PHYSICAL REVIEW D VOLUME 12, NUMBER 2 15 JULY 1975

Trace and Ward-Takahashi identity anomalies in an SU(3) current model with energy-momentum tensor*

Douglas B. Zacrep and Bing-Lin Young
Ames Laboratory–ERDA and Department of Physics, Iowa State University, Ames, Iowa 50010
(Received 3 February 1975; revised manuscript received 21 April 1975)

We discuss the validity of the naive Ward-Takahashi identities and trace identities for arbitrary n-point functions (n-pf's) of scalar, pseudoscalar, vector, and axial-vector currents and the improved energy-momentum tensor, thus extending the previous investigations in a unified way. We show that the validity of the naive Ward-Takahashi identities of the energy-momentum tensor implies the satisfaction of those of the vector currents. This removes an ambiguity concerning the minimal sets of anomalous current Ward-Takahashi identities. We find that all the anomalous Ward-Takahashi identities for the broad structure of n-pf's are again restricted to the axial-vector current of n-pf's of abnormal parity in a well-defined pattern, and the trace identity anomalies occur only in normal-parity n-pf's. We give all these anomalies. Our results show that there are no new anomalies associated with the inclusion of the energy-momentum tensor in the n-pf's.

I. INTRODUCTION

It has been known since 1969[1] that canonical field theories exhibit anomalous behavior. To state it more explicitly, relations derived from canonical rules, such as the Ward-Takahashi identities of currents, trace identities of the energy-momentum tensor,[2] etc., may not hold in explicit calculations because of the singular nature of products of field variables. The singularities are the usual ultraviolet divergences in the momentum-space approach or the short-distance singularities in the configuration-space approach.[3] Thus, modifications to the canonically derived relations, which are called the anomalies or canonical anomalies, are necessary.

The existence of canonical anomalies has many implications both on phenomenological considerations and on basic properties of field theories involving fermion fields.[1-6] Independently of whether or not anomalies are a basic property of certain fundamental physical theories, they place constraints on the structure of field theories, the major theoretical laboratory available to us so far. A prominent example is the construction of a renormalizable theory of weak and electromagnetic interactions. The requirement of the absence of an axial-vector anomaly through cancellations puts constraints, admittedly loose, on the minimum number of sets of fermions and on their relative coupling strengths to other fields involved.[6]

Anomalies of n-point functions of currents with arbitrary internal symmetry have been investigated extensively in various approaches.[7-10] The main ingredients of most of the approaches are regularization and renormalization.[11] A consistent scheme of regularization is needed to define the divergent Feynman integrals of the n-point functions (n-pf's). This may result in a large number of anomalies which are called the naive anomalies. But not all of the naive anomalies can be taken seriously, as they are regularization dependent (see Sec. III). The next step is renormalization, that is, introduction of counterterms, either directly to the Lagrangian[7] or to the momentum-space representation of the n-pf's themselves.[8] The counterterms, satisfying all the general properties of the original amplitudes,[12] are chosen so that there is a minimum number of Ward-Takahashi identities (WTI's) which are anomalous. This last criterion is not sufficient to eliminate all the ambiguities since there exists more than one minimal set. The different minimal sets have different counterterms leading to different anomalies. This has been discussed in detail in Ref. 8, in which two minimal sets are obtained. One set has anomalies restricted to the axial-vector current WTI's (AWTI's), agreeing with the result of Ref. 7. The other contains vector current WTI (VWTI) anomalies as well. In both minimal sets, all the anomalies are constrained by those of the 3-pf's $\langle AAA\rangle$ and $\langle AVV\rangle$. Therefore, all the canonical anomalies of currents are related to the two basic ones in $\langle AAA\rangle$ and $\langle AVV\rangle$.

In this paper, we shall extend the previous work to include in the n-pf's of the improved energy-momentum tensor,[13] its trace, and the currents of an internal-symmetry group. Our purpose is threefold:

(1) By extending n-pf's to include the energy-momentum tensor, etc., which give rise to many more WTI's, we can investigate the anomalies for a broader structure in a unified way.

(2) We want to determine whether or not one of the minimal sets is selected in this broader structure of n-pf's so that the ambiguity concern-

ing the minimal set is eliminated.

(3) We want to see whether or not there are new anomalies. By this we mean the anomalies which are not constrained by those of the n-pf's involving currents only as determined in Refs. 7 and 8.

In Sec. II, we state the model field theory employed, including the regularization scheme, and list various WTI's and trace identities (TI's). Using the regularization scheme, we calculate the anomalous WTI's and TI's. In Sec. III, the counterterms are determined so that the "physical" amplitudes have minimal anomalous WTI's and TI's. The anomalies are listed in Sec. IV. Here we also discuss their general properties, and compare our results on trace anomalies with those of Ref. 2. Concluding remarks are drawn in Sec. V. General expressions of naive WTI's and TI's containing the energy-momentum tensor are listed in an appendix.

We have left untouched the phenomenological applications of anomalies in this work. Rich sources of information for the broad range of applications can be found in the literature.[2,4,5] We also ignore the problem of higher-order corrections to the anomalies.[14] Our attitude towards this is similar to that of Ref. 2, to which we refer the reader for a discussion.

II. REGULARIZATION AND THE ANOMALOUS WARD-TAKAHASHI AND TRACE IDENTITIES

We take the free spinor field theory with SU(3) internal symmetry. The SU(3) currents are defined by

$$j_i^a(x) \equiv \bar{\psi}(x)\tfrac{1}{2}\lambda^a \Gamma_i \psi(x)\,, \quad a=0,1,\ldots,8\,, \tag{1}$$

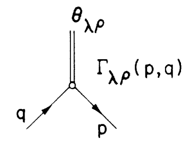

FIG. 1. Momentum assignment of $\theta_{\lambda\rho}$-ψ-$\bar{\psi}$ vertex. The Feynman rule is given in Eq. (4).

where $\psi(x)$ is the free spinor field, a column matrix in the internal symmetry space, and λ^a are the Gell-Mann matrices. The currents are $j_i^a(x) \equiv S^a(x)$, $P^a(x)$, $V_\mu^a(x)$, and $A_\mu^a(x)$ for $\Gamma_i = 1$, $i\gamma_5, \gamma_\mu$, and $\gamma_\mu\gamma_5$, respectively. The symmetric energy-momentum tensor is

$$\theta_{\lambda\rho} \equiv \tfrac{1}{4} i[\bar{\psi}\gamma_\lambda \partial_\rho \psi - \partial_\rho \bar{\psi}\gamma_\lambda \psi + (\lambda \leftrightarrow \rho)] - \tfrac{1}{2} g_{\lambda\rho}(i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\partial^\mu \bar{\psi}\gamma_\mu \psi - 2m\bar{\psi}\psi)\,, \tag{2}$$

with the trace

$$\theta \equiv \theta_\mu^\mu \equiv m\bar{\psi}\psi = m\sqrt{6}\, S^0\,, \tag{3}$$

where S^0 is the unitary singlet scalar current. The Feynman rule for the $\theta_{\lambda\rho} - \psi - \bar{\psi}$ vertex is given in Ref. 15. For completeness, we give it here in Fig. 1, with

$$\Gamma_{\lambda\rho}(p,q) \equiv \tfrac{1}{4}\gamma_\lambda(p_\rho + q_\rho) + \tfrac{1}{4}\gamma_\rho(p_\lambda + q_\lambda) - \tfrac{1}{2} g_{\lambda\rho}(\not{p} + \not{q} - 2m)\,. \tag{4}$$

The Feynman rule for the trace of the energy-momentum tensor, as indicated in Eq. (3), is proportional to the scalar-current-ψ-$\bar{\psi}$ vertex.

An n-pf involving $\theta_{\lambda\rho}(x)$ and $j_i^a(x)$ is defined by the connected part of

$$(2\pi)^4\delta(k+q_1+\cdots+q_n)\langle \theta_{\lambda\rho}(k) j_1^a(q_1)\cdots j_n^d(q_n)\rangle \equiv \int dx\, dy_1\cdots dy_n\, e^{-ikx}e^{-iq_1y_1}\cdots e^{-iq_ny_n}\langle 0|\, T^*\{\theta_{\lambda\rho}(x) j_1^a(y_1)\cdots j_n^d(y_n)\}\,|0\rangle\,, \tag{5}$$

where T^* is the covariant T product.[16] The loop momentum integral representations of the n-pf's $\langle \theta_{\lambda\rho}(k) j_1^a(q_1)\cdots j_n^d(q_n)\rangle$ can be easily found using Eq. (4). For examples,[17]

1-pf

$$\langle \theta_{\lambda\rho}(0)\rangle = -i\int_l \mathrm{Tr}[\Gamma_{\lambda\rho}(l,l)S(l)]\,, \tag{6a}$$

2-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(-k)\rangle = \tfrac{1}{4}\mathrm{Tr}(\lambda_a + c_1\lambda_a)\int_l \mathrm{Tr}[\Gamma_{\lambda\rho}(l, l+k)S(l+k)\Gamma_1 S(l)]\,, \tag{6b}$$

3-pf

$$\langle \theta_{\lambda\rho}(k) j_1^a(p) j_2^b(q)\rangle = \tfrac{1}{8} i\,\mathrm{Tr}(\lambda_a\lambda_b + c_1c_2\lambda_a^T\lambda_b^T) \times \int_l \mathrm{Tr}[\Gamma_{\lambda\rho}(l-q, l+p)S(l+p)\Gamma_1 S(l)\Gamma_2 S(l-q) + c_1c_2\Gamma_{\lambda\rho}(l-p, l+q)S(l+q)\Gamma_2 S(l)\Gamma_1 S(l-p)]\,, \tag{6c}$$

etc., where

$$S(l)=(\not{l}-m)^{-1}. \tag{7}$$

c_i is the charge-conjugation parity for j_i^a: $+1$ for A, S, and P and -1 for V. In the above expressions, charge-conjugation invariance is assumed and incorporated. The above representations of the n-pf's (6a)–(6c) are not well defined since the integrals are divergent. We shall discuss shortly the regularization scheme which makes them finite.

The general expressions of the naive WTI's and TI's for arbitrary n-pf's, by (5), are given in the Appendix. They are derived in Ref. 15. A few examples of simple cases are shown below.

Tensor Ward-Takahashi identity (TWTI)

$$\begin{aligned}k^\lambda\langle\theta_{\lambda\rho}(k)V_\mu^a(p)V_\nu^b(q)\rangle=&-i(k+p)_\rho\langle V_\mu^a(p+k)V_\nu^b(q)\rangle-i(k+q)_\rho\langle V_\mu^a(p)V_\nu^b(q+k)\rangle\\&+\tfrac{1}{2}ik_\mu\langle V_\rho^a(p+k)V_\nu^b(q)\rangle-\tfrac{1}{2}ig_{\rho\mu}k^\sigma\langle V_\sigma^a(p+k)V_\nu^b(q)\rangle\\&+\tfrac{1}{2}ik_\nu\langle V_\mu^a(p)V_\rho^b(q+k)\rangle-\tfrac{1}{2}ig_{\rho\nu}k^\sigma\langle V_\mu^a(p)V_\sigma^b(q+k)\rangle.\end{aligned} \tag{8a}$$

Vector Ward-Takahashi identity (VWTI)

$$\begin{aligned}p^\mu\langle\theta_{\lambda\rho}(k)V_\mu^a(p)V_\nu^b(q)\rangle=&\tfrac{1}{2}ip_\lambda\langle V_\rho^a(p+k)V_\nu^b(q)\rangle+\tfrac{1}{2}ip_\rho\langle V_\lambda^a(p+k)V_\nu^b(q)\rangle\\&-ig_{\lambda\rho}p^\sigma\langle V_\sigma^a(p+k)V_\nu^b(q)\rangle+f^{abb'}\langle\theta_{\lambda\rho}(k)V_\nu^{b'}(p+q)\rangle.\end{aligned} \tag{8b}$$

Axial-vector Ward-Takahashi identity (AWTI)

$$\begin{aligned}p^\mu\langle\theta_{\lambda\rho}(k)A_\mu^a(p)A_\nu^b(q)\rangle=&\tfrac{1}{2}ip_\lambda\langle A_\rho^a(p+k)A_\nu^b(q)\rangle+\tfrac{1}{2}ip_\rho\langle A_\lambda^a(p+k)A_\nu^b(q)\rangle-ig_{\lambda\rho}p^\sigma\langle A_\sigma^a(p+k)A_\nu^b(q)\rangle\\&-2mi\langle\theta_{\lambda\rho}(k)P^a(p)A_\nu^b(q)\rangle+2mg_{\lambda\rho}\langle P^a(p+k)A_\nu^b(q)\rangle+f^{abb'}\langle\theta_{\lambda\rho}(k)V_\nu^{b'}(p+q)\rangle.\end{aligned} \tag{8c}$$

Trace identity (TI)

$$g^{\lambda\rho}\langle\theta_{\lambda\rho}(k)V_\mu^a(p)V_\nu^b(q)\rangle=\langle\theta(k)V_\mu^a(p)V_\nu^b(q)\rangle-3i[\langle V_\mu^a(p+k)V_\nu^b(q)\rangle+\langle V_\mu^a(p)V_\nu^b(q+k)\rangle]. \tag{8d}$$

In each of the expressions above, it is understood that $k+p+q=0$.

Following Ref. 8, we shall use the scheme of universal regularization, which is of the Pauli-Villars type,[18] with, however, all the logarithmic terms being discarded. This simple scheme eliminates all the divergences which occur in the n-pf's for $n\leq 6$ and is sufficient for the discussion of WTI's and TI's. We refer to Ref. 8 for the details. Let us further remark that our choice of this particular scheme is due to the fact that we shall make use of the counterterms of Ref. 8. We shall come back to this point later. Now, the WTI's and TI's are relations among the universally regularized Feynman amplitudes.

It has been shown in Ref. 8 that, in the universally regularized scheme, the resultant anomaly of an axial-vector WTI, which is called the naive anomaly, is proportional to the coefficient of the $1/m$ term of a certain n-pf in the WTI. The naive VWTI's are satisfied automatically. In the present case the navie TWTI's and VWTI's are satisfied, with anomalies occurring only in AWTI's and TI's. To be precise, let $\Delta_T'(\theta_{\lambda\rho}j_1^a\cdots j_n^d)$ and $\Delta_A'(\theta_{\lambda\rho}(k)A_\mu^a(p)\cdots j_n^d)$ denote the respective naive anomalies of the TI, $g^{\lambda\rho}\langle\theta_{\lambda\rho}j_1^a\cdots j_n^d\rangle$, and the AWTI, $p^\mu\langle\theta_{\lambda\rho}(k)A_\mu^a(p)\cdots j_n^d\rangle$; then

$$\Delta_T'(\theta_{\lambda\rho}j_1^a\cdots j_n^d)=-\sqrt{6}\langle S^0j_1^a\cdots j_n^d\rangle|_{1/m} \tag{9a}$$

and

$$\begin{aligned}\Delta_A'(\theta_{\lambda\rho}(k)A_\mu^a(p)j_1^b\cdots j_n^d)&\\=&2i\langle\theta_{\lambda\rho}(k)P^a(p)j_1^b\cdots j_n^d\rangle|_{1/m}\\&-2g_{\lambda\rho}\langle P^a(k+p)j_1^b\cdots j_n^d\rangle|_{1/m},\end{aligned} \tag{9b}$$

where $\langle\cdots\rangle|_{1/m}$ is the coefficient of the $1/m$ term in a large-m expansion of $\langle\cdots\rangle$. Now, in the universal regularization scheme, the canonically derived (naive) relations, such as Eqs. (8c) and (8d), need modifications, namely Δ_A' or Δ_T' added to their right-hand sides.

The calculation of the $1/m$ terms of the tensor $(n+1)$-pf $\langle\theta_{\lambda\rho}j_1^a\cdots j_n^d\rangle$, Eq. (9b), is similar to that of the corresponding current n-pf $\langle j_1^a\cdots j_n^d\rangle$, given in Ref. 8, with the insertion of the $\theta_{\lambda\rho}$ vertex, Eq. (4), in the momentum loop integration of the latter. Further, they both have the same internal-symmetry structure, as $\theta_{\lambda\rho}$ is a unitary singlet. The $1/m$ terms which contribute to the TI anomalies can be taken from Appendix A of Ref. 8, pp. 1502–1503. Tables I and II list respcetively all the anomalous AWTI's and TI's obtained in the universal regularization.

TABLE I. Universally regularized n-pf's with tensor $\theta_{\lambda\rho}$ which have naive AWTI anomalies (i.e., $1/m$ term contributions).

3-pf	$\langle\theta AA\rangle$				
4-pf	$\langle\theta AAV\rangle$	$\langle\theta ASP\rangle$	$\langle\theta AVV\rangle$	$\langle\theta AAA\rangle$	
5-pf	$\langle\theta AAVV\rangle$ $\langle\theta AVVV\rangle$	$\langle\theta APSV\rangle$ $\langle\theta AAAV\rangle$	$\langle\theta AAAA\rangle$	$\langle\theta AASS\rangle$	$\langle\theta AAPP\rangle$
6-pf	$\langle\theta AAVVV\rangle$ $\langle\theta AAVPP\rangle$ $\langle\theta AAAAA\rangle$	$\langle\theta AAAAV\rangle$ $\langle\theta APSSS\rangle$ $\langle\theta AAAVV\rangle$	$\langle\theta AAVSS\rangle$ $\langle\theta APPPS\rangle$ $\langle\theta AVVVV\rangle$	$\langle\theta APVVS\rangle$	$\langle\theta AAAPS\rangle$

III. DETERMINATION OF THE COUNTERTERMS

The large number of anomalous WTI's and TI's obtained in Sec. II by means of the universal regularization method cannot be taken seriously. The anomalies thus obtained are in fact regularization-dependent; different regularization schemes lead to different sets of naive anomalies. For instance, in the universal regularization scheme, the 3-pf $\langle\theta_{\lambda\rho}(k)A^a_\mu(p)P^b(q)\rangle$ satisfies the naive AWTI. It is anomalous,[19] however, when calculated in the dimensional regularization scheme.[20] Therefore a "renormalization" procedure is needed to redefine the anomalies to make them at least independent of the regularization scheme. As discussed in Ref. 8 this procedure consists of the following steps: (1) add counterterms to the n-pf's and (2) adjust them so that there are a minimal number of anomalous WTI's and TI's. The counterterms are local polynomials in the fermion mass, m, and the external momenta involved in the given n-pf. They have to have the same dimensionality, the same evenness and oddness in m, and all other general properties, including the internal-symmetry structure and the crossing properties, as the n-pf's themselves.

Let us denote the counterterms for $\langle\theta_{\lambda\rho}j^a_1\cdots j^d_n\rangle$ and $\langle j^a_1\cdots j^d_n\rangle$ by $\delta(\theta_{\lambda\rho}j^a_1\cdots j^d_n)$ and $\delta(j^a_1\cdots j^d_n)$, respectively. Then we define the "physical" amplitudes as

$$\text{``}\langle\theta_{\lambda\rho}j^a_1\cdots j^d_n\rangle\text{''}=\langle\theta_{\lambda\rho}j^a_1\cdots j^d_n\rangle_R+\delta(\theta_{\lambda\rho}j^a_1\cdots j^d_n) \tag{10a}$$

and

$$\text{``}\langle j^a_1\cdots j^d_n\rangle\text{''}=\langle j^a_1\cdots j^d_n\rangle_R+\delta(j^a_1\cdots j^d_n), \tag{10b}$$

where $\langle j^a_1\cdots j^d_n\rangle_R$ is the universally regularized amplitude.[8] Now, expressing the WTI's and TI's in terms of the physical amplitudes, we obtain a new set of anomalies for all the identities:

$$\Delta_A(\theta_{\lambda\rho}\cdots)=\Delta'_A(\theta_{\lambda\rho}\cdots)+\delta(\text{LHS})-\delta(\text{RHS}), \tag{11}$$

$$\Delta_T(\theta_{\lambda\rho}\cdots)=\Delta'_T(\theta_{\lambda\rho}\cdots)+\delta(\text{LHS})-\delta(\text{RHS}), \tag{12}$$

where Δ'_A and Δ'_T are given by Eqs. (9a) and (9b) and δ(LHS) and δ(RHS) are the sum of the counterterms of the individual n-pf's entering respectively the left- and right-hand sides of the WTI's and TI's.

We determine the counterterms, $\delta(j^a_1\cdots j^d_n)$ and $\delta(\theta_{\lambda\rho}j^a_1\cdots j^d_n)$, by attempting to satisfy as many naive WTI's and TI's as possible expressed in terms of the physical amplitudes. We begin with the 2-pf $\langle\theta_{\lambda\rho}(k)S^a(-k)\rangle$ and work up to the 6-pf's. Let us remark that, for the dimensionality reason and the fact that they are local polynomials, $\delta(\theta_{\lambda\rho}j^a_1\cdots j^d_n)$ and $\delta(j^a_1\cdots j^d_n)$ vanish for $n\geq 5$.

The actual process of determining the counterterms by minimizing the number of anomalies is tedious but straightforward. Its logical steps have been discussed in Ref. 8; we shall omit all the details here and present only the results and a few remarks.

TABLE II. Universally regularized n-pf's with tensor $\theta_{\lambda\rho}$ which have naive TI anomalies.

3-pf	$\langle\theta VV\rangle$	$\langle\theta AA\rangle$	$\langle\theta SS\rangle$	$\langle\theta PP\rangle$	
4-pf	$\langle\theta VVV\rangle$	$\langle\theta VAA\rangle$	$\langle\theta VSS\rangle$	$\langle\theta VPP\rangle$	$\langle\theta ASP\rangle$
5-pf	$\langle\theta VVVV\rangle$ $\langle\theta AASS\rangle$ $\langle\theta PPPP\rangle$	$\langle\theta VVAA\rangle$ $\langle\theta AAPP\rangle$	$\langle\theta VVSS\rangle$ $\langle\theta VASP\rangle$	$\langle\theta VVPP\rangle$ $\langle\theta SSSS\rangle$	$\langle\theta AAAA\rangle$ $\langle\theta SSPP\rangle$

We find that all the naive TWTI's and VWTI's can be satisfied with the "physical" amplitudes. Let us elaborate on this point. The TWTI and the corresponding VWTI are intimately related; the appearance or absence of an anomaly in one has a similar effect on the other. Hence, the satisfaction of the naive TWTI's, which can be thought of as expressions derived from Poincaré invariance,[21] requires that all the VWTI's be satisfied in their naive forms. It turns out that we can use directly the expressions of $\delta(j_1^a \cdots j_n^d)$ of Ref. 8 except for a constraint relating their parameters a_1 and a_4, which we found here [see the remark (2) below].

A few remarks are called for:

(1) All the counterterms can be expressed in terms of four parameters a_1, a_2, a_3, and a_4 introduced in Ref. 8 through the consideration of the current WTI's only. Hence the extension of the n-pf's to include the energy-momentum tensor and its trace does not further complicate the structure of the counterterms.[22]

(2) The requirement of the naive TI's for $\langle \theta_{\lambda\rho}(k)S^a(-k)\rangle$ in order to minimize the number of anomalous TI's implies that $a_4 = 3a_1$. Therefore only three undetermined constants appear in all the counterterms.

(3) For the abnormal-parity series, the requirement of the naive tensor WTI's implies that all their counterterms, $\delta\langle AVV\rangle$, $\delta\langle AAAV\rangle$, and $\delta\langle AVVV\rangle$, vanish.[22]

IV. DETERMINATION OF THE ANOMALIES

Using the results of Secs. II and III, i.e., the $1/m$ terms and the counterterms, we are equipped to determine the WTI and TI anomalies from Eqs. (11) and (12).[23] We discuss them in turn.

A. The Ward-Takahashi identity anomalies

As discussed in Sec. III, the tensor WTI's and hence the vector WTI's satisfy the naive forms expressed in terms of the physical amplitudes defined by Eqs. (10a) and (10b). Explicit calculation also shows that all the normal-parity n-pf's retain their naive forms of AWTI's. Anomalies occur only in the AWTI's of the following two sets of abnormal-parity n-pf's: the set containing the energy-momentum tensor, $\langle \theta AVV\rangle$, $\langle \theta AAA\rangle$, $\langle \theta AAAV\rangle$, $\langle \theta AVVV\rangle$, $\langle \theta AAAAA\rangle$, $\langle \theta AAAVV\rangle$, and $\langle \theta AVVVV\rangle$, and the set containing internal symmetry currents only, $\langle AVV\rangle$, $\langle AAA\rangle$, $\langle AAAV\rangle$, $\langle AVVV\rangle$, $\langle AAAAA\rangle$, $\langle AAAVV\rangle$, and $\langle AVVVV\rangle$. Their anomalies arise simply from the $1/m$ terms of $\langle \theta PVV\rangle$ etc., and $\langle PVV\rangle$ etc.

The second set above is Bardeen's minimal set. We discard another minimal set of Ref. 8 because it contains anomalous VWTI's and hence anomalous TWTI's. In the following, we list all the 14 anomalies[24,25]:

$$
\begin{aligned}
&\Delta_A(AVV) = -3yd^{abc}\epsilon_{\nu\sigma\alpha\beta}q^\alpha s^\beta ,\\
&\Delta_A(\theta AVV) = 3iyd^{abc}\{g_{\lambda\rho}\epsilon_{\nu\sigma\alpha\beta}p^\alpha(q-s)^\beta - \tfrac{1}{2}[g_{\nu\lambda}\epsilon_{\sigma\rho\alpha\beta}p^\alpha s^\beta + g_{\sigma\lambda}\epsilon_{\nu\rho\alpha\beta}p^\alpha q^\beta + (\lambda \leftrightarrow \rho)]\} ,\\
&\Delta_A(AAA) = \tfrac{1}{3}\Delta_A(AVV) ,\\
&\Delta_A(\theta AAA) = \tfrac{1}{3}\Delta_A(\theta AVV) ,\\
&\Delta_A(AAAV) = -\tfrac{1}{8}iy\epsilon_{\nu\sigma\tau\alpha}X^\alpha ,\\
&\Delta_A(\theta AAAV) = -\tfrac{1}{8}y\{g_{\lambda\rho}\epsilon_{\nu\sigma\tau\alpha}(2X^\alpha + 6\overline{W}_2 k^\alpha) + \tfrac{1}{2}[\epsilon_{\nu\sigma\tau\lambda}X_\rho + (\lambda \leftrightarrow \rho)]\\
&\qquad - \tfrac{1}{2}[g_{\lambda\nu}\epsilon_{\rho\sigma\tau\alpha}k^\alpha(-\overline{W}_1 + \overline{W}_2) + g_{\lambda\sigma}\epsilon_{\nu\rho\tau\alpha}k^\alpha(\overline{W}_3 + \overline{W}_2) + g_{\lambda\tau}\epsilon_{\nu\sigma\rho\alpha}k^\alpha(\overline{W}_1 + 4\overline{W}_2 - \overline{W}_3) + (\lambda \leftrightarrow \rho)]\} ,\\
&\Delta_A(AVVV) = -\tfrac{3}{8}iy\epsilon_{\nu\sigma\tau\alpha}Y^\alpha ,\\
&\Delta_A(\theta AVVV) = -\tfrac{3}{8}y\{2g_{\lambda\rho}\epsilon_{\nu\sigma\tau\alpha}[Y^\alpha + (-\overline{W}_1 + \overline{W}_2 + \overline{W}_3)k^\alpha] + \tfrac{1}{2}[\epsilon_{\nu\sigma\tau\lambda}Y_\rho + (\lambda \leftrightarrow \rho)]\\
&\qquad - \tfrac{1}{2}[g_{\lambda\nu}\epsilon_{\sigma\tau\rho\sigma}k^\alpha(-\overline{W}_1 + \overline{W}_2) + g_{\lambda\sigma}\epsilon_{\nu\rho\tau\alpha}k^\alpha(\overline{W}_2 + \overline{W}_3) + g_{\lambda\tau}\epsilon_{\nu\sigma\rho\alpha}k^\alpha(\overline{W}_3 - \overline{W}_1) + (\lambda \leftrightarrow \rho)]\} ,\\
&\Delta_A(AAAAA) = \tfrac{1}{32}y\epsilon_{\nu\sigma\tau\eta}Z^{abcde} + 23 \text{ more terms with } \nu\sigma\tau\eta \text{ permuted with } bcde ,\\
&\Delta_A(\theta AAAAA) = -3ig_{\lambda\rho}\Delta_A(AAAAA) ,\\
&\Delta_A(AAAVV) = \tfrac{1}{16}y\epsilon_{\nu\sigma\tau\eta}[Z^{abcde} + Z^{abdce} + Z^{acbed} + Z^{acebd} + 3Z^{abdec} + Z^{adcbe} - (d \leftrightarrow e)] ,\\
&\Delta_A(\theta AAAVV) = -3ig_{\lambda\rho}\Delta_A(AAAVV) ,\\
&\Delta_A(AVVVV) = -\tfrac{3}{32}y\epsilon_{\nu\sigma\tau\eta}Z^{abcde} + 23 \text{ more terms with } \nu\sigma\tau\eta \text{ permuted with } bcde ,\\
&\Delta_A(\theta AVVVV) = -3ig_{\lambda\rho}\Delta_A(AVVVV) ,
\end{aligned}
\tag{13}
$$

where

$$y \equiv \frac{1}{24\pi^2 i},$$

$$X_\mu \equiv \overline{W}_1(t-q)_\mu + \overline{W}_2(4t+q+s)_\mu + \overline{W}_3(s-t)_\mu,$$

$$Y_\mu \equiv -\overline{W}_1(t+q)_\mu + \overline{W}_2(q+s)_\mu + \overline{W}_3(s+t)_\mu,$$

$$\overline{W}_1 \equiv \mathrm{Tr}[\lambda_a\lambda_b\lambda_c\lambda_d - \lambda_a^T\lambda_b^T\lambda_c^T\lambda_d^T], \quad (14)$$

$$\overline{W}_2 \equiv \mathrm{Tr}[\lambda_a\lambda_b\lambda_d\lambda_c - \lambda_a^T\lambda_b^T\lambda_d^T\lambda_c^T],$$

$$\overline{W}_3 \equiv \mathrm{Tr}[\lambda_a\lambda_c\lambda_b\lambda_d - \lambda_a^T\lambda_c^T\lambda_b^T\lambda_d^T],$$

$$Z^{abcde} = \mathrm{Tr}[\lambda_a\lambda_b\lambda_c\lambda_d\lambda_e + \lambda_a^T\lambda_b^T\lambda_c^T\lambda_d^T\lambda_e^T], \text{ etc.}$$

In the above expressions the energy-momentum tensor carries the tensor indices $\lambda\rho$ and momentum k; the vector and axial-vector currents carry the vector indices μ, ν, σ, τ, and η; the momenta are p, q, s, t, and u; and the internal-symmetry indices are a, b, c, d, and e in that order from left to right.

One can show that the tensor-related anomalies are completely constrained by those without the tensor $\theta_{\lambda\rho}$. The presence of the energy-momentum tensor does not introduce any new anomalies to the AWTI's,[26] and the well-defined pattern of anomalies in the AWTI's containing only internal symmetry currents holds in the broader structure when the symmetric energy-momentum tensor is included. In this sense, we argue that all the axial-vector anomalies are determined by those of $\langle AVV\rangle$ and $\langle AAA\rangle$.

The axial-vector anomalies of n-pf's involving only internal symmetry currents have been succinctly summarized in Ref. 7 by means of the anomalous divergence. One is tempted to do the same in the present case. However, we found that the presence of the energy-momentum tensor vastly complicates the structure of the anomalous divergence; the result is unilluminatingly involved. We shall not present it here.

B. The trace identity anomalies

The trace identity anomalies are a completely different story. We first summarize the main features and then list all the trace anomalies (there are no trace anomalies occurring in n-pf's for $n \geq 6$ because of their lack of $1/m$ terms and counterterms):

(1) There are two sources for the trace anomalies, the $1/m$ terms defined in Eq. (9a) and the counterterms.

(2) Except for the trivial cases, $\langle \theta_{\lambda\rho}(0)\rangle$ and $\langle \theta_{\lambda\rho}(k)S^0(-k)\rangle$, in which we can adjust the counterterms so that they are free of trace anomalies, the absence of trace anomalies is a result of the absence of both $1/m$ terms and counterterms, e.g., $\langle \theta_{\lambda\rho}AAA\rangle$, $\langle \theta_{\lambda\rho}AAAV\rangle$, etc.

(3) Trace anomalies occur in all the following three possibilities: (a) from the $1/m$ terms alone, e.g., $\langle \theta_{\lambda\rho}VV\rangle$, $\langle \theta_{\lambda\rho}AVV\rangle$, etc; (b) from the counterterms alone, e.g., $\langle \theta_{\lambda\rho}AP\rangle$, $\langle \theta_{\lambda\rho}VAP\rangle$, etc; (c) from both the $1/m$ terms and the counterterms, e.g., $\langle \theta_{\lambda\rho}SS\rangle$, etc.

(4) In contrast to the axial-vector anomalies, which are restricted to abnormal-parity n-pf's, the trace anomalies occur only in n-pf's of normal parity.

There are totally 25 anomalous TI's; we list their anomalies below[25]:

$n=3$

$$\Delta_T(\theta VV) = 2iy\delta^{ab}(g_{\mu\nu}p\cdot q - p_\nu q_\mu),$$

$$\underline{\Delta}_T(\theta AA) = \Delta_T(\theta VV) + 12iy\delta^{ab}m^2 g_{\mu\nu},$$

$$\hat{\Delta}_T(\theta AP) = -6y\delta^{ab}mq_\mu, \quad (15a)$$

$$\underline{\Delta}_T(\theta PP) = -3iy\delta^{ab}p\cdot q,$$

$$\underline{\Delta}_T(\theta SS) = -iy\delta^{ab}(12m^2 + 3p\cdot q).$$

$n=4$

$$\Delta_T(\theta VVV) = 2iyf^{abc}[g_{\mu\nu}(q-p)_\sigma + g_{\mu\sigma}(p-s)_\nu + g_{\nu\sigma}(s-q)_\mu],$$

$$\Delta_T(\theta VAA) = \Delta_T(\theta VVV),$$

$$\hat{\Delta}_T(\theta VAP) = 6yf^{abc}mg_{\mu\nu},$$

$$\hat{\Delta}_T(\theta AAS) = -12yd^{abc}mg_{\mu\nu}, \quad (15b)$$

$$\Delta_T(\theta VSS) = 3iyf^{abc}(q-s)_\mu,$$

$$\Delta_T(\theta VPP) = \Delta_T(\theta VSS),$$

$$\underline{\Delta}_T(\theta ASP) = 3iyd^{abc}(q-s)_\mu,$$

$$\hat{\Delta}_T(\theta SSS) = 18yd^{abc}m,$$

$$\hat{\Delta}_T(\theta SPP) = \tfrac{1}{3}\Delta_T(\theta SSS).$$

$n=5$

$$\Delta_T(\theta VVVV) = \tfrac{1}{4}iy[g_{\mu\nu}g_{\sigma\tau}(W_1+W_2-2W_3) + g_{\mu\tau}g_{\nu\sigma}(W_1-2W_2+W_3) + g_{\mu\sigma}g_{\nu\tau}(-2W_1+W_2+W_3)],$$

$$\Delta_T(\theta VVAA) = \Delta_T(\theta VVVV),$$

$$\Delta_T(\theta AAAA) = \Delta_T(\theta VVVV),$$

$$\Delta_T(\theta VVSS) = -\tfrac{3}{8}iyg_{\mu\nu}(W_1+W_2-2W_3),$$

$$\Delta_T(\theta VVPP) = \Delta_T(\theta VVSS), \quad (15c)$$

$$\Delta_T(\theta VASP) = \tfrac{3}{8}yg_{\mu\nu}(\overline{W}_1 - \overline{W}_2 + 2\overline{W}_3),$$

$$\Delta_T(\theta AASS) = -\tfrac{3}{8}iyg_{\mu\nu}(W_1+W_2+2W_3),$$

$$\Delta_T(\theta AAPP) = \Delta_T(\theta AASS),$$

$$\Delta_T(\theta SSSS) = \tfrac{3}{4}iy(W_1+W_2+W_3),$$

$$\Delta_T(\theta SSPP) = \tfrac{3}{4}iy(W_1+W_2-W_3),$$

$$\Delta_T(\theta PPPP) = \Delta_T(\theta SSSS),$$

where

$$W_1 \equiv \mathrm{Tr}[\lambda_a\lambda_b\lambda_c\lambda_d + \lambda_a^T\lambda_b^T\lambda_c^T\lambda_d^T],$$
$$W_2 \equiv \mathrm{Tr}[\lambda_a\lambda_b\lambda_d\lambda_c + \lambda_a^T\lambda_b^T\lambda_d^T\lambda_c^T], \quad (16)$$
$$W_3 \equiv \mathrm{Tr}[\lambda_a\lambda_c\lambda_b\lambda_d + \lambda_a^T\lambda_c^T\lambda_b^T\lambda_d^T].$$

The assignments for the vector and internal-symmetry indices and the momenta in the above are the same as those given following Eq. (14). The anomalies arising from the counterterms and from the mixture of the $1/m$ terms and the counterterms are indicated with carets and underlines, respectively. The rest are from the $1/m$ terms only.

The trace anomalies listed in (15) are related to the canonical trace anomalies of Chanowitz and Ellis.[2] Their naive trace identities obtained directly from the scale transformation are of the following form:

$$\langle \theta(0) j_{\mu_1}^{a_1}(p_1)\cdots j_{\mu_n}^{a_n}(p_n)\rangle = i\left(4-n-\sum_{l=1}^{n-1} p_l \frac{\partial}{\partial p_l}\right)\langle j_{\mu_1}^{a_1}(p_1)\cdots j_{\mu_n}^{a_n}(p_n)\rangle - \Delta_T(\theta j_{\mu_1}^{a_1}\cdots j_{\mu_n}^{a_n}), \quad (17)$$

where

$$p_n = -(p_1 + \cdots + p_{n-1}).$$

Equation (17) can be obtained from the tensor WTI's and TI's as follows: Differentiate the tensor WTI with respect to k_ρ, the momentum carried by the energy-momentum tensor, and set $k_\rho = 0$; then eliminate the term $g^{\lambda\rho}\langle \theta_{\lambda\rho} j_{\mu_1}^{a_1}\cdots j_{\mu_n}^{a_n}\rangle$ by means of the corresponding trace identity. The canonical trace anomaly of Ref. 2 is just $-\Delta_T(\theta j_{\mu_1}^{a_1}\cdots j_{\mu_n}^{a_n})$.[27] In the derivation of the canonical trace anomalies in Ref. 2, it has been assumed that the canonical anomalies are constrained by the relations derived from the chiral symmetry. In the present calculation we did not make this assumption, calculating directly from the definition of the anomaly according to our regularization procedure, independent of the chiral symmetry. Our result, nevertheless, verifies the assumption of Chanowitz and Ellis. In the absence of new anomalies, the agreement of the present results and those of Ref. 2 is, of course, expected in view of the general result of Wilson,[28] which is valid to all orders in perturbation theory.

V. CONCLUSION

By introducing the symmetric energy-momentum tensor into the free spinor field theory with internal symmetry, we have extended the investigation of canonical anomalies in both WTI's and TI's. In this broader structure we have found that the validity of the tensor WTI's, which we argue is a consequence of the Poincaré invariance,[21] forces the validity of the vector WTI's. This removes the ambiguity present in the literature concerning the presence of anomalies in the vector WTI's, and provides a basis for Bardeen's minimal set of anomalous axial-vector WTI's involving symmetry currents only. The inclusion of the energy-momentum tensor does not complicate the structure of the axial-vector WTI anomalies; the anomalies possess the same well-defined pattern as those without it, which are restricted to the abnormal-parity n-pf's containing only vectors and axial vectors. Further, we have found that all the axial-vector anomalies are constrained by those of $\langle AAA\rangle$ and $\langle AVV\rangle$. Therefore, the inclusion of the improved energy-momentum tensor does not give rise to new anomalies.

The trace anomalies are a different story. There are many anomalous trace identities, totally 25, which are restricted to normal-parity n-pf's only. The sources of the trace anomalies are the counterterms, the $1/m$ terms defined in Eq. (9a), or both, in contrast with the case of the axial-vector anomalies, defined in (9a), which come from the $1/m$ terms alone. Similarly to the case of the axial-vector anomalies, the presence of the trace anomalies cannot be blamed on the failure of the symmetry transformation which generates the formal relations (in the present case, the scale transformation). This can be easily seen by noticing the fact that 19 of the trace anomalies are still present when the fermion mass m is set equal to zero.

Finally, let us notice that there exists an anomaly-free trace relation,

$$g^{\lambda\rho}\langle \theta_{\lambda\rho}(0) j_{\mu_1}^{a_1}(p_1)\cdots j_{\mu_n}^{a_n}(p_n)\rangle = -i\left[4(n-1) + \sum_{l=0}^{n-1} p_l \frac{\partial}{\partial p_l}\right] \times \langle j_{\mu_1}^{a_1}(p_1)\cdots j_{\mu_n}^{a_n}(p_n)\rangle, \quad (18)$$

which can be obtained from the anomalous trace relation of Ref. 2 [i.e., Eq. (17)] and that used in the present work by eliminating the term $\langle\theta(0) j_{\mu_1}^{a_1}(p_1)\cdots j_{\mu_n}^{a_n}(p_n)\rangle$. This relation follows, of course, from the tensor WTI by differentiating it with respect to k_ρ as explained following Eq. (17).

ACKNOWLEDGMENTS

We would like to thank Professor H. Suura and Dr. Ivan Muzinich for helpful discussions and Professor A. C. Hearn for the use of the REDUCE 2 Program. One of us (B.-L. Y.) would also like to thank Dr. R. F. Peierls, Dr. T. L. Trueman, and Dr. J. Weneser for the hospitality extended to him during a summer visit in 1974 at Brookhaven, where this work was completed.

APPENDIX

In this appendix we list the general expressions of the WTI's and TI's of the n-pf's involving the energy-momentum tensor and internal symmetry currents.[15]

Tensor Ward-Takahashi identities

$$\begin{aligned} k^{\lambda}\langle\theta_{\lambda\rho}(k)j_{\mu}^{a}(p)\cdots j_{\nu}^{d}(q)\rangle = & -i(k+p)_{\rho}\langle j_{\mu}^{a}(p+k)\cdots j_{\nu}^{d}(q)\rangle - \cdots - i(k+q)_{\rho}\langle j_{\mu}^{a}(p)\cdots j_{\nu}^{d}(q+k)\rangle \\ & + \tfrac{1}{2}ik^{\lambda}(\bar{\Sigma}_{\lambda\rho})_{\mu}^{\tau}\langle j_{\tau}^{a}(k+p)\cdots j_{\nu}^{d}(q)\rangle + \cdots + \tfrac{1}{2}ik^{\lambda}(\bar{\Sigma}_{\lambda\rho})_{\nu}^{\tau}\langle j_{\mu}^{a}(p)\cdots j_{\tau}^{d}(q+k)\rangle, \end{aligned} \tag{A1}$$

where

$$(\bar{\Sigma}_{\lambda\rho})_{\mu}^{\tau} = \begin{cases} g_{\lambda\mu}g_{\rho}^{\tau} - g_{\lambda}^{\tau}g_{\rho\mu} & \text{for } V \text{ and } A, \\ 0 & \text{for } S \text{ and } P. \end{cases} \tag{A2}$$

Vector Ward-Takahashi identity

$$\begin{aligned} p^{\mu}\langle\theta_{\lambda\rho}(k)V_{\mu}^{a}(p)\cdots j_{\nu}^{b}(q)\cdots\rangle = & \tfrac{1}{2}i(g_{\lambda\alpha}g_{\rho\beta} + g_{\lambda\beta}g_{\rho\alpha} - 2g_{\lambda\rho}g_{\alpha\beta})p^{\alpha}\langle V^{a}(k+p)^{\beta}\cdots j_{\nu}^{b}(q)\cdots\rangle \\ & + \cdots + f^{abb'}\langle\theta_{\lambda\rho}(k)\cdots j_{\nu}^{b'}(p+q)\cdots\rangle + \cdots . \end{aligned} \tag{A3}$$

Axial-vector Ward-Takahashi identity

$$\begin{aligned} p^{\mu}\langle\theta_{\lambda\rho}(k)A_{\mu}^{a}(p)\cdots j_{\nu}^{b}(q)\cdots\rangle = & \tfrac{1}{2}i[g_{\lambda\alpha}g_{\rho\beta} + g_{\lambda\beta}g_{\rho\alpha} - 2g_{\alpha\beta}g_{\lambda\rho}]p^{\alpha}\langle A^{a}(k+p)^{\beta}\cdots j_{\nu}^{b}(q)\cdots\rangle \\ & + 2mg_{\lambda\rho}\langle P^{a}(p+k)\cdots j_{\nu}^{b}(q)\cdots\rangle - 2mi\langle\theta_{\lambda\rho}(k)P^{a}(p)\cdots j_{\nu}^{b}(q)\cdots\rangle \\ & + \cdots + \left\langle \theta_{\lambda\rho}(k)\cdots \begin{Bmatrix} f^{abb'} \begin{pmatrix} A_{\nu}^{b'}(p+q) \\ V_{\nu}^{b'}(p+q) \end{pmatrix} \\ d^{abb'} \begin{pmatrix} P^{b'}(p+q) \\ -S^{b'}(p+q) \end{pmatrix} \end{Bmatrix} \cdots \right\rangle + \cdots \quad \text{for } j_{\nu}^{b} = \begin{Bmatrix} V_{\nu}^{b} \\ A_{\nu}^{b} \\ S^{b} \\ P^{b} \end{Bmatrix}. \end{aligned} \tag{A4}$$

Trace identity

$$g^{\lambda\rho}\langle\theta_{\lambda\rho}(k)j_{\mu}^{a}(p)\cdots j_{\nu}^{b}(q)\rangle = \sqrt{6}\,m\langle S^{0}(k)j_{\mu}^{a}(p)\cdots j_{\nu}^{b}(q)\rangle - 3i[\langle j_{\mu}^{a}(p+k)\cdots j_{\nu}^{b}(q)\rangle + \cdots + \langle j_{\mu}^{a}(p)\cdots j_{\nu}^{b}(q+k)\rangle]. \tag{A5}$$

In all the above n-pf's, 4-momentum conservation is understood, i.e.,

$$k + p + \cdots + q + \cdots = 0. \tag{A6}$$

---

*Prepared for the U. S. Energy Research and Development Administration under Contract No. W-7405-eng-82.

[1]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969); J. Schwinger, Phys. Rev. 82, 664 (1951).

[2]M. S. Chanowitz and J. Ellis, Phys. Lett. 40B, 397 (1972); Phys. Rev. D 7, 2490 (1973). The second article contains also a good deal of discussion on the applications of the trace anomalies.

[3]K. G. Wilson, Phys. Rev. 181, 1909 (1969).

[4]For a review and a recent discussion on the theoretical implications of anomalies, we refer to S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971), Vol. 1; S. B. Treiman, R. Jackiw and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton Univ. Press, Princeton, N. J., 1972), pp. 97–254; M. Gell-Mann, in *Elementary Particle Physics, proceedings of the XI Schladming Conference* (Acta Phys. Austriaca Suppl. IX), edited by P. Urban (Springer, New York, 1972), p. 733; H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973); P. G. O. Freund and S. Nandi, Phys. Rev. Lett. 32, 181 (1974); S. D. Drell, Phys. Rev. D 7, 2190 (1973); J. L. Newmeyer and S. D. Drell, *ibid.* 8, 4070 (1973).

[5]The literature on the phenomenological implications of the anomaly is abundant. An earlier summary is given by A. Zee, in *Proceedings of the XVI International Con-*

ference on High Energy Physics, Chicago-Batavia, Ill., 1972, edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 132. There does not seem to exist an up-to-date comprehensive review. The following is at best an incomplete list (we apologize for any work being omitted): S. Okubo, Phys. Rev. 179, 1629 (1969); S. L. Glashow, R. Jackiw, and S.-S. Shei, *ibid.* 187, 1916 (1969); C. S. Lai, Nucl. Phys. B14, 348 (1969); G. J. Gounaris, Phys. Rev. D 1, 1426 (1970); Bing-Lin Young, *ibid.* 2, 606 (1970); S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, *ibid.* 4, 3497 (1971); R. Aviv, N. D. Hari Dass, and R. F. Sawyer, Phys. Rev. Lett. 26, 591 (1971); Tsu Yao, Phys. Lett. 35B, 225 (1971); F. Strocchi and R. V. Caffarelli, *ibid.* 35B, 595 (1971); M. V. Terent'ev, Zh. Eskp. Teor. Fiz. Pis'ma Red. 14, 140 (1971) [JETP Lett. 14, 94 (1971)]; M. Schweda, Nuovo Cimento 4A, 111 (1971); C. S. Lai and S. Y. Lo, *ibid.* 4A, 198 (1971); N. D. Hari Dass, Phys. Rev. D 5, 1542 (1972); R. Aviv and A. Zee, *ibid.* 5, 2372 (1972); M. D. Scadron and R. L. Thews, *ibid.* 6, 844 (1972); A. Zee, *ibid.* 6, 900 (1972); J. Smith and N. Stanko, *ibid.* 6, 1332 (1972); M. V. Terent'ev, Phys. Lett. 38B, 419 (1972); H. Terazawa, Phys. Rev. D 7, 3123 (1973); V. M. Raval and R. Ramachandran, *ibid.* 8, 1144 (1973); J. Smith and Z. E. S. Uy, *ibid.* 8, 3056 (1973); D. A. Akyeampong and R. Delbourgo, Nuovo Cimento 17A, 578 (1973); G. Furlan, F. Legovini, and N. Paver, *ibid.* 17A, 635 (1973); H. Terazawa, Rev. Mod. Phys. 45, 615 (1973).

[6]D. J. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972); C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Lett. 38B, 519 (1972); W. A. Bardeen, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 295.

[7]W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

[8]R. W. Brown, C.-C. Shih, and B.-L. Young, Phys. Rev. 186, 1491 (1969).

[9]I. S. Gerstein and R. Jackiw, Phys. Rev. 181, 1955 (1969); D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento 65A, 55 (1970).

[10]J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971); J. H. Lowenstein and B. Schroer, Phys. Rev. D 6, 1553 (1972); P. Breitenlohner and H. Mitter, Nuovo Cimento 10A, 655 (1972).

[11]In order to emphasize the importance of a consistent regularization procedure in defining all the n-pf's considered, we separate it from the renormalization procedure, which is practically, in our approach, to redefine the n-pf's. Ordinarily, the normalization program in quantum field theories contains both procedures.

[12]See Adler (Ref. 1).

[13]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[14]See Adler (Ref. 4) and S. L. Adler, R. W. Brown, T. F. Wong, and B.-L. Young, Phys. Rev. D 6, 1788 (1971) and the references cited therein.

[15]H. Suura and B.-L. Young, Phys. Rev. D 8, 4354 (1973).

[16]See the discussion given in Ref. 15, p. 4357, and the references cited therein.

[17]Our basic notation and conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). In particular, $\epsilon_{0123}=-\epsilon^{0123}=1$, $\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho)=4i\,\epsilon_{\mu\nu\lambda\rho}$. We also use the notation $\int_l \equiv (1/2\pi)^4\int d^4l$.

[18]W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[19]Before renormalization, almost all the AWTI's are anomalous in the dimensional regularization, in contrast to the criterion (9b) derived from the universal Pauli-Villars scheme [H. Suura and B.-L. Young (unpublished)].

[20]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

[21]S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971). We are aware of the fact that the form of the symmetric energy-momentum tensor is not unique and is subject to modification as long as it gives the canonical energy-momentum 4-vector, as was pointed out by Callan, Coleman, and Jackiw, Ref. 13. However, a change of the form of $\theta_{\lambda\rho}$ will result in a change of the TWTI. An example of this has been given in Ref. 15, pp. 4364–4365. Once a form of $\theta_{\lambda\rho}$ is chosen, the naive TWTI is determined as a consequence of the Poincaré invariance. The presence of anomalies in TWTI's may violate Poincaré invariance, if it contributes to the integrated WTI of the commutation relations between the field operators and the Poincaré group generators.

[22]See Eq. (55) of Ref. 8.

[23]The $1/m$ terms and the counterterms are given by D. B. Zacrep and Bing-Lin Young, Iowa State University report, 1974 (unpublished). In this work a few errors in the counterterm expressions of Ref. 8 are pointed out and corrected. They are

$$\delta(SSS)=\tfrac{3}{2}i(a_4-a_1+4y)d^{abc}m,$$

$$\delta(SSPP)=-\tfrac{1}{16}(a_4-a_1)(W_1+W_2-W_3)-\tfrac{3}{8}y(W_1+W_2),$$

and

$$\delta(SSSS)=-\tfrac{1}{16}(a_4-a_1+16y)(W_1+W_2+W_3).$$

[24]There should be an extra minus sign in front of the y's in both $\Delta(AAAV)$ and $\Delta(AVVV)$ given in Ref. 8, Eq. (56). There should also be an extra minus sign on the b's of $\Delta(AVVV)$. They are correct here.

[25]Some of the trace calculations of the $1/m$ terms were done with the help of REDUCE 2, an algebraic program by Hearn [A. C. Hearn, REDUCE 2 User's Manual, 2nd ed., Univ. of Utah, 1973 (unpublished)].

[26]The new anomalies, if existent, could come from two sources: (a) those which are not expected from the axial-vector anomalies in $\langle AVV\rangle$, $\langle AAA\rangle$, and $\langle AAAAA\rangle$, and (b) those due to the noncommutativity of the order in taking the double WTI's, e.g.,

$$k^\lambda p^\mu\langle\theta_{\lambda\rho}(k)A^a_\mu(p)\cdots\rangle\neq p^\mu k^\lambda\langle\theta_{\lambda\rho}(k)A^a_\mu(p)\cdots\rangle.$$

The possibility of type (a) is limited by the tensor structure and the dimensionality of the n-pf.

[27]Several of the trace anomalies listed in Eq. (15) are given in Ref. 2, with which our results agree except for the $\langle\theta_{\lambda\rho}AA\rangle$. According to Eq. (15), the anomaly is

$$\frac{R}{4\pi^2}(p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{g_{\mu\nu}}{\pi^2} m^2$$

instead of

$$\frac{R}{8\pi^2}(p^2 g_{\mu\nu} - p_\mu p_\nu) - \frac{g_{\mu\nu}}{\pi^2} m^2$$

as given in Eq. (14.12) of Ref. 2, where $R = \frac{2}{3}$.

[28]K. Wilson, Phys. Rev. 179, 1499 (1969).