions, as illustrated in Fig. 5. In the gap $-m<E<m$, q is positive. The Γ-function factors were chosen to satisfy the Wronskian normalization (34) in a way that proves convenient. Note that bound states appear as poles in the Green's function. These poles arise from the Γ function in Eq. (35). Thus, the bound-state energies are determined by

$$\eta(E_{\pm\lambda,n_r})=\lambda+\tfrac{1}{2}\pm\tfrac{1}{2}+n_r\ , \quad (39)$$

in which

$$n_r=0,1,2,\ldots \quad (40)$$

is the radial quantum number. Something of the nature of these functions is illustrated by their limits as $r\to\infty$,

$$A_{\pm\lambda}(r)\sim\Gamma(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2})(2qr)^{\eta}e^{-qr}\ , \quad (41a)$$

$$B_{\pm\lambda}(r)\sim\Gamma(\lambda-\eta+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}(2q)^{-1}(2qr)^{-\eta}e^{+qr} \quad (41b)$$

and as $r\to0$,

$$A_{\pm\lambda}(r)\sim\Gamma(2\lambda\pm1)(2qr)^{-\lambda+1/2\mp1/2}\ , \quad (42a)$$

$$B_{\pm\lambda}(r)\sim\Gamma(2\lambda+1\pm1)^{-1}(2q)^{-1}(2qr)^{\lambda+1/2\pm1/2}\ . \quad (42b)$$

Although we will not need an explicit construction of the projection operators $P_{\mathcal{L}'}$, we will need some of their properties. We can write the projection operators in terms of eigenspinors $\chi^m_{K',\mathcal{L}'}$ of the commuting operators K, $\mathcal{L}$, and J_z,

$$K\chi^m_{K',\mathcal{L}'}(\hat r)=K'\chi^m_{K',\mathcal{L}'}(\hat r)\ ,$$
$$\mathcal{L}\chi^m_{K',\mathcal{L}'}(\hat r)=\mathcal{L}'\chi^m_{K',\mathcal{L}'}(\hat r)\ , \quad (43)$$
$$J_z\chi^m_{K',\mathcal{L}'}(\hat r)=m\chi^m_{K',\mathcal{L}'}(\hat r)\ .$$

The operator $\mathcal{L}$ is not Hermitian, but $\mathcal{L}^\dagger\beta=\beta\mathcal{L}$, so that $\bar\chi=\chi^\dagger\beta$ is a left eigenspinor of $\mathcal{L}$,

$$\bar\chi^m_{K',\mathcal{L}'}(\hat r)\,\mathcal{L}=\mathcal{L}'\bar\chi^m_{K',\mathcal{L}'}(\hat r)\ . \quad (44)$$

The other operators K and J_z are Hermitian and they commute with β, and so the $\bar\chi^m_{K',\mathcal{L}'}$ are also their left eigenspinors. We write $\gamma_r=\vec\gamma\cdot\hat r$ and note that this matrix commutes with K and J_z but anticommutes with $\mathcal{L}$ and squares to -1:

$$\{\mathcal{L},\gamma_r\}=0,\quad \gamma_r{}^2=-1\ . \quad (45)$$

Thus γ_r transforms $\chi_{\pm\lambda}$ into $\chi_{\mp\lambda}$ with a phase factor which squares to -1. We adopt the phase convention

$$\bar\chi^m_{K',\pm\lambda}\gamma_r=-i\bar\chi^m_{K',\mp\lambda}\ . \quad (46)$$

Using this convention and requiring that the eigenspinors diagonalize $\mathcal{L}$, which we may write as

$$\mathcal{L}=-K\beta+iZ\alpha\gamma_r\beta\ , \quad (47)$$

gives the two-dimensional matrix representation

$$\begin{pmatrix}\bar\chi^m_{K',+\lambda}\\ \bar\chi^m_{K',-\lambda}\end{pmatrix}\beta=\begin{pmatrix}-\frac{K'}{\lambda} & \frac{Z\alpha}{\lambda}\\ -\frac{Z\alpha}{\lambda} & \frac{K'}{\lambda}\end{pmatrix}\begin{pmatrix}\bar\chi^m_{K',+\lambda}\\ \bar\chi^m_{K',-\lambda}\end{pmatrix}\ . \quad (48)$$

The familiar technique of computing $\bar\chi\cdots\chi$ matrix elements of K, $\mathcal{L}$, J_z, considering the operators to act either to the right or to the left, shows that the scalar product of one $\bar\chi$ with another χ vanishes unless they have identical eigenvalues. Hence, with a convenient choice of normalization,

$$\int d\Omega\,\bar\chi^{m'}_{K',\mathcal{L}'}(\hat r)\,\chi^{m''}_{K'',\mathcal{L}''}(\hat r)=-\frac{K'\mathcal{L}'}{k\lambda}\,\delta_{K'K''}\delta_{\mathcal{L}'\mathcal{L}''}\,\delta^{m'm''}\ . \quad (49)$$

The metric factor $-K'\mathcal{L}'/k\lambda=\pm1$ is necessary to satisfy Eq. (48) and the positivity of $\chi^\dagger\chi$. We now have the representation

$$P_{\pm\lambda}(\hat r,\hat r')=\sum_{K'=\pm k}\sum_m\left(\frac{-K'\mathcal{L}'}{k\lambda}\right)\chi^m_{K',\pm\lambda}(\hat r)\,\bar\chi^m_{K',\pm\lambda}(\hat r')\ , \quad (50)$$

where, it should be recalled, k and λ are related by $\lambda=[k^2-(Z\alpha)^2]^{1/2}$.

In our later work, we shall need the spinor traces

$$\int d\Omega\,\mathrm{tr}P_{\pm\lambda}(\hat r,\hat r)=4k\ , \quad (51)$$

$$\int d\Omega\,\mathrm{tr}\beta P_{\pm\lambda}(\hat r,\hat r)=0\ , \quad (52)$$

and

$$\int d\Omega\,\mathrm{tr}\alpha_r P_{\pm\lambda}(\hat r,\hat r)=\mp4k\,\frac{Z\alpha i}{\lambda}\ . \quad (53)$$

These results are easily derived with the aid of the formulas of the previous paragraph. The factor $4k=2(2j+1)$ arises from the K' and m sums in Eq. (50). The remaining numbers follow from Eqs. (46), (48), and (49).

In the present paper we shall make use only of the second-order Green's function, $\mathcal{G}$. However,

we will use the first-order version, G, in a later paper. For the sake of completeness, we shall give some discussion of this function here. If we make use of the definition of $\mathfrak{L}$, Eq. (21), in Eq. (18), we can write the relationship (12) between the first- and second-order Green's functions as[7]

$$G(\vec{r},\vec{r}';E)=\left[m+\beta E+i\gamma_r\left(\frac{\partial}{\partial r}+\frac{\mathfrak{L}+1}{r}\right)\right]\mathcal{G}(\vec{r},\vec{r}';E)\ . \tag{54}$$

In order to carry out the indicated differentiation, we observe that

$$\left[-\frac{d^2}{dr^2}+\frac{\lambda(\lambda\pm1)}{r^2}+m^2-E^2-\frac{2EZ\alpha}{r}\right]\left[\frac{d}{dr}\mp\frac{\lambda}{r}\pm\frac{EZ\alpha}{\lambda}\right]=\left[\frac{d}{dr}\mp\frac{\lambda}{r}\pm\frac{EZ\alpha}{\lambda}\right]\left[-\frac{d^2}{dr^2}+\frac{\lambda(\lambda\mp1)}{r^2}+m^2-E^2-\frac{2EZ\alpha}{r}\right]. \tag{55}$$

The right-hand side of this equation annihilates $A_{\mp\lambda}$ or $B_{\mp\lambda}$. Hence the operator $[d/dr\mp\lambda/r\pm EZ\alpha/\lambda]$ applied to $A_{\mp\lambda}$ or $B_{\mp\lambda}$ must produce, up to an overall constant factor, the same functions but with the sign of their index reversed, because this operation does not alter the regular or irregular character of the functions, and it does give a solution to the appropriate differential equation. The constant factor is readily determined by referring to the long-distance limits Eqs. (41), with the results

$$\left[\frac{d}{dr}-\frac{\lambda}{r}+\frac{EZ\alpha}{\lambda}\right]\begin{Bmatrix}A_{-\lambda}(r;E)\\ B_{-\lambda}(r;E)\end{Bmatrix}=\begin{Bmatrix}-\frac{q}{r}A_{+\lambda}(r;E)\\ (\lambda^2-\eta^2)\frac{q}{\lambda}B_{+\lambda}(r;E)\end{Bmatrix}, \tag{56}$$

$$\left[\frac{d}{dr}+\frac{\lambda}{r}-\frac{EZ\alpha}{\lambda}\right]\begin{Bmatrix}A_{+\lambda}(r;E)\\ B_{+\lambda}(r;E)\end{Bmatrix}=\begin{Bmatrix}-(\lambda^2-\eta^2)\frac{q}{\lambda}A_{-\lambda}(r;E)\\ \frac{q}{\lambda}B_{-\lambda}(r;E)\end{Bmatrix}. \tag{57}$$

Recalling the structure of Eq. (54), we define the spinors

$$\psi^m_{K'}(\vec{r};E)\equiv\left[m+\beta E+i\gamma_r\left(\frac{\partial}{\partial r}-\frac{\lambda}{r}\right)\right]\times A_{-\lambda}(r;E)\,\chi^m_{K',-\lambda}(\hat{r}) \tag{58}$$

and

$$\phi^m_{K'}(\vec{r};E)\equiv\left[m+\beta E+i\gamma_r\left(\frac{\partial}{\partial r}+\frac{\lambda}{r}\right)\right]\times B_{+\lambda}(r;E)\,\chi^m_{K',+\lambda}(\hat{r})\ . \tag{59}$$

These spinor functions, when multiplied by r^{-1}, become solutions of the first-order Dirac equation. We may use Eqs. (56), (57), (46), and the adjoint of Eq. (48) to compute

$$\psi^m_{K'}(\vec{r};E)=\left[m+\frac{EK'}{\lambda}\right]A_{-\lambda}(r;E)\,\chi^m_{K',-\lambda}(\hat{r})+\frac{q}{\lambda}A_{+\lambda}(r;E)\,\chi^m_{K',+\lambda}(\hat{r})\ , \tag{60}$$

$$\phi^m_{K'}(\vec{r};E)=\left[m-\frac{EK'}{\lambda}\right]B_{+\lambda}(r;E)\,\chi^m_{K',+\lambda}(\hat{r})-\frac{q}{\lambda}B_{-\lambda}(r;E)\,\chi^m_{K',-\lambda}(\hat{r})\ , \tag{61}$$

and

$$\left[m+\beta E+i\gamma_r\left(\frac{\partial}{\partial r}+\frac{\lambda}{r}\right)\right]A_{+\lambda}(r;E)\,\chi^m_{K',+\lambda}(\hat{r})=\frac{\lambda}{q}\left[m-\frac{EK'}{\lambda}\right]\psi^m_{K'}(\vec{r};E)\ , \tag{62}$$

$$\left[m+\beta E+i\gamma_r\left(\frac{\partial}{\partial r}-\frac{\lambda}{r}\right)\right]B_{-\lambda}(r;E)\,\chi^m_{K',-\lambda}(\hat{r})=-\frac{\lambda}{q}\left[m+\frac{EK'}{\lambda}\right]\phi^m_{K'}(\vec{r};E)\ . \tag{63}$$

Finally, we can insert the partial-wave decompositions (30) and (50) into Eq. (54) and use Eqs. (58)–(63) to secure the expansions

$$r>r':\ G(\vec{r},\vec{r}';E)=\sum_{K'}\sum_{m}\left(-\frac{\lambda}{q}\frac{K'}{k}\right)\frac{1}{r}\,\psi^m_{K'}(\vec{r};E)\,\bar{\phi}^m_{K'}(\vec{r}';E)\,\frac{1}{r'},$$

$$r<r':\ G(\vec{r},\vec{r}';E)=\sum_{K'}\sum_{m}\left(-\frac{\lambda}{q}\frac{K'}{k}\right)\frac{1}{r}\,\phi^m_{K'}(\vec{r};E)\,\bar{\psi}^m_{K'}(\vec{r}';E)\,\frac{1}{r'}\ . \tag{64}$$

We should remark that the adjoint spinor wave functions $\bar{\psi}^m_{K'}$ and $\bar{\phi}^m_{K'}$ that appear here are analytic continuations of the wave functions from the gap $-m<E<m$, where the functions $A_{\pm\lambda}(r;E)$ and $B_{\pm\lambda}(r;E)$ are real. Thus, these adjoint wave functions are obtained from Eqs. (60) and (61) by simply replacing the spinor factors $\chi^m_{K',\pm\lambda}$ by their (Dirac) adjoints $\bar{\chi}^m_{K',\pm\lambda}$.

III. INFINITE DETERMINANTS

The calculation of the vacuum-polarization induced point charge is considerably simplified if we make use of operator techniques involving an infinite determinant. We shall discuss the determinant in this section so as to have all the mathematical tools we need when we perform the calculation in the following section. In order to introduce this determinant, we write the radial Green's function equation (32) in an operator notation,

$$\left[p_r^2+\frac{\lambda(\lambda\pm1)}{r^2}+m^2-E^2-\frac{2EZ\alpha}{r}\right]\mathcal{G}_{\pm\lambda}(E)=1\ . \tag{65}$$

We denote the corresponding free Green's function where $Z\alpha=0$ by $\mathcal{G}^{(0)}_{\pm\lambda}(E)$, so that Eq. (65) may be written in the form

$$\left[1-\mathcal{G}^{(0)}_{\pm\lambda}(E)\,\frac{2EZ\alpha}{r}\right]\mathcal{G}_{\pm\lambda}(E)=\mathcal{G}^{(0)}_{\pm\lambda}(E)\ . \tag{66}$$

The infinite determinant which we shall employ is the Fredholm determinant[14] associated with Eq. (66),

$$F_{\pm\lambda}(E)=\mathrm{Det}\left[1-\mathcal{G}^{(0)}_{\pm\lambda}(E)\,\frac{2EZ\alpha}{r}\right]\ . \tag{67}$$

It is the Jost function[15] of relativistic Coulomb scattering: The ratio $F_{\pm\lambda}(E)^*/F_{\pm\lambda}(E)$ is the partial-wave scattering function $e^{2i\delta_{\pm\lambda}(E)}$, and $F_{\pm\lambda}(E)^{-1}$ is the ratio of the interacting to noninteracting wave functions evaluated at the origin.

An infinite determinant involving an operator of the form $(1-\lambda A)$, where λ is a numerical parameter, may be defined by

$$\mathrm{Det}[1-\lambda A]=\exp[\mathrm{Tr}\ln(1-\lambda A)]\ . \tag{68}$$

Here the trace of the operator logarithm is understood to be the infinite sum of traces that result from a formal expansion of the logarithm in powers of λ. If the operator A can be diagonalized in some basis, then formula (68) would express the determinant as an infinite product of eigenvalues, just as a finite determinant can be expressed as a finite product of eigenvalues. If $|\mathrm{Tr}A|<\infty$ and Tr $\mathrm{Tr}A^\dagger A<\infty$, then[14] the power series for the exponent in Eq. (68) converges for sufficiently small $|\lambda|$. One can then easily prove that the definition (68) obeys all the usual rules of operation for finite determinants such as the determinant of a product of operators being the product of determinants of the individual operators. When $|\lambda|$ becomes large, the determinant may vanish somewhere on the circle $|\lambda|=$ constant, and the power series for the exponent will diverge. However, if the exponential is expanded and the resulting double power series rearranged so as to form a single power series in λ for the determinant itself rather than its logarithm, then this new series converges absolutely[14] for any value of λ. Thus, if $|\mathrm{Tr}A|<\infty$ and $\mathrm{Tr}A^\dagger A<\infty$, the infinite determinant exists and is well-defined. In the situation where $\mathrm{Tr}A$ does not converge but $\mathrm{Tr}A^\dagger A$ is finite, a modified determinant may be defined by simply deleting the first trace $\mathrm{Tr}A$ in its expansion. This modified determinant will then possess an absolutely converging power series in λ and has essentially the same properties as an ordinary determinant. We conclude this brief summary with a remark that will be crucial for our later calculation: It follows from the definition (68) and the cyclic symmetry of the trace ($\mathrm{Tr}AB=\mathrm{Tr}BA$) that

$$\delta\ln\mathrm{Det}X=\mathrm{Tr}X^{-1}\delta X\ . \tag{69}$$

We may easily compute the infinite determinant (67) if the following method[8] is employed. We take the energy E to be initially fixed at some place in the bound-state interval $0<E<m$. After the calculation is performed, the result can be analytically continued to any energy value. Now, with the energy fixed in the bound-state interval, solutions to the homogeneous counterpart of Eq. (65), or its equivalent (66), will exist for critical values of the coupling constant $Z\alpha$, which we denote generically by $(Z\alpha)'$. These critical coupling constant values are those for which a bound state exists at energy E. They are given by the relativistic Balmer formula (39)

$$(Z\alpha)'=(\lambda+\tfrac{1}{2}\pm\tfrac{1}{2}+n_r)\left(\frac{m^2-E^2}{E^2}\right)^{1/2}\ , \tag{70}$$

where we now treat λ as an independent parameter. When the coupling constant assumes a critical value $(Z\alpha)'$ the determinant (67) must vanish, for otherwise the homogeneous counterpart of Eq. (66) would have no solution. Hence, the eigenvalues of the kernel $\mathcal{G}^{(0)}_{\pm\lambda}(E)\,2E/r$ are the reciprocals of the eigenvalues of $Z\alpha$,

$$\left[\mathcal{G}^{(0)}_{\pm\lambda}(E)\,\frac{2E}{r}\right]'=(\lambda+\tfrac{1}{2}\pm\tfrac{1}{2}+n_r)^{-1}\left(\frac{E^2}{m^2-E^2}\right)^{1/2}\ . \tag{71}$$

We may now compute the determinant (67) by eval-

uating the definition (68) in the basis which diagonalizes the kernel. Since the energy lies in the bound-state interval, the kernel is square-integrable, and this discrete basis is complete. Thus

$$\ln F_{\pm\lambda}(E) = \sum_{n_r=0}^{\infty} \ln\left[1 - \frac{\eta(E)}{\lambda + \frac{1}{2} \pm \frac{1}{2} + n_r}\right] , \tag{72}$$

where, we recall,

$$\eta(E) = Z\alpha\left(\frac{E^2}{m^2 - E^2}\right)^{1/2} . \tag{38}$$

The first term in the expansion of the logarithm, which is the first trace term, diverges,

$$\sum^{\infty} \frac{1}{n_r} = \ln\infty . \tag{73}$$

This divergence arises from the infinite range of the Coulomb potential, $\mathrm{Tr}\, 1/r \sim \int^{\infty} dr/r = \ln\infty$. It is the familiar, divergent phase that appears in the Coulomb scattering amplitude. If we delete this first trace, defining the modified determinant, then the remaining sum converges for sufficiently small $\eta(E)$, and it can be analytically continued to any value of $\eta(E)$. The divergence in the first trace is independent of λ and appears only to first order in $Z\alpha$ in $\ln F_{\pm\lambda}(E)$, the quantity that will later be used in the induced point charge calculation. Since we are interested only in the terms of order $(Z\alpha)^3$ and higher in the induced point charge, the divergence will not affect our calculation, and we can discard it. The argument of the logarithm in Eq. (72) has poles when $\lambda + \frac{1}{2} \pm \frac{1}{2}$ is a negative integer and zeros when $\lambda + \frac{1}{2} \pm \frac{1}{2} - \eta(E)$ is a negative integer. Thus, the result of performing[8] the sum (72) is a natural one,

$$F_{\pm\lambda}(E) = \mathrm{Det}\left[1 - \mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2EZ\alpha}{r}\right] = \frac{\Gamma(\lambda + \frac{1}{2} \pm \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2} \pm \frac{1}{2} - \eta(E))} . \tag{74}$$

We should emphasize again that the result (74) is a slightly formal one in the sense that the divergent part of the first trace is implicitly understood to have been discarded.

We shall need the evaluation of certain traces for our subsequent calculation which we now derive with the aid of the variational formula (69) and our result (74).

We consider first the derivative with respect to $Z\alpha$ treating λ as an independent parameter. From the variational formula

$$\begin{aligned}\frac{\partial}{\partial(Z\alpha)} \ln F_{\pm\lambda}(E) &= -\mathrm{Tr}\left[1 - \mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2EZ\alpha}{r}\right]^{-1} \mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2E}{r} \\ &= -\mathrm{Tr}\left[\mathcal{G}^{(0)}_{\pm\lambda}(E)^{-1} - \frac{2EZ\alpha}{r}\right]^{-1}\frac{2E}{r} \\ &= -\mathrm{Tr}\,\mathcal{G}_{\pm\lambda}(E)\frac{2E}{r} ,\end{aligned} \tag{75}$$

while from our result

$$\frac{\partial}{\partial(Z\alpha)} \ln F_{\pm\lambda}(E) = (Z\alpha)^{-1}\eta(E)\,\psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2} - \eta(E)) . \tag{76}$$

Hence, comparing the two forms, we conclude that

$$\begin{aligned}\int_0^{\infty} dr\, \mathcal{G}_{\pm\lambda}(r, r; E)\frac{2EZ\alpha}{r} &\equiv \mathrm{Tr}\,\mathcal{G}_{\pm\lambda}(E)\frac{2EZ\alpha}{r} \\ &= -\eta(E)\,\psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2} - \eta(E)) .\end{aligned} \tag{77}$$

Next, again treating λ as an independent variable, we use

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathcal{G}^{(0)}_{\pm\lambda}(E) &= -\mathcal{G}^{(0)}_{\pm\lambda}(E)\left[\frac{\partial}{\partial\lambda}\mathcal{G}^{(0)}_{\pm\lambda}(E)^{-1}\right]\mathcal{G}^{(0)}_{\pm\lambda}(E) \\ &= -\mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2\lambda \pm 1}{r^2}\mathcal{G}^{(0)}_{\pm\lambda}(E)\end{aligned} \tag{78}$$

to calculate

$$\begin{aligned}\frac{\partial}{\partial\lambda}\ln F_{\pm\lambda}(E) &= \mathrm{Tr}\,\mathcal{G}_{\pm\lambda}(E)\frac{2\lambda \pm 1}{r^2}\mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2EZ\alpha}{r} \\ &= \mathrm{Tr}\,\mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2EZ\alpha}{r}\mathcal{G}_{\pm\lambda}(E)\frac{2\lambda \pm 1}{r^2} \\ &= \mathrm{Tr}\left[\mathcal{G}_{\pm\lambda}(E) - \mathcal{G}^{(0)}_{\pm\lambda}(E)\right]\frac{2\lambda \pm 1}{r^2} .\end{aligned} \tag{79}$$

On the other hand,

$$\frac{\partial}{\partial\lambda}\ln F_{\pm\lambda}(E) = -\psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2} - \eta(E)) + \psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2}) . \tag{80}$$

Hence

$$\begin{aligned}&\mathrm{Tr}\left[\mathcal{G}_{\pm\lambda}(E) - \mathcal{G}^{(0)}_{\pm\lambda}(E)\right]\frac{2\lambda \pm 1}{r^2} \\ &\qquad = -\psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2} - \eta(E)) + \psi(\lambda + \tfrac{1}{2} \pm \tfrac{1}{2}) .\end{aligned} \tag{81}$$

Finally, using entirely similar methods, we compute the energy derivative

$$\frac{\partial}{\partial E}\ln F_{\pm\lambda}(E) = \mathrm{Tr}\,\mathcal{G}_{\pm\lambda}(E)\left(2E + \frac{2Z\alpha}{r}\right)\mathcal{G}^{(0)}_{\pm\lambda}(E)\frac{2EZ\alpha}{r} \tag{82}$$

to derive

$$\mathrm{Tr}\,[\mathcal{G}_{\pm\lambda}(E)-\mathcal{G}^{(0)}_{\pm\lambda}(E)]\left(2E+\frac{2Z\alpha}{r}\right) = -\frac{\partial}{\partial E}\ln\Gamma(\lambda+\tfrac{1}{2}\pm\tfrac{1}{2}-\eta(E)) . \tag{83}$$

IV. INDUCED POINT CHARGE

The vacuum-polarization induced charge density can be expressed formally as

$$\rho_{\mathrm{vac\,pol}}(r) = -e\langle\psi^\dagger(\vec{r},t)\,\psi(\vec{r},t)\rangle , \tag{84}$$

where ψ is the electron field operator, and the expectation value is taken in the external point Coulomb field of the nucleus. This matrix element is the electron Green's function with common space-time coordinates. The equal-time Green's function may be expressed as an energy integral over its Fourier transform with the contour illustrated in Fig. 5. This contour may be deformed to run along the imaginary axis without crossing any singularities, and we secure the formal expression

$$\rho_{\mathrm{vac\,pol}}(r) = -ie\int_{-i\infty}^{i\infty}\frac{dE}{2\pi}\,\mathrm{tr}\,G(\vec{r},\vec{r};E)\,\gamma^0 . \tag{85}$$

We shall use the subscript k to denote a projection into the union of subspaces having eigenvalues $K'=\pm k$ of the operator K [Eq. (16)]. This gives a precise meaning to our previous definition,

$$C_k(\rho,m) = -ie\int_{-i\infty}^{+i\infty}\frac{dE}{2\pi}\int d\Omega\int_0^\rho r^2dr\,\mathrm{tr}\,G_k(\vec{r},\vec{r};E)\,\gamma^0 . \tag{2}$$

As we remarked in Sec. I, the $\rho\to\infty$ limit of this quantity does not vanish, and we obtain a nonvanishing total unrenormalized induced vacuum polarization charge in order $(Z\alpha)^3$ and higher. This total charge must be removed by the subtraction of a constant in momentum space or, equivalently, by the subtraction of a constant times the δ function in configuration space. Thus, we must identify the renormalized induced point charge $\delta Q'$,

$$\rho_{\mathrm{vac\,pol}}(r) = \delta Q'\,\delta(\vec{r})+\cdots , \tag{86}$$

as being

$$\delta Q' = \sum_{k=1}^{\infty}\left[\lim_{\rho\to 0}C_k(\rho,m)-\lim_{\rho\to\infty}C_k(\rho,m)\right] . \tag{3}$$

And, as we remarked in Sec. I, since m is the only parameter which carries a dimension, the limit $\rho\to 0$ is equivalent to the limit $m\to 0$,

$$\delta Q' = \sum_{k=1}^{\infty}\left[C_k(\rho,m=0)-C_k(\rho=\infty,m)\right] . \tag{4}$$

We write the first-order Green's function G in terms of the second-order function $\mathcal{G}$ using a symmetrical average of the two forms in Eq. (12), for this makes the derivative terms combine into a total derivative. Taking the expansion for $\mathcal{G}$ given in Eq. (30), we have

$$C_k(\rho,m) = -ie\int_{-i\infty}^{i\infty}\frac{dE}{2\pi}\int d\Omega_{\hat{r}}\,\mathrm{tr}\left\{\frac{i}{2}\,\alpha_r\left[\sum_{\mathfrak{L}'=\pm\lambda}P_{\mathfrak{L}'}(\hat{r},\hat{r})\,\mathcal{G}_{\mathfrak{L}'}(\rho,\rho;E)\right]\right. + \left.\int_0^\rho dr\left(\gamma^0 m+E+\frac{Z\alpha}{r}\right)\left[\sum_{\mathfrak{L}'=\pm\lambda}P_{\mathfrak{L}'}(\hat{r},\hat{r})\,\mathcal{G}_{\mathfrak{L}'}(r,r;E)\right]\right\} . \tag{87}$$

In writing Eq. (87), we have used the equivalence of the union of the subspaces with $K'=\pm k$ to the union of the subspaces with $\mathfrak{L}'=\pm\lambda=\pm[k^2-(Z\alpha)^2]^{1/2}$. Using the traces of the projection operators given by Eqs. (51)–(53), we find that

$$C_k(\rho,m) = \bar{C}_k(\rho,m)+\tilde{C}_k(\rho,m) , \tag{88}$$

with

$$\bar{C}_k(\rho,m) = -ie\,4k\int_{-i\infty}^{+i\infty}\frac{dE}{2\pi}\int_0^\rho dr\left(E+\frac{Z\alpha}{r}\right)[\mathcal{G}_{+\lambda}(r,r;E)+\mathcal{G}_{-\lambda}(r,r;E)] \tag{89}$$

and

$$\tilde{C}_k(\rho,m) = -ie\,2k\,\frac{Z\alpha}{\lambda}\int_{-i\infty}^{+i\infty}\frac{dE}{2\pi}[\mathcal{G}_{+\lambda}(\rho,\rho;E)-\mathcal{G}_{-\lambda}(\rho,\rho;E)] . \tag{90}$$

We demonstrate in Appendix B that the quantity $\bar{C}_k(\rho, m)$, which came from the total derivative term, does not contribute to the induced point charge. Thus, we need consider only the quantity $\bar{C}_k(\rho, m)$.

With the aid of our previous determinantal development, the computation of the charge renormalization is trivial, for it immediately appears as a trace,

$$\bar{C}_k(\infty, m) = -2ik\frac{e}{\pi}\int_{-i\infty}^{+i\infty} dE\,\mathrm{Tr}\left[\mathcal{G}_{+\lambda}(E)+\mathcal{G}_{-\lambda}(E)\right]\left(E+\frac{Z\alpha}{r}\right). \tag{91}$$

We may regularize the trace by subtracting the free Green's functions $\mathcal{G}^{(0)}_{\pm\lambda}(E)$ inside the square brackets since such a subtraction affects only terms of order $(Z\alpha)$ which we omit. Thus, we obtain the trace (83) and hence the result

$$\bar{C}_k(\infty, m) = -\frac{2ek}{\pi}\,\mathrm{Im}\left[\ln\Gamma(\lambda+1-iZ\alpha)+\ln\Gamma(\lambda-iZ\alpha)\right], \tag{92}$$

since $\eta(\pm i\infty) = \pm iZ\alpha$. We noted briefly in the previous section that the determinant which we use is a Jost function $F_{\pm\lambda}(E)$ and that for real energies $F_{\pm\lambda}(E)^*/F_{\pm\lambda}(E) = e^{2i\delta_{\pm\lambda}(E)}$. We now find that the partial-wave charge renormalization is related to the infinite-energy limit of the phase shifts,

$$\bar{C}_k(\infty, m) = e2k\,\frac{1}{\pi}\left[\delta_\lambda(\infty)+\delta_{-\lambda}(\infty)\right].$$

In a normal nonrelativistic problem, phase shifts vanish in the infinite-energy limit. Note that $2k = 2j+1$ is simply the multiplicity of the states with angular momentum j.

The calculation of the remaining limit $\bar{C}_k(\rho, 0)$ requires more work. We recall that the Green's function has the bilinear form

$$\mathcal{G}_{\pm\lambda}(r, r'; E) = A_{\pm\lambda}(r_>; E)\,B_{\pm\lambda}(r_<; E), \tag{33}$$

with the functions $A_{\pm\lambda}$ and $B_{\pm\lambda}$ satisfying the homogeneous counterpart of Eq. (32),

$$\left[-\frac{d^2}{dr^2}+\frac{\lambda(\lambda\pm1)}{r^2}+m^2-E^2-\frac{2EZ\alpha}{r}\right]\begin{Bmatrix}A_{\pm\lambda}(r; E)\\ B_{\pm\lambda}(r; E)\end{Bmatrix}=0. \tag{93}$$

The radial integration in Eq. (89) can be performed by the following device: Consider the radial derivative of a Wronskian-like structure involving regular and irregular solutions with different energies. Using Eq. (93) we get

$$\frac{d}{dr}\left[A_{\pm\lambda}(r; E')\left(\frac{\overleftarrow{\partial}}{\partial r}-\frac{\overrightarrow{\partial}}{\partial r}\right)B_{\pm\lambda}(r; E)\right]=\left[\left(E'^2+\frac{2E'Z\alpha}{r}\right)-\left(E^2+\frac{2EZ\alpha}{r}\right)\right]A_{\pm\lambda}(r; E')\,B_{\pm\lambda}(r; E). \tag{94}$$

When E' approaches E, the right-hand side of Eq. (94) approaches $2(E'-E)$ times $(E+Z\alpha/r)A_{\pm\lambda}B_{\pm\lambda}$, which are the terms in the integrand of Eq. (89). Thus, the device (94) expresses the integrand as a total radial derivative when the limit $E'\to E$ is taken, and we get

$$\bar{C}_k(\rho, m) = -i\frac{ek}{\pi}\int_{-i\infty}^{+i\infty} dE\left[\frac{\partial A_\lambda(\rho; E)}{\partial E}\left(\frac{\overleftarrow{\partial}}{\partial\rho}-\frac{\overrightarrow{\partial}}{\partial\rho}\right)B_\lambda(\rho; E)+(\lambda\to-\lambda)\right]. \tag{95}$$

The $r=0$ limit of the radial integration vanishes by virtue of the limiting behavior given in Eqs. (42).

We may now pass to the massless limit $m=0$. In this limit $A_{\pm\lambda}(\rho; E)$ and $EB_{\pm\lambda}(\rho; E)$ become scale-invariant functions of the single variable (Er) [cf. Eqs. (35) and (36)], and thus the radial derivatives in Eq. (95) can be replaced by energy derivatives,

$$\frac{\partial}{\partial\rho}A_{\pm\lambda}(\rho; E) = \frac{E}{\rho}\frac{\partial}{\partial E}A_{\pm\lambda}(\rho; E), \tag{96a}$$

$$\frac{\partial}{\partial\rho}\left[EB_{\pm\lambda}(\rho; E)\right] = \frac{E}{\rho}\frac{\partial}{\partial E}\left[EB_{\pm\lambda}(E; \rho)\right]. \tag{96b}$$

We make use of these replacements and perform two partial integrations to put the result in a convenient form:

$$\bar{C}_k(\rho, 0) = -i\frac{ek}{\pi}\int_{-i\infty}^{+i\infty} dE\left\{-2\frac{\partial^2 A_\lambda(\rho; E)}{\partial E^2}\frac{E}{\rho}B_\lambda(\rho; E)-\frac{1}{2E^2}A_\lambda(\rho; E)\frac{E}{\rho}B_\lambda(\rho; E)\right.$$
$$\left.-\frac{1}{2E\rho}\left[\frac{\partial A_\lambda(\rho; E)}{\partial E}EB_\lambda(\rho; E)-A_\lambda(\rho; E)\frac{\partial EB_\lambda(\rho; E)}{\partial E}\right]+(\lambda\to-\lambda)\right\}. \tag{97}$$

The end points of the partial integrations vanish by virtue of Eqs. (41). It should be emphasized that we

are now using the $m=0$ limit of the functions A and EB. Since these are functions of the single variable (Er), they are not altered by the interchange $\rho \to E$, $E \to \rho$, and they must obey the correspondingly interchanged version of Eq. (93) in the $m=0$ limit,

$$\left[-\frac{\partial^2}{\partial E^2}+\frac{\lambda(\lambda\pm1)}{E^2}-\rho^2-\frac{2\rho Z\alpha}{E}\right]\begin{Bmatrix} A_{\pm\lambda}(\rho;E) \\ EB_{\pm\lambda}(\rho;E)\end{Bmatrix}=0\ . \tag{98}$$

This enables us to replace the double derivative of A in Eq. (97) by terms involving no derivatives. Moreover, the Wronskian of the two solutions of Eq. (98) is independent of E and can thus be evaluated in the $E\to 0$ limit displayed in Eqs. (42),

$$\frac{\partial A_{\pm\lambda}(\rho;E)}{\partial E}EB_{\pm\lambda}(\rho;E)-A_{\pm\lambda}(\rho;E)\frac{\partial EB_{\pm\lambda}(\rho;E)}{\partial E}=-\rho\ . \tag{99}$$

The terms in the square bracket in Eq. (97) contribute, therefore, a term $1/2E$ to the integrand which is independent of $Z\alpha$. Hence, it can be omitted, and we have now achieved the result

$$\bar{C}_k(\rho,0)=-\frac{2ek}{\pi}\,\mathrm{Im}\int_0^\infty d\epsilon\left\{A_\lambda(\rho;i\epsilon)\frac{\epsilon}{\rho}B_\lambda(\rho;i\epsilon)\left[2\frac{\lambda(\lambda+1)+\frac14}{\epsilon^2}+2\rho^2-\frac{4i\rho Z\alpha}{\epsilon}\right]+(\lambda\to-\lambda)\right\}\ . \tag{100}$$

In order to proceed in a lucid manner, we note that by Eq. (36)

$$\frac{\epsilon}{\rho}B_{\pm\lambda}(\rho;i\epsilon)=B_{\pm\lambda}(\epsilon;i\rho)\ , \tag{101}$$

and we accordingly change variables, denoting ϵ by r and ρ by E:

$$\epsilon=r,\quad \rho=E\ . \tag{102}$$

Now

$$\begin{aligned}\bar{C}_k(\rho,0)&=-\frac{4ek}{\pi}\,\mathrm{Im}\int_0^\infty dr\left\{\mathcal{G}_\lambda(r,r;iE)\left[\frac{\lambda(\lambda+1)+\frac14}{r^2}+E^2-\frac{2iEZ\alpha}{r}\right]+(\lambda\to-\lambda)\right\}\\&=-\frac{4ek}{\pi}\,\mathrm{Im}\left\{\mathrm{Tr}\,\mathcal{G}_\lambda(iE)(-iE)\left(iE+\frac{Z\alpha}{r}\right)+\mathrm{Tr}\,\mathcal{G}_\lambda(iE)\frac{\lambda(\lambda+1)+\frac14}{r^2}+\mathrm{Tr}\,\mathcal{G}_\lambda(iE)\frac{-iEZ\alpha}{r}+(\lambda\to-\lambda)\right\}\ .\end{aligned} \tag{103}$$

We may regularize the first two traces above by subtracting the same quantities with the Green's function $\mathcal{G}_\lambda(iE)$ replaced by the free Green's function $\mathcal{G}^{(0)}_\lambda(iE)$. Because $\mathcal{G}^{(0)}_\lambda(iE)$ is real, this subtraction does not change the terms of order $(Z\alpha)^3$ and higher after the imaginary part has been taken. Using the $m=0$ limit of the formulas at the end of the previous section, Eqs. (77), (81), and (83), we find that

$$\mathrm{Tr}\left[\mathcal{G}_{\pm\lambda}(iE)-\mathcal{G}^{(0)}_{\pm\lambda}(iE)\right]\left(iE+\frac{Z\alpha}{r}\right)=0\ , \tag{104}$$

$$\begin{aligned}\mathrm{Im}\,\mathrm{Tr}\left[\mathcal{G}_{\pm\lambda}(iE)-\mathcal{G}^{(0)}_{\pm\lambda}(iE)\right]\frac{\lambda(\lambda\pm1)+\frac14}{r^2}&\\=-\mathrm{Im}\,\tfrac12(\lambda\pm\tfrac12)\,\psi(\lambda+\tfrac12\pm\tfrac12-iZ\alpha)\ ,&\end{aligned} \tag{105}$$

$$\mathrm{Im}\,\mathrm{Tr}\,\mathcal{G}_{\pm\lambda}(iE)\frac{-iEZ\alpha}{r}=\mathrm{Im}\,\frac{i}{2}\,Z\alpha\psi(\lambda+\tfrac12\pm\tfrac12-iZ\alpha)\ . \tag{106}$$

Using

$$\psi(\lambda+1-iZ\alpha)=\frac{1}{\lambda-iZ\alpha}+\psi(\lambda-iZ\alpha) \tag{107}$$

and adding up the numbers gives

$$\bar{C}_k(\rho,0)=\frac{4ek}{\pi}\,\mathrm{Im}(\lambda-iZ\alpha)\,\psi(\lambda-iZ\alpha) \tag{108}$$

plus terms of order $Z\alpha$ that are of no interest.

Subtracting Eq. (92) from Eq. (108), deleting terms of order $Z\alpha$, and summing over k gives the result (5) quoted in Sec. I.

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APPENDIX A

The result Eq. (5) for the point charge induced by the vacuum polarization may be expaned in powers of $Z\alpha$,

$$\delta Q' = a_3(Z\alpha)^3 + a_5(Z\alpha)^5 + O((Z\alpha)^7) \ , \tag{A1}$$

where the coefficients are given by

$$a_3 = \frac{4e}{\pi}\sum_{k=1}^{\infty}\left[\frac{1}{2}\psi'(k) + \frac{5k}{6}\psi''(k) + \frac{k^2}{6}\psi'''(k) + \frac{1}{12k^2}\right] \tag{A2}$$

and

$$a_5 = \frac{4e}{\pi}\sum_{k=1}^{\infty}\left[\frac{1}{8k^2}\psi'(k) - \frac{1}{8k}\psi''(k) - \frac{3}{8}\psi'''(k) - \frac{7k}{60}\psi^{\mathrm{iv}}(k) - \frac{k^2}{120}\psi^{\mathrm{v}}(k) + \frac{3}{80k^4}\right] \ . \tag{A3}$$

Here a prime denotes a derivative. By employing the formula

$$\psi'(z) = \sum_{l=0}^{\infty}\frac{1}{(l+z)^2} \ , \tag{A4}$$

we shall reduce the coefficients to terms involving the Reimann ζ function,

$$\zeta(p) = \sum_{l=1}^{\infty}\frac{1}{l^p} \ . \tag{A5}$$

The infinite sums of the individual terms in Eq. (A2) do not separately converge. Hence, we must exercise some care and write

$$a_3 = \frac{e}{3\pi}\zeta(2) + \lim_{K\to\infty}\lim_{L\to\infty}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{4e}{\pi}\left[\frac{1}{2}\frac{1}{(k+l)^2} - \frac{5}{3}\frac{k}{(k+l)^3} + \frac{k^2}{(k+l)^4}\right] \ , \tag{A6}$$

where the limits must be performed in the order indicated. We write $k+l=m$ and interchange the order of the sums (but not the limits) to secure

$$a_3 = \frac{e}{3\pi}\zeta(2) + \lim_{K\to\infty}\lim_{L\to\infty}\frac{4e}{\pi}\left(\sum_{m=1}^{K}\sum_{k=1}^{m} + \sum_{m=K+1}^{K+L}\sum_{k=1}^{K}\right)\left[\frac{1}{2}\frac{1}{m^2} - \frac{5}{3}\frac{k}{m^3} + \frac{k^2}{m^4}\right] \ . \tag{A7}$$

The k sum is now elementary, giving

$$a_3 = \frac{e}{3\pi}\zeta(2) + \lim_{K\to\infty}\lim_{L\to\infty}\frac{4e}{\pi}\left\{\sum_{m=1}^{K}\left[-\frac{1}{3}\frac{1}{m^2} + \frac{1}{6}\frac{1}{m^3}\right] + \sum_{m=K+1}^{K+L}\left[\frac{1}{2}\frac{K}{m^2} - \frac{5}{6}\frac{K(K+1)}{m^3} + \frac{\frac{1}{3}K^3 + \frac{1}{2}K^2 + \frac{1}{6}K}{m^4}\right]\right\} . \tag{A8}$$

Since the second sum involves large numbers, it can be replaced by an integral, and we find

$$a_3 = \frac{e}{3\pi}\zeta(2) + \frac{4e}{\pi}\left\{-\tfrac{1}{3}\zeta(2) + \tfrac{1}{6}\zeta(3) + \tfrac{7}{36}\right\}$$

$$= \frac{e}{3\pi}\left\{2\zeta(3) - \frac{\pi^2}{2} + \tfrac{7}{3}\right\} \ . \tag{A9}$$

We need some double sums to evaluate a_5. An extensive tabulation of such sums is given in an appendix to the work of Wichmann and Kroll.[3] For the sake of completeness, we derive here the results which are needed below. We have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{k^p(k+l)^p} = \sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty}\frac{1}{k^p m^p}$$

$$= \tfrac{1}{2}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{k^p m^p} - \tfrac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{2p}}$$

$$= \tfrac{1}{2}\zeta(p)^2 - \tfrac{1}{2}\zeta(2p) \ , \tag{A10}$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(k+l)^p}=\sum_{m=2}^{\infty}\sum_{l=1}^{m-1}\frac{1}{m^p}$$
$$=\zeta(p-1)-\zeta(p)\ , \tag{A11}$$
$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{k}{(k+l)^p}=\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}\frac{k}{m^p}$$
$$=\tfrac{1}{2}\zeta(p-2)-\tfrac{1}{2}\zeta(p-1)\ , \tag{A12}$$

and

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{k^2}{(k+l)^p}=\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}\frac{k^2}{m^p}$$
$$=\tfrac{1}{3}\zeta(p-3)-\tfrac{1}{2}\zeta(p-2)+\tfrac{1}{6}\zeta(p-1)\ . \tag{A13}$$

Using (A10) with $p=2$, we derive

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{k(k+l)^3}=\tfrac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{k}+\frac{1}{l}\right)\frac{1}{(k+l)^3}$$
$$=\tfrac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(k+l)^2}\left[\frac{(k+l)^2}{2k^2l^2}-\frac{1}{2k^2}-\frac{1}{2l^2}\right]$$
$$=\tfrac{1}{4}\zeta(4)\ . \tag{A14}$$

Using these relations and the formula (A4), we compute the list:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\psi'(k)=\tfrac{1}{2}\left[\zeta(4)+\zeta(2)^2\right]\ , \tag{A15}$$
$$\sum_{k=1}^{\infty}\frac{1}{k}\psi''(k)=-\tfrac{5}{2}\zeta(4)\ , \tag{A16}$$
$$\sum_{k=1}^{\infty}\psi'''(k)=6\zeta(3)\ , \tag{A17}$$
$$\sum_{k=1}^{\infty}k\psi^{\mathrm{iv}}(k)=-12\left[\zeta(3)+\zeta(4)\right]\ , \tag{A18}$$
$$\sum_{k=1}^{\infty}k^2\psi^{\mathrm{v}}(k)=120\left[\tfrac{1}{3}\zeta(3)+\tfrac{1}{2}\zeta(4)+\tfrac{1}{6}\zeta(5)\right]. \tag{A19}$$

It is now a simple matter to add up the sums in Eq. (A3) and obtain

$$a_5=\frac{e}{3\pi}\left[\tfrac{47}{240}\pi^4-\tfrac{71}{5}\zeta(3)-2\zeta(5)\right]\ . \tag{A20}$$

The results Eqs. (A9) and (A20) are those quoted in Eq. (10a) of the text.

APPENDIX B

We prove here that the quantity

$$\tilde{C}_k(\rho,m)=-ie2k\frac{Z\alpha}{\lambda}\int_{-i\infty}^{+i\infty}\frac{dE}{2\pi}\left[\mathcal{G}_{+\lambda}(\rho,\rho;E)-\mathcal{G}_{-\lambda}(\rho,\rho;E)\right] \tag{90}$$

does not contribute to the induced point charge. It follows from the asymptotic forms (41) and the construction (33) that, as $\rho\to\infty$,

$$\mathcal{G}_{\pm\lambda}(\rho,\rho;E)\to(2q)^{-1}\ . \tag{B1}$$

The difference $\mathcal{G}_{+\lambda}(\rho,\rho;E)-\mathcal{G}_{-\lambda}(\rho,\rho;E)$ vanishes as $\rho\to\infty$. Moreover, on dimensional grounds, the leading correction must involve $(q^2\rho)^{-1}$, producing a finite energy integral that vanishes as $\rho\to\infty$. Hence, the charge-renormalization term vanishes,

$$\tilde{C}_k(\infty,m)=0\ . \tag{B2}$$

We can proceed as in the text, interchanging the roles of E and ρ, to obtain

$$\tilde{C}_k(\rho,0)=\frac{ek}{\pi\lambda}\,\mathrm{Im\,Tr}\left[\mathcal{G}_{\lambda}(iE)-\mathcal{G}_{-\lambda}(iE)\right]\frac{2iEZ\alpha}{r}, \tag{B3}$$

where $\mathcal{G}_{\pm\lambda}(iE)$ is now understood to be the zero-mass Green's function. With the aid of Eq. (77), we obtain

$$\tilde{C}_k(\rho,0)=-\frac{ek}{\pi\lambda}\,\mathrm{Im}\left[iZ\alpha\psi(\lambda+1-iZ\alpha)-iZ\alpha\psi(\lambda-iZ\alpha)\right], \tag{B4}$$

which, employing Eq. (107), reduces to

$$\tilde{C}_k(\rho,0)=-\frac{ek}{\pi\lambda}\,\mathrm{Im}\,\frac{iZ\alpha}{\lambda-iZ\alpha}$$
$$=-\frac{ekZ\alpha}{\pi k^2}\ . \tag{B5}$$

This quantity is only first order in $Z\alpha$ and can be, therefore, omitted.

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