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Rarita-Schwinger particles in homogeneous magnetic fields, and inconsistencies of spin-$\frac{3}{2}$ theories

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The Rarita-Schwinger equation for a spin-$\frac{3}{2}$ particle with minimal electromagnetic coupling is solved completely in the case when a constant homogeneous external magnetic field $\mathcal{H}$ is present. It is shown that the spectrum of energy eigenvalues includes complex values if $\mathcal{H}$ is such that $\eta \equiv (2e\mathcal{H}/3m^2) > 1$, and further that the norm of the Rarita-Schwinger wave function (i.e., the total "charge" integral defined from the Lagrangian) which is positive definite for $\eta < 1$ becomes indefinite (even after taking account of the constraints) when η exceeds unity. These results confirm that the difficulties in quantization first discovered by Johnson and Sudarshan are a reflection of the indefiniteness of the norm which appears already at the c-number level, and suggest that the nature of the energy spectrum (whether or not complex values are present) in the presence of very large magnetic fields would provide a quick means of predicting whether such difficulties would arise in quantization.

I. INTRODUCTION

Relativistic wave equations for particles of arbitrary spin have been under investigation for nearly four decades now,[1] but the question of consistency of higher-spin theories in the presence of interactions still remains a live problem. Though one learned long ago to formulate equations free of algebraic inconsistencies[2] by deriving them (including constraints necessary to ensure uniqueness of mass and spin) from a suitable Lagrangian, more subtle types of inconsistencies have come to light in recent years. The earliest of these was the demonstration by Johnson and Sudarshan[3] that in certain manifestly covariant spin-$\frac{3}{2}$ theories with minimal electromagnetic coupling, quantization via Schwinger's procedure[4] leads to an indefinite sign for certain formally positive-definite quantities (anticommutators of fields) and to apparent violation of Lorentz invariance.[5] A similar result has since been shown by Hagen[6] to hold in the case of a spin-$\frac{3}{2}$ field coupled linearly to a spinor field and a scalar field.[7] Another surprising type of difficulty which manifests itself even in the context of c-number theory was brought out by Velo and Zwanziger,[8] who showed by using the method of characteristics[9] that the propagation of the Rarita-Schwinger field[10] for spin-$\frac{3}{2}$ minimally coupled to the electromagnetic field is noncausal. Extensive work[11] following this discovery has led to the identification of a variety of higher-spin theories wherein this type of difficulty occurs and a couple of spin-$\frac{3}{2}$ theories which are free of it.[12,13] Yet another type of inconsistency is the appearance of "normal modes" (or "stationary states" in the quantum particle picture[14]) whose frequencies cease to be real when the magnitude of the external fields exceeds some critical value. This had been shown earlier to happen in the case of spin-1 particles with anomalous magnetic moment moving in a homogeneous magnetic field[15,16] (h.m.f). The method developed by one of us[15] for tackling that problem is easily extended to higher spins, and one of our objectives in this paper is to make a systematic analysis of the behavior of spin-$\frac{3}{2}$ particles (described by the Rarita-Schwinger equation with minimal electromagnetic coupling) in homogeneous magnetic fields. Apart from its intrinsic interest (especially as the first nontrivial example of an explicit solution for spin > 1) the analysis is of value in that it sheds a great deal of light on the circumstances with which the various types of inconsistencies are associated. In particular, it brings out the fact that the Johnson-Sudarshan inconsistency, the indefiniteness of the norm of the c-number wave functions, and the appearance of complex energy eigenvalues in the c-number theory—all occurring for $\eta > 1$—are all closely interrelated phenomena. The relation between the first two was shown by Velo and Zwanziger[8] and is confirmed by our explicit analysis. The connection between these and the occurrence of complex energy modes, demonstrated in this paper, does not seem to have been known so far.[17] Our results reinforce the inference from the work of Velo and Zwanziger that valuable insights into possible inconsistencies in quantization can be gained from appropriate studies of the c-number theory.[18]

The presentation of the material of this paper is as follows. We consider first, in Sec. II, the problem of a Dirac particle with anomalous magnetic moment in a h.m.f. This section serves to display the method of solution clearly in a relatively simple context before we go to the main problem. We reduce the solution of the equation to that of an eigenvalue problem for a 4×4 matrix by exploiting the intimate connection of the problem of a charged particle in a magnetic field to that of a harmonic oscillator. The energy eigenvalues found do coincide with earlier determinations,[16,19,20] but our treatment is vastly simpler. In Sec. III we take up the Rarita-Schwinger equation for spin-$\frac{3}{2}$, with minimal coupling. Observing that the equations for certain components decouple from the others (in the case of a h.m.f.) we reduce these equations to a matrix eigenvalue problem, and solve it to find the stationary-state wave functions and energies E of the spin-$\frac{3}{2}$ particle. The matrix in this case turns out to be non-Hermitian, and its eigenvalues E are found to be all real if and only if $\mathcal{H}$ is below a critical value $\mathcal{H}_c = (3m^2/2e)$. Above $\mathcal{H}_c$ complex eigenvalues appear (besides the crossing over of positive eigenvalues to the negative side and vice versa in the case of some states). At the same time, the Lorentz-invariant norm (the total "charge") of the Rarita-Schwinger field ceases to be positive definite. This fact is demonstrated in Sec. IV, where we first consider the norm of energy eigenfunctions, taking due account of the constraints which cause some components of the wave function to be expressible in terms of the others. It is pointed out that though any state corresponding to a complex value of E has zero norm, its scalar product with a state belonging to E^* is nonzero, and such pairs of states do contribute to physical quantities (the contribution to the total charge is of indefinite sign). The final section (Sec. V) is devoted to a discussion of the results.

II. DIRAC PARTICLE WITH ANOMALOUS MAGNETIC MOMENT IN A H.M.F.

We consider first the Dirac equation with an anomalous magnetic moment coupling[21]

$$\left(\gamma\cdot\pi + m - \frac{e\kappa}{4m}\sigma^{\mu\nu}F_{\mu\nu}\right)\psi = 0\ , \tag{1}$$

where $\pi_\mu = p_\mu + eA_\mu = i\partial/\partial x^\mu + eA_\mu$; A_μ is the electromagnetic potential and κ is the strength of the Pauli interaction. We are interested in solving this equation in the particular case of a constant h.m.f. along the z direction, so we set

$$F_{12} = -F_{21} = \mathcal{H}\ \text{(a constant)}\ , \tag{2}$$

with all the other components of $F_{\mu\nu}$ vanishing. Equation (1) then reduces to

$$\left(\gamma\cdot\pi + m - \frac{e\kappa\mathcal{H}}{2m}\Sigma_3\right)\psi = 0\ , \tag{3}$$

where

$$\Sigma_3 = i\gamma_1\gamma_2\ . \tag{4}$$

Since we are seeking stationary-state solutions characterized by the time dependence e^{-iEt}, we replace $\pi^0 \equiv p^0$ by E. Then writing ψ in the partitioned form

$$\psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix} \tag{5}$$

we can write (3) as the following pair of equations for ϕ and χ:

$$\left(\epsilon - 1 + \frac{\xi\kappa}{2}\sigma_3\right)\phi - \frac{1}{m}(\vec{\sigma}\cdot\vec{\pi})\chi = 0\ , \tag{6a}$$

$$\left(\epsilon + 1 - \frac{\xi\kappa}{2}\sigma_3\right)\chi - \frac{1}{m}(\vec{\sigma}\cdot\vec{\pi})\phi = 0\ , \tag{6b}$$

where

$$\epsilon = (E/m)\ \text{and}\ \xi = (e\mathcal{H}/m^2)\ . \tag{7}$$

As indicated in the Introduction, we shall solve the (partial differential) equations (6) by converting them into an ordinary matrix eigenvalue equation. This is done by exploiting the fact that π_+ and π_- ($\pi_\pm = \pi_1 \pm i\pi_2$) obey an algebra equivalent to that of a simple harmonic oscillator.[22] In fact, with

$$a = (2m^2\xi)^{-1/2}\pi_+,\quad a^\dagger = (2m^2\xi)^{-1/2}\pi_-,\quad N \equiv a^\dagger a \tag{8}$$

we have

$$\begin{aligned}&[a, a^\dagger] = 1,\quad [a, \pi_3] = 0,\quad [a^\dagger, \pi_3] = 0\ ,\\ &[N, a] = -a,\quad [N, a^\dagger] = a^\dagger\ .\end{aligned} \tag{9}$$

Since π_3 commutes with all other operators appearing in (6), we can replace it by its eigenvalue $p_3 = (2m^2\xi)^{1/2}a_3$, where a_3 is any real number. In terms of a and $a^\dagger$, Eqs. (6) may then be written as

$$\left(\epsilon - 1 + \frac{\xi\kappa}{2}\sigma_3\right)\phi - (2\xi)^{1/2}(a^\dagger\sigma_+ + a\sigma_- + a_3\sigma_3)\chi = 0\ , \tag{10a}$$

$$\left(\epsilon + 1 - \frac{\xi\kappa}{2}\sigma_3\right)\chi - (2\xi)^{1/2}(a^\dagger\sigma_+ + a\sigma_- + a_3\sigma_3)\phi = 0\ , \tag{10b}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$.

Let us now define states $|n, \alpha\rangle$ and $|n, \beta\rangle$ by

$$\begin{aligned}&N|n,\alpha\rangle = n|n,\alpha\rangle\ ,\quad N|n,\beta\rangle = n|n,\beta\rangle,\\ &\sigma_3|n,\alpha\rangle = |n,\alpha\rangle\ ,\quad \sigma_3|n,\beta\rangle = -|n,\beta\rangle,\\ &\quad n = 0, 1, 2, \ldots.\end{aligned} \tag{11}$$

If we imagine ϕ and χ to be expanded in terms of these states, it is readily seen by inspection that Eqs. (10) couple any state $|n, \beta\rangle$ to $|n+1, \alpha\rangle$ ($n=0, 1, 2, \ldots$), while the state $|0, \alpha\rangle$ is decoupled from all others. So the solutions of (10) are found to be of the general form

$$\begin{aligned} \phi &= c_1|n+1, \alpha\rangle + c_2|n, \beta\rangle , \\ \chi &= c_3|n+1, \alpha\rangle + c_4|n, \beta\rangle , \\ & n=0, 1, 2, \ldots \end{aligned} \tag{12a}$$

apart from the special solution

$$\phi = c_1'|0, \alpha\rangle , \quad \chi = c_3'|0, \alpha\rangle . \tag{12b}$$

The c's here are constants which are as yet undetermined.

Substituting (12) in (10) and separately equating the coefficients of $|n+1, \alpha\rangle$ and $|n, \beta\rangle$ on the two sides we obtain four linear algebraic equations for the c_i which may be written as

$$Hc = \epsilon c , \tag{13a}$$

with c = column vector (c_1, c_2, c_3, c_4) and

$$H = \begin{bmatrix} 1-\frac{\xi\kappa}{2} & 0 & (2\xi)^{1/2}a_3 & \rho_{n+1} \\ 0 & 1+\frac{\xi\kappa}{2} & \rho_{n+1} & -(2\xi)^{1/2}a_3 \\ (2\xi)^{1/2}a_3 & \rho_{n+1} & -\left(1-\frac{\xi\kappa}{2}\right) & 0 \\ \rho_{n+1} & -(2\xi)^{1/2}a_3 & 0 & -\left(1+\frac{\xi\kappa}{2}\right) \end{bmatrix}, \tag{13b}$$

where $\rho_n = (2\xi n)^{1/2}$. Since H is Hermitian its eigenvalues are all real, and they may be obtained from the characteristic equation

$$[(\epsilon^2 - 2\xi a_3^2) - (\tfrac{1}{4}\xi^2\kappa^2 + 1 + \rho_{n+1}^2)]^2 = \xi^2\kappa^2(1 + \rho_{n+1}^2) .$$

We get

$$\epsilon^2 = 2\xi a_3^2 + [(1+\rho_{n+1}^2)^{1/2} \pm \tfrac{1}{2}\xi\kappa]^2 , \quad n=0, 1, 2, \ldots . \tag{14a}$$

In the special case (12b), the value of ϵ^2 turns out to be

$$\epsilon^2 = 2\xi a_3^2 + (1 - \tfrac{1}{2}\xi\kappa)^2 , \tag{14b}$$

which is the same as (14a) but with $n=-1$ and the lower sign taken in ±. Clearly $\epsilon^2 > 0$ for all n and ξ, as it must be. Our results agree with those given in the earlier literature,[16,19,20] but the derivation here is far simpler.

III. THE SPIN-$\frac{3}{2}$ (RARITA-SCHWINGER) PARTICLE IN A H.M.F.

The Rarita-Schwinger equation for spin-$\frac{3}{2}$ particles employs a 16-component vector-spinor ψ_μ for the wave function. In the presence of minimal coupling to an external electromagnetic field $F_{\mu\nu}$, the Lagrangian density is given by

$$\mathcal{L} = -\bar{\psi}^\mu(\Gamma\cdot\pi + mB)_\mu{}^\nu \psi_\nu . \tag{15}$$

Here $\bar{\psi}_\mu = \psi_\mu^\dagger\gamma^0$ and the matrices $(\Gamma^\alpha)_\mu{}^\nu$ and $B_\mu{}^\nu$ are given by

$$\begin{aligned} (\Gamma^\alpha)_\mu{}^\nu &= \gamma^\alpha g_\mu{}^\nu + W(\gamma_\mu g^{\nu\alpha} + g_\mu{}^\alpha\gamma^\nu) - K\gamma_\mu\gamma^\alpha\gamma^\nu , \\ B_\mu{}^\nu &= -(g_\mu{}^\nu - T\gamma_\mu\gamma^\nu) , \end{aligned} \tag{16}$$

with

$$K = -\tfrac{1}{2}(3W^2 + 2W + 1) ,$$

$$T = (3W^2 + 3W + 1) ,$$

where W is arbitrary (except that $W \neq -\frac{1}{2}$). Since the parameter W is without physical significance[3,23] and can be chosen at will, we shall henceforth set $W=0$.[24] With this choice, the Euler-Lagrange equations obtained from (15) by variation with respect to $\bar{\psi}^\mu$ are

$$(\gamma\cdot\pi - m)\psi_\mu + \tfrac{1}{2}\gamma_\mu(\gamma\cdot\pi + 2m)\gamma\cdot\psi = 0 . \tag{17}$$

On contracting (17) successively with γ^μ and π^μ and comparing the resulting expressions, one gets the constraint relation

$$\gamma\cdot\psi = \frac{2}{3}\frac{ie}{m^2}\gamma\cdot F\cdot\psi \equiv \frac{2}{3}\frac{ie}{m^2}\gamma^\mu F_\mu{}^\nu\psi_\nu . \tag{18}$$

Feeding (18) into (17), we obtain the true equation of motion:

$$(\gamma\cdot\pi - m)\psi_\mu + \frac{1}{3}\frac{ie}{m^2}\gamma_\mu(\gamma\cdot\pi + 2m)\gamma\cdot F\cdot\psi = 0 . \tag{19}$$

It is a simple matter to verify that (19) and (18) are together equivalent to (17).

When the external field is a homogeneous magnetic field, Eq. (19) reduces to the form

$$(\gamma\cdot\pi - m)\psi_\mu - \tfrac{1}{2}\eta\gamma_\mu(\gamma\cdot\pi + 2m)(\gamma_+\psi_- - \gamma_-\psi_+) = 0 , \tag{20}$$

where

$$\eta = \tfrac{2}{3}\xi \equiv \frac{2}{3}\frac{e\mathcal{H}}{m^2} , \quad \gamma_\pm = \tfrac{1}{2}(\gamma_1 \pm i\gamma_2) , \tag{21}$$

and

$$\psi_\pm = \psi_1 \pm i\psi_2 .$$

It is easy to see from (20) that the equations for $\psi_\pm$ do not involve ψ_3 and ψ_0. The coupled equations for the former may be seen to be

$$\Gamma_0^{(+)}\pi_0\psi_+=\gamma_0(\vec{\gamma}\cdot\vec{\pi}+m)\psi_+ - \eta\gamma_0\gamma_+(\vec{\gamma}\cdot\vec{\pi}-2m)(\gamma_+\psi_- - \gamma_-\psi_+) \quad (22a)$$

and

$$\Gamma_0^{(-)}\pi_0\psi_-=\gamma_0(\vec{\gamma}\cdot\vec{\pi}+m)\psi_- - \eta\gamma_0\gamma_-(\vec{\gamma}\cdot\vec{\pi}-2m)(\gamma_+\psi_- - \gamma_-\psi_+)\,, \quad (22b)$$

where

$$\Gamma_0^{(+)}=1+\tfrac{1}{2}\eta(1+\Sigma_3)\,, \quad (23a)$$

$$\Gamma_0^{(-)}=1-\tfrac{1}{2}\eta(1-\Sigma_3)\,. \quad (23b)$$

The solution of Eqs. (22) is facilitated by partitioning each of the $\psi_\pm$ into two-component entities $\phi_\pm$ and $\chi_\pm$:

$$\psi_\pm=\begin{pmatrix}\phi_\pm\\ \chi_\pm\end{pmatrix}\,. \quad (24)$$

Introducing this in Eqs. (22) one obtains

$$[(E-m)+\tfrac{1}{2}\eta(E-2m)(1+\sigma_3)]\phi_+ - [\vec{\sigma}\cdot\vec{\pi}-\eta\sigma_+(\vec{\sigma}\cdot\vec{\pi})\sigma_-]\chi_+ - \eta\sigma_+(\vec{\sigma}\cdot\vec{\pi})\sigma_+\chi_-=0\,, \quad (25a)$$

$$[(E-m)-\tfrac{1}{2}\eta(E-2m)(1-\sigma_3)]\phi_- + \eta\sigma_-(\vec{\sigma}\cdot\vec{\pi})\sigma_-\chi_+ - [\vec{\sigma}\cdot\vec{\pi}+\eta\sigma_-(\vec{\sigma}\cdot\vec{\pi})\sigma_+]\chi_-=0\,, \quad (25b)$$

$$[\vec{\sigma}\cdot\vec{\pi}-\eta\sigma_+(\vec{\sigma}\cdot\vec{\pi})\sigma_-]\phi_+ + \eta\sigma_+(\vec{\sigma}\cdot\vec{\pi})\sigma_+\phi_- - [(E+m)+\tfrac{1}{2}\eta(E+2m)(1+\sigma_3)]\chi_+=0\,, \quad (25c)$$

$$\eta\sigma_-(\vec{\sigma}\cdot\vec{\pi})\sigma_-\phi_+ - [\vec{\sigma}\cdot\vec{\pi}+\eta\sigma_-(\vec{\sigma}\cdot\vec{\pi})\sigma_+]\phi_- + [(E+m)-\tfrac{1}{2}\eta(E+2m)(1-\sigma_3)]\chi_-=0\,. \quad (25d)$$

In writing Eqs. (25), we have replaced π^0 by E. As in the Dirac case, $\phi_\pm$ and $\chi_\pm$ can be expanded in terms of $|n,\alpha\rangle$ and $|n,\beta\rangle$. Knowledge of the effect of the operators a and $a^\dagger$, Eq. (8), and of $\sigma_\pm$ and σ_3 on these states enables us to infer from Eqs. (25) that the following admixtures of the states $|n,\alpha\rangle$, $|n,\beta\rangle$ occur in $\phi_\pm$ and $\chi_\pm$:

$$\begin{aligned}\phi_+&=u_1|n,\alpha\rangle+v_1|n-1,\beta\rangle\\ \phi_-&=u_2|n+2,\alpha\rangle+v_2|n+1,\beta\rangle\\ \chi_+&=u_3|n-1,\beta\rangle+v_3|n,\alpha\rangle\\ \chi_-&=u_4|n+1,\beta\rangle+v_4|n+2,\alpha\rangle\\ &(n=1,2,\ldots)\,.\end{aligned} \quad (26)$$

Besides the above expressions for general n, the following special cases also exist:

$$\phi_+=u_1|0,\alpha\rangle\,,\quad \phi_-=u_2|2,\alpha\rangle+v_2|1,\beta\rangle\,,\quad \chi_+=v_3|0,\alpha\rangle\,,\quad \chi_-=u_4|1,\beta\rangle+v_4|2,\alpha\rangle\,; \quad (27a)$$

$$\phi_+=0,\quad \phi_-=u_2|1,\alpha\rangle+v_2|0,\beta\rangle\,,\quad \chi_+=0,\quad \chi_-=u_4|0,\beta\rangle+v_4|1,\alpha\rangle\,; \quad (27b)$$

$$\phi_+=0,\quad \phi_-=u_2|0,\alpha\rangle\,,\quad \chi_+=0,\quad \chi_-=v_4|0,\alpha\rangle\,. \quad (27c)$$

On substituting (26) [or (27)] into (25) and equating coefficients of $|n,\alpha\rangle$, etc., we get a set of linear equations for the u's and v's. If $p_3=0$ the u's and v's get completely decoupled in the equations. Considering now only the equations for the u_i's, obtained by substituting (26) in (25) with p_3 taken to be zero,[26] we find that they can be put in the form

$$Hu=\epsilon u\quad(\epsilon=E/m)\,, \quad (28a)$$

where $u=\text{col.}(u_1,u_2,u_3,u_4)$ and

$$H=\begin{bmatrix}\dfrac{1+2\eta}{1+\eta} & 0 & \dfrac{\rho_n}{1+\eta} & \dfrac{\eta\rho_{n+1}}{1+\eta}\\ 0 & 1 & 0 & \rho_{n+2}\\ \rho_n & 0 & -1 & 0\\ -\dfrac{\eta\rho_{n+1}}{1-\eta} & \dfrac{\rho_{n+2}}{1-\eta} & 0 & -\dfrac{1-2\eta}{1-\eta}\end{bmatrix}, \quad (28b)$$

with $\rho_n=(3n\eta)^{1/2}$.

It is important to note that, in contrast with the spin-$\frac{1}{2}$ case, this matrix is *not* Hermitian and hence its eigenvalues are not constrained to be real. In fact, when $\eta>1$, complex eigenvalues ϵ do occur for some n. To see this let us consider the characteristic equation for H, which may be verified to be

$$(1-\eta^2)\epsilon^4+2\eta\epsilon^3-[2(1+\rho_{n+1}{}^2)+\eta^2(1-\rho_{n+1}{}^2)]\epsilon^2 -2\eta(1+\rho_{n+1}{}^2)\epsilon+(1+\rho_{n+1}{}^2)(1+\rho_{n+1}{}^2-\eta^2)=0\,. \quad (29)$$

Though the solutions of this quartic equation could in principle be given in a closed form from general theory,[27] their expressions would be so complicated as to be of little use. So we shall content ourselves with determining whether any of the roots of (29) are complex. It is known[27] that for a general quartic of the form

$$ax^4+4bx^3+6cx^2+4dx+e=0 \quad (30)$$

all the four roots will be real if and only if the inequalities

$$\Delta>0,\ a^2I-12K^2<0,\ \text{and}\ K<0 \tag{31}$$

hold simultaneously, where K, I, and Δ are functions of the coefficients a, b, c, d, and e, defined by

$$K=ac-b^2,\quad I=ae-4bd+3c^2\ ,$$
$$\Delta=I^3-27J^2 \text{ with } J=ace+2bcd-ad^2-eb^2-c^3\ . \tag{32}$$

Calculating these for the case of the quartic equation (29), we find after some labor that

$$K=\frac{1}{6}\left[\left(\eta^2+\frac{(33)^{1/2}-1}{4}\right)\left(\eta^2-\frac{(33)^{1/2}+1}{4}\right)-\rho_{n+1}{}^2(1-\eta^2)(2-\eta^2)\right]\ , \tag{33a}$$

$$a^2I-12K^2=\tfrac{1}{4}\eta^2\left[(-16+20\eta^2-8\eta^4+3\eta^6)+2\rho_{n+1}{}^2(-8+17\eta^2-12\eta^4+3\eta^6)-\rho_{n+1}{}^4\eta^2(1-\eta^2)^2\right]\ , \tag{33b}$$

and

$$\Delta=\tfrac{1}{16}\eta^4\sum_{j=0}^{6}A_j(\rho_{n+1}{}^2)^j\ , \tag{33c}$$

where

$$A_0=(-16+8\eta^2+3\eta^4)^2\ ,$$
$$A_1=(1024-1216\eta^2+432\eta^4-132\eta^6+45\eta^8)\ ,$$
$$A_2=(1536-2096\eta^2+1160\eta^4-443\eta^6+66\eta^8)\ ,$$
$$A_3=2(512-776\eta^2+434\eta^4-131\eta^6+9\eta^8)\ , \tag{33d}$$
$$A_4=(256-400\eta^2+145\eta^4+12\eta^6-11\eta^8)\ ,$$
$$A_5=\eta^2(16-26\eta^2+10\eta^4+\eta^6)\ ,$$
$$A_6=\eta^4(1-\eta^2)\ .$$

These quantities are seen to be linear combinations of polynomials in η (with ρ's as coefficients). It is easy to verify that each of these polynomials has a unique sign throughout the interval $\eta^2<1$ and to show therefrom that the conditions (31) are satisfied for $\eta^2<1$. On the other hand, one sees from (33a) that the condition $K<0$ fails for $1<\eta^2<2$ if n is sufficiently large, so that some of the eigenvalues ϵ are complex under these conditions. Further, for any $\eta^2>\frac{4}{3}$ the special solutions (27b) are associated with complex ϵ [see Eq. (35) below]. Thus, the eigenvalues ϵ will be real for *all* the eigenstates if and only if $\eta^2<1$, i.e., the magnetic field $\mathcal{H}$ has a value less than a critical value $\mathcal{H}_c=\frac{3}{2}(m^2/e)$. In the special cases (27a)–(27c), the equations for ϵ are found to be

$$(1-\eta^2)\epsilon^3+(1+2\eta-\eta^2)\epsilon^2-(1+4\eta+2\eta^2-3\eta^3)\epsilon-(1+6\eta+8\eta^2-3\eta^3)=0\ , \tag{34a}$$

$$(1-\eta)\epsilon^2-\eta\epsilon-(1+\eta)=0\ , \tag{34b}$$

$$\epsilon-1=0\ , \tag{34c}$$

respectively. Equation (34c) shows that for the state given by (27c), ϵ is independent of $\mathcal{H}$. From (34b) we find that for states of the type (27b),

$$\epsilon=\frac{\eta\pm(4-3\eta^2)^{1/2}}{2(1-\eta)}\ . \tag{35}$$

Both of these values of ϵ are real for $\eta<2/\sqrt{3}$, though one of them crosses over from the positive to the negative side (passing through infinity) as η increases through the value $\eta=1$. Finally, one can convince oneself with the aid of the theory of cubic equations that all the roots of (34a) are also real if $\eta<1$.

The equations for the coefficients v_i in (26) and (27) can be set up in an entirely analogous manner and solved. The energy eigenvalues obtained are found to be just the negatives of those discussed above.

Returning to the wave function ψ_μ, we observe that the spinors ψ_3 and ψ_0 can be expressed completely in terms of ψ_1, ψ_2, and their space derivatives. In fact, by manipulating (20) and (18), one can readily show that (with $\pi_3=0$)

$$(\gamma_1\pi_1+\gamma_2\pi_2-m)\gamma_0\psi_0=-(\pi_1\psi_1+\pi_2\psi_2)+\tfrac{1}{2}i\eta m(\gamma_1\psi_2-\gamma_2\psi_1) \tag{36a}$$

$$(\gamma_1\pi_1+\gamma_2\pi_2-m)\gamma_3\psi_3=\eta\Sigma_3(\pi_1\psi_1+\pi_2\psi_2)+i(\eta+\Sigma_3)(\pi_1\psi_2-\pi_2\psi_1)+m(1-\tfrac{3}{2}\eta\Sigma_3)(\gamma_1\psi_1+\gamma_2\psi_2)\ . \tag{36b}$$

Inspection of Eqs. (36) shows that expansions analogous to (26) exist for ψ_0 and ψ_3 in the following forms:

$$\psi_0=\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}\ ,\quad \psi_3=\begin{pmatrix}\phi_3\\ \chi_3\end{pmatrix}\ ; \tag{37}$$

$$\phi_3=u_5|n,\beta\rangle+v_5|n+1,\alpha\rangle\ ,$$
$$\chi_3=u_6|n+1,\alpha\rangle+v_6|n,\beta\rangle\ ,$$
$$\phi_0=u_7|n+1,\alpha\rangle+v_7|n,\beta\rangle\ , \tag{38}$$
$$\chi_0=u_8|n,\beta\rangle+v_8|n+1,\alpha\rangle$$
$$(n=1,2,\ldots)\ .$$

On substituting these in (36) one obtains the coefficients $u_5,\ldots,u_8$ explicitly in terms of $u_1,\ldots,u_4$:

$$(1+\rho_{n+1}{}^2)u_5=\left[(1+\tfrac{3}{2}\eta)+\tfrac{1}{2}\rho_{n+1}{}^2(1+2\eta)\right]u_1-\tfrac{1}{2}\rho_{n+1}\rho_{n+2}u_2+\tfrac{1}{2}\rho_n u_3+\tfrac{1}{2}(1-\eta)\rho_{n+1}u_4\ , \tag{39a}$$

$$(1+\rho_{n+1}{}^2)u_6=\tfrac{1}{2}(1+\eta)\rho_{n+1}u_1+\tfrac{1}{2}\rho_{n+2}u_2+\tfrac{1}{2}\rho_n\rho_{n+1}u_3-[\tfrac{1}{2}(1-2\eta)\rho_{n+1}{}^2+(1-\tfrac{3}{2}\eta)]u_4\,, \tag{39b}$$

$$(1+\rho_{n+1}{}^2)u_7=\tfrac{1}{2}(1+\eta)\rho_{n+1}u_1+\tfrac{1}{2}\rho_{n+2}u_2+\tfrac{1}{2}\rho_n\rho_{n+1}u_3+\tfrac{1}{2}(\eta+\rho_{n+1}{}^2)u_4\,, \tag{39c}$$

$$(1+\rho_{n+1}{}^2)u_8=-\tfrac{1}{2}(\eta-\rho_{n+1}{}^2)u_1+\tfrac{1}{2}\rho_{n+1}\rho_{n+2}u_2-\tfrac{1}{2}\rho_n u_3-\tfrac{1}{2}(1-\eta)\rho_{n+1}u_4\,. \tag{39d}$$

The solutions for ψ_3 and ψ_0 corresponding to the special cases (27) as well as the expressions for $v_5,\ldots,v_8$ in terms of $v_1,\ldots,v_4$ can be obtained in an analogous way. These expressions together with Eqs. (39), (28), (26), and (38) determine, for each n, a solution of the wave equation (20) for $p_3=0$. The set of all solutions for all n [including the special solutions (27)] form a complete set of eigensolutions of the problem.

The above discussion holds good for any $\eta\neq 1$. When $\eta=1$ something peculiar happens: $\Gamma_0^{(-)}$ in (22b) reduces to the singular matrix $\frac{1}{2}(1+\Sigma_3)$, so that on projecting Eq. (22b) to the singular subspace of this matrix by applying the projection operator $\frac{1}{2}(1-\Sigma_3)$, one gets a *constraint* relation

$$(1-\Sigma_3)(\vec{\gamma}\cdot\vec{\pi}+m)\psi_- -(1-\Sigma_3)\gamma_-(\vec{\gamma}\cdot\vec{\pi}-2m)(\gamma_+\psi_- -\gamma_-\psi_+)=0\,, \tag{40}$$

which may be rewritten as the pair of equations

$$\sigma_-\pi_-\chi_+ -\sigma_-\pi_+\chi_- +\tfrac{1}{2}m(1-\sigma_3)\phi_-=0\,, \tag{41a}$$

$$\sigma_-\pi_-\phi_+ -\sigma_-\pi_+\phi_- -\tfrac{1}{2}m(1-\sigma_3)\chi_-=0\,. \tag{41b}$$

On substituting (26) in these equations one obtains

$$\rho_{n+1}u_1-\rho_{n+2}u_2-u_4=0 \tag{42}$$

and a similar relation involving v_3, v_4, and v_2. These seem to lead to the paradoxical situation of there being only three instead of four independent spinors for the spin-$\frac{3}{2}$ particle (and an equal number for antiparticles) at $\eta=1$. What actually happens is that the eigenvalue ϵ corresponding to one of the four states (for any given n) tends to infinity as $\eta\to 1$; this may be seen from inspection of Eq. (29). Simultaneously the norm of this state tends to zero, as we shall see in the next section, but its contribution to the energy of the spin-$\frac{3}{2}$ field remains finite.

IV. THE SCALAR PRODUCT AND INDEFINITE METRIC

The calculations of the previous section show that the spectrum of values of E for the Rarita-Schwinger particle in a h.m.f. cannot be purely real for arbitrary values of the external field. However, for $\mathcal{H}<\mathcal{H}_c$, the energy spectrum is real for all n. This behavior of H is closely paralleled by the behavior of the scalar product in the theory: For $\mathcal{H}<\mathcal{H}_c$, the scalar product is positive, while it becomes indefinite if $\mathcal{H}$ exceeds $\mathcal{H}_c$. This will now be demonstrated.

The scalar product in the Rarita-Schwinger theory is

$$(\psi,\psi)=-\int d^3x[\psi^{\dagger\mu}\psi_\mu+\tfrac{1}{2}(\gamma\cdot\psi)^\dagger(\gamma\cdot\psi)]\,. \tag{43}$$

The expression on the right-hand side can be reduced to a sum of contributions from the eigensolutions determined in the preceding section. (There are no cross terms involving eigensolutions characterized by different values of n.) The contribution from a typical eigensolution is

$$\tfrac{1}{2}(|u_2|^2+|u_3|^2)+\tfrac{1}{2}(1-\eta^2)(|u_1|^2+|u_4|^2)+(|u_5|^2+|u_6|^2-|u_7|^2-|u_8|^2)\,, \tag{44}$$

plus an expression of identical form involving the v's. We can rewrite this, after eliminating $u_5,\ldots,u_8$ using (39), as

$$(u,u)=u^\dagger M u\frac{1}{2(1+\rho_{n+1}{}^2)}\,,\quad u=\text{column vector}(u_1,u_2,u_3,u_4) \tag{45a}$$

where

$$M=\begin{bmatrix}(1+\eta)^2(3+\rho_{n+1}{}^2) & -\rho_{n+1}\rho_{n+2}(1+\eta) & \rho_n(1+\eta) & 0\\ -\rho_{n+1}\rho_{n+2}(1+\eta) & (1+\rho_{n+1}{}^2) & 0 & -\rho_{n+2}(1-\eta)\\ \rho_n(1+\eta) & 0 & (1+\rho_{n+1}{}^2) & -\rho_n\rho_{n+1}(1-\eta)\\ 0 & -\rho_{n+2}(1-\eta) & -\rho_n\rho_{n+1}(1-\eta) & (1-\eta)^2(3+\rho_{n+1}{}^2)\end{bmatrix}. \tag{45b}$$

This "metric matrix" is real and symmetric (so that all its eigenvalues are real), and further, the matrix H of Eq. (28b) is Hermitian with respect to the metric M, i.e.,

$$MH = H^{\dagger} M . \tag{46}$$

However, M is not, in general, positive definite. A necessary and sufficient condition for a matrix to be positive definite is[28] that all its leading principal minors be positive. In the case of the matrix (45b) it is easy to see, by computing the leading principal minors, that all of them are positive for $\eta < 1$, while for any $\eta > 1$ this is not true. We see thus that the scalar product is positive definite for $\eta < 1$ and indefinite for $\eta > 1$. For $\eta < 1$, the positive definiteness of M enables us to express it in the form $M = R^{\dagger} R$. It follows then from Eq. (46) that the matrix $H' \equiv RHR^{-1}$ is Hermitian. This means that the eigenvalues of H are all real for $\eta < 1$, which is what we have shown in the preceding section.

It is useful to note here that if $u^{(\alpha)}$ is an eigenvector belonging to an eigenvalue ϵ_α of H as defined in Eqs. (28), then

$$u^{(\alpha)\dagger} M u^{(\beta)} = 0 \text{ for } \epsilon_\alpha \neq \epsilon_\beta^* . \tag{47}$$

This result is obtained directly from Eq. (46) by multiplying on the left and right by $u^{(\alpha)\dagger}$ and $u^{(\beta)}$, respectively, and using (28a) and its Hermitian conjugate. In particular, the "norm" of $u^{(\alpha)}$, i.e., $u^{(\alpha)\dagger} M u^{(\alpha)}$, is zero if ϵ_α is not real. But

$$\mathfrak{M}_{\alpha\alpha^*} \equiv u^{(\alpha)\dagger} M u^{(\alpha^*)} \tag{48}$$

is always nonzero. (Here α^* is used as a label for the eigenvector belonging to ϵ_{α^*}). If we now expand u as

$$u = \sum_{\alpha=1}^{4} a_\alpha u^{(\alpha)} , \tag{49}$$

we get

$$u^{\dagger} M u = \sum_{\alpha} a_\alpha^* a_{\alpha^*} \mathfrak{M}_{\alpha\alpha^*} . \tag{50}$$

For $\eta > 1$, the quantity (50) need not be positive, as already noted. Indefinite contributions arise from terms corresponding to any complex conjugate pair of eigenvalues which H may have. Also, any mode for which ϵ at the particular value of η (> 1) has a sign opposite to that at $\eta = 0$ can make a negative contribution to (50). These facts are easiest to see explicitly in the case of the special solutions characterized by Eqs. (27b) and (34b) where H and M reduce to 2×2 matrices [consisting of the second and fourth rows and columns in (28b) and (45b), with n set equal to -1]. If $u^{(\epsilon)}$ belongs to one of the eigenvalues (35) of this H, one can readily see that

$$u^{(\epsilon)\dagger} M u^{(\epsilon)} = \epsilon(1-\eta)(\eta-2) + 2(1-\eta^2) + \eta \tag{51}$$

(to within a positive normalization factor). This quantity is negative (for $1 < \eta^2 < \frac{4}{3}$) is ϵ is taken with the upper sign in (35). At $\eta = 1$ this value of ϵ is infinite, while the norm (51) becomes zero—a fact which has been referred to in the preceding section. As noted there, despite the vanishing norm, the contribution of this state to the energy (which can be shown to be $u^{(\epsilon)\dagger} M H u^{(\epsilon)} = \epsilon u^{(\epsilon)\dagger} M u^{(\epsilon)}$) remains nonzero and finite as $\eta \to 1$.

V. DISCUSSION

It may be useful to summarize here the salient points of what is known about inconsistencies in higher-spin theories, taking the results of this paper in conjunction with those of earlier investigations. When relativistic wave equations for spin > 1 were first formulated in manifestly covariant form, the supplementary conditions needed to eliminate redundant components and ensure a unique spin were found, on introduction of interaction with external electromagnetic fields, to lead to inadmissible restrictions on the external fields. One learned[1,2] to overcome such difficulties through a Lagrangian approach wherein both the equations of motion and constraints were derived from a Lorentz-invariant Lagrangian density. After eliminating algebraic inconsistencies in this fashion, it came as a surprise when Johnson and Sudarshan[3] showed that there is no possibility of consistent quantization in the case of certain spin-$\frac{3}{2}$ theories with minimal electromagnetic interaction. The difficulty they encountered, namely failure of the positivity of the anticommutator of field components, has now been shown to have its roots in the c-number theory itself where, as seen in Sec. IV, the sign of the Lorentz-invariant norm of the wave function becomes indefinite when the external field is made sufficiently large. Though in early work the positivity of the norm has been considered an essential requirement of *free-particle* wave equations (and in fact has led to the rejection or modification of certain theories[29,30]), it does not seem to be known widely that even in theories satisfying the above requirement the positivity might fail in the presence of interactions. Efforts to devise simple criteria for checking positivity in the presence of interactions (in the c-number context) should therefore be an essential part of any program to construct theories which are consistently quantizable with a positive-definite metric. The observation in the present work that breakdown of positivity goes hand in hand with the appearance of complex energies[31] constitutes partial progress in this direction: It is very easy to set up the equations in the presence of a homogeneous magnetic field and to check, in the limit of large $\mathcal{H}$, whether any of the eigenvalues become complex. Theories which pass this test (no complex energies) may, however, be unsatisfactory for other reasons.

It may be noted incidentally that the existence (or the nonexistence) of the type of difficulty discussed here does not affect the Heisenberg equation of motion, as has been shown by Mainland and Sudarshan[32] for spin $\frac{3}{2}$ and noted by Hagen[33] for spin 1. It is also unrelated to the fact that Eq. (20) ceases to be hyperbolic when η exceeds unity: This may be inferred from the parallel case of spin-1 particles with anomalous magnetic moment, where also imaginary energies appear for magnetic fields exceeding a certain value, but the equation remains not only hyperbolic but also causal for all $\mathcal{H}$. It would be useful to remember in this connection that the question of causality is determined by the behavior of waves with wave numbers tending to infinity (infinite-momentum limit) while the appearance of complex energies (as the external field is increased) starts with levels at the *lower* end of the energy spectrum.

Finally, we wish to call attention to the fact that the only half-integer higher-spin theories now known (as far as we are aware) in which causality problems do not arise in the presence of electromagnetic interaction are theories in which the norm of wave functions is indefinite even in the free case and states of more than one mass or spin are present. The Bhabha equation,[29,34] the Bhabha-Gupta equation[35,12] (with a choice of free parameters which makes the norm indefinite), and the Fisk-Tait equation[13,12] are examples. In the last of these cases we have verified also[36] that complex energies do not appear even with arbitrarily high magnetic fields. In view of these advantages, such theories would appear to deserve serious consideration, especially since quantization with an indefinite metric is no longer as unacceptable as it used to be.[37]

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[18]See also M. Hortaçsu [Phys. Rev. D 9, 928 (1974)] who has calculated explicitly in *2+1 dimensions* the propagator for the Rarita-Schwinger field interacting with a constant magnetic field and has shown that if there is noncausality in the manner of Velo and Zwanziger (see Ref. 8) then the propagator of the field (calculated from the c-number solutions) does not vanish for all spacelike separations.

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$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

[22]It must be remembered, however, that the harmonic-oscillator operators a and $a^\dagger$ defined here involve *two* degrees of freedom pertaining to the orbital motion projected on to the x-y plane. For fuller details and explicit forms of the wave functions, see Ref. 15.

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