PHYSICAL REVIEW D VOLUME 12, NUMBER 12 15 DECEMBER 1975

Covariance, causality, and the refined infinite-momentum limit

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(Received 29 July 1975)

The refined infinite-momentum limit approach is used in the derivation of sum rules for the real parts of current-particle forward scattering amplitudes. The sum rules are based on covariance, causality, and fixed-q^2 unsubtracted dispersion relations in ν for the amplitudes. The fixed-mass sum rules so obtained are found to include those of the light-cone algebra. For amplitudes where the unsubtracted-dispersion-relation assumption is doubtful, the results are shown to be unaltered if once-subtracted dispersion relations are assumed instead. Convergence of the sum rules and inclusion of class-II contributions are also discussed.

I. INTRODUCTION

Recently Taha[1] has proposed a refinement in the conventional infinite-momentum technique[2] which, when coupled with Bjorken scaling, yields from equal-time current algebra sum rules that were previously derived from the light-cone commutators of the vector-gluon–fermion-quark model.[3] In particular, it was shown in Ref. 1 that the application of this refined procedure to the diagonal matrix element of the commutator of conserved vector currents between spinless states gives, for the absorptive parts of scalar amplitudes, the same fixed-mass sum rules as those obtained by Dicus, Jackiw, and Teplitz[4] from the $(+,\nu)$ light-cone commutators. Subsequent works applied the refined technique first[5] to the diagonal matrix element of the commutator of the conserved vector currents between spin-$\frac{1}{2}$ fermions and secondly[6] to the nonforward spin-averaged matrix element of the same commutator. In both cases the complete sets of sum rules for the absorptive parts were found to include the light-cone results. Further, it was noted that a number of these sum rules do in fact follow from causality and scaling alone and that in the nonforward case[6] the refined technique produces sum rules that are not obtained by the light-cone method.[7]

It is our purpose in this paper to use the refined infinite-momentum approach in order to derive fixed-mass sum rules for the real parts of current-particle scattering amplitudes, restricting ourselves to conserved currents and external spin-$\frac{1}{2}$ fermions of equal momenta. Such sum rules were obtained by Heimann, Hey, and Mandula.[8] Their considerations, which were based on causality, covariance of the connected retarded commutator matrix element, and the use of the quark-model light-cone algebra, were subsequently extended to the nonforward case by Lo.[9] It is interesting, in our opinion, to find out whether the light-cone results of Heimann *et al.*[8] can also be arrived at through the application of the refined infinite-momentum technique.

Assuming noncovariant terms in the retarded commutator of two conserved vector currents to be c numbers, we use, in Sec. II A, the causality of the commutator of these currents in order to write Jost-Lehmann-Dyson (JLD) representations for the invariant amplitudes that characterize the Fourier transform of the retarded commutator. From these representations we obtain general covariance-causality sum rules, the convergence of which depends on certain assumptions about the asymptotic behavior of the JLD spectral functions.

In Sec. II B we adopt, on dimensional grounds, scaling hypotheses for the full amplitudes in the covariant expansion of the retarded commutator. Anticipating the application of the refined infinite-momentum technique to the sum rules of Sec. II A, we first assume, on the basis of the Regge-pole model, fixed-q^2 unsubtracted dispersion relations in ν for all the invariant amplitudes except one. The one exception is an amplitude having $(1/q^2)W_L^{ij}$ as an absorptive part and denoted by R_1^{ij}. In the scaling limit these dispersion formulas transform into relations between the real and imaginary parts of the scaling functions for the full amplitudes and hence enable us to eliminate the real parts, which appear in the refined infinite-momentum limit of the sum rules for the real parts of the amplitudes, in favor of the imaginary parts. The fixed-mass limit of the sum rules is studied in Sec. II C, whereas Secs. II D and II E treat, respectively, the derivations of sum rules for R_1^{ij} and for the amplitude defined in the expression for the Fourier transform of the spinless matrix element of the axial-vector–vector-current retarded commutator. At the end of Sec. II we identify the "light-cone" sum rules of Heimann *et al.*[8] to be among those derived in this section. These sum rules—collected for convenience in the Appendix—are then consequences of covariance, causality, and scaling rather than equal-time or light-cone algebras.

In Sec. III we discuss, for the spin-dependent amplitudes, the validity of the unsubtracted-dispersion-relation assumption, and point out that this assumption might be invalidated by the possible presence of dominant Regge cuts in the Regge asymptotic forms of the absorptive parts of these amplitudes. We then reexamine our derivation when it is assumed instead that the spin-dependent amplitudes obey once-subtracted dispersion relations, and find that our results remain unchanged.

Finally, in Sec. IV, we study the difficulties associated with the passage to the fixed-mass limit in the sum rules using a recently introduced[10] improved version of the refined infinite-momentum technique. We discuss the application of the improved version to the type of sum rules derived in this paper and note that the method allows us, in principle, to obtain, for the real parts of the amplitudes, convergent fixed-mass sum rules in which contributions from class-II intermediate states are exhibited explicitly. As remarked in Ref. 6, the refined infinite-momentum limit in its original form misses these contributions, as does the light-cone approach.

II. DERIVATION OF SUM RULES

A. General formulation

We consider the amplitude for forward scattering of a conserved isovector current by a nucleon and assume that its connected part may be written as the Fourier transform of the retarded commutator of the two currents:

$$R^{ij}_{\mu\nu}(p,q,s)=\frac{i}{\pi}\int e^{iqx}\theta(x_0)\langle p,s|[V^i_\mu(x),V^j_\nu(0)]|p,s\rangle_c d^4x, \tag{2.1}$$

where p ($p^2=1$) and s are the nucleon 4-momentum and covariant polarization, respectively.

Assuming noncovariant terms to be c numbers we write $R^{ij}_{\mu\nu}$ in the form

$$R^{ij}_{\mu\nu}=\sum_{k=1}^{4}L^{(k)}_{\mu\nu}R^{ij}_k+\frac{i}{\pi}f^{ijk}F^k\frac{1}{q^2}[(p_\mu q_\nu+q_\mu p_\nu)-\nu g_{\mu\nu}], \tag{2.2}$$

where R^{ij}_k are invariant functions of $\nu=p\cdot q$ and q^2, F^k is defined by

$$\langle p,s|V^k_\mu(0)|p,s\rangle=F^kp_\mu, \tag{2.3}$$

and the covariants $L^{(k)}_{\mu\nu}$ are given by

$$\begin{aligned}L^{(1)}_{\mu\nu}&=q_\mu q_\nu-q^2g_{\mu\nu},\\ L^{(2)}_{\mu\nu}&=\nu(p_\mu q_\nu+q_\mu p_\nu)-q^2p_\mu p_\nu-\nu^2g_{\mu\nu},\\ L^{(3)}_{\mu\nu}&=i\epsilon_{\mu\nu\alpha\beta}s_\alpha q_\beta,\\ L^{(4)}_{\mu\nu}&=iq\cdot s\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta.\end{aligned} \tag{2.4}$$

The absorptive part of $R^{ij}_{\mu\nu}$ is

$$C^{ij}_{\mu\nu}(p,q,s)=\frac{1}{2\pi}\int e^{iqx}\langle p,s|[V^i_\mu(x),V^j_\nu(0)]|p,s\rangle_c d^4x=\sum_{k=1}^{4}L^{(k)}_{\mu\nu}V^{ij}_k, \tag{2.5}$$

where V^{ij}_k also depend on ν and q^2 only and are related to R^{ij}_k by

$$V^{ij}_k=\mathrm{Im}R^{ij}_k.$$

Meyer and Suura[11] have shown that the function V^{ij}_1 is causal, whereas we[5] have recently demonstrated that V^{ij}_3 and V^{ij}_4 are also causal. The function V^{ij}_2, however, has a noncausal component $V^{ij,\mathrm{nc}}_2$ which is identified in Ref. 11 as

$$V^{ij,\mathrm{nc}}_2=if^{ijk}F^k\frac{1}{q^2}[\epsilon(p_0+q_0)\delta(q^2+2\nu)+\epsilon(p_0-q_0)\delta(q^2-2\nu)]. \tag{2.6}$$

Thus on writing JLD representations for the causal parts $V^{ij,c}_k$ we have

$$V^{ij}_k=\int_0^\infty ds\int\epsilon(q_0-u_0)\delta((q-u)^2-s)\psi^{ij}_k(u,s)d^4u+\delta_{k,2}V^{ij,\mathrm{nc}}_2, \tag{2.7}$$

where the spectral function ψ^{ij}_2 corresponds to the causal part in V^{ij}_2. If we then make the assumption that[11,5,12]

$$\lim_{s\to\infty}\psi^{ij}_k(u,s)=0, \tag{2.8}$$

which, as can be seen from (2.7), is equivalent to

$$\lim_{q_0\to\infty}V^{ij,c}_k(q)=0, \tag{2.9}$$

we obtain the causality sum rules

$$\int_{-\infty}^{\infty}V^{ij,c}_k(q)dq_0=0. \tag{2.10}$$

The causality properties of the V^{ij}_k can be used to express the invariant amplitudes R^{ij}_k in a form in which the dependence on q^2 is explicit, which will subsequently enable one to integrate these amplitudes over q_0. To do this one first notes that $R^{ij}_{\mu\nu}$ is obtained from $C^{ij}_{\mu\nu}$ by the transformation

$$R^{ij}_{\mu\nu}(q)\equiv R[C^{ij}_{\mu\nu}(q)]=\frac{1}{\pi}\int dq_0'\frac{C^{ij}_{\mu\nu}(q_0',\vec{q})}{q_0'-q_0-i\epsilon}. \tag{2.11}$$

Using (2.10) one obtains

$$R[q_0V^{ij,c}_k(q)]=q_0R[V^{ij,c}_k(q)] \tag{2.12}$$

and

$$R[q_0{}^2V^{ij,c}_k(q)]=q_0{}^2R[V^{ij,c}_k(q)]+b^{ij}_k, \tag{2.13}$$

where, provided $\lim_{s\to\infty}s\psi^{ij}_k(u,s)=0$, the quantities b^{ij}_k are constants given by

$$b_k^{ij}=\frac{1}{\pi}\int q_0 V_k^{ij,c}(q)dq_0$$
$$=\int d^4u\int_0^\infty \psi_k^{ij}(u,s)ds\,, \tag{2.14}$$

and, owing to the symmetry properties of $\psi_k^{ij}(u,s)$ in u, the b_k^{ij} ($k=1,2,3$) are symmetric in ij, viz., $b_k^{ij}=b_k^{(ij)}$ ($k=1,2,3$), and b_4^{ij} is antisymmetric, $b_4^{ij}=b_4^{[ij]}$. These constants are coefficients of possible Schwinger terms in the equal-time commutators. One can show[13,14,15,5] that under certain general conditions $b_2^{ij}=b_4^{ij}=0$. If one were to assume, further, the absence of scalar operator Schwinger terms in the time-space isovector equal-time commutator (ETC) one would also have[13] $b_1^{ij}=0$. A vanishing b_3^{ij}, however, would obtain[5] in the algebra-of-fields model, but not in a free-quark model.

Making use of Eqs. (2.6), (2.7), and (2.11)–(2.14) one obtains

$$R_{\mu\nu}^{ij}=\sum_{k=1}^{4}\left[g_k^{ij}(q)+\tfrac{1}{2}(b_k^{ij}-b_3^{ij}\delta_{k,3})\frac{\partial^2}{\partial q_0^{\,2}}\right]L_{\mu\nu}^{(k)}$$
$$+\frac{i}{\pi}f^{ijk}F^k\left[\left(\frac{1}{q^2+2\nu+i\epsilon}-\frac{1}{q^2-2\nu-i\epsilon}\right)p_\mu p_\nu+\frac{1}{2}\left(\frac{1}{q^2+2\nu+i\epsilon}+\frac{1}{q^2-2\nu-i\epsilon}\right)[(p_\mu q_\nu+q_\mu p_\nu)-\nu g_{\mu\nu}]\right], \tag{2.15}$$

where

$$g_k^{ij}(q)=-\frac{1}{\pi}\int\frac{\psi_k^{ij}(u,s)d^4u\,ds}{(q-u)^2-s+i\epsilon}. \tag{2.16}$$

Hence, in order to have a covariant $R_{\mu\nu}^{ij}$ we must set[16]

$$b_1^{ij}=b_2^{ij}=b_4^{ij}=0, \tag{2.17}$$

which is equivalent to absence of operator Schwinger terms in the isovector-current time-space ETC. Equations (2.2), (2.4), and (2.15) will then give

$$R_1^{ij}=g_1^{ij}(q), \tag{2.18}$$

$$R_2^{ij}=g_2^{ij}(q)-\frac{i}{\pi}f^{ijk}\frac{F^k}{q^2}\left(\frac{1}{q^2+2\nu+i\epsilon}-\frac{1}{q^2-2\nu-i\epsilon}\right), \tag{2.19}$$

$$R_3^{ij}=g_3^{ij}(q), \tag{2.20}$$

$$R_4^{ij}=g_4^{ij}(q). \tag{2.21}$$

Using Eqs. (2.16) and (2.17) together with the identity

$$\lim_{\epsilon\to 0_+}\frac{1}{(x+i\epsilon)}=\mathrm{P}\frac{1}{x}-i\pi\delta(x), \tag{2.22}$$

and assuming the possibility of exchanging integrals, we find that the functions R_k^{ij} ($k\neq 2$), satisfy the following sum rules:

$$\int R_k^{ij}dq_0=0, \tag{2.23}$$

$$\int q_0 R_k^{ij}dq_0=ib_3^{ij}\delta_{k,3}. \tag{2.24}$$

These sum rules, which are written for $k\neq 2$, hold provided $\lim_{s\to\infty}s\psi_k^{ij}(u,s)=0$. They follow from causality and the assumption that noncovariant terms in the retarded product are c numbers. In establishing them *no recourse has been made to light-cone algebra*.

B. Scaling

We assume that in the scaling limit $\nu\to\infty$, $q^2\to-\infty$ at fixed $\omega=-q^2/2\nu$, the functions V_k^{ij} exhibit the usual behavior[4]:

$$\begin{aligned}\nu V_1^{ij}&\sim\frac{1}{4\omega^2}F_L^{ij}(\omega),\\ \nu^2V_2^{ij}&\sim\frac{1}{2\omega}F_2^{ij}(\omega),\\ \nu V_3^{ij}&\sim F_3^{ij}(\omega),\\ \nu^2V_4^{ij}&\sim F_4^{ij}(\omega).\end{aligned} \tag{2.25}$$

As will become evident later, we shall need to go beyond Eqs. (2.25) and know the scaling behavior of the full amplitudes R_k^{ij} in terms of the scaling functions $F_k^{ij}(\omega)$ ($k=L,2,3,4$). We therefore appeal to the concept of generalized scaling discussed in Ref. 17 and assume, on dimensional grounds, the scaling behavior

$$\begin{aligned}\nu R_1^{ij}&\sim\phi_1^{ij}(\omega),\\ \nu^2R_2^{ij}&\sim\phi_2^{ij}(\omega),\\ \nu R_3^{ij}&\sim\phi_3^{ij}(\omega),\\ \nu^2R_4^{ij}&\sim\phi_4^{ij}(\omega).\end{aligned} \tag{2.26}$$

We next seek to relate the ϕ_k^{ij} to the F_k^{ij} by first writing fixed-q^2 dispersion relations for the amplitudes R_k^{ij} and then considering the large-ν-at-fixed-ω limit of these relations. Unlike the work of Ref. 4, where the question of subtractions in the dispersion relations was unimportant since the object there was to derive sum rules for the imaginary parts of the amplitudes, the present work demands a definite attitude towards this question since we shall primarily be concerned with sum rules for the real parts of the amplitudes.

On the basis of the Regge-pole model one would expect[18] R_2^{ij} to satisfy an unsubtracted dispersion relation, whereas for R_1^{ij} one subtraction is needed. The spin-dependent amplitudes R_3^{ij} and R_4^{ij}, on the other hand, are expected[4] to satisfy unsubtracted dispersion relations provided that Regge cuts do not interfere with the large-ν-at-fixed-q^2 behavior of their absorptive parts. To begin with we shall assume that this proviso is met, but we shall reexamine this question in Sec. III. Deferring also for the moment the discussion of R_1^{ij}, we therefore write the unsubtracted dispersion relations ($k=2, 3, 4$)

$$R_k^{ij}(\nu, q^2)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{V_k^{ij}(\nu', q^2)d\nu'}{\nu'-\nu-i\epsilon}. \tag{2.27}$$

Changing the integration variable in Eq. (2.27) to $\omega'=-q^2/2\nu'$, we obtain

$$\nu R_k^{ij}[\omega, q^2]=\frac{1}{\pi}\int_{-1}^{1}\frac{\nu' V_k^{ij}[\omega', q^2]d\omega'}{\omega-\omega'-i\epsilon}, \tag{2.28}$$

$$\nu^2 R_k^{ij}[\omega, q^2]=\frac{1}{\pi\omega}\int_{-1}^{1}\frac{\omega'\nu'^2 V_k^{ij}[\omega', q^2]d\omega'}{\omega-\omega'-i\epsilon}, \tag{2.29}$$

from which we obtain, on account of (2.25) and (2.26), the asymptotic representations

$$\phi_2^{ij}(\omega)=\frac{1}{2\pi\omega}\int_{-1}^{1}\frac{F_2^{ij}(\omega')d\omega'}{\omega-\omega'-i\epsilon}, \tag{2.30}$$

$$\phi_3^{ij}(\omega)=\frac{1}{\pi}\int_{-1}^{1}\frac{F_3^{ij}(\omega')d\omega'}{\omega-\omega'-i\epsilon}, \tag{2.31}$$

$$\phi_4^{ij}(\omega)=\frac{1}{\pi\omega}\int_{-1}^{1}\frac{\omega' F_4^{ij}(\omega')d\omega'}{\omega-\omega'-i\epsilon}. \tag{2.32}$$

An alternative form for these equations may be obtained on taking advantage of the identity

$$\lim_{\epsilon\to 0_+}\frac{1}{(x-i\epsilon)}=\mathrm{P}\frac{1}{x}+i\pi\delta(x) \tag{2.33}$$

and noting that

$$\phi_k^{ij}(\omega)=\mathrm{Re}\,\phi_k^{ij}(\omega)+i\,\mathrm{Im}\,\phi_k^{ij}(\omega),$$

where

$$\begin{aligned}\mathrm{Im}\,\phi_2^{ij}(\omega)&=\frac{1}{2\omega}F_2^{ij}(\omega),\\ \mathrm{Im}\,\phi_3^{ij}(\omega)&=F_3^{ij}(\omega),\\ \mathrm{Im}\,\phi_4^{ij}(\omega)&=F_4^{ij}(\omega).\end{aligned} \tag{2.34}$$

We get

$$\mathrm{Re}\,\phi_2^{ij}(\omega)=\frac{1}{2\pi\omega}\mathrm{P}\int_{-1}^{1}\frac{F_2^{ij}(\omega')d\omega'}{\omega-\omega'}, \tag{2.35}$$

$$\mathrm{Re}\,\phi_3^{ij}(\omega)=\frac{1}{\pi}\mathrm{P}\int_{-1}^{1}\frac{F_3^{ij}(\omega')d\omega'}{\omega-\omega'}, \tag{2.36}$$

$$\mathrm{Re}\,\phi_4^{ij}(\omega)=\frac{1}{\pi\omega}\mathrm{P}\int_{-1}^{1}\frac{\omega' F_4^{ij}(\omega')d\omega'}{\omega-\omega'}. \tag{2.37}$$

The results (2.35)–(2.37) are consequences of scaling and unsubstracted dispersion relations. It is our aim in what follows to make use of these results in the sum rules (2.23) and (2.24). Towards this end we find it convenient to introduce the variables

$$\begin{aligned}\alpha&=p_0^{-1},\\ \xi&=-p_0^{-2}\vec{\mathrm{p}}\cdot\vec{\mathrm{q}},\\ \eta&=\vec{\mathrm{q}}^2-p_0^{-2}(\vec{\mathrm{p}}\cdot\vec{\mathrm{q}})^2.\end{aligned} \tag{2.38}$$

These parameters vary such that

$$\begin{aligned}&0\leq\alpha\leq 1,\\ &-\infty<\xi<\infty,\\ &\eta\geq\frac{\xi^2}{1-\alpha^2}.\end{aligned} \tag{2.39}$$

and when $\alpha=1$, $\xi=0$ and $\eta\geq 0$.

A change of the integration variable in (2.23) and (2.24) from q_0 to ν allows us to recast these equations in the form

$$\int R_k^{ij}(\nu, \alpha^2\nu^2-2\xi\nu-\eta)d\nu=0, \tag{2.40}$$

$$\alpha^2\int \nu R_k^{ij}(\nu, \ldots)d\nu=ib_3^{ij}\delta_{k,3} \tag{2.41}$$

for $k=3, 4$.

In the refined infinite-momentum technique[1] one considers the general integral

$$\begin{aligned}I&=\int A(\nu, \alpha^2\nu^2-2\xi\nu-\eta)d\nu\\ &=\left(\int_{-\infty}^{-R}+\int_{-R}^{R}+\int_{R}^{\infty}\right)A\,d\nu,\end{aligned} \tag{2.42}$$

and assumes that the limit $\alpha\to 0_+$ can be taken inside the integral over $[-R,R]$. In the other intervals the variable ν is changed to $\xi'=\xi-\frac{1}{2}\alpha^2\nu$. One then obtains

$$\lim_{\alpha^2\to 0}I=\int_{-R}^{R}A(\nu, -2\xi\nu-\eta)d\nu+\lim_{\alpha^2\to 0}\left(\int_{-\infty}^{\xi-\alpha^2R/2}+\int_{\xi+\alpha^2R/2}^{\infty}\right)[2\alpha^{-2}A(-2\alpha^{-2}(\xi'-\xi), 4\alpha^{-2}\xi'(\xi'-\xi)-\eta)]d\xi'. \tag{2.43}$$

Clearly, in the last term the integral is evaluated in the scaling region provided that one chooses $R \geq R_0$, where R_0 is the lowest value of ν at which scaling behavior sets in. Thus, on sending $\alpha^2 \to 0$ and then letting $R \to \infty$ one has

$$\lim_{\alpha^2\to 0} I = \int_{-\infty}^{\infty} A(\nu, -2\xi\nu - \eta)d\nu - \mathrm{P}\int \frac{F(\xi')d\xi'}{\xi' - \xi}, \tag{2.44}$$

where $\nu A \sim F$ in the scaling limit.

Next we apply Eq. (2.44) to the sum rules (2.40) and (2.41) taking scaling behavior into account. We obtain

$$\int R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = \mathrm{P}\int \frac{\phi_3^{ij}(\xi')d\xi'}{\xi' - \xi}, \tag{2.45}$$

$$\int R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0, \tag{2.46}$$

from (2.40), and

$$\int \nu R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu + 2\alpha^{-2}\int \phi_3^{ij}(\xi')d\xi' = \alpha^{-2} i b_3^{ij}, \tag{2.47}$$

$$\int \nu R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = \mathrm{P}\int \frac{\phi_4^{ij}(\xi')d\xi'}{\xi' - \xi}, \tag{2.48}$$

from (2.41).

Assuming that the integral

$$\int \nu R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu \tag{2.49}$$

exists, we observe that Eq. (2.47) is equivalent to the relations

$$i b_3^{ij} = 2\int \phi_3^{ij}(\xi')d\xi', \tag{2.50}$$

$$\int \nu R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0. \tag{2.51}$$

The results (2.45), (2.46), (2.48), (2.50), and (2.51) are scaling sum rules for the complete amplitudes and include, expectedly, the corresponding sum rules for the imaginary parts that were derived in Ref. 5. For example, the imaginary part of Eq. (2.50), i.e.,

$$b_3^{ij} = 2\int F_3^{ij}(\xi')d\xi', \tag{2.52}$$

is identical to Eq. (3.30) of Ref. 5.

We now consider the real parts of Eqs. (2.45), (2.46), (2.48), and (2.51), and using the representations (2.35)–(2.37) and the identities (2.33) and (2.22) we obtain

$$\int \mathrm{Re} R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = \pi F_3^{ij}(\xi) + \frac{1}{\pi}\int \frac{F_3^{ij}(\omega')d\omega' d\xi'}{(\xi' - \xi - i\epsilon)(\xi' - \omega' - i\epsilon)} \tag{2.53}$$

from (2.45),

$$\int \mathrm{Re} R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0 \tag{2.54}$$

from (2.46),

$$\int \nu\, \mathrm{Re} R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = \pi F_4^{ij}(\xi) + \frac{2i}{\xi}\int F_4^{ij}(\omega')d\omega' + \frac{1}{\pi\xi}\int \frac{\omega' F_4^{ij}(\omega')}{\xi' - \omega' - i\epsilon}\left(\frac{1}{\xi' - \xi - i\epsilon} - \frac{1}{\xi' - i\epsilon}\right)d\omega' d\xi' \tag{2.55}$$

from (2.48), and

$$\int \nu\, \mathrm{Re} R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0 \tag{2.56}$$

from (2.51). In Eqs. (2.53) and (2.55) the ξ' integrals have the form (α and β real and unequal parameters)

$$G(\alpha,\beta) = \int_{-\infty}^{\infty} \frac{d\xi'}{(\xi' - \alpha - i\epsilon)(\xi' - \beta - i\epsilon)}, \tag{2.57}$$

which with the aid of (2.33) is given by

$$G(\alpha,\beta) = \frac{1}{\alpha - \beta}\left(\mathrm{P}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \alpha} - \mathrm{P}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \beta}\right) = 0. \tag{2.58}$$

We also observe that the second term on the right-hand side of Eq. (2.55) becomes singular in the fixed-mass limit $\xi \to 0$ unless

$$\int F_4^{ij}(\omega')d\omega' = 0. \tag{2.59}$$

Thus this equation is a necessary condition for the existence of the integral on the left-hand side of (2.55) at $\xi = 0$. It was also derived in Refs. 4 and 5.

Finally, on account of (2.58) and (2.59) the sum rules (2.53)–(2.56) assume the form

$$\int \mathrm{Re} R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = \pi F_3^{ij}(\xi), \tag{2.60}$$

$$\int \mathrm{Re} R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0, \tag{2.61}$$

$$\int \nu\, \mathrm{Re} R_4^{ij}(\nu, -2\xi\nu - \eta)d\nu = \pi F_4^{ij}(\xi), \tag{2.62}$$

$$\int \nu\, \mathrm{Re} R_3^{ij}(\nu, -2\xi\nu - \eta)d\nu = 0. \tag{2.63}$$

C. Fixed-mass sum rules

On simply setting $\xi=0$ in Eqs. (2.60)–(2.63) and remembering that $\mathrm{Re}R_k^{ij}$ have the opposite crossing properties to those of the corresponding imaginary parts V_k^{ij}, we obtain the nontrivial fixed-mass sum rules

$$\int_{-\infty}^{\infty}\mathrm{Re}R_3^{(ij)}(\nu,q^2)d\nu=\pi F_3^{(ij)}(0), \tag{2.64}$$

$$\int_{-\infty}^{\infty}\mathrm{Re}R_4^{[ij]}(\nu,q^2)d\nu=0, \tag{2.65}$$

$$\int_{-\infty}^{\infty}\nu\,\mathrm{Re}R_4^{(ij)}(\nu,q^2)d\nu=\pi F_4^{ij}(0), \tag{2.66}$$

$$\int_{-\infty}^{\infty}\nu\,\mathrm{Re}R_3^{[ij]}(\nu,q^2)d\nu=0, \tag{2.67}$$

where $q^2<0$.

In addition, since the symmetric component, $R_2^{(ij)}$, of R_2^{ij} receives no contribution from the (antisymmetric) noncausal part of V_2^{ij}, so that $R_2^{(ij)}=g_2^{(ij)}$ [see Eq. (2.19)], our procedure also yields the sum rule

$$\int_{-\infty}^{\infty}\mathrm{Re}R_2^{(ij)}(\nu,q^2)d\nu=0, \tag{2.68}$$

where $q^2<0$.

D. Sum rules for $\mathrm{Re}R_1^{ij}$

It was stated earlier that R_1^{ij} satisfies a once-subtracted dispersion relation in ν at fixed q^2. We therefore write

$$\mathrm{Re}R_1^{ij}(\nu,q^2)=\mathrm{Re}R_1^{ij}(0,q^2)+\frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}V_1^{ij}(\nu',q^2)\left(\frac{1}{\nu'-\nu}-\frac{1}{\nu'}\right)d\nu'. \tag{2.69}$$

Changing the integration variable to ω' we obtain

$$\nu\,\mathrm{Re}R_1^{ij}[\omega,q^2]=\nu\,\mathrm{Re}R_1^{ij}[\infty,q^2]+\frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}\nu' V_1^{ij}[\omega',q^2]\left(\frac{1}{\omega-\omega'}-\frac{1}{\omega}\right)d\omega', \tag{2.70}$$

which in the scaling limit becomes

$$\mathrm{Re}\phi_1^{ij}(\omega)=\lim_{q^2\to-\infty}\left(\frac{-q^2}{2\omega}\mathrm{Re}R_1^{ij}[\infty,q^2]\right)+\frac{\mathrm{P}}{4\pi}\int_{-\infty}^{\infty}\frac{F_L^{ij}(\omega')}{\omega'^2}\left(\frac{1}{\omega-\omega'}-\frac{1}{\omega}\right)d\omega'. \tag{2.71}$$

The second integral on the right-hand side of this equation is proportional to the scalar operator Schwinger term[13]

$$\frac{\mathrm{P}}{4\pi}\int_{-\infty}^{\infty}\frac{F_L^{ij}(\omega')d\omega'}{\omega'^2}=\pi b_1^{ij}=0, \tag{2.72}$$

where the last equality follows from (2.17). Hence,

$$F_L^{ij}(\omega)=0,\quad \omega\neq 0. \tag{2.73}$$

Furthermore, on using Eqs. (2.18), (2.16), and (2.14) we obtain in the limit $q^2\to-\infty$ with ν finite and fixed

$$\mathrm{Re}R_1^{ij}(\nu,q^2)\sim-\frac{1}{\pi}b_1^{ij}q^{-2}-\frac{2}{\pi}c_1^{ij}\nu q^{-4}, \tag{2.74}$$

where c_1^{ij} is a constant defined by[5,11,19]

$$\int d^4u\,ds\,u_\lambda\psi_1^{ij}(u,s)=c_1^{ij}p_\lambda. \tag{2.75}$$

Hence,[20]

$$\begin{aligned}\lim_{q^2\to-\infty}\left(\frac{-q^2}{2\omega}\mathrm{Re}R_1^{ij}[\omega=\infty,q^2]\right)&=\lim_{q^2\to-\infty}\left[\frac{-q^2}{2\omega}\mathrm{Re}R_1^{ij}(\nu=0,q^2)\right]\\&=\frac{1}{2\pi\omega}b_1^{ij}\\&=0,\end{aligned} \tag{2.76}$$

since $b_1^{ij}=0$ by Eq. (2.17). Thus, going back to (2.71) we can finally write

$$\mathrm{Re}\phi_1^{ij}(\omega)=0. \tag{2.77}$$

With this result the procedure of the present work yields the sum rule

$$\int_{-\infty}^{\infty}\mathrm{Re}R_1^{(ij)}(\nu,q^2)d\nu=0, \tag{2.78}$$

where $q^2<0$.

E. Axial-vector-current sum rules

Our method may also be applied to the amplitude R_5^{ij} defined in the expression for the Fourier transform of the spin-averaged connected matrix element of the axial-vector–vector–current retarded commutator:

$$\begin{aligned}R_{\mu\nu}^{(5)ij}(p,q)&=\frac{i}{\pi}\int e^{iqx}\theta(x_0)\langle p|[A_\mu^i(x),V_\nu^j(0)]|p\rangle_c d^4x\\&=i\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta R_5^{ij}(\nu,q^2).\end{aligned} \tag{2.79}$$

The imaginary part, V_5^{ij}, of R_5^{ij} is defined by

$$\begin{aligned}C_{\mu\nu}^{(5)ij}(p,q)&=\frac{1}{2\pi}\int e^{iqx}\langle p|[A_\mu^i(x),V_\nu^j(0)]|p\rangle_c d^4x\\&=i\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta V_5^{ij}(\nu,q^2),\end{aligned} \tag{2.80}$$

and is causal.[12]

Assuming that R_5^{ij} satisfies[21] a fixed-q^2 unsubtracted dispersion relation in ν and that R_5^{ij} and V_5^{ij} exhibit the scaling behavior

$$\nu R_5^{ij}(\nu,q^2) \sim \phi_5^{ij}(\omega) = \mathrm{Re}\,\phi_5^{ij}(\omega) + iF_5^{ij}(\omega),$$
$$\nu V_5^{ij}(\nu,q^2) \sim F_5^{ij}(\omega), \tag{2.81}$$

we obtain the sum rules ($q^2 < 0$)

$$\int_{-\infty}^{\infty} \mathrm{Re}R_5^{ij}(\nu,q^2)d\nu = \pi F_5^{ij}(0), \tag{2.82}$$

$$\int_{-\infty}^{\infty} \nu\, \mathrm{Re}R_5^{[ij]}(\nu,q^2)d\nu = 0, \tag{2.83}$$

$$\int d^4u\, ds\, \psi_5^{ij}(u,s) = b_5^{ij} = 2\int F_5^{ij}(\omega)d\omega. \tag{2.84}$$

Among the fixed-mass sum rules derived in this section and summarized in the Appendix are those obtained by Heimann *et al.*[8] from a technique based on the use of the $(-,i)$, $(-,+)$, (i,j), and $(+,i)$ light-cone commutators of a free-quark model. To see this, first note that the amplitudes T_2^{ij}, S_1^{ij}, S_2^{ij}, and T_3^{ij} of Ref. 8 are related to ours by

$$R_2^{ij} = -\frac{1}{q^2}T_2^{ij},$$
$$R_3^{ij} = -S_1^{ij} - \nu S_2^{ij},$$
$$R_4^{ij} = S_2^{ij}, \tag{2.85}$$
$$R_5^{ij} = T_3^{ij}.$$

With this identification we recognize Eqs. (A5), (A2), (A3), and (A7) (see the Appendix) as identical, respectively, to the sum rules A, B, D, and E of Ref. 8. Moreover, the sum of Eqs. (A1), and (A3) is equivalent to the sum rule C of that reference.

In obtaining sum rules in this paper we assumed that all the relevant amplitudes, with the exception of R_1^{ij}, obey fixed-q^2 unsubtracted dispersion relations in ν. While this seems[18,21] to be a sensible assumption for R_2^{ij} and R_5^{ij}, it would be incorrect for the spin-dependent amplitudes R_3^{ij} and R_4^{ij} if Regge cuts dominate the Regge limits of their absorptive parts. We therefore investigate in the next section the dependence of our results for R_3^{ij} and R_4^{ij} on the unsubtracted-dispersion-relation assumption.

III. DISPERSION RELATIONS IN THE PRESENCE OF DOMINANT REGGE CUTS

Restricting ourselves to the consideration of the isospin-symmetric functions $V_3^{(ij)}(\nu,q^2)$ and $V_4^{(ij)}(\nu,q^2)$, we note that their Regge behavior in the presence of dominant Regge cuts is predicted to be[22]

$$V_3^{(ij)} \overset{R}{\sim} \beta_3^{(ij)}(q^2)\epsilon(\nu)|\nu|^{\alpha_{A_1}-1} + \beta_{3c}^{(ij)}(q^2)\epsilon(\nu)|\nu|^{\alpha_c}, \tag{3.1}$$

$$V_4^{(ij)} \overset{R}{\sim} \beta_4^{(ij)}(q^2)|\nu|^{\alpha_{A_1}-2} + \beta_{4C}^{(ij)}(q^2)|\nu|^{\alpha_c-1}, \tag{3.2}$$

where R denotes the Regge limit $\nu\to\infty$ at fixed q^2, $\alpha_{A_1} \equiv \alpha_{A_1}(0) \sim -0.1$ is the intercept of the A_1 trajectory or A_1-Pomeron cut, and α_c is $\alpha(0)$ of the following possible cuts: (i) Pomeron-f, $\alpha_c \sim 0.5$; (ii) Pomeron-A_2, $\alpha_c \sim 0.5$; (iii) Pomeron-Pomeron, $\alpha_c = 1$. Thus, in the presence of dominant cuts the unsubtracted-dispersion-relation assumption, at least for $R_3^{(ij)}$, should be relinquished.

We next write for $R_3^{(ij)}$ a once-subtracted dispersion relation whose form is akin to that of Eq. (2.69) for R_1^{ij}. By steps similar to those leading from (2.69) to (2.71), the scaling limit of this relation is found to be

$$\mathrm{Re}\,\phi_3^{(ij)}(\omega) = \lim_{q^2\to-\infty}\left(\frac{-q^2}{2\omega}\mathrm{Re}R_3^{(ij)}[\infty,q^2]\right) + \frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty} F_3^{(ij)}(\omega')\left(\frac{1}{\omega-\omega'} - \frac{1}{\omega}\right)d\omega'. \tag{3.3}$$

Furthermore, from Eqs. (2.20), (2.16), and (2.14) we have in the limit $q^2 \to -\infty$ with ν finite and fixed

$$\mathrm{Re}R_3^{(ij)}(\nu,q^2) \sim -\frac{1}{\pi}b_3^{(ij)}q^{-2} - \frac{2}{\pi}c_3^{(ij)}\nu q^{-4}, \tag{3.4}$$

where the constant $c_3^{(ij)}$, defined by an equation analogous to (2.75), is zero[5] by crossing symmetry. Thus

$$\lim_{q^2\to-\infty}\left(\frac{-q^2}{2\omega}\mathrm{Re}R_3^{(ij)}[\omega=\infty,q^2]\right) = \lim_{q^2\to-\infty}\left[\frac{-q^2}{2\omega}\mathrm{Re}R_3^{(ij)}(\nu=0,q^2)\right] = \frac{1}{2\pi\omega}b_3^{(ij)}. \tag{3.5}$$

Using (3.5) in (3.3) we therefore write the latter in the form

$$\mathrm{Re}\,\phi_3^{(ij)}(\omega) = \frac{1}{\omega}d_3^{(ij)} + \frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}\frac{F_3^{(ij)}(\omega')d\omega'}{\omega-\omega'}, \tag{3.6}$$

where

$$d_3^{(ij)} = \frac{1}{2\pi}\left[b_3^{(ij)} - 2\int_{-\infty}^{\infty}F_3^{(ij)}(\omega')d\omega'\right]. \tag{3.7}$$

On comparing Eqs. (3.6) and (2.36) for $\mathrm{Re}\,\phi_3^{(ij)}(\omega)$ we see that the subtraction assumption modifies the right-hand side of (2.36) by giving rise to the extra term $(1/\omega)d_3^{(ij)}$. If Eq. (2.52), whose validity hinges on the condition that the imaginary part of the integral (2.49) exists, does hold, then $d_3^{(ij)} = 0$ and (3.6) and (2.36) coincide. However, even if we were to keep $d_3^{(ij)} \neq 0$ and repeat the steps leading to the result (2.53) we would still get precisely that result. Thus the unsubtracted-dispersion-relation assumption is *not* necessary[23] for deriving the sum rules for $\mathrm{Re}R_3^{ij}$.

Next let us consider $R_4^{(ij)}$ and again write a once-subtracted dispersion relation resembling (2.69).

Its scaling form would be

$$\mathrm{Re}\phi_4^{(ij)}(\omega)=\lim_{q^2\to-\infty}\left(\frac{q^4}{4\omega^2}\mathrm{Re}R_4^{(ij)}[\infty,q^2]\right)+\frac{\mathrm{P}}{\pi\omega}\int_{-\infty}^{\infty}\frac{\omega' F_4^{(ij)}(\omega')d\omega'}{\omega-\omega'}, \tag{3.8}$$

where we have made use of the fact that

$$\int_{-\infty}^{\infty}\omega' F_4^{(ij)}(\omega')d\omega'=0\quad [F_4^{(ij)}(\omega)=F_4^{(ij)}(-\omega)].$$

Apart from the first term on its right-hand side, Eq. (3.8) is identical to the ij-symmetric part of (2.37). From the relation corresponding to (3.4) and the observation that $b_4^{(ij)}$ vanishes by crossing symmetry we have as $q^2\to-\infty$ for ν finite and fixed

$$\mathrm{Re}R_4^{(ij)}(\nu,q^2)\sim-\frac{2}{\pi}c_4^{(ij)}\nu q^{-4}, \tag{3.9}$$

so that

$$\lim_{q^2\to-\infty}[q^4\mathrm{Re}R_4^{(ij)}(\nu=0,q^2)]=0, \tag{3.10}$$

implying that the first term on the right-hand side of (3.8) is equal to zero. Hence (3.8) is indeed identical to the ij-symmetric part of (2.37), and the unsubtracted-dispersion-relation results (A3) and (2.59) remain unchanged on using instead a once-subtracted dispersion relation.

In deriving fixed-mass sum rules we have simply set $\xi=0$, which corresponds to taking $q_-=0$ inside the integral in the light-cone approach of Heimann *et al.*[8] A consequence of this is that some of the sum rules will in general be invalid in the sense that they are expected to receive class-II state contributions which are missed on naively setting $\xi=0$. The parallel step of taking $q_-=0$ inside the integral in the light-cone approach also results in the neglect of these contributions, as is evident from Eqs. (B3)–(B5) of Ref. 8.

In a recent extension of his original refined infinite-momentum technique, Taha[10] has developed a procedure for proper handling of the fixed-mass limit which renders possible the inclusion of contributions from all classes of intermediate states. His procedure is based on the isolation of the possible divergences that may arise in this limit and is therefore essentially a recipe for regularization of divergent sum rules. The next section is devoted to a discussion of this aspect of the problem.

IV. REGULARIZATION OF DIVERGENT SUM RULES AND CLASS-II CONTRIBUTIONS

A. Review of Taha's regularization method[10]

The starting point in this method is the simple observation that an integral of the type

$$I^{ij}=p_0\int_{-\infty}^{\infty}A^{ij}(\nu,q^2)dq_0 \tag{4.1}$$

may be rewritten as

$$I^{ij}=p_0\int_{-\infty}^{\infty}\bar{A}^{ij}(\nu,q^2)dq_0, \tag{4.2}$$

where

$$\bar{A}^{ij}(\nu,q^2)=A^{ij}(\nu,q^2)-a^{ij}(\nu,q^2), \tag{4.3}$$

with the function $a^{ij}(\nu,q^2)$ satisfying the condition

$$\int_{-\infty}^{\infty}a^{ij}(\nu,q^2)dq_0=0. \tag{4.4}$$

If $a^{ij}(\nu,q^2)$ is a causal function, Eq. (4.4) will hold provided the spectral function $\psi^{ij}(u,s)$ in the JLD representation of $a^{ij}(\nu,q^2)$,

$$a^{ij}(\nu,q^2)=\int_0^{\infty}ds\int\epsilon(q_0-u_0)\delta((q-u)^2-s)\times\psi^{ij}(u,s)d^4u, \tag{4.5}$$

satisfies

$$\lim_{s\to\infty}\psi^{ij}(u,s)=0. \tag{4.6}$$

Changing the integration variable in (4.2) from q_0 to ν and introducing the parameters of Eq. (2.38), one can then apply the theorem (2.44) to the integral I^{ij} and obtain

$$\lim_{\alpha^2\to0}I^{ij}=K^{ij}-\mathrm{P}\int_{-\infty}^{\infty}\frac{\bar{F}^{ij}(\xi')d\xi'}{\xi'-\xi}, \tag{4.7}$$

where

$$K^{ij}=\int_{-\infty}^{\infty}\bar{A}^{ij}(\nu,-2\xi\nu-\eta)d\nu, \tag{4.8}$$

and in the scaling limit,

$$\nu\bar{A}^{ij}(\nu,q^2)\sim\bar{F}^{ij}(\omega). \tag{4.9}$$

To pass to the fixed-mass limit in (4.7) one first rewrites K^{ij}, for $\xi>0$, as

$$K^{ij}=\int_{-R}^{R}\bar{A}^{ij}(\nu,-2\xi\nu-\eta)d\nu+\left(\int_{-\infty}^{-2\xi R-\eta}+\int_{2\xi R-\eta}^{\infty}\right)\left[(2\xi)^{-1}\bar{A}^{ij}\left(-\frac{\eta+q^2}{2\xi},q^2\right)dq^2\right], \tag{4.10}$$

where $R\geq R_0$, the lowest value of ν at which Regge behavior sets in. Then taking the limit $\xi\to0$ in complete analogy with the steps leading from (2.43) to (2.44) one obtains

$$\lim_{\xi\to0_+}K^{ij}=\int_{-\infty}^{\infty}\bar{A}^{ij}(\nu,-\eta)d\nu+\mathrm{P}\int_{-\infty}^{\infty}\frac{G^{ij}(q^2)dq^2}{q^2+\eta} \tag{4.11}$$

where, in the Regge limit,[24]

$$\nu\bar{A}^{ij}(\nu,q^2)\sim\epsilon(\nu)G^{ij}(q^2). \tag{4.12}$$

Moreover,

$$\lim_{\xi\to 0}\mathrm{P}\int_{-\infty}^{\infty}\frac{\bar{F}^{ij}(\xi')d\xi'}{\xi'-\xi}=\mathrm{P}\int_{-\infty}^{\infty}\frac{\bar{F}^{ij}(\xi')d\xi'}{\xi'}, \tag{4.13}$$

provided

$$\bar{F}^{ij}(\xi') \text{ is finite at } \xi'=0. \tag{4.14}$$

Hence one finally has

$$\lim_{\xi\to 0}\lim_{\alpha^2\to 0}I^{ij}=\int_{-\infty}^{\infty}\bar{A}^{ij}(\nu,-\eta)d\nu + \mathrm{P}\int_{-\infty}^{\infty}\frac{G^{ij}(q^2)dq^2}{q^2+\eta}-\mathrm{P}\int_{-\infty}^{\infty}\frac{\bar{F}^{ij}(\xi')d\xi'}{\xi'}. \tag{4.15}$$

Formula (4.15) is the basic result in the recent method of Taha[10] and it is valid provided the conditions (4.4), (4.9), (4.12), and (4.14) hold. The subtracted form of the integrand in the first term on the right-hand side guarantees, through condition (4.12), the Regge convergence of the fixed-mass sum rules. The third term on the right-hand side represents the Z-graph contributions, and the second term, which was absent in the original version[1] of the refined infinite-momentum-limit technique, denotes contributions from class-II intermediate states. The application of the theorem (4.15) to the type of sum rules discussed in this paper will be given in the next subsection.

B. Application to sum rules for real parts of the amplitudes

Consider, for example, the sum rule (A1), the derivation of which springs from the general causality sum rule for $\mathrm{Re}R_3^{(ij)}$, i.e., from the real part of the ij-symmetric $k=3$ component of Eq. (2.23),

$$\int \mathrm{Re}R_3^{(ij)}dq_0=0. \tag{4.16}$$

Regardless of the Regge behavior of $V_3^{(ij)}$ the integral in (4.16) converges provided $\lim_{s\to\infty}\psi_3^{(ij)}(u,s)=0$. In what follows we apply Taha's regularization method to show how, in principle, a convergent fixed-mass sum rule can be obtained from (4.16).

Comparing (4.16) and (4.1) we write

$$A^{(ij)}(\nu,q^2)=\mathrm{Re}R_3^{(ij)}(\nu,q^2). \tag{4.17}$$

The Bjorken limit (B limit) of this function is, according to (2.26),

$$\nu A^{(ij)}\overset{B}{\sim}\mathrm{Re}\phi_3^{(ij)}(\omega). \tag{4.18}$$

For current-particle scattering amplitudes Regge behavior is usually assumed only for the absorptive parts.[25] If, however, the amplitude obeys a fixed-q^2 dispersion relation in ν, the large-ν behavior of the real part can nevertheless be predicted from knowledge of the number of subtractions in that relation. This is accomplished with the aid of an asymptotic theorem[25] which states that, given an analytic regular function $f(\nu)$ satisfying the requirement

$$f(\nu)\sim\text{const}\times\nu^{\alpha},\quad \nu\to\infty \tag{4.19}$$

$\mathrm{Re}\alpha=-n-\epsilon$ (n a non-negative integer, $0<\epsilon<1$), the asymptotic expansion for large ν of its Stieltjes-Hilbert transform,

$$F(\nu)=\frac{\mathrm{P}}{\pi}\int_{-\infty}^{\infty}\frac{f(\nu')d\nu'}{\nu'-\nu}, \tag{4.20}$$

is given by

$$F(\nu)\sim\sum_{p=1}^{n}\frac{a_p}{\nu^p}+\text{const}\times\nu^{\alpha}, \tag{4.21}$$

where

$$a_p=-\frac{1}{\pi}\int_{-\infty}^{\infty}\nu^{p-1}f(\nu)d\nu. \tag{4.22}$$

As a corollary one also deduces that if $f(\nu)$ and $F(\nu)$ are connected by a once-subtracted dispersion relation then they both have the same asymptotic behavior in ν.

In the presence of dominant Regge cuts $\mathrm{Re}R_3^{(ij)}$ obeys a once-subtracted dispersion relation in ν, and consequently we predict, on the strength of the preceding theorem, that $\mathrm{Re}R_3^{(ij)}$ has the same large-ν behavior as $V_3^{(ij)}$. The next-to-leading behavior is obtained on subtracting the leading behavior from $\mathrm{Re}R_3^{(ij)}$ and noting that the remainder satisfies an unsubtracted dispersion relation. Then the asymptotic theorem, together with the fact that $V_3^{(ij)}(\nu,q^2)$ is odd in ν, enables us to infer that this behavior is also the same for both real and imaginary parts of $R_3^{(ij)}$. Thus we are finally able to write in the Regge limit (R limit)

$$\mathrm{Re}R_3^{(ij)}\overset{R}{\sim}\gamma_3^{(ij)}(q^2)|\nu|^{\alpha_{A1}-1}+\gamma_{3c}^{(ij)}(q^2)|\nu|^{\alpha_c}. \tag{4.23}$$

Next, define $\bar{A}^{(ij)}(\nu,q^2)$ by

$$\bar{A}^{(ij)}(\nu,q^2)=A^{(ij)}(\nu,q^2)-a^{(ij)}(\nu,q^2), \tag{4.24}$$

where $a^{(ij)}(\nu,q^2)$ is an even function in ν that satisfies (4.4) and is such that $\nu a^{(ij)}$ scales and

$$a^{(ij)}(\nu,q^2)\to\gamma_{3c}^{(ij)}(q^2)|\nu|^{\alpha_c} \tag{4.25}$$

in the Regge limit. Then

$$\nu\bar{A}^{(ij)}(\nu,q^2)\xrightarrow{B}\mathrm{Re}\bar{\phi}_3^{(ij)}(\omega) \tag{4.26}$$

and

$$\nu\bar{A}^{(ij)}(\nu,q^2)\xrightarrow{R}\gamma_3^{(ij)}(q^2)|\nu|^{\alpha_{A1}}. \tag{4.27}$$

Although we have previously indicated that $\alpha_{A1}\sim -0.1$, we nevertheless remark that this value should not be taken too seriously since the A_1, to be charitable, is not too well known. If α_{A1} is not positive then Eqs. (4.15), (4.16), (4.26), and (4.27) yield the convergent fixed-mass sum rule ($q^2 < 0$)

$$\int_{-\infty}^{\infty} \mathrm{Re}\bar{R}_3^{(ij)}(\nu, q^2)d\nu - \mathrm{P}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\bar{\phi}_3^{(ij)}(\xi')d\xi'}{\xi'} + \lambda \mathrm{P}\int_{-\infty}^{\infty} \frac{\gamma_3^{(ij)}(q^2)dq^2}{q^2+\eta} = 0, \quad (4.28)$$

where

$$\lambda = 1, \quad \text{if } \alpha_{A1} = 0$$
$$= 0, \quad \text{if } \alpha_{A1} < 0. \quad (4.29)$$

The function $\mathrm{Re}\bar{\phi}_3^{(ij)}$ can be related to $\bar{F}_3^{(ij)}$ by the techniques of Sec. II. If $\alpha_{A1} = 0$ the last term on the left-hand side of (4.28) will be present and one would then need to know $\gamma_3^{(ij)}(q^2)$. Likewise, a knowledge of $\gamma_{3c}^{(ij)}(q^2)$ is essential for the calculation of the appropriate subtraction function $a^{(ij)}(\nu, q^2)$ [see the Appendix in Ref. 10.] Owing to the doubtful status of the A_1 any calculation of $\gamma_3^{(ij)}(q^2)$ is bound to be uncertain, whereas quantitative deductions about $\gamma_{3c}^{(ij)}$ will depend on models for Regge cuts. If, on the other hand, $\alpha_{A1} < 0$, the sum rule will not receive class-II contributions unless there is an $\alpha = 0$ fixed pole in $\mathrm{Re}R_3^{(ij)}$. If such a pole does exist then it will give rise to class-II contributions of the form of the last term on the left-hand side of (4.28) (with $\lambda = 1$) but with the residue of the pole replacing $\gamma_3^{(ij)}(q^2)$. One hopes that the experimental check of the sum rule could then throw light on the nature of the residue of the pole, e.g., whether or not it is a polynomial in q^2.

Finally, we remark that the procedure we followed in the regularization of the sum rule (A1) can also be repeated for the remaining sum rules in the Appendix.

ACKNOWLEDGMENTS

The author is grateful to Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He would also like to thank the Swedish International Development Authority for making possible his associateship at the ICTP.

APPENDIX

The following is a summary of the fixed-mass sum rules.

$$\int \mathrm{Re}R_3^{(ij)}d\nu = \pi F_3^{(ij)}(0), \quad (\mathrm{A1})$$

$$\int \mathrm{Re}R_4^{[ij]}d\nu = 0, \quad (\mathrm{A2})$$

$$\int \nu\, \mathrm{Re}R_4^{(ij)}d\nu = \pi F_4^{(ij)}(0), \quad (\mathrm{A3})$$

$$\int \nu\, \mathrm{Re}R_3^{[ij]}d\nu = 0, \quad (\mathrm{A4})$$

$$\int \mathrm{Re}R_2^{(ij)}d\nu = 0, \quad (\mathrm{A5})$$

$$\int \mathrm{Re}R_1^{(ij)}d\nu = 0, \quad (\mathrm{A6})$$

$$\int \mathrm{Re}R_5^{(ij)}d\nu = \pi F_5^{(ij)}(0), \quad (\mathrm{A7})$$

$$\int \nu\, \mathrm{Re}R_5^{[ij]}d\nu = 0, \quad (\mathrm{A8})$$

$$\int d^4u\, ds\, \psi_5^{ij}(u,s) = b_5^{ij} = 2\int F_5^{ij}(\omega)d\omega. \quad (\mathrm{A9})$$

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$$\mathrm{Re}R_3^{[ij]}(\nu,q^2) \underset{\substack{q^2 \to -\infty \\ \nu \text{ fixed}}}{\sim} -\frac{2}{\pi} c_3^{[ij]} \nu q^{-4}.$$

Hence

$$q^4 \mathrm{Re}R_3^{[ij]}(0,q^2) \to 0,$$

and the ij-antisymmetric component of Eq. (2.36) is unchanged.

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