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Renormalization of non-Abelian gauge theories in a background-field gauge. I. Green's functions

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The renormalization of non-Abelian gauge theories is studied in the background-field gauge by means of Ward-Slavnov identities derived from supergauge transformations and with use of 't Hooft and Veltman's renormalization. As a result, the counterterm depending on the background field only is shown to be $(g/g_0)^2\mathcal{L}(\vec{A}^{cl})$.

I. INTRODUCTION

DeWitt[1] suggested replacing the study of the generating functional of Green's functions $Z(\eta)$,

$$Z(\vec{\eta}_\mu)=\int dA\exp\left\{i\int[\mathcal{L}(\vec{A}_\mu)+\vec{\eta}_\mu\cdot\vec{A}_\mu]\right\},$$

with the study of the functional $Z[\eta(A^{cl})]$, where the background field $A^{cl}(x)$ satisfies the classical equation of motion: $\delta\mathcal{L}(\vec{A}_\mu^{cl})/\delta\vec{A}_\mu+\vec{\eta}_\mu=0$. The two approaches are in fact equivalent except for gauge theories, for which DeWitt quantizes in a gauge specified by the classical background field $\vec{A}_\mu^{cl}(x)$. The interesting feature of this method is the fact that this gauge-fixing term does not break gauge invariance relative to the transformations of the background field $\vec{A}_\mu^{cl}$; it fixes only the gauge of the quantized field. This property should simplify the discussion of renormalization in these gauges. Nevertheless, since $\vec{A}_\mu^{cl}$ simultaneously denotes the source of quantized fields and fixes the gauge of the latter, the equivalence between the Green's functions derived in this way and the usual Green's functions is not clear. DeWitt argued that on the mass shell, the functional $Z(\vec{A}_\mu^{cl})$ is gauge-independent and provides the same S-matrix elements as in Fermi-type gauges. These arguments were developed for similar source-dependent gauges by Honerkamp,[2] Kallosh,[3] and by Arefieva, Faddeev, and Slavnov,[4] who claim that the formal invariance property of $Z(\vec{A}_\mu^{cl})$ can be preserved by renormalization, and were supported in the one-loop approximation by effective computations of Honerkamp,[2] 't Hooft,[5] and Crewther.[6] In our view, the problems raised by this quantization are threefold. First, the solution of the classical equation of motion for $\vec{A}_\mu^{cl}$ in terms of the source η_μ requires a constraint equation for η_μ (Ref. 2) and is, furthermore, nonunique. Second, mass-shell arguments are very dangerous in nonbroken gauge theories and do not exhibit the symmetries of the problem. Third, the graphs with only external $\vec{A}_\mu^{cl}$ legs cannot be renormalized by themselves, since subgraphs of the latter are graphs with external quantum legs $\vec{A}_\mu$. Thus for the purpose of renormalization one must introduce an auxiliary functional $Z(\vec{A}_\mu^{cl},\vec{\eta}_\mu)$ where $\vec{\eta}_\mu$ is an $\vec{A}_\mu^{cl}$-independent source for the quantum field $\vec{A}_\mu$ and one cannot restrict the discussion as in Refs. 1–4 to the functional $Z(\vec{A}_\mu^{cl})$.

In this paper we study the renormalization of the functional $Z(\vec{A}_\mu^{cl},\vec{\eta}_\mu)$, which is obviously renormalizable by power counting, and we show that the renormalized functional $Z^R(\vec{A}_\mu^{cl},\vec{\eta}_\mu)$ exhibits the same symmetry properties as the bare functional. This is achieved by investigation of Slavnov identities, which are derived from supergauge transformations introduced first by Becchi, Rouet, and Stora.[7] This method has been extended by the authors[8] to derive Slavnov identities relating Green's functions in different gauges. A gauge-invariant regularization and subtraction procedure, for which we can use the dimensional regularization and the "minimal subtraction" of 't Hooft and Veltman,[9] implies that the Slavnov identities hold for the renormalized functional. To derive the symmetry properties of the renormalized action from the Slavnov identities satisfied by this action, a very elegant method has been developed by Zinn-Justin[10]; it avoids the investigation of the divergences of each individual superficially divergent Green's function. Thus we shall adopt this method here.

We also prove that the counterterm depending on the background field only is $(g/g_0)^2\mathcal{L}(\vec{A}_\mu^{cl})$; this result is known in the one-loop approximation.[5] In the last section, we investigate the dependence of the Green's functions with respect to the gauge parameter α and discuss its implication for the practical computation of g/g_0 at higher orders.

II. WARD-SLAVNOV IDENTITIES AND THE RENORMALIZATION OF THE GREEN'S FUNCTIONS

A. Properties of the bare functional

After a change of variable, $\vec{A}_\mu = \vec{A}_\mu^{\rm cl} + \vec{Q}_\mu$, the generating functional for Green's functions in the covariant gauge reads

$$Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \overline{\xi}_i) = \int dQ\, dC\, d\overline{C} \exp\left(i \int \left\{\mathcal{L}(A+Q) - \frac{1}{2\alpha}[D_\mu^{ij}(A)Q_j^\mu]^2 + \overline{C}_i D_\mu^{ij}(A) D_\mu^{jk}(A+Q) C_k + \vec{\eta}_\mu \cdot \vec{Q}^\mu + \overline{\xi}_i C_i + \overline{C}_i \xi_i\right\}\right), \tag{2.1}$$

where we suppress from now on the superscript of $A^{\rm cl}$ and where we introduce the following notations:

$$\mathcal{L}(V) = -\tfrac{1}{4}(\partial_\mu \vec{V}_\nu - \partial_\nu \vec{V}_\mu + g\vec{V}_\mu \times \vec{V}_\nu)^2, \tag{2.2}$$

$$D_\mu^{ij}(V)C^j = \partial_\mu C^i + g(\vec{V}_\mu \times \vec{C})^i = \partial_\mu C_i + g f_{ijk} V_\mu^j C^k. \tag{2.3}$$

$\vec{Q}_\mu$ denotes the quantized gauge field and the Faddeev-Popov ghosts C_i and $\overline{C}_i$ are associated with the gauge transformation of $\vec{Q}_\mu$ for fixed $\vec{A}_\mu$:

$$\delta Q_\mu^i = D_\mu^{ij}(A+Q)\delta\omega^j, \qquad \delta A_\mu^i = 0 \quad \text{(type II transformation)}, \tag{2.4}$$

which leaves $\mathcal{L}(\vec{A}_\mu + \vec{Q}_\mu)$ invariant. The peculiarity of this gauge is that the gauge-fixing term introduced for quantization, $-(1/2\alpha)[D_\mu^{ij}(A)Q_j^\mu]^2$, is invariant for gauge transformations of $\vec{A}_\mu$:

$$\delta A_\mu^i(x) = D_\mu^{ij}(A)\delta\omega^j(x) \quad \text{(type I transformation)}, \tag{2.5a}$$

provided that one also performs a homogeneous transformation of $\vec{Q}_\mu$:

$$\delta\vec{Q}_\mu(x) = g\vec{Q}_\mu(x) \times \delta\vec{\omega}(x). \tag{2.5b}$$

Notice that $\mathcal{L}(\vec{A}_\mu + \vec{Q}_\mu)$, as well as the Faddeev-Popov term, $\det D_\mu^{ij}(A) D_{jk}^\mu(A+Q)$, is invariant under the type I transformation. Thus we assign the ghosts to the adjoint representation of the type I group:

$$\delta C^i = g f_{ijk} C^j \delta\omega^k, \qquad \delta\overline{C}^i = g f_{ijk} \overline{C}^j \delta\omega^k. \tag{2.6}$$

One must also notice that for vanishing sources of quantized fields, $\vec{\eta}_\mu = \xi_i = \overline{\xi}_i = 0$, the change in $Z(\vec{A}_\mu, 0, 0, 0)$ due to a small variation $\delta\vec{A}_\mu$ of the background field reads

$$\delta Z(\vec{A}_\mu, 0,0,0) = \int \left[\frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j D_\nu^{ik}(A+Q)\delta A_\nu^k + f_{ikl}\overline{C}^k \delta A_\mu^l D_\mu^{ij}(A+Q) C_j\right] \times \exp(iS)\, dQ\, dC\, d\overline{C} \tag{2.7}$$

after a translation of the variable $\vec{Q}_\mu$: $\vec{Q}_\mu' = \vec{Q}_\mu - \delta\vec{A}_\mu$. In this formula S denotes the action:

$$S = \int d^4x \left\{\mathcal{L}(A+Q) - \frac{1}{2\alpha}[D_\mu^{ij}(A)Q_j^\mu]^2 + \overline{C}_i D_\mu^{ij}(A) D_\mu^{jk}(A+Q) C_k\right\}.$$

By the gauge transformation on $\vec{Q}_\mu$ (Ref. 11),

$$\delta Q_\mu^i(x) = \int d^4y\, D_{\mu x}^{ij}(A+Q) M^{-1}{}_{jk}(x,y) \times D_{\nu y}^{kl}(A+Q)\delta A_\nu^l(y), \tag{2.8}$$

where M denotes

$$M_{ij}(x,y) = D_{\mu x}^{ik}(A) D_{\mu x}^{kj}(A+Q)\delta(x-y), \tag{2.9}$$

one shows that $\delta Z/\delta\vec{A}_\mu = 0$ in the absence of sources $\eta_\mu, \xi, \overline{\xi}$. This property turns out to be essential for the study of the counterterms associated with $\mathcal{L}(\vec{A}^{\rm cl})$.

Let us remark that for $\vec{A}_\mu = 0$ the functional Z reduces to the usual generating functional in Fermi-type gauges. Much information concerning renormalization can be obtained from this simple fact and from the power counting of $\vec{A}_\mu$. $\vec{A}_\mu$ appears as a source term for various composite operators constructed from $\vec{Q}_\mu$; for power counting, $\vec{A}_\mu$ is of dimension one. Thus, as for an operator $j\Phi^3$ in $\lambda\Phi^4$ theory, the counterterms will be of degree less than or equal to 4 in $\vec{A}_\mu$. Finally, we emphasize that $\vec{A}_\mu$ and $\vec{\eta}_\mu$ are independent variables and that the source for $\vec{Q}_\mu$ is $\vec{\eta}_\mu$.

B. Derivation of the Slavnov identities

We introduce the generating functional $Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \overline{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu)$:

$$Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \overline{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) = \int dQ\, dC\, d\overline{C} \exp\left(i\int d^4x \left\{\mathcal{L}(A+Q) - \frac{1}{2\alpha}[D_\mu^{ij}(A)Q_j^\mu]^2 + \overline{C}_i D_\mu^{ij}(A) D_\mu^{jk}(A+Q) C_k + \vec{\eta}_\mu \cdot \vec{Q}^\mu + \overline{\xi}_i C_i + \overline{C}_i \xi_i + J_\mu^i D_{ij}^\mu(A+Q) C^j + \tfrac{1}{2} g \vec{K} \cdot \vec{C} \times \vec{C} + L_\mu^i [D_\mu^{ij}(A+Q)\overline{C}_j - J_\mu^i]\right\}\right). \tag{2.10}$$

The use of sources $\vec{J}_\mu$ and $\vec{K}$ is explained in Ref. 8: they allow us to linearize the Slavnov identities relative to type II transformations. The anticommuting source $\vec{L}_\mu(x)$ serves as a generating device for the change of gauge due to a change $\delta\vec{A}_\mu$ of the background field $\vec{A}_\mu$. Namely, the gauge transformation of $\vec{Q}_\mu$ [Eq. (2.8)] is obtained by simultaneous insertions of the operators associated with $\vec{J}_\mu$ and $\vec{L}_\mu$:

$$\int dQ\,dC\,d\overline{C}\left[\int d^4y\, D_\mu^{ij}(A+Q)(x)M^{-1}{}_{jk}(xy)D_\nu^{kl}(A+Q)(y)\delta A_\nu^l(y)\right]\exp(iS)$$

$$=\int dQ\,dC\,d\overline{C}[D_\mu^{ij}(A+Q)C_j](x)\int d^4y\,\overline{C}_k(y)D_\nu^{kl}(A+Q)\delta A_\nu^l(y)\exp(iS)\,. \quad (2.11)$$

The functional $Z(\vec{A}_\mu,\vec{\eta}_\mu,\xi_i,\overline{\xi}_i,\vec{J}_\mu,\vec{K},\vec{L}_\mu)$ is clearly invariant under type I transformations [Eqs. (2.5), (2.6)], if all sources, except $\vec{A}_\mu$, belong to the adjoint representation

$$\left.\begin{aligned}&\delta\vec{\eta}_\mu(x)=g\vec{\eta}_\mu(x)\times\delta\vec{\omega}(x)\,,\\&\delta\vec{\xi}(x)=g\vec{\xi}(x)\times\delta\vec{\omega}(x)\,,\\&\cdots\\&\delta\vec{L}_\mu(x)=g\vec{L}_\mu(x)\times\delta\vec{\omega}(x)\end{aligned}\right\}\ \text{(type I)}\,. \quad (2.12)$$

We introduce the Legendre transform $\Gamma(\vec{A}_\mu,\vec{Q}_\mu,C_i,\overline{C}_i,\vec{J}_\mu,\vec{K},\vec{L}_\mu)$ of $W=i\ln Z$ with respect to $\vec{\eta}_\mu$, ξ_i, and $\overline{\xi}_i$, with the same conventions as in Ref. 8:

$$\Gamma+W+\int d^4x(\vec{\eta}_\mu\cdot\vec{Q}^\mu+\overline{\xi}_iC_i+\overline{C}_i\xi_i)=0\,. \quad (2.13)$$

Type I invariance of the 1PI (one-particle irreducible) functional is then expressed by the following Ward identity:

$$D_\mu^{ij}(A)\frac{\delta\Gamma}{\delta A_\mu^j}+gf^{ijk}\left(Q_\mu^j\frac{\delta\Gamma}{\delta Q_\mu^k}+C^j\frac{\delta\Gamma}{\delta C^k}+\overline{C}^j\frac{\delta\Gamma}{\delta\overline{C}^k}+J_\mu^j\frac{\delta\Gamma}{\delta J_\mu^k}+K^j\frac{\delta\Gamma}{\delta K^k}+L_\mu^j\frac{\delta\Gamma}{\delta L_\mu^k}\right)=0\,. \quad (2.14)$$

Under the following change of variables, which is in fact the combination of an infinitesimal type II gauge transformation, associated with an anticommuting parameter,[7,8,10] and of a variation of the background field, the change of the functional Z arises only through the source terms $\vec{\eta}_\mu\cdot\vec{Q}^\mu$, $\overline{\xi}_iC_i$, and $\overline{C}_i\xi_i$:

$$\begin{aligned}&\delta A_\mu^i(x)=-L_\mu^i(x)\delta\lambda\,,\\&\delta Q_\mu^i(x)=D_\mu^{ij}(A+Q)C_j(x)\delta\lambda+L_\mu^i(x)\delta\lambda\,,\\&\delta\vec{C}(x)=\frac{g}{2}\vec{C}\times\vec{C}\delta\lambda\,,\\&\delta\overline{C}_i(x)=-\frac{1}{\alpha}D_\mu^{ij}(A)Q_j^\mu(x)\delta\lambda\,.\end{aligned} \quad (2.15)$$

This leads to the identity for Z,

$$\int d^4x\left[\eta_\mu^i\frac{\delta}{i\delta J_\mu^i}+\overline{\xi}_i\frac{\delta}{i\delta K_i}+\frac{1}{\alpha}\xi_iD_\mu^{ij}(A)\frac{\delta}{i\delta\eta_\mu^j}+L_\mu^i\frac{\delta}{i\delta A_\mu^i}\right]Z=0\,. \quad (2.16)$$

This identity is integrated over space because the parameter $\delta\lambda$ of the transformation is x-independent. As in Ref. 8, we also need the equation of motion of the ghost field obtained by a change $\delta\overline{C}_i$:

$$\left[D_\mu^{ij}(A)\frac{\delta}{i\delta J_\mu^j(x)}+\xi^i(x)+gf^{ijk}\frac{\delta}{i\delta\eta_\mu^j(x)}L_\mu^k(x)\right]Z=0\,. \quad (2.17)$$

The Slavnov identity and the ghost equation of motion for Γ read

$$\int d^4x\left[\frac{\delta\Gamma}{\delta Q_\mu^i}\frac{\delta\Gamma}{\delta J_\mu^i}-\frac{\delta\Gamma}{\delta C^i}\frac{\delta\Gamma}{\delta K^i}+\frac{1}{\alpha}\frac{\delta\Gamma}{\delta\overline{C}^i}D_\mu^{ij}(A)Q_j^\mu-L_\mu^i\frac{\delta\Gamma}{\delta A_\mu^i}\right]=0\,, \quad (2.18)$$

$$D_\mu^{ij}(A)\frac{\delta\Gamma}{\delta J_\mu^j}-\frac{\delta\Gamma}{\delta\overline{C}_i}+gf^{ijk}Q_\mu^jL_\mu^k=0\,. \quad (2.19)$$

Equation (2.18) can be simplified further by introducing

$$\hat{\Gamma}=\Gamma+\int d^4x\frac{1}{\alpha}[D_\mu^{ij}(A)Q_j^\mu]^2\,. \quad (2.20)$$

Then $\hat{\Gamma}$ satisfies Eq. (2.19) and the identity

$$\int d^4x\left(\frac{\delta\hat{\Gamma}}{\delta Q_\mu^i}\frac{\delta\hat{\Gamma}}{\delta J_\mu^i}-\frac{\delta\hat{\Gamma}}{\delta C^i}\frac{\delta\hat{\Gamma}}{\delta K^i}-L_\mu^i\frac{\delta\hat{\Gamma}}{\delta A_\mu^i}\right)=0\,. \quad (2.21)$$

Reciprocally, if we start with a functional Γ satisfying Eqs. (2.14), (2.18), and (2.19), we can show immediately that the associated action S,

$$\exp[iS(\vec{A}_\mu, \vec{Q}_\mu, C_i, \overline{C}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu)] = \exp\left\{i\hat{S} + \frac{i}{2\alpha}\int [D_\mu^{ij}(A)Q_j^\mu]^2 d^4x\right\}$$

$$= \int d\eta_\mu d\xi\, d\bar{\xi}\, \exp\left\{-i\left[W(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) + \int d^4x(\vec{\eta}_\mu\cdot\vec{Q}^\mu + \bar{\xi}_i C_i + \overline{C}_i\xi_i)\right]\right\}, \tag{2.22}$$

satisfies corresponding identities:

$$D_\mu^{ij}(A)\frac{\delta\hat{S}}{\delta A_\mu^j} + gf^{ijk}\left(Q_\mu^j\frac{\delta\hat{S}}{\delta Q_\mu^k} + C^j\frac{\delta\hat{S}}{\delta C^k} + \overline{C}^j\frac{\delta\hat{S}}{\delta\overline{C}^k} + J_\mu^j\frac{\delta\hat{S}}{\delta J_\mu^k} + K^j\frac{\delta\hat{S}}{\delta K^k} + L_\mu^j\frac{\delta\hat{S}}{\delta L_\mu^k}\right) = 0, \tag{2.23a}$$

$$D_\mu^{ij}(A)\frac{\delta\hat{S}}{\delta J_\mu^j} - \frac{\delta\hat{S}}{\delta\overline{C}^i} + gf^{ijk}Q_\mu^j L_\mu^k = 0, \tag{2.23b}$$

$$\int d^4x\left(\frac{\delta\hat{S}}{\delta Q_\mu^i}\frac{\delta\hat{S}}{\delta J_\mu^i} - \frac{\delta\hat{S}}{\delta C^i}\frac{\delta\hat{S}}{\delta K^i} - L_\mu^i\frac{\delta\hat{S}}{\delta A_\mu^i}\right) = 0. \tag{2.23c}$$

C. Slavnov identities for renormalized Green's functions

We recall that we use the gauge-invariant regularization of Ref. 9 (or eventually that of Ref. 12); therefore the identities (2.14), (2.19), and (2.21) are valid for the bare regularized functional Γ^b. The recursive proof on the number of loops starts with the renormalized functional $\Gamma^{R[n](k)}$, for Green's functions associated with graphs containing k loops, which is supposed to be finite for $k \leq n$ and which is assumed to verify identities (2.14), (2.19), and (2.21) to all orders. The first step of our renormalization procedure[8] consists of subtracting the pole part at $d=4$, denoted by $\Gamma_{\text{div}}^{R[n](n+1)}$ arising from graphs with $(n+1)$ loops; this may be summarized by the formula

$$\Gamma^{R[n+1](k)} = \Gamma^{R[n](k)} - \Gamma_{\text{div}}^{R[n](k)}, \tag{2.24}$$

with the convention $\Gamma_{\text{div}}^{R[n](k)} = 0$ for $k < n+1$.

The linear identities (2.14) and (2.19) are automatically preserved by this renormalization procedure. For identity (2.18), introducing

$$\hat{\Gamma}^{R[n]} = \Gamma^{R[n]} + \frac{1}{2\alpha}\int d^4x[D_\mu^{ij}(A)Q_j^\mu]^2,$$

we obtain the identity

$$\int d^4x\left[\frac{\delta\hat{\Gamma}_{\text{div}}^{R[n](n+1)}}{\delta Q_\mu^i(x)}\frac{\delta\hat{S}}{\delta J_\mu^i(x)} + \frac{\delta\hat{S}}{\delta Q_\mu^i(x)}\frac{\delta\hat{\Gamma}_{\text{div}}^{R[n](n+1)}}{\delta J_\mu^i(x)} - \frac{\delta\hat{\Gamma}_{\text{div}}^{R[n](n+1)}}{\delta C^i(x)}\frac{\delta\hat{S}}{\delta K^i(x)} - \frac{\delta\hat{S}}{\delta C^i(x)}\frac{\delta\hat{\Gamma}_{\text{div}}^{R[n](n+1)}}{\delta K^i(x)} - L_\mu^i(x)\frac{\delta\hat{\Gamma}_{\text{div}}^{R[n](n+1)}}{\delta A_\mu^i(x)}\right] = 0,$$

where $\hat{S}$ denotes the bare transverse action $\hat{S} = \hat{\Gamma}^{R[n](0)}$. Thus the functional $\hat{\Gamma}^{R[n+1]}$ satisfies identity (2.21) up to order $(n+1)$, and the corresponding equation for $\hat{S}^{R[n+1]}$, Eq. (2.23), also holds to the same order.

The second step of the proof is the adjustment of the counterterms of $\hat{S}^{R[n+1]}$ at order $k \geq n+2$ in a way which ensures the validity of identity (2.21) for $\hat{\Gamma}^{R[n+1]}$ at all orders. This program is performed in Sec. IID by imposing the validity of Eqs. (2.23) for $\hat{S}^{R[n+1]}$ at all orders.

D. Solution of the Slavnov identities for the action

To simplify notations, we delete the superscripts $[n+1]$, which play no role in the argument, and we are now about to derive the constraints on the structure of the counterterms of S^R implied by Eqs. (2.23). The subsequent discussion follows exactly the analysis performed by Zinn-Justin[10] for Fermi-type gauges. By power counting and ghost-number conservation, the counterterms in $\vec{J}_\mu$ can only take the forms $\vec{J}_\mu\cdot\partial^\mu\vec{C}$, $\vec{J}_\mu\cdot\vec{Q}^\mu\times\vec{C}$, $\vec{J}_\mu\cdot\vec{A}^\mu\times\vec{C}$, $\vec{J}_\mu\cdot\vec{L}^\mu$, and those in $\vec{K}$ can only take the form $\vec{K}\cdot\vec{C}\times\vec{C}$. Owing to invariance of S^R with respect to type I transformations, these counterterms must be type-I–invariant:

$$\frac{\delta\hat{S}^R}{\delta J_\mu^i} = \tilde{Z}_3 D_\mu^{ij}(A)C_j + Y_1 g(\vec{Q}_\mu\times\vec{C})_i + Y_2 L_\mu^i, \tag{2.25}$$

$$\frac{\delta\hat{S}^R}{\delta K^i} = \frac{g}{2}\tilde{Z}_1 f_{ijk}C^jC^k. \tag{2.26}$$

The counterterms in $\vec{J}_\mu$ and $\vec{K}$ being linear in these sources, we may write

$$\hat{S}^R = \int\left[J_\mu^i\frac{\delta\hat{S}^R}{\delta J_\mu^i} + K^i\frac{\delta\hat{S}^R}{\delta K^i} + \hat{S}'(\vec{A}_\mu, \vec{Q}_\mu, C_i, \overline{C}_i, \vec{L}_\mu)\right]. \tag{2.27}$$

The dependence of $\hat{S}^R$ on $\bar{C}$ is given by the equation of motion (2.23b):

$$\hat{S}^R = \int [J^i_\mu + \bar{C}^j D^{ji}_\mu(\vec{A})] \frac{\delta \hat{S}^R}{\delta J^i_\mu} + K^i \frac{\delta \hat{S}^R}{\delta K^i} + g\bar{C}_i(\vec{Q}_\mu \times \vec{L}_\mu)_i + \Sigma(\vec{A}_\mu, \vec{Q}_\mu) . \quad (2.28)$$

In fact Σ cannot depend on C_i or on $\vec{L}_\mu$ due to power counting and ghost-number conservation. Inserting the expression for $\hat{S}^R$ in the Slavnov identity (2.23c), we obtain the following system:

$$\int dy \frac{\delta^2 \hat{S}^R}{\delta C^k(y)\delta K^i(x)} \frac{\delta \hat{S}^R}{\delta K^k(y)} = 0 , \quad (2.29a)$$

$$\int dy \left[\frac{\delta^2 \hat{S}^R}{\delta Q^k_\nu(y)\delta J^i_\mu(x)} \frac{\delta \hat{S}^R}{\delta J^k_\nu(y)} + \frac{\delta^2 \hat{S}^R}{\delta C^k(y)\delta J^i_\mu(x)} \frac{\delta \hat{S}^R}{\delta K^k(y)} - L^k_\nu(y) \frac{\delta^2 \hat{S}^R}{\delta A^k_\nu(y)\delta J^i_\mu(x)} \right] = 0 , \quad (2.29b)$$

$$\int dy \left[\frac{\delta \Sigma}{\delta Q^k_\nu(y)} \frac{\delta \hat{S}^R}{\delta J^k_\nu(y)} - L^k_\nu(y) \frac{\delta \Sigma}{\delta A^k_\nu(y)} \right] = 0 . \quad (2.29c)$$

Equation (2.29a) is trivially satisfied by $\delta \hat{S}^R/\delta K$, since it is the Jacobi identity for the structure constants f_{ijk}. Similarly, Eq. (2.29b) gives

$$Y_1 = \tilde{Z}_1 \text{ and } Y_2 = \frac{\tilde{Z}_3}{\tilde{Z}_1} . \quad (2.30)$$

The identity (2.29c) for Σ yields two equations:

$$\frac{\delta \Sigma}{\delta A^i_\mu(x)} - \frac{\tilde{Z}_3}{\tilde{Z}_1} \frac{\delta \Sigma}{\delta Q^i_\mu(x)} = 0 , \quad (2.31)$$

$$\int dy \frac{\delta \Sigma}{\delta Q^i_\mu(y)} \left(D^{ij}_\mu(A) + g\frac{\tilde{Z}_1}{\tilde{Z}_3} f_{ikj} Q^k_\mu \right) C^j . \quad (2.32)$$

These equations imply that Σ is a function of the variable $\vec{A}_\mu + (\tilde{Z}_1/\tilde{Z}_3)\vec{Q}_\mu$, invariant under the gauge transformation

$$\delta Q^i_\mu = D^{ij}_\mu \left(\vec{A}_\mu + \frac{\tilde{Z}_1}{\tilde{Z}_3} \vec{Q}_\mu \right) \delta\omega^j . \quad (2.33)$$

Thus Σ takes the form

$$\Sigma = Z_3 \left(\frac{\tilde{Z}_3}{\tilde{Z}_1} \right)^2 \mathcal{L} \left(\vec{A}_\mu + \frac{\tilde{Z}_1}{\tilde{Z}_3} \vec{Q}_\mu , g \right) , \quad (2.34)$$

which can also be cast as

$$\Sigma = \mathcal{L} \left(Z_3^{1/2} \vec{Q}_\mu + \frac{g}{g_0} \vec{A}_\mu , g_0 \right) , \quad (2.35)$$

where g_0 denotes the bare coupling constant

$$g_0 = g \frac{\tilde{Z}_1}{Z_3^{1/2} \tilde{Z}_3} . \quad (2.36)$$

In conclusion, the renormalized action takes the expression

$$S^R = \mathcal{L} \left(Z_3^{1/2} \vec{Q}_\mu + \frac{g}{g_0} \vec{A}_\mu , g_0 \right) - \frac{1}{2\alpha} [D^{ij}_\mu(A) Q^\mu_j]^2 + g\bar{C}_i(\vec{Q}_\mu \times \vec{L}_\mu)_i + \frac{g}{2} \tilde{Z}_1 K_i (\vec{C} \times \vec{C})^i$$
$$+ [J^i_\mu + \bar{C}^j D^{ji}_\mu(A)] \left[\tilde{Z}_3 D^{ik}_\mu(A) C^k + \frac{\tilde{Z}_3}{\tilde{Z}_1} L^i_\mu + g\tilde{Z}_1 (\vec{Q}_\mu \times \vec{C})^i \right] . \quad (2.37)$$

After a wave-function renormalization of the quantized fields,

$$\vec{Q}'_\mu = Z_3^{1/2} \vec{Q}_\mu , \quad \bar{C}'_i = \tilde{Z}_3^{1/2} \bar{C}_i , \, C'_i = \tilde{Z}_3^{1/2} C_i , \quad (2.38)$$

and the renormalization of parameters g and α,

$$g_0 = g \frac{\tilde{Z}_1}{\tilde{Z}_3 Z_3^{1/2}} , \quad \alpha_0 = \alpha Z_3 , \quad (2.39)$$

the renormalization boils down to a multiplicative renormalization of all other sources:

$$\vec{J}'_\mu = \tilde{Z}_3^{1/2} \vec{J}_\mu , \quad \vec{L}'_\mu = \vec{L}_\mu \frac{\tilde{Z}_3^{1/2}}{\tilde{Z}_1} , \quad \vec{K}' = Z_3^{1/2} \vec{K} , \quad \vec{A}'_\mu = \frac{g}{g_0} \vec{A}_\mu . \quad (2.40)$$

The action as a function of these new variables takes on the form of the bare action, and the renormalized action is clearly invariant under type I and type II transformations with parameter g_0.

This discussion can be readily extended to the case where matter fields are introduced.[8,10]

E. Dependence of the Green's functions on the gauge parameter α

To study the variation of the Green's functions with respect to the gauge parameter α, we introduce as in Ref. 8 a new source L for the insertion of the operator $\int d^4x \bar{C}_i(x) D^{ij}_\mu(A) Q^\mu_j$. Let us consider the generating functional

$$Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu, L) = \int dQ\, dC\, d\bar{C} \exp[i(S + \eta^\mu_i Q^i_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i)] , \quad (2.41)$$

where the bare action S now denotes

$$S=\int d^4x\Big\{\mathcal{L}(A+Q)-\frac{1}{2\alpha}[D_\mu^{ij}(A)Q_j^\mu]^2+\overline{C}^i D_\mu^{ij}(A)D_{jk}^\mu(A+Q)C_k+\frac{g}{2}\vec{K}\cdot(\vec{C}\times\vec{C})$$
$$+J_\mu^i D_{ij}^\mu(A+Q)C^j+L_\mu^i[D_{ij}^\mu(A+Q)\overline{C}_j-J_i^\mu]+L[\overline{C}_i D_\mu^{ij}(A)Q_j^\mu+J_\mu^i Q_i^\mu+a\vec{K}\cdot\vec{C}]\Big\}. \quad (2.42)$$

We recall[8] that L is an x-independent anticommuting source and possesses no group index; thus it satisfies $L^2=0$. As we discuss below, the operators $LJ_\mu^i Q_i^\mu$ and $L\vec{K}\cdot\vec{C}$ are needed as counterterms of the operator $\overline{C}_i D_\mu^{ij}(A)Q_j^\mu$.

As one can see by mere computation, the action S in Eq. (2.42) is invariant:

(a) under type I transformations [Eqs. (2.5), (2.6), and (2.12)];

(b) under the following change of variables which is a combination of a type II transformation, of a change of gauge δA_μ, and of a change of gauge parameter α:

$$\delta\alpha=2\alpha L\delta\lambda\,,\quad \delta A_\mu^i=-L_\mu^i\delta\lambda\,,\quad \delta Q_\mu^i=D_\mu^{ij}(A+Q)C_j\delta\lambda+L_\mu^i\delta\lambda\,,$$
$$\delta\vec{C}=\frac{g}{2}\vec{C}\times\vec{C}\delta\lambda\,,\quad \delta\overline{C}_i=-\frac{1}{\alpha}[D_\mu^{ij}(A)Q_j^\mu]\delta\lambda-\overline{C}_i L\delta\lambda\,, \quad (2.43)$$
$$\delta J_\mu^i=-J_\mu^i L\delta\lambda\,,\quad \delta K_i=-aK_i L\delta\lambda\,,\quad \delta L_\mu^i=\delta L=0\,.$$

The 1PI functional still obeys the type I Ward identity (2.14). The invariance under the transformation (2.43) is expressed by the following Ward-Slavnov identity:

$$L\Big\{2\alpha\frac{\partial\Gamma}{\partial\alpha}+\int d^4x\Big[Q_\mu^i\frac{\delta\Gamma}{\delta Q_\mu^i}-J_\mu^i\frac{\delta\Gamma}{\delta J_\mu^i}-\overline{C}^i\frac{\delta\Gamma}{\delta\overline{C}^i}+a\Big(C^i\frac{\delta\Gamma}{\delta C^i}-K^i\frac{\delta\Gamma}{\delta K^i}\Big)\Big]\Big\}$$
$$+\int d^4x\Big[\frac{\delta\Gamma}{\delta Q_\mu^i}\frac{\delta\Gamma}{\delta J_\mu^i}-\frac{\delta\Gamma}{\delta C^i}\frac{\delta\Gamma}{\delta K^i}+\frac{1}{\alpha}\frac{\delta\Gamma}{\delta\overline{C}^i}D_\mu^{ij}(A)Q_j^\mu-L_\mu^i\frac{\delta\Gamma}{\delta A_\mu^i}\Big]=0\,. \quad (2.44)$$

We make use of the ghost equation of motion (2.19) which remains unchanged in the presence of the source L. The type II Slavnov identity (2.44) for the functional $\hat{\Gamma}$ defined in Eq. (2.20) then takes the expression

$$L\Big\{2\alpha\frac{\partial\hat\Gamma}{\partial\alpha}+\int d^4x\Big[Q_\mu^i\frac{\delta\hat\Gamma}{\delta Q_\mu^i}-J_\mu^i\frac{\delta\hat\Gamma}{\delta J_\mu^i}-\overline{C}^i\frac{\delta\hat\Gamma}{\delta\overline{C}^i}+a\Big(C^i\frac{\delta\hat\Gamma}{\delta C^i}-K^i\frac{\delta\hat\Gamma}{\delta K^i}\Big)\Big]\Big\}+\int d^4x\Big[\frac{\delta\hat\Gamma}{\delta Q_\mu^i}\frac{\delta\hat\Gamma}{\delta J_\mu^i}-\frac{\delta\hat\Gamma}{\delta C^i}\frac{\delta\hat\Gamma}{\delta K^i}-L_\mu^i\frac{\delta\hat\Gamma}{\delta A_\mu^i}\Big]=0\,. \quad (2.45)$$

The minimal renormalization procedure preserves the identities (2.14) and (2.45) and the ghost equation of motion (2.19). The renormalized action $\hat{S}^R$ satisfies thus the identities (2.23a), (2.23b), and (2.45) ($\hat\Gamma\to\hat{S}^R$). Owing to type I invariance and to ghost number conservation, the counterterms linear in L are of the form

$$\hat{S}^R=\hat{S}^R(L=0)+L[Z_{11}\overline{C}_i D_\mu^{ij}(A)Q_j^\mu+Z'_{11}J_\mu^i Q_i^\mu+Z_{12}K^iC^i]\,.$$

The ghost equation of motion (2.23b) implies $Z'_{11}=Z_{11}$, and by power counting the operator $L\vec{K}\cdot\vec{C}$ is multiplicatively renormalizable. Consequently a can be chosen to ensure L to be multiplicatively renormalizable:

$$\hat{S}^R=\hat{S}^R(L=0)+LZ_3^{1/2}\tilde{Z}_3^{1/2}Z_{11}[\overline{C}_i D_\mu^{ij}(A)Q_j^\mu+J_\mu^i Q_i^\mu+a(g_0,\alpha_0)K^iC^i]\,. \quad (2.46)$$

The insertion of this expression in identity (2.45) immediately shows

$$\alpha\frac{\partial}{\partial\alpha}g_0=0\,, \quad (2.47)$$

and relates the coupling matrix $\{Z_{ij}\}$ of the operators $[\overline{C}_i D_\mu^{ij}(A)Q_j^\mu+J_\mu^i Q_i^\mu]$ and $\vec{K}\cdot\vec{C}$ to the variation with respect to α of Z_3 and $\tilde{Z}_3$, the wave-function renormalizations of the fields $\vec{Q}_\mu$ and C_i.

The result (2.47), already derived in Ref. 8, implies that the counterterms in $\hat{S}^R$ depending only on the field $\vec{A}_\mu$, namely $(g/g_0)^2\mathcal{L}(\vec{A}_\mu)$, are α-independent. From identity (2.45) for $\hat{S}^R$ it is evident that the α dependence of the counterterms $\mathcal{L}(\vec{A}_\mu)$ can arise only through a counterterm $L\vec{J}_\mu\cdot\vec{A}^\mu$ which is not type-I–invariant. However, one can construct many nonlocal functions linear in L and $\vec{J}_\mu$ which possess this invariance, as for example

$$\int dx\,dy\,LJ_\mu^i(x)[D_\rho(A)D^\rho(A)]^{-1}{}_{ij}(x,y)D_{\nu y}^{jk}(A)\times(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+g\vec{A}_\mu\times\vec{A}_\nu)_k(y)\,. \quad (2.48)$$

Thus the function $\delta^3\hat{\Gamma}/\delta A^j_\nu \delta J^i_\mu \delta L$ is finite but nonvanishing, and its contribution to the following identity derived from Eq. (2.45) implies that the finite part of the Green's functions with external A legs only depends on the gauge parameter α:

$$2\alpha \frac{\partial \hat{\Gamma}}{\partial \alpha}(A) = \int d^4x \frac{\delta \hat{\Gamma}}{\delta Q^i_\mu}(A) \frac{\delta^2 \hat{\Gamma}}{\delta J^i_\mu \delta L}(A).$$

Indeed an explicit one-loop computation shows that $\delta^3\hat{\Gamma}/\delta L \delta J^i_\mu \delta A^j_\nu$ does not vanish.

The dependence on α of the Green's functions with external A fields only has a practical implication for the computation of g/g_0. To compute the latter to order $2n$, one must know to order $(2n-2)$ both g/g_0 and the renormalized two-point $\vec{A}^{cl}_\mu$ function $\Gamma^{(2)}_R$. $(g/g_0)^{[n]}$ receives contributions from the primitive divergences of $\Gamma^{(2)}_b$ at order n and contributions from the divergences at order n of the renormalized function $\Gamma^{(2)[n-1]}_R$. This method seems to avoid the computation of non-gauge-invariant counterterms such as Z_3, $\tilde{Z}_3$, and Z_1. However, the renormalized function $\Gamma^{(2)}$, needed to order $(2n-2)$, is a finite α-dependent function and thus the renormaliza ion of α, namely Z_3, is also needed:

$$\Gamma^{(2)[n-1]}_R = \left(\frac{g}{g_0}\right)^{[n-1]} \Gamma^{(2)[n-1]}_b(g_0, \alpha_0),$$
$$\alpha_0^{[n-1]} = Z_3^{[n-1]}\alpha . \tag{2.49}$$

The contribution of $\alpha_0^{[n-1]}$ to the divergence at order n of $\Gamma^{(2)[n-1]}_R$ vanishes only in the Landau gauge $\alpha = 0$.

III. CONCLUSION

It was shown that the renormalized generating functional in the background-field gauge is invariant under type I and type II transformations, confirming or generalizing thereby the conclusions of Refs. 1–5. The interest of this gauge seems obvious: For the computation of g/g_0 to order n, it seems possible to avoid the calculation of non-gauge-invariant counterterms. However, we have shown that it is true only in the Landau gauge.

It seems that this "background field" approach can be applied usefully to the study of the renormalization of gauge-invariant operators, as is shown by a one-loop computation for operators of twist-two performed by Crewther[6] and by an explicit verification to all orders on our favorite operator $\vec{F}_{\mu\nu}{}^2$.[8] This will be the topic of a forthcoming paper.

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