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Vacuum symmetry and the pseudo-Goldstone phenomenon*

Howard Georgi†
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138

A. Pais
Rockefeller University, New York, New York 10021
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In the context of gauge theories with spontaneous symmetry breakdown induced by the Higgs phenomenon, a simple theorem is stated which gives conditions under which pseudo-Goldstone particles may occur. These conditions are broader than the occurrence of accidental symmetry and contain the latter case as a special instance. We analyze an example in which the gross features of the vector- and the scalar-meson mass spectra are determined by quantum effects.

I. INTRODUCTION

Field theories which contain spinless particles which are massless to zeroth order in a "natural" way[1] but which acquire mass due to radiative corrections may be of physical interest for several reasons. First, they may give a clue as to the occurrence of low-mass particles such as pions.[2] Secondly, they may possibly play a role in the understanding of CP violation as a natural quantum effect.[3] The natural occurrence of such particles has been realized in the context of gauge theories with a spontaneous symmetry breakdown generated by the Higgs mechanism. Namely, as was noted by Weinberg,[2] such particles may occur if the scalar field potential of the Lagrangian exhibits an accidental symmetry, i.e., a natural symmetry which is larger than the gauge symmetry of the theory. We shall denote this potential by $V(\phi, \alpha)$, where ϕ is a vector whose components ϕ_k are the set of scalar fields and where α denotes the set of parameters which enters in the potential. Note that the presence of accidental symmetry is a property of V which is independent of α. (Of course the α are subject to the condition of strict renormalizability of the theory.)

It is the purpose of this paper to state a theorem which broadens the options for obtaining naturally particles of this kind and to give a few examples. Once again, gauge theories with Higgs particles provide the frame; the accidental-symmetry case as defined by Weinberg[2] is contained as a subclass in our case. In order not to be overcrowded with new nomenclature we shall again denote such particles as pseudo-Goldstone bosons (PGB's), the term coined originally[2] for the more specific accidental-symmetry case. Our general theorem covers in particular the instances noted by Halpern,[4] by Danskin,[5] and by Lieberman[6] where such PGB's were obtained seemingly without any deeper reason, and also the local unlocking discussed by Duncan.[7] As will soon be evident, our arguments simply derive from an examination of the content of the Goldstone theorem itself.

Let G be the local gauge group of the theory with dimension $D(G)$. Let S be the surface in ϕ space (a finite-dimensional vector space) on which the potential $V(\phi, \alpha)$ assumes its minimum value in the tree approximation. That is to say, when the scalar fields ϕ_i develop a zeroth-order vacuum expectation value $\langle\phi_i\rangle_{\text{vac}} = \lambda_i$, then $\lambda \in S$. Any transformation $\in G$ carries S into itself. The vacuum expectation value breaks G down to a subgroup G_λ [with dimension $D(G_\lambda)$] which is the little group of the vector λ. Let S_λ be the $D(S_\lambda)$-dimensional subspace of ϕ space spanned by the tangents to the surface S at the point λ. Observe that the structure of S depends in general on the α in $V(\phi, \alpha)$ (see Sec. III). Moreover for fixed α, hence fixed S, $D(S_\lambda)$ may depend on the particular choice of $\lambda \in S$ (see Sec. IV).

Theorem. The number of spinless particles which are massless in the tree approximation and which are not absorbed by the vector gauge fields is at least as large as the number n defined by

$$n = D(S_\lambda) - D(G) + D(G_\lambda) \ . \tag{1}$$

n is potentially the number of PGB's. It depends on the detailed particle content of the theory whether all n are PGB's or whether some (or all) of them are in fact true surviving Goldstone bosons (GB's). (Examples are known[8] where a theory contains both PGB's and a true GB.) With this provision well in mind we shall, for brevity only, refer to n as the number of PGB's.

In a large subclass of theories, the surface S can be generated by acting on any λ in S with a group G_{vac} [with dimension $D(G_{\text{vac}})$] of linear transformations mapping S into itself. Let $G_{\text{vac},\lambda}$ [with dimension $D(G_{\text{vac},\lambda})$] be the subgroup which leaves

λ unchanged. Then

$$D(S_\lambda) = D(G_{\text{vac}}) - D(G_{\text{vac},\lambda}) , \tag{2}$$

so that

$$n = D(G_{\text{vac}}) - D(G_{\text{vac},\lambda}) - D(G) + D(G_\lambda) . \tag{3}$$

Often [as for the familar $SU(2)\times U(1)$ models] $G_{\text{vac}} = G$, $G_{\text{vac},\lambda} = G_\lambda$, but this is not always so. Indeed it is our main allegation (and examples to be given will illustrate this) that G_{vac}, which we will call the vacuum symmetry, *may be a larger group than the symmetry G of the theory as a whole*.

As is well known, for the case of accidental symmetry there exists a formula[2] analogous to Eq. (3) for the number n' of (potential) PGB's, namely

$$n' = D(G') - D(G'_\lambda) - D(G) + D(G_\lambda) , \tag{4}$$

where G' is the accidental-symmetry group and G'_λ the corresponding little group. Since G' is a symmetry group of the entire potential $V(\phi, \alpha)$, it clearly contains G. In turn, G_{vac} must contain G'. The existence of a nontrivial G', that is a G' which is larger than G, corresponds, for the cases analyzed in the past, to $G_{\text{vac}} = G' \supset G$, in obvious notation. In this sense, the accidental-symmetry equation (4) is a special case of Eq. (3).

We shall next do three things: First, we shall prove the theorem (Sec. II). Secondly, we shall exhibit situations where $G_{\text{vac}} \supset G$ even though there is no accidental symmetry at all, thereby establishing that accidental symmetry is a sufficient but not a necessary condition for the presence of PGB's (Sec. III). Thirdly, we will analyze an example where $n > 0$ but where the surface S is *not* generated by a vacuum symmetry G_{vac}, so that Eq. (1) applies but not Eq. (3) (Sec. IV).

II. PROOF OF THE THEOREM

The tangent space S_λ may be defined by considering the set of differentiable curves $\mu(t)$ [for every t, $\mu(t)$ is a vector in ϕ space with components $\mu_k(t)$] such that $\mu(t) \in S$ for $t = [0, 1]$, $\mu(0) = \lambda$, and the limit

$$\mu'(0) = \lim_{t\to 0} \mu'(t) \tag{5}$$

exists and is nonvanishing. The vector $\mu'(0)$ is a tangent to S at λ, and the set of all such curves generates all tangents. Now since S is the surface of minimum potential, $V_i(\phi, \alpha) = 0$ for $\phi \in S$ [we use the notation $V_{i_1 \cdots i_n}(\phi, \alpha) = \partial^n V(\phi, \alpha)/\partial\phi_{i_1} \cdots \partial\phi_{i_n}$]. In particular $V_i(\mu(t), \alpha) = 0$. Differentiating with respect to t we obtain

$$V_{ij}(\mu(t), \alpha)\, \mu_j'(t) = 0 ,$$

and therefore

$$V_{ij}(\lambda, \alpha)\, \mu_j'(0) = 0 . \tag{6}$$

But $V_{ij}(\lambda, \alpha)$ is the zeroth-order mass matrix of the spinless mesons, so Eq. (6) implies that each tangent to S at λ is an eigenvector of the meson mass matrix with eigenvalue zero. Therefore there are at least $D(S_\lambda)$ massless spinless mesons in the tree approximation. Since $D(G) - D(G_\lambda)$ Goldstone bosons are absorbed by the Higgs mechanism, there are at least $D(S_\lambda) - D(G) + D(G_\lambda)$ spinless mesons left massless in the tree approximation, and the theorem is proved.

Consider a change of the parameters α in the potential, so that $V(\phi, \alpha) \to V(\phi, \alpha')$. This change maintains the symmetry G (and any global symmetries of the full theory). The surface of minimum potential will change to S'. We will say that S is a *natural* surface of minimum potential if for any sufficiently small change in α, the resulting surface S' is similar to S in the following sense: There exists a one-to-one and onto map of $\lambda \in S \to \lambda' \in S'$ such that $G_\lambda = G_{\lambda'}$ and $D(S_\lambda) = D(S'_{\lambda'})$. In other words, S is natural if its shape is maintained for a range of α, even though details (such as the length of vectors $\lambda \in S$) may change. If S is natural, the masslessness of the spinless mesons associated with S_λ is also natural. In general, it may be possible to find special choices of the parameters α for which there are additional massless mesons, not associated with S_λ. The second derivative of V may vanish in some direction even though V is not constant over any finite interval (for instance, consider a massless scalar field with quartic interactions). This is why we have been careful to state our theorem only as an inequality. But we suspect that any such situation is unnatural in the sense that if the parameters in V are slightly changed, the spinless mesons not associated with S_λ will not remain massless.

III. VACUUM SYMMETRY

In this section we give an example of a model with no accidental symmetry but a nontrivial G_{vac}. The models with PGB's mentioned earlier[4-7] will readily be seen as belonging to this subclass $G_{\text{vac}} \supset G$. The example we have chosen to discuss next is not constructed to resemble physics but rather to be as simple as possible as an illustration of vacuum symmetry.

The gauge group is $SO(3)\times SO(3)\times SO(3)=G$. The scalar mesons are three real 3×3 matrix fields ϕ_{12}, ϕ_{23}, and ϕ_{31} which transform as $\phi_{ij}\to U_i\phi_{ij}U_j^T$, where U_1, U_2, and U_3 are independent real 3×3 orthogonal matrices. In other words, the scalars transform like $(3,3,1)+(1,3,3)+(3,1,3)$ under the gauge group. The potential (with I the 3×3 unit matrix) is

$$\begin{aligned}V(\phi,\alpha)=&\ \alpha_1\{[\mathrm{tr}(\phi_{12}\phi_{12}^T-\mu^2I)]^2+[\mathrm{tr}(\phi_{23}\phi_{23}^T-\mu^2I)]^2+[\mathrm{tr}(\phi_{31}\phi_{31}^T-\mu^2I)]^2\}\\&+\alpha_2\{[\mathrm{tr}(\phi_{12}\phi_{12}^T-\mu^2I)][\mathrm{tr}(\phi_{23}\phi_{23}^T-\mu^2I)]+[\mathrm{tr}(\phi_{23}\phi_{23}^T-\mu^2I)][\mathrm{tr}(\phi_{31}\phi_{31}^T-\mu^2I)]\\&\quad+[\mathrm{tr}(\phi_{31}\phi_{31}^T-\mu^2I)][\mathrm{tr}(\phi_{12}\phi_{12}^T-\mu^2I)]\}\\&+\alpha_3\{\mathrm{tr}[(\phi_{12}\phi_{12}^T-\mu^2I)^2]+\mathrm{tr}[(\phi_{23}\phi_{23}^T-\mu^2I)^2]+\mathrm{tr}[(\phi_{31}\phi_{31}^T-\mu^2I)^2]\}\\&+\alpha_4\{\mathrm{tr}[(\phi_{12}\phi_{12}^T-\mu^2I)(\phi_{31}^T\phi_{31}-\mu^2I)]+\mathrm{tr}[(\phi_{23}\phi_{23}^T-\mu^2I)(\phi_{12}^T\phi_{12}-\mu^2I)]\\&\quad+\mathrm{tr}[(\phi_{31}\phi_{31}^T-\mu^2I)(\phi_{23}^T\phi_{23}-\mu^2I)]\}\ .\end{aligned}\tag{7}$$

This is the most general quartic potential consistent with gauge invariance and with the additional discrete symmetries $\phi_{12}\to\phi_{23}\to\phi_{31}\to\phi_{12}$ and $\phi_{12}\to-\phi_{12}$.

If $\alpha_1>|\alpha_2|$ (or $\alpha_1>\frac{1}{2}\alpha_2>0$) and $\alpha_3>|\alpha_4|$, the potential is minimized $[V(\phi)=0]$ when

$$\phi_{12}\phi_{12}^T=\phi_{12}^T\phi_{12}=\phi_{23}\phi_{23}^T=\phi_{23}^T\phi_{23}=\phi_{31}\phi_{31}^T=\phi_{31}^T\phi_{31}=\mu^2I.\tag{8}$$

The condition (8) is unaffected by multiplication of any ϕ_{ij} on the left or right by an orthogonal matrix. Thus $G_{\mathrm{vac}}=SO(3)\times SO(3)\times SO(3)\times SO(3)\times SO(3)\times SO(3)$: The symmetry of the surface of minimum potential is larger than the symmetry of the potential as a whole. A particular choice of vacuum expectation values λ satisfying (8) breaks the vacuum symmetry down to the diagonal subgroups $G_{\mathrm{vac},\lambda}=SO(3)\times SO(3)\times SO(3)$ (if $\langle\phi\rangle=\mu O$, where O is orthogonal, multiplication on the left by a general orthogonal matrix U and on the right by O^TUO leaves $\langle\phi\rangle$ unchanged). Thus $D(S_\lambda)=D(G_{\mathrm{vac}})-D(G_{\mathrm{vac},\lambda})=18-9=9$. So nine spinless mesons are massless in the tree approximation. For example, if $\langle\phi_{12}\rangle=\langle\phi_{23}\rangle=\langle\phi_{31}\rangle=I$, the antisymmetric parts of the ϕ's are the nine massless fields.

Just as in an accidental-symmetry situation, the minimization of the zeroth-order potential is not sufficient to determine the physical structure of the theory. In the present case, this can be seen as follows. We can use condition (8) and G and the discrete symmetries to choose

$$\langle\phi_{12}\rangle=\langle\phi_{23}\rangle=\mu I.\tag{9}$$

Then $\langle\phi_{31}\rangle$ cannot in general be completely diagonalized, but we can put it in the form

$$\langle\phi_{31}\rangle=\mu\begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix},\tag{10}$$

where the angle θ is *not* determined by extremal conditions in the tree approximation. If $\theta=0$, $G_\lambda=SO(3)$ and there are three massless vector mesons and three PGB's. For other values of θ, $G_\lambda=SO(2)$, and there is only a single massless "photon" and one PGB.

Thus the angle θ has to be determined by the quantum corrections to the potential, as discussed by Coleman and Weinberg.[9] We have described earlier[3] another situation where an angle was to be determined by radiative corrections; it was a case of accidental symmetry, and we had occasion to note at that time that the qualitative properties of the vector-meson mass spectrum depended on quantum corrections to the potential. Not only is this also true here, but in addition something new happens. Indeed, now not only the vector-meson masses but the scalar-meson masses as well depend on θ. Hence scalar-meson loops must be included in the Coleman-Weinberg sum to determine θ.

It should be clear from this example that vacuum symmetry is a straightforward extension of accidental symmetry which considerably expands the options for producing PGB's. We have exhibited the simplest of an infinite class of theories of this kind. Such models can be constructed with almost any gauge group of the form $g\times g\times g\times\cdots$. The cyclic symmetry is not essential; some of the subgroups need not be gauged (as in the pseudo-Goldstone pion examples), etc. There are also many examples which do not have the simple cyclic structure of the model discussed here.

IV. A PECULIAR EXAMPLE

In this section we give a particular example (its main virtue is again its simplicity) in which the surface of minimum potential is not obtainable from a vacuum symmetry. The model[10] has a U(1) gauge group and four complex scalar fields ϕ_1, ϕ_2, ϕ_3, and ϕ_4 transforming in the same way under a gauge transformation: $\phi_k \to e^{i\theta}\phi_k$. The potential is

$$V(\phi, \alpha) = \alpha_1 \sum_{k=1}^{4} (\phi_k^* \phi_k - \mu^2)^2 + \alpha_2 \left[\sum_{k=1}^{4} (\phi_k^* \phi_k - \mu^2)\right]^2 + \alpha_3 \left(\sum_{k=1}^{4} \phi_k^*\right) \left(\sum_{k=1}^{4} \phi_k\right) . \quad (11)$$

This is the most general gauge-invariant quartic potential with the following properties: It is symmetric under permutations of the four fields; the quartic terms are invariant under independent U(1) transformations of each of the four fields, $\phi_k \to e^{i\theta_k}\phi_k$. The last condition is renormalizable because the gauge couplings are also invariant under the larger global symmetry U(1)×U(1)×U(1)×U(1). Thus all terms of dimension 4 in the Lagrangian have this symmetry and it is only broken by one of the mass terms,[11] the α_3 term in Eq. (11).

If α_1, α_2, and α_3 are positive, the surface of minimum potential for $V(\phi, \alpha)$ given by Eq. (11) is defined by

$$\phi_k^* \phi_k = \mu^2, \quad k = 1 \text{ to } 4$$
$$\sum_{k=1}^{4} \phi_k = 0 . \quad (12)$$

We can think of the ϕ's as vectors in the complex plane. Then Eq. (12) implies that the four vectors have the same length μ and that their vector sum is zero. Thus the four vectors must form a parallelogram. At first glance, one might expect two massless mesons in the tree approximation corresponding to the two degrees of freedom of the parallelogram, rotation and deformation. But as we will see, more careful analysis is required.

There is, of course, always one massless meson arising from the breakdown of the U(1) gauge symmetry. We can "factor" the corresponding degree of freedom out of the surface S of minimum potential by using the gauge freedom to choose

$$\langle \phi_1 \rangle = \mu ,$$
$$\langle \phi_k \rangle = \mu e^{i\theta_k}, \quad k = 2 \text{ to } 4 . \quad (13)$$

To understand what is going on, we can look at a part of the surface of minimum potential in the plane $\theta_2 = \theta_3 + \theta_4 - 2\pi$ shown in Fig. 1. The two solid lines are distinct branches of S, $\theta_3 = \pi$, $\theta_2 = \theta_4 - \pi$ and $\theta_4 = \pi$, $\theta_2 = \theta_3 - \pi$. The dotted point $(0, \pi, \pi)$ is the intersection of these branches. The other dotted points are intersection points of the branches in the $\theta_2 = \theta_3 + \theta_4 - 2\pi$ plane with other branches not in the plane. All the dotted points are physically equivalent because of the permutation symmetry.

Since we have already removed the degree of freedom associated with the gauge symmetry by choosing $\langle \phi_1 \rangle = \mu$, the dimension of the slice of the tangent space shown in Fig. 1 is $D(S_\lambda) - D(G) + D(G_\lambda) = D(S_\lambda) - 1$. In other words, it is the number of PGB's. Except at the dotted points, the slice is one-dimensional and there is one PGB in accord with our naive counting. But at $(0, \pi, \pi)$ the slice is two-dimensional and there are two PGB's. The dotted points correspond to the degenerate situation in which all four vectors are parallel, and at these special points there are two different directions along the surface of minimum potential.

The existence of separate branches of S intersecting at isolated points is what keeps us from being able to define a vacuum symmetry in this situation. Any single branch is generated by a U(1) symmetry [plus the U(1) of over-all rotation], but this U(1) is not a vacuum symmetry because it does not map the other branches into themselves.

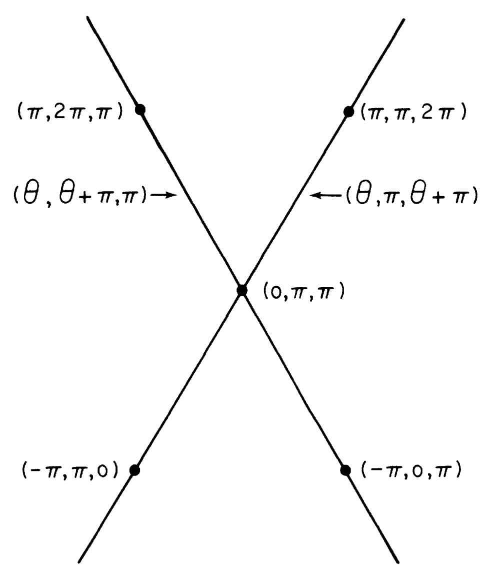

FIG. 1. S in the $\theta_2 = \theta_3 + \theta_4 - 2\pi$ plane. Coordinates are labeled $(\theta_2, \theta_3, \theta_4)$.

V. CONCLUDING QUESTIONS

The present investigation raises two questions, one new, one old. The first one is: Now that broader conditions for PGB's have been obtained, how near are we to having not only sufficient but also necessary conditions for the naturalness of particles whose mass vanishes, but only to leading order? The second one is: Are such particles needed in the description of such physical phenomena as mentioned in Sec. I? The answer to either question is presently beyond us.

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†Junior Fellow, Society of Fellows, Harvard University.

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