Ward identities and some clues to the renormalization of gauge-invariant operators

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The problem of the renormalization of gauge-invariant operators in the non-Abelian Yang-Mills theory is tackled through the study of a specific example, $\vec{F}_{\mu\nu}{}^2$, for which the explicit solution can be derived from renormalization-group considerations. It is shown that the operator $\vec{F}_{\mu\nu}{}^2$ mixes with non-gauge-invariant operators and that this mixing must be taken into account for the computation of the anomalous dimension of the renormalized gauge-invariant operator. The explicit solution is examined with the help of Ward identities derived from a new type of gauge transformations which appear very convenient from a technical point of view. The multiplicatively renormalizable gauge-invariant operator is shown to satisfy Ward identities and to possess an α-independent anomalous dimension. As a by-product, we analyze the gauge dependence of the Callan-Symanzik function β.

I. INTRODUCTION

Since the first observation of scaling in electroproduction, much effort has been devoted to understanding this phenomenon in the framework of field theory, which was the indication of a vanishing strong interaction in the deep Euclidean region. A major success in this respect was the discovery[1] that gauge theories, at the exclusion of other Lagrangian theories, exhibit "asymptotic freedom" in the absence of scalar fields; in gauge theories for strong interactions, the effective coupling constant vanishes logarithmically in the deep Euclidean region. For leptoproduction, the moments of structure functions depend logarithmically on the square of the momentum transfer from incident to outgoing lepton.[2] The success of the quark-parton model as a qualitative description of deep-inelastic experiments has made the idea of quarks as elementary fields quite attractive, in spite of the lack of experimental observation of quark particles; Weinberg[3] has suggested the possibility of an unbroken gauge symmetry which would prohibit the existence of physical states corresponding to quantum numbers of the quarks and of the gauge fields. However, for such a theory, the absence of an S matrix, at least in the conventional sense, obscures our understanding of gauge invariance. In fact, results[4] concerning gauge invariance were derived for S-matrix elements in spontaneously broken theories. The latter are shown to be independent of the gauge parameter α which is introduced through the gauge-fixing term $-(1/2\alpha)(\partial_\mu \vec{A}^\mu)^2$ necessary for quantization. For quantum electrodynamics, probability amplitudes evaluated at a given order of perturbation theory are themselves α-independent; furthermore, it was conjectured and shown on some examples that any operator invariant under a gauge transformation leaves the physical subspace invariant and for suitable normalization possesses α-independent matrix elements between physical states.[5] No properties of this sort are established for non-Abelian unbroken gauge theories.

The study of asymptotic properties in leptoproduction requires two tools: Wilson's expansion of the product of two currents, weak or electromagnetic, which are invariant under transformations of the strong gauge group, and the renormalization group which gives the behavior of the coefficients occurring in this expansion in the deep Euclidean region ($q^2 < 0$ and large). We do not deal with the important problem of the possible occurrence of non-gauge-invariant operators in the Wilson expansion. Instead we have started to treat the questions of gauge invariance connected with the renormalization group which were initially pointed out by Gross and Wilczek[6] in the context of a one-loop computation of the high-energy behavior in leptoproduction[6,7]: the renormalization of a bare operator invariant under gauge transformations, the dependence of the Callan-Symanzik function β on the gauge parameter α, and the study of the Ward identities satisfied by a renormalized gauge-invariant operator. As an example we have picked the operator $\vec{F}_{\mu\nu}{}^2$ at zero momentum; the renormalization of this operator, which does not give a dominant contribution to the light cone, is simply related to the counterterms of the Lagrangian. In Sec. II we prove that this operator is coupled by renormalization to non-gauge-invariant operators; from the renormalization-group equations for ordinary Green's functions we derive the expression of the multiplicatively renormalizable operator associated with an α-independent anomalous dimension. The study of the properties of this operator is postponed until we introduce Ward identities. Section III is devoted to the study of a new kind of Ward-Slavnov identities derived from a supergauge transformation introduced by Becchi, Rouet, and Stora.[8] These Ward identities, more tractable, are equivalent to the conventional ones; the derivation of the usual relations between counterterms of the Lagrangian is carried out in Appendix A, and the introduction of matter fields is briefly discussed in Appendix B. With the help of different Ward identities, we examine in Sec. IV the dependence of the β function on the gauge parameter α which is determined by the detailed renormalization prescription and for which the infrared divergences due to the absence of a mass scale are crucial. For a pure Yang-Mills theory we derive a result first established by Caswell and Wilczek[9] under specific assumptions: The function β can be chosen to be α-independent by proper renormalization conditions. Finally in Sec. V we show that the renormalized operator derived in Sec. II and corresponding to an α-independent anomalous dimension satisfies Ward identities analogous to those of the gauge-invariant bare operator $\vec{F}_{\mu\nu}{}^2$.

II. RENORMALIZATION OF A GAUGE-INVARIANT OPERATOR

We call a formally gauge-invariant operator a bare operator which is invariant under conventional gauge transformations of the vector-meson or fermion field. The renormalization of these operators raises two questions:

Are the counterterms also formally gauge-invariant?

If not, is it legitimate to neglect the coupling to formally non-gauge-invariant operators when one computes anomalous dimensions of gauge-invariant operators?

The answer to both questions is negative. An explicit counterexample is given by the study of the renormalization of operators of dimension four and spin zero, such as $\vec{F}_{\mu\nu}{}^2(x)$, in a pure Yang-Mills theory. We first define our renormalized operators and derive the renormalization matrix Z, the results at the one-loop level are then discussed, and finally, we derive to all orders in perturbation theory the expression of a multiplicatively renormalizable operator with an α-independent anomalous dimension.

A. Definition of renormalized operators

We define the generating functional for bare Green's functions,

$$Z(\vec{\eta}_\mu, \xi_i, \bar{\xi}_i) = \int dA\, dC\, d\bar{C} \times \exp\left[i \int d^4x (\mathcal{L} + \vec{\eta}_\mu \cdot \vec{A}^\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i)\right], \tag{2.1}$$

for the Yang-Mills Lagrangian, $\mathcal{L}$,

$$\mathcal{L} = -\tfrac{1}{4}\vec{F}_{\mu\nu}{}^2 - \frac{1}{2\alpha_0}(\partial_\mu \vec{A}^\mu)^2 + \bar{C}_i M_{ij} C_j, \tag{2.2}$$

where C_i and $\bar{C}_i$ are the well-known Faddeev-Popov ghosts[10] and where M_{ij} denotes

$$\begin{aligned} M_{ij}(x,y,\vec{A}_\nu) &= \partial_\mu D^\mu_{ij}\delta(x-y), \\ D^\mu_{ij}C_j &= (\partial^\mu \vec{C} + g_0 \vec{A}^\mu \times \vec{C})_i \\ &= \partial^\mu C_i + g_0 f_{ijk} A^\mu_j C_k, \\ \vec{F}_{\mu\nu} &= \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g \vec{A}_\mu \times \vec{A}_\nu. \end{aligned} \tag{2.3}$$

According to established results[11-14] the renormalized generating functional reads

$$\begin{aligned} Z^R(\vec{\eta}_\mu, \xi_i, \bar{\xi}_i) = \int dA\, dC\, d\bar{C} \exp\Big\{ i \int \Big[&-\tfrac{1}{4} Z_3 (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)^2 - \tfrac{1}{2} Z_1 g (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)\cdot(\vec{A}^\mu \times \vec{A}^\nu) \\ &- \tfrac{1}{4} Z_4 g^2 (\vec{A}_\mu \times \vec{A}_\nu)^2 - \frac{1}{2\alpha}(\partial_\mu \vec{A}^\mu)^2 + \tilde{Z}_3 \bar{C}_i \partial^2 C_i + \tilde{Z}_1 g \bar{C}_i \partial_\mu (\vec{A}^\mu \times \vec{C})_i \\ &+ \vec{\eta}_\mu \cdot \vec{A}^\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i \Big] \Big\}, \end{aligned} \tag{2.4}$$

where the counterterms satisfy the Ward identities (WI)

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3}, \quad \frac{Z_4}{Z_3} = \left(\frac{Z_1}{Z_3}\right)^2, \quad g_0 = g\frac{Z_1}{Z_3^{3/2}}, \quad \text{and } \alpha_0 = Z_3 \alpha. \tag{2.5}$$

Let us perform in the generating functional Z of (2.1) the following change of variables g_0, α_0, and $\vec{\eta}_\mu$:

$$\begin{aligned} Z(\vec{\eta}'_\mu, \xi_i, \bar{\xi}_i; g'_0, \alpha'_0) &= Z(\vec{\eta}_\mu(1-\tfrac{1}{2}\epsilon), \xi_i, \bar{\xi}_i; g_0(1-\tfrac{1}{2}\epsilon), \alpha_0(1+\epsilon)) \\ &= Z(\vec{\eta}_\mu, \xi_i, \bar{\xi}_i; g_0, \alpha_0) + \epsilon\left[\alpha_0 \frac{\partial}{\partial \alpha_0} - \frac{g_0}{2}\frac{\partial}{\partial g_0} - \frac{1}{2}\int d^4x\, \eta^i_\nu(x)\frac{\delta}{\delta \eta^i_\nu(x)}\right] Z(\vec{\eta}_\mu, \xi_i, \bar{\xi}_i; g_0, \alpha_0) + O(\epsilon^2). \end{aligned}$$

A further change of variable, $\vec{A}_\mu \to \vec{A}_\mu(1+\tfrac{1}{2}\epsilon)$, shows that the term linear in ϵ in the above expression is just the generating functional with one insertion of the operator $\vec{F}_{\mu\nu}{}^2$ at zero momentum:

$$Z(\vec{\eta}'_\mu, \xi_i, \bar{\xi}_i; g'_0, \alpha'_0) = \int dA\, dC\, d\bar{C} \exp\left\{ i \int d^4x \left[-\tfrac{1}{4}(1+\epsilon)\vec{F}_{\mu\nu}{}^2 - \frac{1}{2\alpha_0}(\partial_\mu \vec{A}^\mu)^2 + \bar{C}_i M_{ij} C_j + \bar{C}_i \xi_i + \bar{\xi}_i C_i + \vec{\eta}_\mu \cdot \vec{A}_\mu \right] \right\}. \tag{2.7}$$

Thus in the tree approximation ($\alpha_0 = \alpha$, $g_0 = g$) the insertion of the operator $O_1 = -\frac{1}{4}\int d^4x\, \vec{F}_{\mu\nu}{}^2(x)$ in a one-particle irreducible Green's function (1PI) $\Gamma^{(n,p)}$, with n external vector-meson legs and p ghost and antighost external legs, can be expressed as

$$\Gamma_{O_1}^{(n,p)} = \left(\alpha \frac{\partial}{\partial \alpha} - \frac{g}{2}\frac{\partial}{\partial g} + \frac{n}{2}\right)\Gamma^{(n,p)} \tag{2.8}$$

because the insertion of the operator $\int \vec{\eta}^i_\mu(\delta/\delta\vec{\eta}^i_\mu) d^4x$ in a graph just counts the number of external $\vec{A}_\mu$ legs. The renormalized operator O_1 may be defined according to Eq. (2.8), where $\Gamma^{(n,p)}$ and $\Gamma_{O_1}^{(n,p)}$ should be replaced by the renormalized functions $\Gamma^{R(n,p)}$ and $\Gamma_{O_1}^{R(n,p)}$, provided we impose on it the corresponding renormalization conditions[15]: for example,

$$I_{O_1}(q^2 = -\mu^2) = \left(\alpha\frac{\partial}{\partial\alpha} - \frac{g}{2}\frac{\partial}{\partial g} + 1\right) I(q^2 = -\mu^2), \tag{2.9}$$

where I_{O_1} and I denote

$$\Gamma^{R(2,0)}_{O_1\,\mu\nu} = -(q^2 g_{\mu\nu} - q_\mu q_\nu) I^R_{O_1}(q^2),$$
$$\Gamma^{R(2,0)}_{\mu\nu} = -(q^2 g_{\mu\nu} - q_\mu q_\nu) I^R(q^2) - \frac{q_\mu q_\nu}{\alpha}. \tag{2.10}$$

Similarly we define the renormalized operators O_2, O_3, and O_4 which in the tree approximation correspond respectively to the operators

$$\int d^4x g \frac{\delta \mathcal{L}(x)}{\delta g},$$

$$\int d^4x \bar{C}_i M_{ij} C_j(x),$$

and

$$\frac{1}{2\alpha}\int [\partial_\mu \vec{A}^\mu(x)]^2 d^4x,$$

according to

$$\Gamma^{R(n,p)}_{O_2} = g\frac{\partial}{\partial g}\Gamma^{R(n,p)},$$
$$\Gamma^{R(n,p)}_{O_3} = p\Gamma^{R(n,p)}, \tag{2.11}$$
and
$$\Gamma^{R(n,p)}_{O_4} = \alpha\frac{\partial}{\partial\alpha}\Gamma^{R(n,p)}.$$

As a consequence of the equations of motion of the ghost field, the insertion of O_3 in a graph simply counts the number of external ghost lines of the graph.[16,17] Operator O_4, which is not needed for the renormalization of $\vec{F}_{\mu\nu}{}^2$, is introduced for convenience. With these definitions (2.8) and (2.11) the counterterms for the insertion of this operator are easily related to those of the Lagrangian:

$$\frac{\partial}{\partial\epsilon}\Big[-\tfrac{1}{4}Z_3(g',\alpha')(1+\epsilon)(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)^2 - \tfrac{1}{2}g Z_1(g',\alpha')(1+\epsilon)(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)\cdot(\vec{A}^\mu\times\vec{A}^\nu)$$
$$-\tfrac{1}{4}g^2 Z_4(g',\alpha')(1+\epsilon)(\vec{A}_\mu\times\vec{A}_\nu)^2 + \tilde{Z}_3(g',\alpha')\bar{C}_i\partial^2 C_i + g\tilde{Z}_1(g',\alpha')\bar{C}_i\partial_\mu(\vec{A}^\mu\times\vec{C})_i\Big]$$

$$= \left[1 + \left(\alpha\frac{\partial}{\partial\alpha} - \frac{g}{2}\frac{\partial}{\partial g}\right)\ln Z_3\right] O_1^{\rm IR} + \left[\left(\alpha\frac{\partial}{\partial\alpha} - \frac{g}{2}\frac{\partial}{\partial g}\right)\ln\frac{Z_1}{Z_3}\right] O_2^{\rm IR} + \left[\left(\alpha\frac{\partial}{\partial\alpha} - \frac{g}{2}\frac{\partial}{\partial g}\right)\ln\tilde{Z}_3\right] O_3^{\rm IR}, \tag{2.12}$$

where an intermediate renormalization was performed for $O_i^{\rm IR}$,

$$O_1^{\rm IR} = -\tfrac{1}{4}Z_3(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)^2 - \tfrac{1}{2}gZ_1(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)\cdot(\vec{A}^\mu\times\vec{A}^\nu) - \tfrac{1}{4}g^2 Z_4(\vec{A}_\mu\times\vec{A}_\nu)^2,$$
$$O_2^{\rm IR} = -\tfrac{1}{2}gZ_1(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)\cdot(\vec{A}^\mu\times\vec{A}^\nu) - \tfrac{1}{2}g^2 Z_4(\vec{A}_\mu\times\vec{A}_\nu)^2 + g\tilde{Z}_1\bar{C}_i\partial_\mu(\vec{A}^\mu\times\vec{C})_i, \tag{2.13}$$
$$O_3^{\rm IR} = \tilde{Z}_3\bar{C}_i\partial^2 C_i + \tilde{Z}_1 g\bar{C}_i\partial_\mu(\vec{A}^\mu\times\vec{C})_i,$$

and where use has been made of WI (2.5). The expression of the matrix $\{Z_{ij}\}$,

$$\Gamma^{R(n,p)}_{O_i} = Z_3^{n/2}\tilde{Z}_3^{p} Z_{ij}\Gamma^{b(n,p)}_{O_j}, \quad i,j = 1,2,3 \tag{2.14}$$

is readily obtained:

$$\{Z_{ij}\}=\begin{bmatrix} 1+\left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\ln Z_3 & \left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\ln\frac{Z_1}{Z_3} & \left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\ln\tilde{Z}_3 \\ g\frac{\partial}{\partial g}\ln Z_3 & 1+g\frac{\partial}{\partial g}\ln\frac{Z_1}{Z_3} & g\frac{\partial}{\partial g}\ln\tilde{Z}_3 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.15)$$

We are now in position to discuss the renormalization of $\vec{F}_{\mu\nu}{}^2$ at the one-loop level.

B. One-loop computation

The result for $\{Z_{ij}\}$ in this approximation is

$$\{Z_{ij}\}=\begin{pmatrix} 1+\frac{13}{3}A & -\frac{3}{2}A & \frac{3}{2}A \\ -2(\frac{13}{3}-\alpha)A & 1+(3+\alpha)A & -(3-\alpha)A \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.16)$$

where A stands for

$$A=\frac{C_2(G)g^2}{16\pi^2}\ln\frac{\mu}{\Lambda}, \quad (2.17)$$

with the conventions of Ref. 6:

$$f_{ikl}f_{jkl}=C_2(G)\delta_{ij}. \quad (2.18)$$

Thus, the anomalous-dimension matrix $\{\gamma_{ij}\}$ reads

$$\{\gamma_{ij}\}=\left\{\left(\mu\frac{\partial}{\partial\mu}Z_{ik}\right)Z_{kj}{}^{-1}\right\}$$
$$=\frac{C_2(G)g^2}{16\pi^2}\begin{pmatrix} \frac{13}{3} & -\frac{3}{2} & \frac{3}{2} \\ -2(\frac{13}{3}-\alpha) & 3+\alpha & -(3-\alpha) \\ 0 & 0 & 0 \end{pmatrix} \quad (2.19)$$

and possesses the following eigenvalues: $(0, \frac{22}{3}$, and $\alpha)\times[C_2(G)g^2/16\pi^2]$.

This example exhibits all the above-mentioned properties. First, the operator O_1 mixes through renormalization with non-formally-gauge-invariant operators like the operator O_2. Second, the anomalous-dimension matrix γ has, apart from the trivial zero eigenvalue, one α-independent and one α-dependent eigenvalue. As expected, the α-independent anomalous dimension belongs to a multiplicatively renormalizable gauge-invariant operator which is derived and discussed in the sections which follow: It is a combination of O_1, O_2, and O_3. However, it should be noticed that a naive calculation of this anomalous dimension, where one ignores the coupling of O_1 to the non-formally-gauge-invariant operator O_2, gives a wrong value, although it is still α-independent:

$$\gamma=\frac{13}{3}\frac{C_2(G)g^2}{16\pi^2}.$$

This procedure where one neglects non-gauge-invariant operators was followed by Georgi and Politzer[7] and by Gross and Wilczek[6] in their computation of anomalous dimensions of twist-two operators. In this case, however, the result for the anomalous dimension of the operators $F^{i_1}_{\rho\mu_1}D^{i_1i_2}_{\mu_2}\cdots D^{i_{n-2}i_{n-1}}_{\mu_{n-1}}F^{i_{n-1}\rho}_{\mu_n}$ is supported by a calculation in another gauge, the so-called axial-vector gauge: $n_\mu\vec{A}^\mu=0$, where n_μ is some constant four-vector. When the gauge-fixing term in the Lagrangian takes the form $-(1/2\alpha)(n_\mu\vec{A}^\mu)^2$, the ghosts are believed to decouple in the limit $\alpha\to 0$.[6,18-20] However, the gauge $-(1/2\alpha)(n_\mu\vec{A}^\mu)^2$, $\alpha\neq 0$, is not renormalizable and no consistent prescription for computation in the axial-vector gauge exists, except at the one-loop level,[20] where one finds that all counterterms coincide:

$$Z_1=Z_3 \quad (2.20)$$

(see also Ref. 21 for an approach in a different spirit). The operator $\vec{F}_{\mu\nu}{}^2$, defined in this gauge as

$$\Gamma^{R(n)}_{O_1}=\left(\frac{n}{2}-g\frac{\partial}{\partial g}\right)\Gamma^{R(n)}, \quad (2.21)$$

is multiplicatively renormalizable; and its renormalization factor computed from the result of Gross and Wilczek,[6]

$$Z_3=1-\frac{22}{3}A+O(g^4),$$

is

$$Z_{11}=1-\frac{1}{2}g\frac{\partial}{\partial g}\ln Z_3$$
$$=1+\frac{22}{3}A+O(g^4). \quad (2.22)$$

Thus a calculation in the axial-vector gauge leads to the correct answer for the gauge-independent anomalous dimension. Our discussion concerning renormalization of $\vec{F}_{\mu\nu}{}^2$ will now be extended to all orders in perturbation theory.

C. General considerations

Let us write the renormalization-group equations in the form

$$\left[\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial\alpha}-\frac{n}{2}\gamma-p\gamma_G\right)\delta_{ij}-\gamma_{ij}\right]\Gamma^{(n,p)}_{O_j}=0, \tag{2.23}$$

where γ, γ_G, and δ are defined as (our conventions for γ and γ_G differ by a factor of 2 from those of Ref. 22)

$$\gamma=\mu\frac{\partial}{\partial\mu}\ln Z_3\bigg|_{\alpha_0,g_0},$$

$$\gamma_G=\mu\frac{\partial}{\partial\mu}\ln\tilde{Z}_3\bigg|_{\alpha_0,g_0},$$

and (2.24)

$$\delta=\mu\frac{\partial}{\partial\mu}\alpha\bigg|_{\alpha_0,g_0},$$

and γ_{ij} was defined in Eq. (2.19). According to a well-known result,[22] δ is related to γ: $\delta=-\alpha\gamma$. It is then a matter of simple algebra to derive the expression for the matrix $\{\gamma_{ij}\}$ from Eq. (2.15):

$$\{\gamma_{ij}\}=\begin{bmatrix}\left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\gamma & \left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\left(\frac{\gamma}{2}-\frac{\beta}{g}\right) & \left(\alpha\frac{\partial}{\partial\alpha}-\frac{g}{2}\frac{\partial}{\partial g}\right)\gamma_G\\ g\frac{\partial}{\partial g}\gamma & g\frac{\partial}{\partial g}\left(\frac{\gamma}{2}-\frac{\beta}{g}\right) & g\frac{\partial}{\partial g}\gamma_G\\ 0&0&0\end{bmatrix}. \tag{2.25}$$

We are looking for linear combinations of operators diagonal under renormalization:

$$u_i(\alpha,g)\Gamma^{R(n,p)}_{O_i}(\alpha,g)=ZZ_3^{n/2}\tilde{Z}_3^{\,p}u_i(\alpha_0,g_0)\times\Gamma^{b(n,p)}_{O_i}(\alpha_0,g_0)$$

or equivalently

$$(D-\hat{\gamma})u_i(\alpha,g)\Gamma^{R(n,p)}_{O_i}(\alpha,g)=0\,, \tag{2.26}$$

$$\hat{\gamma}=\mu\frac{\partial\ln Z}{\partial\mu}\bigg|_{\alpha_0,g_0},$$

where D denotes

$$D=\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial\alpha}-\frac{n}{2}\gamma-p\gamma_G\,.$$

In general, Z and $\hat{\gamma}$ are not eigenvalues of the matrices $\{Z_{ij}\}$ and $\{\gamma_{ij}\}$, respectively, neither are u_i the eigenvectors; the solution of Eqs. (2.26) can only be computed order by order in perturbation theory. However, in Sec. IV we shall show that β can be chosen independent of α. For this choice, which is assumed in the following, the matrix $\{Z_{ij}\}$ can be cast in a triangular form by a change of basis independent of g and α, namely the basis O_1, $O_1+\frac{1}{2}O_2$, and O_3. Therefore Z and $\hat{\gamma}$ are indeed eigenvalues of $\{Z_{ij}\}$ and $\{\gamma_{ij}\}$:

$$\gamma_1=-g\frac{\partial}{\partial g}\left(\frac{\beta}{g}\right),\quad \gamma_2=\alpha\frac{\partial\gamma}{\partial\alpha},\quad \gamma_3=0\,. \tag{2.27}$$

There exists also a trivial solution of Eq. (2.26) for $\gamma_3=0$:

$$D\Gamma^{R(n,p)}_{O_3}=0\,.$$

The fact that $O_3=\overline{C}_iM_{ij}C_j$ has a zero anomalous dimension is a consequence of the equation of motion for the ghost field. For the eigenvalue γ_1 the solution of (2.26) can be guessed from the renormalization-group equations for ordinary Green's functions:

$$D\Gamma^{R(n,p)}=0\,,$$

which implies

$$D\mu\frac{\partial}{\partial\mu}\Gamma^{R(n,p)}=0\,,$$

or equivalently

$$D\left[\beta\frac{\partial}{\partial g}-\gamma\left(\alpha\frac{\partial}{\partial\alpha}+\frac{n}{2}\right)-p\gamma_G\right]\Gamma^{R(n,p)}=0\,. \tag{2.28}$$

The equation of motion for the vector field[16,17] leads to a counting identity:

$$2\Gamma^{R(n,p)}_{O_1}+\Gamma^{R(n,p)}_{O_2}-2\Gamma^{R(n,p)}_{O_4}=n\Gamma^{R(n,p)}\,. \tag{2.29}$$

Finally, using this identity together with Eqs. (2.11), we rewrite equation (2.28) as

$$D\left[\frac{\beta}{g}\Gamma^{(n,p)}_{O_2}-\gamma(\Gamma^{R(n,p)}_{O_1}+\tfrac{1}{2}\Gamma^{R(n,p)}_{O_2})-\gamma_G\Gamma^{R(n,p)}_{O_3}\right]=0\,, \tag{2.30}$$

which upon multiplication by $g/2\beta$ yields the solution $\Gamma^{R(n,p)}_{O_1'}$ of Eq. (2.26) relative to the α-independent value $\gamma_1=-g(\partial/\partial g)(\beta/g)$:

$$\Gamma^{R(n,p)}_{O_1'}=\frac{\gamma g}{2\beta}\Gamma^{R(n,p)}_{O_1}+\frac{1}{2}\left(\frac{\gamma g}{2\beta}-1\right)\Gamma^{R(n,p)}_{O_2}+\frac{\gamma_G g}{2\beta}\Gamma^{R(n,p)}_{O_3}\,. \tag{2.31}$$

The normalization of this operator is chosen to be finite at $g=0$.

A blind generalization of our one-loop considerations in the axial gauge leads to the correct eigenvalue γ_1 to all orders in perturbation theory. Indeed, in this gauge β and γ are not independent,

$$\beta - \frac{\gamma g}{2} = g\mu \frac{\partial}{\partial \mu} \ln \frac{Z_3}{Z_1}\Big|_{\alpha_0, \varepsilon_0} = 0 , \qquad (2.32)$$

in view of Eq. (2.20) (Ref. 21), and γ_{O_1} is easily computed to be equal to γ_1. This result is rather surprising, since it shows that the axial-vector gauge gives a sensible answer at all orders in perturbation theory. *A priori*, apart from the problem of a correct computational prescription, one might have expected that the operator $\vec{F}_{\mu\nu}{}^2$ would couple through renormalization with non-Lorentz-invariant operators. This result gives an indication that it should be possible to quantize Yang-Mills theory in the axial gauge. This would avoid considering spurious operators, such as O_2 and O_3 in the usual gauges $-(1/2\alpha)(\partial_\mu \vec{A}^\mu)^2$, for the study of gauge-invariant operators.

As a final comment let us note that the introduction of fermions does not alter our previous considerations, since $\bar{\psi}_a \not{D}_{ab} \psi_b$ at zero momentum is a multiplicatively renormalizable operator.

The operator O_1' corresponding to an α-independent anomalous dimension satisfies the same kind of Ward identities as the bare operator O_1: To show this, we will now introduce Ward identities first for ordinary Green's functions and then for the insertion of the bare operator O_1.

III. THE USE OF SUPERGAUGE TRANSFORMATIONS FOR WARD IDENTITIES

Ward identities are usually obtained by application of a nonlocal gauge transformation,[11,12]

$$\delta A_\mu^i(x) = D_\mu^{ij}(x) M^{-1}{}_{jk}(x, y, A)\delta\omega^k(y) , \qquad (3.1)$$

which leaves the generating functional $Z(0)$ invariant. The nonlocal character of this transformation is a technical complication when one performs the Legendre transform, which leads to the 1PI Green's functions. But noting that

$$M^{-1}{}_{ij}\left(x, y, \frac{1}{i}\frac{\delta}{\delta\vec{\eta}_\mu}\right) Z(\vec{\eta}_\mu) = -i \int dA\, dC\, d\bar{C}\, C_i(x)\bar{C}_j(y) \times \exp\left[i \int (\mathcal{L} + \vec{\eta}_\mu \cdot \vec{A}^\mu)\right] , \qquad (3.2)$$

Becchi, Rouet, and Stora[8] have derived the WI from a gauge transformation containing only the local part of the transformation (3.1). For non-Abelian gauge theories, these supergauge transformations read

$$\delta A_\mu^i(x) = (D_\mu^{ij} C^j)(x)\delta\lambda , \qquad (3.3a)$$

$$\delta C^i(x) = \tfrac{1}{2} g_0 f^{ijk} C^j(x) C^k(x)\delta\lambda = \tfrac{1}{2} g_0 (\vec{C}\times\vec{C})^i(x)\delta\lambda , \qquad (3.3b)$$

$$\delta \bar{C}^i(x) = -\frac{1}{\alpha_0}\partial_\mu A^{\mu i}(x)\delta\lambda , \qquad (3.3c)$$

where $\delta\lambda$ is the x-independent anticommuting infinitesimal parameter of the transformation.[23] With the help of the identity

$$\vec{B}\times(\vec{C}\times\vec{C}) = 2(\vec{B}\times\vec{C})\times\vec{C} , \qquad (3.4)$$

which holds as a trivial consequence of the Jacobi identity for an anticommuting variable $C^i(x)$, independently of the fermionic or bosonic nature of B, one can see that the Lagrangian $\mathcal{L}$ is indeed invariant under the transformation (3.3):

$$\delta[D_\mu^{ij} C^j] = 0, \quad \delta(\vec{C}\times\vec{C}) = 0 , \qquad (3.5)$$

$$(\delta\bar{C})_i \partial_\mu D^\mu_{ij} C_j + \delta\left(-\frac{1}{2\alpha_0}(\partial_\mu \vec{A}^\mu)^2\right) = 0 . \qquad (3.6)$$

The Jacobian of the transformation (3.3) reduces to unity due to the antisymmetry of the structure constants f_{ijk}. As usual,[11-14,24] WI relate ordinary Green's functions and Green's functions where two external legs, a vector-meson leg, and a ghost C_i leg have been contracted; the latter functions are in fact insertions of composite operators for which we find it more convenient to introduce new sources; we thus consider the generating functional Z (of course, the sources ξ_i, $\bar{\xi}_i$, and $\vec{J}_\mu$ are anticommuting objects),

$$Z(\vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}) = \int dC\, d\bar{C}\, dA \exp\left\{ i \int d^4x [\mathcal{L} + \vec{\eta}^\mu(x)\cdot\vec{A}_\mu(x) + \bar{\xi}_i(x) C_i(x) + \bar{C}_i(x)\xi_i(x) + J_\mu^i(x) D^\mu_{ij} C^j(x) + \tfrac{1}{2} g_0 \vec{K}(x)\cdot(\vec{C}\times\vec{C})(x)]\right\} , \qquad (3.7)$$

and the generating functional for connected Green's functions $W = i\ln Z$. We suppose that a gauge-invariant regularization has been introduced to define properly the generating functional Z; this refers either to the regularization proposed by Lee and Zinn-Justin,[12] which makes use of higher covariant derivatives and auxiliary fields, or to the more tractable dimensional regularization of 't Hooft and Veltman,[25] for which, however, the proof of an unambiguous gauge-invariant prescription valid to all orders is lacking at present. The invariance of the integral (3.7) under any local change of the variable $\overline{C}_i(x)$ and under the transformations (3.3) is expressed by the following identities:

$$\int dC\, d\overline{C}\, dA[\partial_\mu D^\mu_{ij} C_j(x) + \xi_i(x)] \exp(iS) = 0\,, \tag{3.8}$$

$$\int dC\, d\overline{C}\, dA \int d^4x \Big[\eta^i_\mu(x) D^\mu_{ij} C^j(x) + \tfrac{1}{2} g_0 \overline{\xi}_i(x)(\vec{C}\times\vec{C})_i(x) + \frac{1}{\alpha_0} \vec{\overline{\xi}}(x)\cdot\partial_\mu \vec{A}^\mu(x)\Big] \exp(iS) = 0\,, \tag{3.9}$$

where $\exp(iS)$ denotes the integrand in Eq. (3.7). Note that in contradistinction with the usual Ward-Slavnov identities,[11,12] identity (3.9) is not local in x. Equations (3.8) and (3.9) can be reexpressed in terms of W as

$$\partial_\mu \frac{\delta W}{\delta \vec{J}_\mu(x)} = \vec{\overline{\xi}}(x)\,, \tag{3.10}$$

$$\int d^4x \Big[\vec{\eta}_\mu(x)\cdot \frac{\delta W}{\delta \vec{J}_\mu(x)} + \overline{\xi}_i(x)\frac{\delta W}{\delta K_i(x)} + \frac{1}{\alpha_0}\vec{\overline{\xi}}(x)\cdot\partial_\mu \frac{\delta W}{\delta \vec{\eta}_\mu(x)}\Big] = 0\,. \tag{3.11}$$

The Legendre transform Γ of W with respect to $\vec{\eta}_\mu$, ξ_i, and $\overline{\xi}_i$ is the generating functional for the 1PI Green's functions:

$$W(\vec{\eta}_\mu, \xi_i, \overline{\xi}_i, \vec{J}_\mu, \vec{K}) + \Gamma(\vec{A}_\mu, \overline{C}_i, C_i, \vec{J}_\mu, \vec{K}) + \int d^4x [\vec{\eta}_\mu(x)\cdot\vec{A}^\mu(x) + \overline{C}_i(x)\xi_i(x) + \overline{\xi}_i(x) C_i(x)] = 0\,, \tag{3.12}$$

where

$$\vec{A}_\mu = -\frac{\delta W}{\delta\vec{\eta}_\mu}\,,\quad \vec{\eta}_\mu = -\frac{\delta W}{\delta\vec{A}_\mu}\,,$$
$$C_i = -\frac{\delta W}{\delta\overline{\xi}_i}\,,\quad \overline{\xi}_i = \frac{\delta\Gamma}{\delta C_i}\,, \tag{3.13}$$
$$\overline{C}_i = \frac{\delta W}{\delta\xi_i}\,,\quad \xi_i = -\frac{\delta\Gamma}{\delta\overline{C}_i}\,,$$

and

$$\frac{\delta W}{\delta\vec{J}_\mu} = -\frac{\delta W}{\delta\vec{J}_\mu}\,,\quad \frac{\delta W}{\delta K_i} = -\frac{\delta\Gamma}{\delta K_i}\,. \tag{3.14}$$

Our Ward identity and equation of motion then read

$$\partial_\mu \frac{\delta\Gamma}{\delta J^i_\mu(x)} = \frac{\delta\Gamma}{\delta\overline{C}_i(x)} \tag{3.15}$$

and

$$\int d^4x \Big[\frac{\delta\Gamma}{\delta\vec{A}_\mu(x)}\circ\frac{\delta\Gamma}{\delta\vec{J}_\mu(x)} + \frac{1}{\alpha_0}\partial_\mu A^\mu_i(x)\frac{\delta\Gamma}{\delta\overline{C}_i(x)} - \frac{\delta\Gamma}{\delta\vec{C}(x)}\cdot\frac{\delta\Gamma}{\delta\vec{K}(x)}\Big] = 0\,. \tag{3.16}$$

The introduction of an auxiliary functional $\hat{\Gamma}$,

$$\hat{\Gamma}(\vec{A}_\mu, C_i, \overline{C}_i, \vec{J}_\mu, \vec{K}) = \Gamma(\vec{A}_\mu, C_i, \overline{C}_i, \vec{J}_\mu, \vec{K}) + \frac{1}{2\alpha_0}\int [\partial_\mu \vec{A}^\mu(x)]^2 d^4x\,, \tag{3.17}$$

and the use of Eq. (3.15) allows to reduce Eq. (3.16) to

$$\int d^4x \Big[\frac{\delta\hat{\Gamma}}{\delta A^i_\mu(x)}\frac{\delta\hat{\Gamma}}{\delta J^i_\mu(x)} - \frac{\delta\hat{\Gamma}}{\delta C^i(x)}\frac{\delta\hat{\Gamma}}{\delta K^i(x)}\Big] = 0\,. \tag{3.18}$$

Equations (3.15) and (3.18) are derived with respect to $\vec{A}_\mu, \vec{C}, \vec{J}_\mu, \ldots$ and yield relations between regularized form factors appearing in the parametrization of various superficially divergent 1PI Green's functions; these relations give the usual information about renormalization of ordinary Green's functions[11-14] and imply, furthermore, that the insertions of operators associated with sources $\vec{J}_\mu$ and $\vec{K}$ are multiplicatively renormalizable; this derivation is performed in Appendix A. The renormalized generating functional Z^R reads

$$\begin{aligned} Z^R(\vec{\eta}_\mu, \xi_i, \overline{\xi}_i, \vec{J}_\mu, \vec{K}) = \int dA\, dC\, d\overline{C} \exp\Big\{ i\int d^4x \Big[&-\tfrac{1}{4}Z_3(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)^2 - \tfrac{1}{2}Z_1 g(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)\cdot(\vec{A}^\mu\times\vec{A}^\nu) \\ &-\tfrac{1}{4}Z_4 g^2(\vec{A}_\mu\times\vec{A}_\nu)^2 - \frac{1}{2\alpha}(\partial_\mu\vec{A}^\mu)^2 + \tilde{Z}_3\overline{C}_i\partial^2 C_i \\ &+\tilde{Z}_1 g\overline{C}_i\partial_\mu(\vec{A}^\mu\times\vec{C})_i + \vec{\eta}_\mu\cdot\vec{A}^\mu + \overline{\xi}_i C_i + \overline{C}_i\xi_i + \tilde{Z}_3\vec{J}_\mu\cdot\partial^\mu\vec{C} \\ &+\tilde{Z}_1 g J^i_\mu(\vec{A}^\mu\times\vec{C})_i + \tilde{Z}_1\tfrac{1}{2}gK_i(\vec{C}\times\vec{C})_i\Big]\Big\}\,, \end{aligned} \tag{3.19}$$

where the counterterms Z_i and $\tilde{Z}_i$ satisfy the WI (2.5). The counterterms, which are determined up to finite renormalizations, are fixed

(a) by imposing renormalization conditions at some Euclidean point $-\mu^2$ (Ref. 12) in agreement with WI on the superficially divergent 1PI Green's functions,

(b) or by imposing in the framework of a specific gauge-invariant regularization[12,25] a "minimal renormalization" for which the counterterms are defined according to $Z_i = 1 +$ singular pieces only.

More precisely, for a regularization involving a dimensional cutoff Λ (Ref. 12), the counterterms take the form

$$1+\sum a_n(g^2,\alpha)\ln^n\left(\frac{\Lambda^2}{\mu^2}\right)$$

(μ is an arbitrary mass scale),

whereas for the dimensional regularization[25] they have the expression

$$1+\sum \frac{a_n(\lambda^2,\alpha)}{\epsilon^n}\,,\quad \epsilon=4-d$$

(d is the dimension of space-time).

Let us recall that in the latter regularization, the mass scale μ enters the theory via the dimension of the coupling constant g which is $\frac{1}{2}\epsilon$ (Ref. 26): $g^2=\mu^\epsilon\lambda^2$.

This derivation of Ward identities is immediately extended to the case where matter fields are introduced (see Appendix B).

The previous Ward identities (3.18) give no information about the variation of Green's functions with respect to the parameter α. We extract this information from another type of Ward identities which we will now derive.

IV. VARIATIONS OF GREEN'S FUNCTIONS WITH RESPECT TO THE GAUGE PARAMETER

The nondependence of β on α was discussed by a number of authors. The arguments were based on S-matrix gauge invariance and neglected infrared divergences, which are important to the issue. The first correct discussion was given by Caswell and Wilczek[9]; however, they suppose the existence of multiplicatively renormalizable gauge-invariant operators, which is hard to establish (see Sec. V) and which is in fact irrelevant to the issue: As we shall show in this section, Ward identities contain all the information relative to the dependence on the gauge parameter α.

Let us recall that Lee and Zinn-Justin[4] have computed, in the absence of the ghost sources ξ_i and $\bar{\xi}_i$, the effect of a change of parameter α_0; only the source term $\vec{\eta}_\mu$ is affected:

$$\eta_\mu^i A_i^\mu \to \eta_\mu^i(x)\Big[A_i^\mu(x) + \frac{\Delta\alpha_0}{\alpha_0}\int d^4y\, D_{ij}^\mu(x)M^{-1jk}(x,y)\partial^\rho A_{\rho k}(y)\Big]. \tag{4.1}$$

We notice that this change can be obtained by a simultaneous insertion of the operator $D_\mu^{ij}C^j$ and of the operator $\bar{C}_i\partial_\mu A_i^\mu$ for which we introduce a new scalar source L:

$$Z(\vec{\eta}_\mu,\xi_i,\bar{\xi}_i,\vec{J}_\rho,\vec{K},L)=\int dA\,dC\,d\bar{C}\exp\Big\{i\int d^4x[\mathcal{L}+\vec{\eta}_\mu(x)\cdot\vec{A}^\mu(x)+\bar{\xi}_i(x)C_i(x)+\bar{C}_i(x)\xi_i(x)+J_\mu^i(x)D_{ij}^\mu C_j(x) + \tfrac{1}{2}g_0K_i(x)(\vec{C}\times\vec{C})_i(x)+L(\bar{C}_i(x)\partial_\mu A_i^\mu(x)+\vec{J}_\mu(x)\cdot\vec{A}^\mu(x))]\Big\}. \tag{4.2}$$

L is an x-independent[27] anticommuting source and the supplementary term $L\vec{J}_\mu\cdot\vec{A}^\mu$ is introduced for convenience as we shall see. The WI satisfied by Z then reads

$$\int d^4x\Big[(\vec{\eta}_\mu+L\vec{J}_\mu)\cdot\Big(\frac{\delta W}{\delta\vec{J}_\mu}+L\frac{\delta W}{\delta\vec{\eta}_\mu}\Big)+\bar{\xi}_i\frac{\delta W}{\delta K_i}+\frac{1}{\alpha_0}\partial_\mu\frac{\delta W}{\delta\vec{\eta}_\mu}\cdot\vec{\bar{\xi}}\Big]\exp(-iW) - \int dC\,d\bar{C}\,dA\,L\int d^4x\Big(\bar{C}_i\partial_\mu D_{ij}^\mu C_j-\frac{1}{\alpha_0}(\partial_\mu\vec{A}^\mu)^2\Big)\exp(iS)=0\,, \tag{4.3}$$

where iS denotes the exponent in (4.2). Applying the operator $\int d^4x[\delta/\delta\xi_i(x)]$ to the equation of motion of $\bar{C}$,

$$\int dC\,d\bar{C}\,dA[\partial_\mu D_{ij}^\mu C_j(x)+\xi_i(x)-L\partial_\mu A_i^\mu(x)]\exp(iS)=0\,, \tag{4.4}$$

we obtain the counting identity[16,17]

$$\int dA\,dC\,d\overline{C}\left[\int d^4x\,\overline{C}_i(x)\partial_\mu D^\mu_{ij}C_j(x)\right]\exp(iS)$$
$$=\int d^4x\left[\xi_i(x)\frac{\delta W}{\delta\xi_i(x)}+L\frac{\delta W}{\delta L}-L\vec{J}_\mu(x)\cdot\frac{\delta W}{\delta\vec{\eta}_\mu(x)}\right]$$
$$\times\exp(-iW)\,. \tag{4.5}$$

The change ΔZ of Z under a variation $\Delta\alpha_0$ of the gauge parameter is given by the functional for one insertion of the operator $-(1/\alpha_0)(\partial_\mu\vec{A}^\mu)^2$:

$$\Delta Z=-i\Delta W\exp(-iW)$$
$$=\int dA\,dC\,d\overline{C}\left(-\frac{i\Delta\alpha_0}{2\alpha_0}\right)\left[\int d^4x\left(-\frac{(\partial_\mu\vec{A}^\mu)^2}{\alpha_0}\right)\right]$$
$$\times\exp(iS)\,. \tag{4.6}$$

Equations (4.3), (4.5), and (4.6) yield the WI

$$2\alpha_0 L\frac{\Delta W}{\Delta\alpha_0}=\int d^4x\left[(\vec{\eta}_\mu+L\vec{J}_\mu)\cdot\left(\frac{\delta W}{\delta\vec{J}_\mu}+L\frac{\delta W}{\delta\vec{\eta}_\mu}\right)\right.$$
$$\left.+\overline{\xi}_i\frac{\delta W}{\delta K_i}+\frac{1}{\alpha_0}\partial_\mu\frac{\delta W}{\delta\eta^i_\mu}\xi_i-L\xi_i\frac{\delta W}{\delta\xi_i}\right]. \tag{4.7}$$

We perform the Legendre transformation, take the derivative of Eq. (4.7) with respect to L, set L equal to zero, and introduce the transverse functional $\hat{\Gamma}$ defined in (3.17). This leads to the Ward identity which expresses the variation of 1PI Green's functions under a change of the parameter α_0:

$$-2\alpha_0\frac{\Delta\hat{\Gamma}}{\Delta\alpha_0}=\int d^4x\left[\vec{A}_\mu\circ\frac{\delta\hat{\Gamma}}{\delta\vec{A}_\mu}-\overline{C}_i\frac{\delta\hat{\Gamma}}{\delta\overline{C}_i}-\vec{J}_\mu\circ\frac{\delta\hat{\Gamma}}{\delta\vec{J}_\mu}+\frac{\delta}{\delta L}\left(\frac{\delta\hat{\Gamma}}{\delta\vec{A}_\mu}\circ\frac{\delta\hat{\Gamma}}{\delta\vec{J}_\mu}-\frac{\delta\hat{\Gamma}}{\delta\vec{K}}\circ\frac{\delta\hat{\Gamma}}{\delta\vec{C}}\right)\right]. \tag{4.8}$$

The Green's functions for two- and three-vectors satisfy the equations

$$-2\alpha_0\frac{\partial}{\partial\alpha_0}\frac{\delta^2\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu}=2\frac{\delta^2\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu}+\left[\frac{\delta^2\hat{\Gamma}}{\delta A^i_\mu\delta A^k_\rho}\frac{\delta^3\hat{\Gamma}}{\delta A^j_\nu\delta L\delta J^k_\rho}+(\mu\leftrightarrow\nu;\,i\leftrightarrow j)\right], \tag{4.9a}$$

$$-2\alpha_0\frac{\partial}{\partial\alpha_0}\frac{\delta^3\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu\delta A^k_\rho}=3\frac{\delta^3\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu\delta A^k_\rho}+\left[\frac{\delta^3\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu\delta A^l_\sigma}\frac{\delta^3\hat{\Gamma}}{\delta A^k_\rho\delta L\delta J^l_\sigma}+(\mu\to\nu\to\rho;\,i\to j\to k)\right]$$
$$+\left[\frac{\delta^2\hat{\Gamma}}{\delta A^k_\rho\delta A^l_\sigma}\frac{\delta^4\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu\delta L\delta J^l_\sigma}+(\mu\to\nu\to\rho;\,i\to j\to k)\right], \tag{4.9b}$$

where all sources are set equal to zero after differentiating. Let us parametrize the Green's functions involved in Eqs. (4.9) (see Fig. 1) as

$$\frac{\delta^2\hat{\Gamma}}{\delta A^i_\mu\delta A^j_\nu}=\Gamma^{ij}_{\mu\nu}(p)$$
$$=\delta_{ij}(p_\mu p_\nu-g_{\mu\nu}p^2)I(p^2)\,, \tag{4.10a}$$

$$\frac{\delta^3\Gamma}{\delta A^i_\mu\delta L\delta J^j_\nu}=-\delta_{ij}[g_{\mu\nu}X_1(p^2)+p_\mu p_\nu X_2(p^2)]\,, \tag{4.10b}$$

and at the symmetric point $p^2=q^2=r^2$

$$\frac{\delta^3\Gamma}{\delta A^i_\mu\delta^j_\nu\delta A^k_\rho}=\Gamma^{ijk}_{\mu\nu\rho}(p,q,r)$$
$$=ig_0f_{ijk}\{[g_{\mu\nu}(q-p)_\rho+g_{\nu\rho}(r-q)_\mu+g_{\rho\mu}(p-r)_\nu]G(p^2)+(q-r)_\mu(r-p)_\nu(p-q)_\rho G_1(p^2)$$
$$+(r_\mu p_\nu q_\rho-r_\nu p_\rho q_\mu)G_2(p^2)\}\,, \tag{4.10c}$$

$$\frac{\delta^4\Gamma}{\delta A^i_\mu\delta A^j_\nu\delta L\delta J^k_\rho}=ig_0f_{ijk}\{g_{\mu\nu}(q-p)_\rho Y_1(p^2)+[g_{\mu\rho}(p-r)_\nu-g_{\nu\rho}(q-r)_\mu]Y_2(p^2)+\cdots\}\,. \tag{4.10d}$$

With this parametrization, Eqs. (4.9a) and (4.9b) for $p^2=q^2=r^2$ yield

$$-2\alpha_0\frac{\partial}{\partial\alpha_0}I(p^2)=2I(p^2)[1-X_1(p^2)]\,, \qquad (4.11a)$$

$$-2\alpha_0\frac{\partial}{\partial\alpha_0}G_1(p^2)=3G_1(p^2)[1-X_1(p^2)] -p^2[Y_1(p^2)+2Y_2(p^2)]\,I(p^2)\,, \qquad (4.11b)$$

where the terms omitted in the parametrization (4.10d) do not contribute to (4.11b) and where only I, G, and X_1 are superficially divergent and $(1-X_1)=O(g^2)$. The variation of α_0 gives rise only to one new superficially divergent Green's function, namely X_1. Gauge invariance connects the insertion of $(\partial_\mu \vec{A}^\mu)^2$, which is of dimension four, and the insertion of $D_\mu^{ij}C^j(x)(\overline{C}_k\partial_\mu A_k^\mu)(y)$, which, for power counting, acts as an operator of dimension one. Thus due to Ward identities (4.11) only one new renormalization, the wave-function renormalization, is necessary: This is achieved by the "minimal" renormalization. In the usual renormalization scheme supplementary counterterms arise: The relation between these counterterms yields a differential equation for β.

The "minimal" renormalization leads to a bare coupling constant g_0 independent of α. This can be seen by a recursive proof. Suppose that we have introduced all counterterms, including those for the source L, needed for the renormalization of graphs with number of loops less than or equal to n, and that the bare coupling constant relative to order n satisfies $\partial g_0^{[n]}/\partial\alpha=\partial g/\partial\alpha_0^{[n]}=0$. The equation of motion for C (3.15) implies that L is multiplicatively renormalizable [in fact another independent counterterm, $ZL\vec{K}\cdot\vec{C}$, is needed; however, this extra term brings no change in Eqs. (4.9)]; thus the bare Green's functions obey Eqs. (4.11), where all functions I, G, Y_i, and X_1 are understood to be bare functions and depend on α_0 and g_0. We introduce the renormalized functions up to order n:

$$I^R(p^2,g,\alpha)=Z_3^{[n]}I^b(p^2,g_0,\alpha_0)\,, \qquad (4.12a)$$

$$gG^R(p^2,g,\alpha)=g_0(Z_3^{[n]})^{3/2}G^b(p^2,g_0,\alpha_0)\,, \qquad (4.12b)$$

$$X_{1,2}^R(p^2,g,\alpha)=Z^{[n]}X_{1,2}^b(p^2,g_0,\alpha_0)\,, \qquad (4.12c)$$

$$gY_{1,2}^R(p^2,g,\alpha)=g_0Z^{[n]}(Z_3^{[n]})^{1/2}\,Y_{1,2}^b(p^2,g_0,\alpha_0)\,, \qquad (4.12d)$$

where the same counterterm Z appears in Eq. (4.12c) and (4.12d). Taking into account $\partial g/\partial\alpha_0^{[n]}=0$ and eliminating $\alpha_0\partial\alpha/\partial\alpha_0$ between Eqs. (4.11a) and (4.11b), we obtain the following identity:

$$\frac{(\partial/\partial\alpha)\ln[G^R/(I^R)^{3/2}]}{(\partial/\partial\alpha)\ln\alpha\, I^R}=\frac{p^2I^R(Y_1^R+2Y_2^R)}{2X_1^RG^R}\,. \qquad (4.13)$$

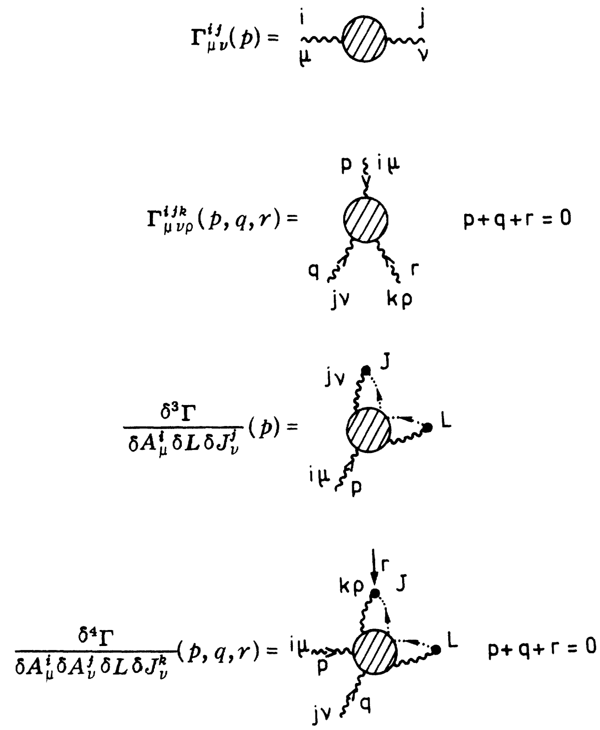

FIG. 1. Graphical representation of the Green's functions involved in Eqs. (4.9).

The divergences of functions I^R, G^R, and X_1^R appear only at order $n+1$, whereas Y_1^R and Y_2^R are divergent only at order $(n+2)$; I^R, G^R, and X_1^R have an expansion $1+O(g^2)$, in contradistinction to Y_1 and Y_2, which are of order g^2. In the "minimal" renormalization, the counterterms gZ_1 and Z_3 are identified order by order in perturbation theory with the singular part of the superficially divergent Green's functions $-gG^R$ and $-I^R$, respectively. Thus Eq. (4.13) says that the quantity

$$\frac{\partial g_0^{(n+1)}}{\partial\alpha}=g\,\frac{\partial}{\partial\alpha}(Z_1^{(n+1)}-\tfrac{3}{2}Z_3^{(n+1)})$$

is of order $(n+2)$ and can be set equal to zero. This completes our proof: $\partial g_0^{(n+1)}/\partial\alpha=0$. The "minimal" renormalization of 't Hooft and Veltman[25] yields in a natural way

$$\frac{\partial\beta}{\partial\alpha}=0\,.$$

What happens in other renormalization schemes? Suppose that we impose the following renormalization conditions:

$$I^R(-\mu^2)=\psi(g,\alpha) =1+O(g^2)\,, \qquad (4.14)$$

$$G^R(-\mu^2,-\mu^2,-\mu^2)=\phi(g,\alpha) =1+O(g^2)\,.$$

Equations (4.11) tell us that the expression

$$\frac{(\partial/\partial\alpha_0)\ln(g\psi/\phi^{3/2})}{(\partial/\partial\alpha_0)\ln\alpha\phi} = -\mu^2\,\frac{\phi(Y_1^R+2Y_2^R)}{2X_1^R\psi} \qquad (4.15)$$

is finite, or equivalently that the function ρ, defined by Caswell and Wilczek,[9] is finite:

$$\rho = \frac{\partial g/\partial\alpha_0}{\partial\alpha/\partial\alpha_0}\,. \qquad (4.16)$$

This is the only information relevant to our discussion which we extract from WI (4.15); Caswell and Wilczek derive it from the renormalization-group equation satisfied by a postulated gauge-invariant and multiplicatively renormalizable operator. The identity

$$\left[\mu\frac{\partial}{\partial\mu}\bigg|_{\alpha_0,g_0},\ \frac{\partial}{\partial\alpha_0}\right]=0$$

expressed in terms of the functions ρ, β, and δ, with $\delta=\mu\partial\alpha_0/\partial\mu|_{g_0}$, yields the result of Ref. 9:

$$\frac{\partial\beta}{\partial\alpha}=\beta\frac{\partial\rho}{\partial g}-\rho\frac{\partial\beta}{\partial g}+\frac{\partial}{\partial\alpha}[\delta\rho]+\rho^2\frac{\partial\delta}{\partial g}\,. \qquad (4.17)$$

Let us make a comment about infrared divergences. The choice of $\mu^2=0$ would trivially realize the independence of β with respect to α [Eq. (4.15)]. However, the functions Y_1 and Y_2, which have dimension -2, are infrared-divergent. The condition $\partial\beta/\partial\alpha=0$ holds if and only if $\rho=0$; at the two-loop level Eq. (4.17) reduces to

$$\frac{\partial\beta^{[2]}}{\partial\alpha}=\frac{\partial}{\partial\alpha}[\rho^{[1]}\delta^{[1]}]\,.$$

Explicit computation for the renormalization conditions

$$G=I=1 \text{ at } p^2=-\mu^2\,,$$

yields a nonzero value of $\rho^{[1]}$.

At last, let us consider the Ward identity (4.8) for Green's functions with external ghost legs. As noted above, we are faced with a new divergence; we must introduce a counterterm $aLKC$ which allows us to ensure the multiplicative renormalizability of the source L and which modifies Eq. (4.8) according to

$$-2\alpha_0\frac{\Delta\hat\Gamma}{\Delta\alpha_0}=\int d^4x\left[\vec{A}_\mu\cdot\frac{\delta\hat\Gamma}{\delta\vec{A}_\mu}-\bar{C}_i\frac{\delta\hat\Gamma}{\delta\bar{C}_i}-\vec{J}_\mu\cdot\frac{\delta\hat\Gamma}{\delta\vec{J}_\mu}+a\left(\vec{C}\cdot\frac{\delta\hat\Gamma}{\delta\vec{C}}-\vec{K}\cdot\frac{\delta\hat\Gamma}{\delta\vec{K}}\right)+\frac{\delta}{\delta L}\left(\frac{\delta\hat\Gamma}{\delta\vec{A}_\mu}\cdot\frac{\delta\hat\Gamma}{\delta\vec{J}_\mu}-\frac{\delta\hat\Gamma}{\delta\vec{K}}\circ\frac{\delta\hat\Gamma}{\delta\vec{C}}\right)\right]. \qquad (4.18)$$

In terms of the functions X_1 and X_3,

$$\frac{\delta^3\hat\Gamma}{\delta C_i\delta K_i\delta L}=X_3\,,$$

the identity (4.18) for Green's functions evaluated at at the symmetric point reads

$$\left(\alpha_0\frac{\partial}{\partial\alpha_0}+\frac{n}{2}\right)\hat\Gamma^{(n,p)}=\left\{\frac{n}{2}X_1+p[1-X_1+(X_3-a)]\right\}\hat\Gamma^{(n,p)}+\cdots\,, \qquad (4.19)$$

where the dots denote less-divergent terms irrelevant to the "minimal" renormalization. This equation gives no new information concerning the variation with respect to α of $\tilde{Z}_1/\tilde{Z}_3$ in view of identity (2.5). However, we shall need it for the study of the properties of our gauge-invariant operator O_1'.

V. WARD IDENTITIES FOR THE OPERATOR O_1'

In this section we first derive the Ward identities satisfied by the bare Green's functions with one insertion of $O_1=\vec{F}_{\mu\nu}{}^2$. Then the bare operators O_2 and O_3 are shown to satisfy the same Ward identities as O_1. Finally, the expression of the multiplicatively renormalizable operator O_1' derived in Sec. II is used in connection with Ward identities of the second type (Sec. IV) to recover the result of Sec. II that the anomalous dimension of O_1' is α-independent.

A. Ward identities for O_1

The bare operator O_1 is invariant under gauge transformations (3.3); thus the Ward identity (3.18), as well as the equation of motion of the ghost (3.15), are unaltered by the presence of the source term N_1 for operator O_1:

$$p_\mu\frac{\delta^3\Gamma}{\delta A^i_\mu\delta A^j_\nu\delta O_1}=0\,, \qquad (5.1)$$

which implies the transversality of this function

$$\frac{\delta^3\Gamma}{\delta A^i_\mu\delta A^j_\nu\delta O_1}=-\delta_{ij}(p^2g_{\mu\nu}-p_\mu p_\nu)I_{O_1}(p^2) \qquad (5.2)$$

and at the symmetric point ($p^2=q^2=r^2$)

$$I_{O_1}\tilde{G}+I\tilde{G}_{O_1}=\tilde{I}_{O_1}G+\tilde{I}G_{O_1}+\cdots\,, \qquad (5.3)$$

where the amplitudes I, G, $\tilde{I}$, and $\tilde{G}$ are the superficially divergent form factors involved in the vector and ghost propagators and vertices (see Appendix A for precise definitions) and where the index O_1 refers to the same functions with one insertion of O_1. Here and in the following, we work with the "minimal" renormalization and neglect in the identities all less-singular terms denoted by dots. Similarly, another Ward identity connects the functions with four vector legs to the previous amplitudes. All the previous identities will be referred to as identities of the first type.

Identities of the second type constrain the variation with respect to α of the Green's functions; they are obtained from Eq. (4.19) by derivation with respect to the source O_1. We eliminate among the equations for different values n and p the unknown form factors corresponding to the insertion of operators L and $\vec{J}_\mu$. Let us quote for $n=2,3$ and $p=0$ and $p=1$ the identities

$$\alpha_0 \frac{\partial}{\partial \alpha_0}\left(\frac{\Gamma_{O_1}^{(3,0)}}{\Gamma^{(3,0)}} - \frac{3}{2}\frac{\Gamma_{O_1}^{(2,0)}}{\Gamma^{(2,0)}}\right) + \cdots = 0\,, \qquad (5.4)$$

$$\alpha_0 \frac{\partial}{\partial \alpha_0}\left(\frac{\Gamma_{O_1}^{(0,1)}}{\Gamma^{(0,1)}} - \frac{\Gamma_{O_1}^{(1,1)}}{\Gamma^{(1,1)}} + \frac{1}{2}\frac{\Gamma_{O_1}^{(2,0)}}{\Gamma^{(2,0)}}\right) + \cdots = 0\,. \qquad (5.5)$$

B. Ward identities for O_2 and O_3

We show here that the bare insertions of operators O_2 and O_3, which are coupled by renormalization to operator O_1, satisfy the same type of Ward identities as those for O_1 [Eqs. (5.1), (5.3), (5.4), and (5.5)] (we emphasize that the latter identities contain no more reference to operators L and $\vec{J}_\mu$). To see this, we take advantage of the equations of motion introduced in Sec. II [(2.11) and (2.29)]:

$$\Gamma_{O_3}^{(n,p)} = p\Gamma^{(n,p)}\,, \qquad (5.6)$$

$$\Gamma_{O_2}^{(n,p)} = \left(n + 2\alpha_0\frac{\partial}{\partial\alpha_0}\right)\Gamma^{(n,p)} - 2\Gamma_{O_1}^{(n,p)}\,.$$

The functions $\Gamma_{O_3}^{(n,p)}$ trivially satisfy Eqs. (5.3) and (5.5). Since O_1 satisfies identities (5.1), (5.3), (5.4), and (5.5) we consider instead of operator O_2 the operator $(n+2\alpha_0\partial/\partial\alpha_0)\Gamma^{(n,p)}$, which reduces to the following expression in view of the WI for $\Gamma^{(n,p)}$ [Eq. (4.19)]:

$$\frac{1}{\Gamma^{(n,p)}}\left(n + 2\alpha_0\frac{\partial}{\partial\alpha_0}\right)\Gamma^{(n,p)} = \{nX_1 + 2p[1 - X_1 + (X_3 - a)]\} + \cdots\,. \qquad (5.7)$$

The linearity in n and p ensures trivially Eqs. (5.1) and (5.3) and the vanishing of the terms in parentheses in Eqs. (5.4) and (5.5). It is important to note that the operators O_2 and O_3, although they satisfy WI, are not invariant under the transformations (3.3).

C. Ward identities for the renormalized operator O_1'

We take for granted the result of Sec. II that operator O_1',

$$O_1' = O_1 + \Psi(O_1 + \tfrac{1}{2}O_2) + \phi O_3\,, \qquad (5.8)$$

is multiplicatively renormalizable for $\Psi = \gamma g/2\beta - 1$ and $\phi = \gamma_G g/2\beta$. The Ward identities of the first type [Eqs. (5.1) and (5.3)] are obviously satisfied by the renormalized Green's functions with one insertion of O_1'. Again the linearity in n and p of the bare insertions of $O_1 + \frac{1}{2}O_2$ and of O_3 implies the vanishing of the contribution of the derivatives $\partial\Psi(\alpha_0, g_0)/\partial\alpha_0$ and $\partial\phi(\alpha_0, g_0)/\partial\alpha_0$ to the WI (5.4) and (5.5) for the bare insertion of O_1'. Thus the bare operator O_1' satisfies the same identities as the bare operator O_1. Equation (5.4) can be used to show that the counterterm Z for the multiplicatively renormalizable operator O_1' is α-independent in the "minimal" renormalization scheme. We introduce the counterterms Z_1, Z_3, and $Z^{[n]}$ and assume $\alpha(\partial/\partial\alpha)Z^{[n]} = 0$. Then the Green's functions for O_1' renormalized to order n satisfy

$$\frac{\partial}{\partial\alpha}\left(\frac{\Gamma_{O_1'}^{R(3,0)}}{\Gamma^{R(3,0)}} - \frac{3}{2}\frac{\Gamma_{O_1'}^{R(2,0)}}{\Gamma^{R(2,0)}}\right) + \cdots = 0\,. \qquad (5.9)$$

The divergences of these functions at order $(n+1)$ are

$$\text{div. part}\left(\frac{\Gamma_{O_1'}^{R(3,0)}}{\Gamma^{R(3,0)}} - \frac{3}{2}\frac{\Gamma_{O_1'}^{R(2,0)}}{\Gamma^{R(2,0)}}\right) = -Z^{(n+1)}(1 - \tfrac{3}{2})\,.$$

Thus it follows that $(\partial/\partial\alpha)Z = 0$ to all orders.

Let us note that there exists another multiplicatively renormalizable operator $(O_1 + \frac{1}{2}O_2 + \phi' O_3)$ whose bare functions satisfy WI (5.4). However, this identity says nothing about the α dependence of its counterterm because $(O_1 + \frac{1}{2}O_2 + \phi' O_3)$ in the tree approximation gives a vanishing contribution to (5.4). Indeed, the corresponding anomalous dimension (2.27), $\gamma_2 = \alpha\partial\gamma/\partial\alpha$, depends explicitly on α.

VI. CONCLUSION

The gauge transformations introduced by Becchi, Rouet, and Stora[8] appeared to be extremely useful for the study of various properties related to gauge invariance. In particular we have obtained by this method the results derived otherwise by Caswell and Wilczek,[9] namely, the α independence of the β function for a specific choice of renormalization prescription.[25] This tool was also well suited for the study of the renormalization of the bare gauge-invariant operator $\vec{F}_{\mu\nu}^{\,2}$, for which a direct method allowed an immediate computation of the corresponding multiplicatively renormalizable operator. All operators coupled through renormalization to $\vec{F}_{\mu\nu}^{\,2}$ satisfy the same Ward identities as $\vec{F}_{\mu\nu}^{\,2}$, where all terms referring to auxiliary ghostlike operators were eliminated, although they are not invariant under gauge transformations. Furthermore, it was shown that there exists one multiplicatively renormalizable operator whose anomalous dimension is α-independent due to Ward identities. Finally, this example shows that the non-gauge-invariant operators coupled to $\vec{F}_{\mu\nu}{}^2$ by renormalization cannot be ignored for the computation of this α-independent anomalous dimension, even at the one-loop level.

We conjecture that the same conclusions hold for any gauge-invariant operator: A bare gauge-invariant operator couples by renormalization to all operators which satisfy the same Ward identities; to each bare gauge-invariant operator corresponds a multiplicatively renormalizable operator which satisfies similar Ward identities and which possesses an α-independent anomalous dimension. It seems hard, however, to derive the expression of this operator on the basis of Ward identities only. For the present, we feel that our analysis through Ward identities could be successfully applied to the study of the renormalization of twist-two gauge-invariant operators. On the other hand, it might be simpler for the study of matrix elements of gauge-invariant operators between physical states to consider Green's functions with insertions of such operators and without external legs.[28] An alternative and attractive approach is provided by the axial gauge: In the example studied here, and in this gauge, $\vec{F}_{\mu\nu}{}^2$ is coupled by renormalization only to itself, and the correct gauge-invariant anomalous dimension is obtained to all orders in perturbation theory; the complications due to the mixing with non-gauge-invariant operators are thus avoided. Both from the fundamental and computational point of view, it would be of great interest to understand the quantization in the axial gauge.

Finally there remains another tough problem: the possible occurrence of non-gauge-invariant operators in the Wilson expansion and, eventually, the vanishing of their matrix elements between physical states. Intuitively, any operator corresponding to an α-dependent anomalous dimension should not be observable; this calls for a clean proof.

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APPENDIX A: USE OF WARD IDENTITIES FOR RENORMALIZATION PROCEDURE

This appendix is intended to show how easily the WI [Eq. (3.18)] and equation of motion (3.15) are exploited for the renormalization procedure:

$$\partial_\mu \frac{\delta\hat{\Gamma}}{\delta J^i_\mu(x)} = \frac{\delta\hat{\Gamma}}{\delta\bar{C}^i(x)} \tag{A1}$$

and

$$\int \left[\frac{\delta\hat{\Gamma}}{\delta A^i_\mu(x)}\,\frac{\delta\hat{\Gamma}}{\delta J^i_\mu(x)} - \frac{\delta\hat{\Gamma}}{\delta C^i(x)}\,\frac{\delta\hat{\Gamma}}{\delta K^i(x)}\right] d^4x = 0\,, \tag{A2}$$

where $\hat{\Gamma}$ denotes

$$\hat{\Gamma} = \Gamma + \frac{1}{2\alpha}\int [\partial_\mu \vec{A}^\mu(x)]^2\, d^4x\,. \tag{A3}$$

We introduce the following parametrization (Fig. 2):

$$\frac{\delta^2\Gamma}{\delta C^i \delta J^j_\mu} = -\,ip_\mu \tilde{I}(p^2)\delta_{ij}\,, \tag{A4}$$

where p_μ is the incoming momentum of the ghost. Here and in the following, the arguments of the functional Γ are set equal to zero after differentiation. Then differentiating equation (A1) with re-

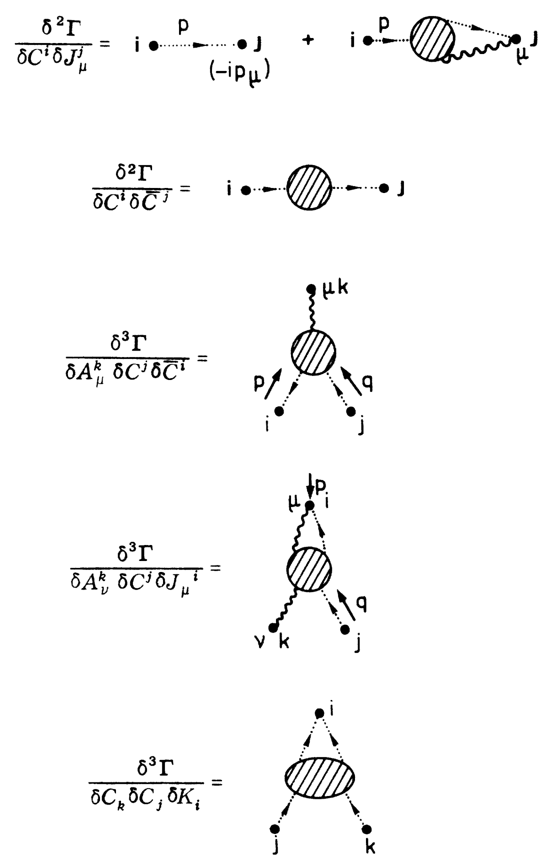

FIG. 2. Graphical representation of the Green's functions involved in Eqs. (A4), (A5) and (A7).

spect to C gives

$$\frac{\delta^2\hat{\Gamma}}{\delta C^i\delta\overline{C}^j}=-p^2\tilde{I}(p^2)\delta_{ij}\,, \tag{A5}$$

which tells us that no ghost mass counterterm is needed. On the other hand, by differentiating Eq. (A2) with respect to both A_ν and C, we obtain

$$\frac{\delta^2\hat{\Gamma}}{\delta A^i_\mu\delta A^i_\nu}p_\mu\tilde{I}(p^2)=0\,. \tag{A6}$$

Thus the two-vector Green's function $\delta^2\hat{\Gamma}/\delta A^i_\mu\delta A^i_\nu$ remains transverse and no counterterm of the form $(\partial_\mu\vec{A}^\mu)^2$ is introduced in the Lagrangian.

We now have to introduce further parametrizations. For the sake of simplicity, and because it is sufficient for the purpose of renormalization, we parametrize the three-leg Green's functions at the symmetric point $p^2=q^2=r^2$. Besides the functions defined in (4.10), we shall use the following notations (see Fig. 2):

$$\frac{\delta^3\Gamma}{\delta A^k_\mu\delta C^j\delta\overline{C}^i}=-igf_{ijk}[p_\mu\tilde{G}(p^2)+q_\mu\tilde{G}_1(p^2)]\,, \tag{A7a}$$

$$\frac{\delta^3\Gamma}{\delta A^k_\nu\delta C^j\delta J^i_\mu}=-gf_{ijk}[g_{\mu\nu}\tilde{H}(p^2)+p_\mu p_\nu\tilde{H}_1(p^2)+p_\mu q_\nu\tilde{H}_2(p^2)+p_\nu q_\mu\tilde{H}_3(p^2)+q_\mu q_\nu\tilde{H}_4(p^2)]\,, \tag{A7b}$$

$$\frac{\delta^3\Gamma}{\delta C_k\delta C_j\delta K_i}=gf_{ijk}\tilde{K}(p^2)\,. \tag{A7c}$$

Apart from the four-vector Green's function, only the Green's functions involved in Eqs. (4.10), (A4), (A5), and (A7) are superficially divergent. From (A1), one now easily derives the relations

$$\tilde{H}+p^2\tilde{H}_1+p\cdot q\tilde{H}_3=\tilde{G}\,, \tag{A8a}$$

$$p^2\tilde{H}^2+p\cdot q\tilde{H}_4=\tilde{G}_1,\ \text{with } p\cdot q=-\tfrac{1}{2}p^2\,. \tag{A8b}$$

Thus $\tilde{G}_1$ is superficially convergent, and the renormalizations of $\tilde{G}$ and $\tilde{H}$ are related.

Similarly, by differentiation of Eq. (A2), we obtain

$$(\tilde{G}+\tilde{G}_1)\cdot\tilde{I}=\tilde{K}\cdot\tilde{I} \tag{A9}$$

and the important relation

$$[\tilde{H}+\tfrac{1}{2}p^2(\tilde{H}_3+\tilde{H}_4)]\cdot I=[G-\tfrac{1}{2}p^2G_2]\tilde{I}\,. \tag{A10}$$

In view of Eq. (A8a), we write it as

$$\tilde{G}\cdot I=G\cdot\tilde{I}+\cdots\,.$$

The dots denote a sum of products of superficially divergent amplitudes, namely I or $\tilde{I}$, by superficially convergent amplitudes, of order $O(g^2)$, such as G_2 or $\tilde{H}_i$. In the "minimal" renormalization procedure described in the end of Sec. II, these terms are of no importance for the determination of the counterterms. In contradistinction, if one chooses to impose some renormalization conditions at a fixed Euclidean point, one then has to take these terms into account in order to have constraints consistent with gauge invariance.

For the sake of completeness, we should also derive the relation between three-vector and four-vector vertices. This is simply obtained by further differentiation of Eq. (A1).

Finally, we obtain the renormalized generating functional of Eq. (3.19).

APPENDIX B: WARD IDENTITIES IN THE PRESENCE OF MATTER FIELDS

The derivation of Ward identities by means of gauge transformations developed in Sec. III is immediately extended to the case where matter fields are introduced. Suppose for example that we consider spinor fields, which transform like some irreducible representation of the gauge group; the infinitesimal transformations of the fields read

$$\begin{aligned}\delta\psi_a(x)&=ig_0T^i_{ab}\delta\omega^i(x)\psi_b(x)\,,\\ \delta\overline{\psi}_a(x)&=-ig_0\overline{\psi}_b(x)T^i_{ba}\delta\omega^i(x)\,.\end{aligned} \tag{B1}$$

The Hermitian matrices T form a representation of the Lie algebra:

$$[T^i,T^j]=if_{ijk}T^k\,. \tag{B2}$$

We now want to use the special gauge transformation (B1) where

$$\delta\omega^i(x)=C^i(x)\delta\lambda \tag{B3}$$

as studied in Sec. III.

Besides the Lagrangian of the ψ field and source terms for the ψ and $\overline{\psi}$ fields, we also introduce, as in Sec. III for the A and C fields, source terms for the variations of ψ and $\overline{\psi}$; we thus add to the action of Eq. (3.7) the term

$$\int d^4x\Big[\overline{\psi}_a(x)(i\not{D}-m)_{ab}\psi_b(x)+\overline{\zeta}_a(x)\psi_a(x)+\overline{\psi}_a(x)\zeta_a(x)+ig_0\overline{\psi}_a(x)T^i_{ab}C^i(x)M_b(x)+ig_0\overline{N}_a(x)T^i_{ab}C^i(x)\psi_b(x)\Big]\,, \tag{B4}$$

where $D^{ab}_\mu\psi_b$ denotes the covariant derivative of ψ:

$$D_\mu^{ab}\psi_b = \partial_\mu \psi_a - ig_0 A_\mu^i T_{ab}^i \psi_b \,. \tag{B5}$$

It is easy to see that under gauge transformations (3.3), (B1), and (B3), the terms $\bar{\psi}_a T^i_{ab} C^i$ and $T^i_{ab} C^i \psi_b$ are invariant. The new Ward identity satisfied by the 1PI generating functional is then readily derived:

$$\int d^4x \left(\frac{\delta \hat{\Gamma}}{\delta A_\mu^i} \frac{\delta \Gamma}{\delta J_\mu^i} - \frac{\delta \Gamma}{\delta C^i} \frac{\delta \Gamma}{\delta K^i} + \frac{\delta \Gamma}{\delta \psi_a} \frac{\delta \Gamma}{\delta N_a} + \frac{\delta \Gamma}{\delta \bar{\psi}_a} \frac{\delta \Gamma}{\delta M_a} \right) = 0 \tag{B6}$$

and the usual relation[11-14] between charge and wave-function renormalization for the ψ_a field follows immediately.

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*Chargée de Recherches au CNRS.

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[1]G. 't Hooft, unpublished; D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973).

[2]C. G. Callan and D. J. Gross, Phys. Rev. D 8, 4383 (1973).

[3]S. Weinberg, in proceedings of the Second International Conference on Elementary Particles, Aix-en-Provence, 1973 [J. Phys. (Paris) Supp. 34, C1-11–12 (1973) and Rev. Mod. Phys. 46, 255 (1974)].

[4]G. 't Hooft and M. Veltman, Nucl. Phys. B50, 318 (1972); R. E. Kallosh and I. V. Tyutin, Yad. Fiz. 17, 190 (1973) [Sov. J. Nucl. Phys. 17, 98 (1973)]; B. W. Lee and J. Zinn-Justin, Phys. Rev. D 7, 1049 (1973).

[5]J. H. Lowenstein and B. Schroer, Phys. Rev. D 6, 1553 (1972).

[6]D. J. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).

[7]H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).

[8]C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. 42, 127 (1975).

[9]W. E. Caswell and F. Wilczek, Phys. Lett. 49B, 291 (1974).

[10]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); Kiev Report No. ITP-67-36 (unpublished).

[11]A. A. Slavnov, Teor. Mat. Fiz. 10, 153 (1972) [Theor. Math. Phys. 10, 99 (1972)].

[12]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972).

[13]G. 't Hooft, Nucl. Phys. B33, 173 (1971).

[14]J. C. Taylor, Nucl. Phys. B33, 436 (1971).

[15]One can easily convince oneself that the insertion of O_1 at zero momentum or of the operators O_i introduced further do not give rise to infrared divergences, provided that external momenta are nonexceptional.

[16]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT Press, Cambridge, Mass., 1970), Vol. 1.

[17]J. H. Lowenstein, Phys. Rev. D 4, 2281 (1970).

[18]S. Coleman, lectures given at the 1973 International Summer School of Physics "Ettore Majorana," Erice, Italy, 1973 (unpublished).

[19]R. N. Mohapatra, Phys. Rev. D 4, 378 (1971); 4, 1007 (1971); 4, 2215 (1971); 5, 417 (1972).

[20]J. M. Cornwall, Phys. Rev. D 10, 500 (1974). Further references may be found in this paper.

[21]R. E. Kallosh, Nucl. Phys. B78, 293 (1974).

[22]D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).

[23]It is easy to convince oneself that the invariance of $\mathcal{L}$ under transformations (3.3) holds even when the parameter $\delta\lambda$ is considered as a commuting object, provided that the signs in the right-hand sides of (3.3b) and (3.3c) are changed. However, we prefer the first formulation because then the transformations (3.3) do not change the commuting or anticommuting nature of the variables; the derivation of Ward identities is then more transparent.

[24]B. W. Lee, Phys. Lett. 46B, 214 (1973); Phys. Rev. D 9, 933 (1974).

[25]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972). The dimensional regularization is a very convenient tool for computations which up to now have been performed only to lowest orders. The existence of unambiguous rules for the analytic continuation of any Feynman graph involving massless particles and their consistency with gauge invariance have not been established so far.

[26]G. 't Hooft, Nucl. Phys. B61, 455 (1973).

[27]The insertion of L in a graph with nonexceptional momenta has no infrared divergences.

[28]S. Coleman, private communication.