Nonrelativistic field theory for strong coupling*

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We study the Hamiltonian for a nonrelativistic nucleon with Yukawa coupling to a scalar meson field. The nucleon self-energy is logarithmically divergent. It is known that one can isolate the divergent part by a canonical transformation, but this technique is feasible only for simple field theories. We study the theory by a systematic strong coupling in inverse powers of the coupling constant e. The leading term in the energy is of order e^4 and corresponds to classical particlelike solutions. The terms of order e^2 represent a generalized meson-pair theory. The zero-point energy shift yields the lorgarithmically divergent energy. The residual Hamiltonian contains scattering resonances, and techniques are developed to study the resonances.

I. INTRODUCTION

In the present paper we exhibit some new features of a nonrelativistic field theory that contains particlelike solutions. The Hamiltonian is ($c=1$, $\hbar=1$)

$$H=\frac{1}{2M}\int \vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi d^3x+e\int \phi(x)\psi^\dagger(x)\psi(x)d^3x+\tfrac{1}{2}\int :\pi^2+(\vec{\nabla}\phi)^2+\mu^2\phi^2:d^3x\,. \tag{1.1}$$

This represents the coupling of a meson field ϕ to a scalar nucleon field ψ. Since we will be interested in the one-nucleon sector $\int\psi^\dagger\psi d^3x=1$, ψ can be either a fermion or boson field.

When H is studied classically, one finds localized "particlelike" solutions for $\psi(x,t)$ of the form $\psi_0(x)e^{-iE_0t}$, with an associated static meson field and with a total finite energy E_0. The spatial extent of the localized solution is a Bohr radius $1/Me^2$ and the energy depression is of order e^4. One can go on to study moving nucleons, coupled small oscillations of meson and nucleon fields, excited self-consistent field solutions, etc. The same procedure can be followed for relativistic field theories.

Our earlier work[1] was devoted to understanding the quantum-mechanical content of this procedure. In particular, there is the apparent paradox that for weak coupling the one-nucleon self-energy E_G is logarithmically divergent,

$$E_G=\frac{-e^2}{\Omega}\sum_k\frac{1}{2w}\frac{1}{k^2/2M+w}, \tag{1.2}$$

where $w(k)=(k^2+\mu^2)^{1/2}$.

We showed that the addition of $E_G\int\psi^\dagger\psi d^3x$ to H yields a divergence-free theory. By performing a canonical transformation on H we cancel the counterterm. The remaining Hamiltonian is free of divergences. This aspect has been put on a mathematically rigorous basis by Nelson.[2] Physically more interesting was the demonstration that the divergence-free Hamiltonian also had particlelike solutions. This was in accord with the intuitive feeling that the quantum divergence arises from fluctuations of very short wavelength. These are insensitive to the existence of any over-all structure of the order of a Bohr radius. However, the treatment of the short-wave fluctuations affects and modifies the particlelike solutions due to long-wave fluctuations, and one would like a systematic procedure for treating these finite strong-coupling effects.

In the present paper we examine the same theory from the point of view of an adiabatic strong-coupling theory where we proceed systematically in inverse powers of the coupling constant. The advantage is that the procedure is directly applicable to more general theories and does not depend on the particular canonical transformation used to handle divergences. The leading term in the one-nucleon energy is again the classical energy of order e^4. We calculate the next term of order e^2. It contains a logarithmically divergent part, just as in weak coupling. The divergent term now arises as zero-point energy for the meson field which is governed by a pair Hamiltonian with attractive interactions. Corresponding to the discrete excited states of the nucleon in the potential well provided by the static part of the meson field, there are scattering resonances of the meson field, with the nucleon remaining in its ground state.

II. CANONICAL TRANSFORMATION FOR ADIABATIC THEORY

We start with Eq. (1.1), using primes in the variables, and perform a scaling transformation

$$\psi'(x)=\frac{1}{\lambda^{3/2}}\psi\left(\frac{x}{\lambda}\right),\quad \phi'(x)=\frac{1}{\lambda}\phi\left(\frac{x}{\lambda}\right),\quad \pi'(x)=\frac{1}{\lambda^2}\pi\left(\frac{x}{\lambda}\right). \tag{2.1}$$

The transformation preserves the canonical commutation relations. We obtain the scaled form of the Hamiltonian

$$H=\frac{1}{2M\lambda^2}\int \vec{\nabla}\psi^\dagger\cdot\vec{\nabla}\psi d^3x+\frac{e}{\lambda}\int \phi(x)\psi^\dagger(x)\psi(x)d^3x + \frac{1}{2\lambda}\int :[\pi^2+(\vec{\nabla}\phi)^2+\mu^2\lambda^2\phi^2]:d^3x\,. \qquad (2.2)$$

For weak coupling the appropriate length is $\lambda=1$, whereas for strong coupling it is $\lambda=1/Me^2$ and goes to zero with infinite coupling strength. We will be interested in the strong-coupling limit and will use the small parameter $\epsilon=1/e$ so that $\lambda=\epsilon^2/M$. The three terms in H are in the ratios $1{:}\epsilon{:}\epsilon^2$.

We now expand the nucleon field in a spatially preferred basis:

$$\psi(x)=\sum_n b_n\psi_n(x)\,. \qquad (2.3)$$

The $\psi_n(x)$ have spatial extent unity and are orthonormal,

$$\int \psi_n^*(x)\psi_m(x)d^3x=\delta_{n,m}\,. \qquad (2.4)$$

We perform a canonical transform that shifts the meson field about a mean static field $\overline{\phi}(x)$:

$$U_0\phi(x)U_0^{-1}=\overline{\phi}(x)+\phi(x)\,. \qquad (2.5)$$

The functions $\psi_n(x)$ are chosen to be eigenfunctions of a particle moving in the potential $\epsilon\overline{\phi}(x)$,

$$[-\tfrac{1}{2}\nabla^2+\epsilon\overline{\phi}(x)]\psi_n(x)=E_n\psi_n(x)\,. \qquad (2.6)$$

To determine $\overline{\phi}(x)$, one examines the diagonal part of $U_0HU_0^{-1}$ with $b_0^\dagger b_0=1$, and then chooses $\overline{\phi}(x)$ to make the linear terms in the meson field ϕ vanish. This yields

$$-\nabla^2\overline{\phi}+\frac{\mu^2\epsilon\overline{\phi}}{M^2}=-\frac{\psi_0^2(x)}{\epsilon}. \qquad (2.7)$$

In the limit of strong coupling $\epsilon\overline{\phi}(x)$ is of order unity. The two equations yield the wave function of the particlelike solution as the function obeying the nonlinear eigenvalue problem:

$$\left\{-\tfrac{1}{2}\nabla^2-\frac{1}{4\pi}\int \psi_0^2(y)\frac{\exp[-(\mu\epsilon^2/M)|x-y|]}{|x-y|}d^3y\right\}\psi_0(y) = E_0\psi_0(y)\,. \qquad (2.8)$$

The potential $\epsilon\overline{\phi}$ is also equal to

$$\epsilon\overline{\phi}(x)=E_0+\frac{1}{\psi_0}\frac{\nabla^2\psi_0}{2}. \qquad (2.9)$$

These equations are the same as those obtained as exact solutions of the classical theory, or alternatively from a variational argument based on a Hartree trial function.

There are several noteworthy features of the primitive strong-coupling theory. The continuum for the $\psi_n(x)$ starts at an energy $|E_0|$ above the ground state. In addition, depending on the value of $\mu\epsilon^2/M$, there may be discrete nucleon excited states in between. For example for $\mu=0$, the effective potential $\epsilon\overline{\phi}(x)$ is of Coulomb type at large distances and tends to a finite value as $|x|\to 0$. So there are infinitely many excited states. These are the excited states of a particle in a potential $\epsilon\overline{\phi}(x)$ and not the self-consistent excited states of the Hartree theory. It has been shown by Meyer[3] that the solution $\psi_0(x)$ has some subtle features that are usually ignored in simple variational estimates. In particular, the Fourier components of $\psi_0(x)$ fall to zero faster than any power of the wave vector, for large wave vectors. We now have

$$U_0\frac{H}{e^4M}U_0^{-1}=\sum E_n b_n^\dagger b_n+\frac{\epsilon^2}{2}\sum w(k)a^\dagger(k)a(k) -\frac{\epsilon}{2}\int \overline{\phi}(x)\psi_0^2d^3x+V\,. \qquad (2.10)$$

Here

$$V=\epsilon\int \phi(x)[\psi^\dagger(x)\psi(x)-\psi_0^2b_0^\dagger b_0]d^3x \qquad (2.11)$$

and we use the usual decomposition

$$\phi(x)=\sum_k\frac{1}{(2w\Omega)^{1/2}}[a(k)e^{ikx}+\text{c.c.}]\,,\qquad w(k)=(k^2+\mu^2\epsilon^4/M^2)^{1/2}\,. \qquad (2.12)$$

Let us write

$$V=V_0+V_1\,,$$
$$V_0=\epsilon\sum_{n\neq 0}(G_{n0}b_n^\dagger b_0+G_{n0}^\dagger b_0^\dagger b_n)\,, \qquad (2.13)$$
$$V_1=\epsilon\sum_{\substack{m\neq 0\\ n\neq 0}} G_{mn}b_m^\dagger b_n\,.$$

Here

$$G_{mn}\equiv\int \phi(x)\psi_m^*(x)\psi_n(x)d^3x$$

depends on the meson field. Because of the character of the E_n it is easy to develop an adiabatic theory based on expansions in powers of ϵ. We remove V_0 to first order by performing a unitary transform $U=e^S$, $S^\dagger=-S$, that satisfies the condition

$$\epsilon[S,\ \sum E_nb_n^\dagger b_n]+V_0=0\,. \qquad (2.14)$$

Note that the adiabatic idea is expressed by the use of only the free part of the nucleon field. The meson free field has a coefficient ϵ^2.

The desired S is simply

$$S=\sum_{n\neq 0}(F_{n0}b_n^\dagger b_0-\text{H.c.})\,,$$

with (2.15)

$$F_{n0}=G_{n0}/(E_n-E_0)\,.$$

Actually this canonical transformation can be performed in closed form with F_{no} an arbitrary functional of the meson field ϕ. We are here, however, only interested in the simplest treatment that calculates the energy of the lowest state of the nucleon and the associated excited states of the meson field to order ϵ^2. The free-meson part of the Hamiltonian is already of order ϵ^2, so that the correction due to transformation of the π variable is of order ϵ^3. We now set down the transformed Hamiltonian, including all terms of order ϵ^2. We find

$$U\frac{H}{e^4M}U^{-1}=-\frac{\epsilon}{2}\int\overline{\phi}(x)\psi_0{}^2d^3x+\sum E_nb_n^\dagger b_n+\frac{\epsilon^2}{2}\int:\left[\pi^2+(\nabla\phi)^2+\frac{\mu^2\epsilon^4}{M^2}\phi^2\right]:d^3x-\epsilon^2\int\int\phi(x)\psi_0(x)g(x|y)\psi_0(y)\phi(y)d^3x\,d^3y+R\,. \tag{2.16}$$

Here

$$g(x|\,y)=\sum_{n\neq 0}\frac{\psi_n^*(x)\psi_n(y)}{E_n-E_0}. \tag{2.17}$$

The residual part of the Hamiltonian is

$$R=\frac{\epsilon}{2}\sum_{m\neq 0}\left(G_{mn}+\epsilon\frac{G_{n0}G_{m0}^*}{E_n-E_0}\right)b_n^\dagger b_m+\text{H.c.}-\epsilon^2\sum_{\substack{m\neq 0\\ n'\neq 0}}\left(\frac{G_{q0}G_{q0}^\dagger}{E_q-E_0}b_m^\dagger b_0+\text{H.c.}\right). \tag{2.18}$$

R contains diagonal terms $b_n^\dagger b_n$ for $n\neq 0$, but we are only interested in the dressed nucleon in the lowest state where $b_0^\dagger b_0=1$. The terms in R of the type $b_m^\dagger b_0$ can be eliminated against the nucleon energy $\sum E_nb_n^\dagger b_n$ by a further unitary transformation. But the generator is of order ϵ^2, so there are no correction terms of order ϵ^2.

We now comment on some notable features of the pair Hamiltonian. In the limit $\mu=0$, there is an over-all factor of ϵ^2, but otherwise the meson Hamiltonian is independent of ϵ. (For $\mu\neq 0$, it is only weakly dependent on ϵ, apart from the finite range for the meson field.) The ground state of the meson Hamiltonian contains a zero-point energy shift. Thus the entire Hamiltonian H in strong coupling for $\mu=0$ contains in addition to the "semiclassical" energy shift of order e^4, an energy depression of order e^2. This latter contribution is logarithmically divergent, as is the case for the same Hamiltonian in the weak-coupling limit. Adding the divergent term $E_G\int\psi^\dagger\psi d^3x$ makes the theory finite, but leaves a finite nonzero contribution of order e^2 to the self-energy.

Even if we do not treat the pair theory accurately, one can use a variational argument to exhibit the logarithmic divergence. We take the expectation value of the Hamiltonian with the meson vacuum Φ_0, such that $a(k)\Phi_0=0$. Then an upper bound to the energy is

$$\Delta H=-e^2\frac{1}{(2\pi)^3}\int\int\int\frac{d^3k}{2w(k)}G(k\,|\,k)\,, \tag{2.19}$$

where

$$G(k\,|\,k')=\int\int e^{ikx}\psi_0(x)g(x|\,y)\psi_0(y)e^{-ik'y}d^3x\,d^3y\,. \tag{2.20}$$

The argument that shows this is logarithmically divergent is similar to the one given in our earlier paper.[1] $g(x|\,y)$ obeys an integral equation [Eq. (3.5)] in which the main inhomogeneous term is

$$g_0(x|\,y)=\frac{1}{(2\pi)^3}\int\frac{e^{i\kappa(x-y)}d^3\kappa}{\frac{1}{2}\kappa^2+|\,E_0|}=\frac{2}{4\pi}\frac{\exp[-(2|\,E_0|)^{1/2}|\,x-y|\,]}{|\,x-y\,|}. \tag{2.21}$$

The divergent contribution comes from g_0; the other contributions are finite. If we define

$$\tilde{\psi}_0(k)=\int e^{ikx}\psi_0(x)d^3x \tag{2.22}$$

we have with $g=g_0$

$$-\frac{e^2}{(2\pi)^6}\int\int\frac{1}{2w(k)}\frac{|\,\tilde{\psi}_0(\kappa)|^2}{[\frac{1}{2}(k-\kappa)]^2+|\,E_0|}d^3k\,d^3\kappa\,. \tag{2.23}$$

Now $\tilde{\psi}_0(\kappa)$ falls to zero for $\kappa=\kappa_0\gg 1$, faster than any power of κ. Thus there is a divergent part

$$-\frac{e^2}{(2\pi)^3}\int_{k>\kappa_0}\frac{1}{2w(k)}\frac{1}{\frac{1}{2}k^2+|E_0|}d^3k\,. \tag{2.24}$$

As an aside we note that the logarithmic divergence occurs in both extreme limits where the $\psi_n(x)$ are plane waves or oscillator functions.

The second point is that in general $g(x|y)$ contains contributions from discrete excited states of the nucleon in the potential well $\epsilon\overline{\phi}(x)$. There is then the possibility of scattering resonances and even of meson bound states associated with each of these excited states. Since the energies of excitations of the free meson field are of order e^0, the meson resonances are also of order e^0 (in conventional units).

III. TREATMENT OF THE PAIR HAMILTONIANS

To set up a practical calculation of the scattering resonances one must do two things. First, useful approximations for $g(x|y)$ have to be developed. Second, the generalized pair Hamiltonian must be approximately diagonalized. While it is questionable as to how much interest attaches to this model Hamiltonian, the procedures are directly useful in associated solid-state problems[4,5] and also for relativistic Hamiltonians.

Consider first the kernel $g(x|y)$. Consider the Hermitian operator

$$L_x=\frac{-\nabla^2}{2}+\epsilon\overline{\phi}(x)\,,\quad \epsilon\overline{\phi}=E_0+\frac{1}{2}\frac{1}{\psi_0}\nabla^2\psi_0\,, \tag{3.1}$$

$$L_x\psi_n(x)=E_n\psi_n(x)\,. \tag{3.2}$$

Then

$$(L_x-E_0)g(x|y)=\delta(x-y)-\psi_0(x)\psi_0(y)\,. \tag{3.3}$$

In terms of the Green's function $g_0(x|y)$, obeying

$$\left(-\frac{\nabla^2}{2}+|E_0|\right)g_0(x|y)=\delta(x-y)\,, \tag{3.4}$$

we have the integral equation

$$g(x|y)=g_0(x-y)-\psi_0(x)\int g_0(x-x_1)\psi_0(x_1)d^3x_1 + \int g_0(x-x_1)\epsilon\overline{\phi}(x_1)g(x_1|y)d^3x_1\,. \tag{3.5}$$

Define $\phi_k(x)$ by

$$\phi_k(x)\psi_0(x)=\int g(x|y)e^{iky}\psi_0(y)d^3y\,, \tag{3.6}$$

where we note that

$$\int\phi_k\psi_0d^3x=0\,.$$

We have the differential equations

$$(L_x-E_0)(\phi_k\psi_0)=\left[e^{ikx}-\int e^{iky}\psi_0^2(y)d^3y\right]\psi_0(x) \equiv\psi_0f_k(x)\,, \tag{3.7}$$

$$-\tfrac{1}{2}\nabla^2\phi_\kappa-\tfrac{1}{2}(\vec{\nabla}\ln\psi_0)\cdot\vec{\nabla}\phi_k=f_k(x)\,, \tag{3.8}$$

and

$$-\tfrac{1}{2}\vec{\nabla}\cdot(\psi_0^2\vec{\nabla}\phi_k)=\psi_0^2f_k(x)\,. \tag{3.9}$$

In terms of $\phi_k(x)$ we can write

$$G(k|k')=\int e^{ikx}\psi_0^2(x)\psi_{k'}^*(x)d^3x =\tfrac{1}{2}\int\psi_0^2(x)[e^{ikx}\phi_{k'}^*(x)+e^{-ik'x}\phi_k(x)]d^3x\,. \tag{3.10}$$

We now set up a stationary variational principle for $\phi_k(x)$. Consider the functional

$$J=\tfrac{1}{2}\int\psi_0^2\vec{\nabla}\phi_k^*\cdot\vec{\nabla}\phi_kd^3x-\int\phi_k^*(x)\psi_0^2f_k(x)d^3x -\int\phi_k\psi_0^2f_k^*(x)d^3x\,, \tag{3.11}$$

where ϕ_k and ϕ_k^* are to be independently varied. The stationary value of J, $[J]$, is $\int\phi_\kappa^*\psi_0^2f_k(x)d^3x$ and allows an estimate of the diagonal element $G(k|k)$ that is important in the self-energy.

The variational principle is useful in handling the difficult sum over the continuum states that occurs in $g(x|y)$. For example, assume the trial function

$$\phi_k(x)=\beta_k[e^{ikx}-\rho(k)]+\phi_k^D(x)\,, \tag{3.12}$$

where β_k is a variational parameter, $\rho(k)=\int e^{ikx}\psi_0^2d^3x$. The term $\phi_k^D(x)$ represents the contribution of the discrete excited states and is given by

$$\phi_k^D(x)\psi_0(x)=\sum_\mu\frac{\psi_\mu^*(x)}{E_\mu-E_0}R_{\mu0}(k)\,, \tag{3.13}$$

where

$$R_{\mu0}(k)=\int e^{ikx}\psi_\mu(x)\psi_0(x)d^3x\,.$$

The discrete excited states ψ_μ with energies E_μ are obtained by a separate analysis of the eigenvalue problem for the potential $\epsilon\overline{\phi}(x)$. The form of the first term in $\phi_k(x)$ is suggested by the plane-wave limit for the $\psi_n(x)$.

Using this trial in J and varying with respect to β_k^*, we find

$$\beta_k\frac{k^2}{2}=(1-\rho_k\rho_{-k})-i\vec{k}\cdot\int e^{-ikx}\psi_0^2\vec{\nabla}\phi_k^D(x)d^3x \tag{3.14}$$

and

$$[J] = -\beta_k(1-\rho_k\rho_{-k}) - \int \phi_k^D \psi_0{}^2 e^{ikx} d^3x . \qquad (3.15)$$

This yields the estimate

$$G(k|k') = \tfrac{1}{2}(\beta_k + \beta_{k'}^*)(\rho_{k-k'} - \rho_k\rho_{k'}) + \sum_{\mu \neq 0} \frac{R_{\mu 0}^*(k')R_{\mu 0}(k)}{E_\mu - E_0} . \qquad (3.16)$$

It is to be noted that the high-k behavior of β_k is the same as in the earlier paper analyzing the divergence of the theory. However, the long-wavelength behavior is different. The form of $G(k|k')$ contains additional separable parts from the continuum, as well as the separable parts from the discrete excited states.

We now turn to the second problem, the treatment of the pair Hamiltonian. For brevity we use the notation of standard expositions of pair theories.[6,7] Define

$$V(x|y) = 2\psi_0(x)g(x|y)\psi_0(y) ,$$
$$V(k|k') = \int V(x|x')e^{-ikx}e^{ik'x'}d^3x\,d^3x' \qquad (3.17)$$

and use the four-dimensional Fourier transforms

$$\phi(\vec{k}, \kappa_0) = \frac{1}{(2\pi)^{3/2}} \int \phi(\vec{x}, t) e^{-i(\vec{k}\cdot\vec{x} - \kappa_0 t)} d^3x\,dt ,$$
$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^{5/2}} \int \phi(\vec{k}, \kappa_0) e^{i(\vec{k}\cdot\vec{x} - \kappa_0 t)} d^3k\,d\kappa_0 . \qquad (3.18)$$

The equation of motion based on the pair Hamiltonian is written in the integral form

$$\phi(\vec{k}, \kappa_0) = \phi^{\text{in}}(\vec{k}, \kappa_0) + \frac{1}{(2\pi)^3} \int d^3k' G_0(k) V(k, k') \phi(\vec{k}', \kappa_0) ,$$
$$G_0(k) = \frac{1}{w^2(k) - (\kappa_0 + i\eta)^2} , \quad \eta \to 0^+ . \qquad (3.19)$$

To exhibit the solution, introduce the resolvent operator $R(k, k', \kappa_0)$

$$\phi(\vec{k}, \kappa_0) = \phi^{\text{in}}(\vec{k}, \kappa_0) + \int R(k, k', \kappa_0) \phi^{\text{in}}(\vec{k}, \kappa_0) d^3k' . \qquad (3.20)$$

In an operator notation

$$R = G_0V + G_0VR , \qquad (3.21)$$

where

$$\langle k|G_0V|k'\rangle = G_0(k)V(k,k') .$$

We now write $V = V_D + V_1$. V_D contains the contributions from the discrete levels and is the sum of separable terms. V_1 is the contribution from the continuum. Let R be the resolvent for V_1, so that

$$R_1 = G_0V_1 + G_0V_1R_1 . \qquad (3.22)$$

We adopt the frequently employed procedure based on the attitude that adequate approximations for R_1 can be found. With

$$Q_1 = 1 + R_1 , \quad Q = 1 + R \qquad (3.23)$$

and using $Q_1(1 - G_0V_1)^{-1}$,

$$Q = Q_1 + Q_1G_0V_DQ . \qquad (3.24)$$

Suppose V_D is a sum of a finite number of separable terms and is written as

$$V_D = \sum_\beta |\beta\rangle W_\beta \langle\beta| . \qquad (3.25)$$

The states $|\beta\rangle$ are not orthogonal in general, as is seen from our expression for $G(k|k')$. Then

$$\langle\alpha|Q - \sum_\beta \langle\alpha|Q_1G_0|\beta\rangle W_\beta\langle\beta|Q = \langle\alpha|Q_1 . \qquad (3.26)$$

Write the solution of this finite set as

$$\langle\alpha|Q = \sum_\alpha \frac{N_{\alpha\alpha'}}{D}\langle\alpha'|Q_1 \qquad (3.27)$$

so that

$$\langle k|Q|k'\rangle = \langle k|Q_1|k'\rangle + \sum_{\alpha\alpha'} \langle k|Q_1G_0|\alpha\rangle \frac{W_\alpha N_{\alpha\alpha'}}{D} \langle\alpha'|Q_1|k'\rangle . \qquad (3.28)$$

The scattering resonances are located at the zeros of the determinant D. The results differ from those of simple pair theories because of the modification arising from the nonzero resolvent for the continuum part of $g(x|y)$, viz. Q_1.

Our variational estimate of the continuum contribution to $G(k|k')$ shows that it is useful to extract separable parts. Thus

$$\langle k|V_1|k'\rangle = (\beta_k + \beta_{k'}^*)(\rho_{k-k'} - \rho_k\rho_{k'}) . \qquad (3.29)$$

Thus we introduce a resolvent R_2 for the nonseparable part and write

$$\langle k|Q_1|k'\rangle = \langle k|Q_2|k'\rangle - \int \langle k|Q_2|k_1\rangle G_0(k_1)d^3k_1(\beta_{k_1} + \beta_{k_2}^*) \times \rho_{k_1}\rho_{k_2}d^3k_2\langle k_2|Q_1|k'\rangle , \qquad (3.30)$$

$$\langle k| Q_2 | k'\rangle = (\beta_k + \beta^*_{k'})\rho_{k-k'} + G_0(k) \int (\beta_k + \beta^*_{k_1})\rho_{k-k_1} d^3k_1 \langle k_1 | Q_2 | k'\rangle . \quad (3.31)$$

After a suitable approximation has been found for the integral equation for Q_2, the calculation of Q_1 and thus of Q is a matter of simple algebra.

This completes our discussion of the technique needed to analyze the extended pair theory.

Summary. The key point of the present paper arises from the fact that the introduction of a breaking of translational invariance of a particle-like solution alters the spectrum of the "unperturbed" part of the Hamiltonian. This in turn makes possible a systematic adiabatic strong-coupling theory in which quantum fluctuations (and divergences) are not lost and can be treated. No specific prior technique is needed to handle the self-energy divergence. There is thus hope that relativistic renormalizable theories can be attacked by an analogous procedure.

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