Impossibility of spontaneously breaking local symmetries

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It is argued that a spontaneous breaking of local symmetry for a symmetrical gauge theory without gauge fixing is impossible. The argument is demonstrated in a simple system of Abelian gauge fields on a lattice.

INTRODUCTION

A mechanism for confining quarks was suggested recently,[1-3] coupling the quarks strongly to gauge fields (either Abelian or non-Abelian). The model was demonstrated with a built-in ultraviolet cutoff being defined on a space-time lattice,[1,2] and treated by the path-integral formalism. A Hamiltonian version of the model was also investigated[3] defining the fields on a space lattice and continuous time. A simple model of this type[2] consists of a field ϕ_i which is an angle variable defined on the sites of a space-time cubic lattice coupled to a gauge field $A_{i,\underline{n}}$, also an angle variable, defined on the link between the neighboring sites i and $i+\underline{n}$, where $\underline{n}$ is a primitive translation of the lattice. The action is given by

$$S=K\sum_{i,\underline{n}}\cos(\phi_i-\phi_{i+\underline{n}}-A_{i,\underline{n}})+\frac{1}{g^2}\sum_{i,\underline{n}\neq\underline{m}}\cos(A_{i,\underline{n}}+A_{i+\underline{n},\underline{m}}-A_{i+\underline{m},\underline{n}}-A_{i,\underline{m}}). \tag{1}$$

The first sum is taken over the sites and the positive translations $\underline{n}$, the second over the sites and the pairs of positive mutually orthogonal primitive translations $\underline{n},\underline{m}$. The action (1) is invariant under the local (i.e., site-dependent) gauge transformation

$$\begin{aligned}\phi_i&\to\phi_i+C_i\\ A_{i,\underline{n}}&\to A_{i,\underline{n}}+C_i-C_{i+\underline{n}}\,.\end{aligned} \tag{2}$$

C_i is an arbitrary c-number function of the sites. The condition for ϕ to be an angle variable is some substitution for self-interaction of the field ϕ. In this way we get a compact Abelian group with simple potential generalization to non-Abelian compact groups. It is emphasized[1,2] that since the orbit of each variable under the gauge group is compact, one can use the action in its symmetrical form without introducing an asymmetrical gauge-fixing term.

The power of the model to confine quarks (in this case, to confine ϕ excitations) was illustrated in the strong-coupling limit $g\to\infty$. In the opposite limit of weak coupling, $g\to 0$, the "magnetic" term in the action proportional to $1/g^2$ becomes important, and one expects for large enough K a behavior similar to ordinary QED. It is natural to suggest a phase transition for an intermediate value of g. The crudest way to look for such a transition is the mean-field approximation, and, indeed, a critical value for g was calculated in this approximation. In that approach one substitutes in the path integral for the true action an action in which each site and link interacts with the mean values of the neighboring fields rather than the fields themselves. So, one sets $\alpha=\langle\cos\phi\rangle$ and $\beta=\langle\cos A\rangle$, and the path integral then splits into a product of independent integrals over each field separately, which is a function of α and β. Using this simple path integral, one then calculates $\langle\cos\phi\rangle$ and $\langle\cos A\rangle$ as a function of α and β, and demands, for self-consistency, that $\langle\cos\phi\rangle=\alpha$ and $\langle\cos A\rangle=\beta$ (see Ref. 2 for a more accurate treatment). It is claimed that the larger d, the dimension of space-time, is the better the assumption is of any site to interact with the mean field only. The self-consistency equation gives, indeed,[2] for large g the unique solution $\beta=0$, but for g smaller than some critical g_c, it gives another solution, which turns out to be closer in some sense to the true situation, in which $\langle\cos A\rangle=\beta\neq 0$. Since $\cos A_{i,\underline{n}}$ is not a gauge-invariant quantity, the phase transition at g_c was related to spontaneous breaking of local gauge symmetry. Our purpose here is to indicate that a local symmetry cannot be spontaneously broken (unless being explicitly broken by a gauge-fixing term), so that such a transition, if it exists, cannot be related to spontaneous breaking of local symmetry. The mean value of the non-gauge-invariant field is not a good order parameter for describing it. The phenomenon of spontaneously broken symmetries is characteristic of large macroscopic systems where the broken symmetry involves a macroscopic number of degrees of freedom. The behavior of the macroscopic system as a whole is, then, classical, and one can confine the system in a small portion of its orbit under the symmetry group with negligible quantum fluctuations. This is not the case for the local gauge symmetry. The number of degrees of freedom is still infinite, but the symmetry involves only very few of them. The quantum fluctuations tend to smear the ground-state wave function of the system homogeneously over the whole orbit under the group, since there is nothing in the invariant Hamiltonian to prefer any portion of it. To be more specific we are going to prove the vanishing of $\langle \cos A_{i,\underline{n}} \rangle$ for any value of K and g in (1). This is done in Sec. I. Then, in Sec. II, the possibility of breaking a global symmetry is explained. The fact that the mean-field approximation gives breaking of the symmetry is discussed in Sec. III, and Sec. IV is a brief conclusion.

I. PROOF OF THE VANISHING OF $\langle \cos A_{i,\underline{n}} \rangle$

The quantity $\langle \cos A_{i,\underline{n}} \rangle$ (analytically continued to imaginary time) is given by the path integral

$$\lim_{J\to 0}\lim_{N\to\infty} W(J,N) \equiv \lim_{J\to 0}\lim_{N\to\infty} \int_{-\pi}^{\pi} \prod_{j,\underline{m}}^{N} d\phi_j dA_{j,\underline{m}} e^{S(\phi,A)} \exp\Big(J\sum_{j,\underline{m}} \cos A_{j,\underline{m}}\Big) \cos A_{i,\underline{n}} Z(J,N)^{-1} . \quad (3)$$

N in (3) is the number of lattice sites, and

$$Z(N,J) = \int_{-\pi}^{\pi} \prod_{j,\underline{m}}^{N} d\phi_j dA_{j,\underline{m}} e^{S(\phi,A)} \exp\Big(J\sum_{j,\underline{m}} \cos A_{j,\underline{m}}\Big).$$

Change, now, the variables $\phi_j, A_{j,\underline{m}}$ to new variables $\phi_j, B_{j,\underline{m}}$ such that all the new variables except one will be invariant under the transformation (2) taken with $C_j = C\delta_{i,j}$. For that, naturally, we have to change only a few variables near the site i. For example, take

$$B_{j,\underline{m}} = A_{j,\underline{m}} - \delta_{i,j}\phi_i + \delta_{i-\underline{m},j}\phi_i . \quad (4)$$

The action will now be

$$S = K\sum_{\underline{m}}\Big[\sum_{j\neq i;\, j\neq i-m} \cos(\phi_j - \phi_{j+\underline{m}} - B_{j,\underline{m}}) + \cos(-\phi_{i+\underline{m}} - B_{i,\underline{m}}) + \cos(\phi_{i-\underline{m}} - B_{i-\underline{m},\underline{m}})\Big]$$

$$+\frac{1}{g^2}\sum_{j,\underline{l}>\underline{m}} \cos(B_{j,\underline{l}} + B_{j+\underline{l},\underline{m}} - B_{j+\underline{m},\underline{l}} - B_{j,\underline{m}}) ; \quad (5)$$

which is independent of ϕ_i.

The path integral (3) will now be

$$\lim_{J\to 0}\lim_{N\to\infty} W(J,N) = \lim_{J\to 0}\lim_{N\to\infty} Z^{-1}\int_{-\pi}^{\pi} \prod_{j\neq i}^{N} d\phi_j \prod_{j,\underline{m}} dB_{j,\underline{m}} e^{S(\phi,B)} \exp\Big(\sum_{\underline{m}}\sum_{j\neq i;\, j\neq i-\underline{m}} J\cos B_{j,\underline{m}}\Big)$$

$$\times\int_{-\pi}^{\pi} d\phi_i \exp\Big\{J\sum_{\underline{m}}[\cos(B_{i,\underline{m}} - \phi_i) + \cos(B_{i-\underline{m},\underline{m}} + \phi_i)]\Big\} \cos(B_{i,\underline{n}} - \phi_i) . \quad (6)$$

Integrating over B from $-\pi$ to π rather than on A means only, by (4), that we are integrating over a different unit cell in the configuration-space lattice (the lattice of the values of A and ϕ, not the space-time lattice). Since the integrand has the periodicity of that lattice (i.e., $A_{j,\underline{m}} \to A_{j,\underline{m}} + 2\pi N_{j,\underline{m}}$, $\phi_j \to \phi_j + 2\pi N_j$) the choice of the unit cell does not matter. Denote the ϕ_i integral in (6) by f, namely

$$f(B_{i,\underline{m}}, B_{i-\underline{m},\underline{m}}, J) = \int_{-\pi}^{\pi} d\phi_i \exp\Big\{J\sum_{\underline{m}}[\cos(B_{i,\underline{m}} - \phi_i) + \cos(B_{i-\underline{m},\underline{m}} + \phi_i)]\Big\} \cos(B_{i,\underline{n}} - \phi_i) . \quad (7)$$

Clearly, $f(B_{i,\underline{m}}, B_{i-\underline{m},\underline{m}}, 0) = 0$. Furthermore, for small enough J, f can be made arbitrarily small: $|f| < \epsilon$. Also, for any value of $B_{i,\underline{m}}, B_{i-\underline{m},\underline{m}}$, the inequality

$$\int_{-\pi}^{\pi} d\phi_i \exp\Big\{J\sum_{\underline{m}}[\cos(B_{i,\underline{m}} - \phi_i) + \cos(B_{i-\underline{m},\underline{m}} + \phi_i)]\Big\} \geq 2\pi e^{-2Jd}$$

is obeyed, so that

$$\frac{1}{2\pi} e^{2Jd}\int_{-\pi}^{\pi} d\phi_i \exp\left\{J\sum_{\underline{m}}\left[\cos(B_{i,\underline{m}}-\phi_i)+\cos(B_{i-\underline{m},\underline{m}}+\phi_i)\right]\right\} \geq 1 . \tag{8}$$

For small enough J, then, for any N, $W(J,N)$ can be bounded, by (8), as follows:

$$\begin{aligned}|W(J,N)| &= Z(J,N)^{-1}\int\prod_{j\neq i} d\phi_j \prod dB_{j,\underline{m}}\, e^{S(\phi,B)}\exp\left(J\sum_{\underline{m}}\sum_{j\neq i;\, j\neq i-\underline{m}}\cos B_{j,\underline{m}}\right)|f(B_{i,\underline{m}},B_{i-\underline{m},\underline{m}},J)|\\
&\leq \frac{\epsilon e^{2Jd}}{2\pi} Z(J,N)^{-1}\int\prod_{j\neq i} d\phi_j \prod dB_{j,\underline{m}}\, e^{S(\phi,B)}\exp\left(J\sum_{\underline{m}}\sum_{j\neq i;\, j\neq i-\underline{m}}\cos B_{j,\underline{m}}\right)\\
&\quad\times\int d\phi_i \exp\left\{J\sum\left[\cos(B_{i,\underline{m}}-\phi_i)+\cos(B_{i-\underline{m},\underline{m}}+\phi_i)\right]\right\}\\
&= \frac{\epsilon e^{2Jd}}{2\pi} Z^{-1}(J,N)\, Z(J,N)\\
&= \epsilon\frac{e^{2Jd}}{2\pi} .\end{aligned} \tag{9}$$

Thus, for small enough J $|\lim_{N\to\infty} W(J,N)| < \epsilon e^{2Jd}/2\pi$, which means that

$$\langle \cos A_{i,\underline{n}}\rangle = \lim_{J\to 0}\lim_{N\to\infty} W(J,N) = 0 .$$

An identical argument holds in the compact non-Abelian case, showing that any local quantity with vanishing mean value on its orbit under the action of the gauge group cannot develop a ground-state expectation value. One separates out some variables, call them x_j, to parametrize the orbit of the fields under the gauge transformation on a single site, i, taking the remaining variables to be invariant to these transformations. The volume remaining to be integrated over the fixed values of x_j does not depend on their actual values, since the gauge group preserves the measure in the configuration space. Also the action is x_j-independent being gauge-invariant. The main part of the external source term is x_j-invariant except for a few variables in the vicinity of the site i. The path integral, then, contains an integral over x_j involving only a few terms of the external source action times the quantity of interest. For $J=0$ this integral vanishes by assumption. For small enough J it can be made arbitrarily small even in the limit $N\to\infty$ depending only on few variables near i. So the path integral vanishes in the limit $J\to 0$.

II. BREAKING A GLOBAL SYMMETRY

To see the difference in the case of global symmetry, let us look at a similar model, an angle-variable field ϕ on a lattice without a gauge field. Let the action be

$$S = K\sum_{j,\underline{m}} V(\phi_j-\phi_{j+\underline{m}}) , \tag{10}$$

where V is some periodic function. The action is invariant under the global transformation

$$\phi_i \to \phi_i + C . \tag{11}$$

The expectation value of $\cos\phi_i$ for some site i is

$$\langle\cos\phi_i\rangle = \lim_{J\to 0}\lim_{N\to\infty} Z^{-1}\int\prod_j^N d\phi_j\, e^{S(\phi)} \times\exp\left(J\sum_j^N\cos\phi_j\right)\cos\phi_i , \tag{12}$$

with

$$Z = \int\prod_j^N d\phi_j\, e^{s(\phi)}\exp\left(J\sum_j^N\cos\phi_j\right),$$

As before we can separate out some variable, say, ϕ_0, and define

$$\phi_j' = \phi_j - \phi_0,\quad j\neq 0 . \tag{13}$$

All the primed variables are invariant under (11) except for ϕ_0. The action (10) is ϕ_0 independent, so

$$\langle\cos\phi_i\rangle = \lim_{J\to 0}\lim_{N\to\infty} Z^{-1}\int_{-\pi}^{\pi}\prod_{j\neq 0} d\phi_j'\, e^{S(\phi')} \times\int_{-\pi}^{\pi} d\phi_0 \exp\left[J\sum_0^N\cos(\phi_j'+\phi_0)\right] \times\cos(\phi_i'+\phi_0) . \tag{14}$$

Again denote

$$\int_{-\pi}^{\pi} d\phi_0 \exp\left[J \sum_{j}^{N} \cos(\phi_j' + \phi_0)\right] \cos(\phi_i' + \phi_0) = f_N(\phi_j, J). \tag{15}$$

The point is that now f depends on a macroscopic number of fields. It is still true that for any finite N there is small enough J for which f_N is bounded by an arbitrarily small number, but this bound depends on N, and the limit $N \to \infty$ must be taken before the limit $J \to 0$. For large N, and given J, there are configuration of $\{\phi_j\}$ for which $\sum \cos\phi_j'$ in (15) is very large, giving large value to $f_N(\phi_j, J)$. Such aligned configurations may get heavy weights in the path integral (14) for appropriate values of parameters in the action S, leading to spontaneously broken global symmetry.

III. THE BROKEN SYMMETRY IN THE MEAN-FIELD APPROXIMATION

In the mean-field approximation one gets a spontaneously broken gauge symmetry for our model (1). But this is only because, dealing in this approximation with a single site and link separately, one loses the information about gauge symmetry which involves several neighboring links at once. Indeed, in an appropriately modified version of the mean-field approximation which preserves that information one gets the conclusion that spontaneously broken gauge symmetry is impossible. In this modified version, rather than assuming every single site and link to interact with the mean value of the neighboring fields, we take every group of the field ϕ at an even site together with the $2d$ A fields at the links adjacent to this even site to interact with the mean fields at the neighboring links and odd sites (a site is even if the sum of its integral coordinates is even). Taking $\alpha = \langle \cos\phi_i \rangle$, $\beta = \langle \cos A_{i,\underline{n}} \rangle$, one gets the self-consistency equation

$$\beta = Z^{-1} \int \prod_{\underline{m}}^{d} (dA_{i,\underline{m}} \, dA_{i-\underline{m},\underline{m}}) \, d\phi_i \, e^{S(\phi_i, A_{i,\underline{m}}, A_{i-\underline{m},\underline{m}}, \alpha, \beta)} \cos A_{j,\underline{m}} \,, \tag{16}$$

where

$$Z = \int \prod_{\underline{m}} (dA_{i,\underline{m}} \, dA_{i-\underline{m},\underline{m}}) \, d\phi_i \, e^{S} \,,$$

$$S(\phi_j, A_{j,\underline{m}}, A_{j-\underline{m},\underline{m}}, \alpha, \beta) = K\alpha \sum_{\underline{m}} [\cos(\phi_j - A_{j,\underline{m}}) + \cos(\phi_j + A_{j-\underline{m},\underline{m}})]$$

$$+ \frac{\beta^2}{2g^2} \sum_{\underline{m} \neq \underline{l}} [\cos(A_{j,\underline{m}} - A_{j,\underline{l}}) + \cos(A_{j-\underline{m},\underline{m}} - A_{j-\underline{l},\underline{l}}) + 2\cos(A_{j,\underline{m}} + A_{j-\underline{l},\underline{l}})] \,. \tag{17}$$

As in (4) pass to the variables

$$B_{i\underline{m}} = A_{i,\underline{m}} - \phi_i \,, \quad B_{i-\underline{m},\underline{m}} = A_{i-\underline{m},\underline{m}} + \phi_i \,.$$

S depends only on $B_{i,\underline{m}}$, not on ϕ_i, and one gets

$$\beta = Z^{-1} \int_{-\pi}^{\pi} \prod (dB_{i,\underline{m}} \, dB_{i-\underline{m},\underline{m}}) \, e^{S(B)} \times \int_{-\pi}^{\pi} d\phi_i \cos(B_{i,\underline{n}} + \phi_i) = 0 \tag{16'}$$

for any value of K and g and any dimension d.

IV. CONCLUSION

In conclusion, breaking of local symmetry such as the Higgs phenomenon, for example, is always explicit, not spontaneous. The local symmetry must be broken first explicitly by a gauge-fixing term leaving only global symmetry. This remaining global symmetry can be broken spontaneously. Thus, in this aspect confinement of the fields to be angle variables is not a substitution for gauge fixing.[1,2] It must be emphasized, however, that the occurrence of spontaneous breaking and phase transition in the one-link mean-field approximation may be a good guide in assuming a possible phase transition in the true symmetrical theory. Action path integrals in the true theory are bounded by similar integrals in the mean-field approximation and discontinuities in the bound may indicate transitions in the true quantities, especially for high dimension.[2] What was proved here is only that such possible transitions in the true theory, unlike those in the one-link approximation, cannot be connected to spontaneous breaking of the local symmetry. To describe such transitions in the symmetrical theory, one has to look for more appropriate order parameters than the expectation value of a noninvariant quantity.

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